ISSN 1472-2739 (on-line) 1472-2747 (printed)

Algebraic & Geometric Topology
Volume 5 (2005) 1451–1469
Published: 30 October 2005

Hyperbolic covering knots

Daniel S. Silver
Wilbur Whitten

Abstract Given any knot k, there exists a hyperbolic knot $\tilde{k}$ with arbitrarily large volume such that the knot group πk is a quotient of $\pi\tilde{k}$ by a map that sends meridian to meridian and longitude to longitude. The knot $\tilde{k}$ can be chosen to be ribbon concordant to k and also to have the same Alexander invariant as k.

AMS Classification 57M25; 20F34

Keywords Alexander module, hyperbolic knot, ribbon concordance, tangle

1 Introduction

The classical problem of topology to find all homotopy classes of maps $M \to N$ between given complexes M and N has been variously expanded in recent years for the case in which M and N are manifolds of the same dimension; for an overview, see [27]. In the spirit of this expanded viewpoint as applied to knot theory, the authors in [23] showed that given any knot k, there exists infinitely many prime knots $\tilde{k}$ admitting an epimorphism of knot groups $\pi\tilde{k} \to \pi k$ sending a meridian-longitude pair for $\tilde{k}$ to a meridian-longitude pair for k. We make use of this result, and go further, proving that the knots $\tilde{k}$ can in fact be chosen to be hyperbolic with arbitrarily large volumes (see Theorem 2.2). The knots $\tilde{k}$ that we construct are ribbon concordant to k, and have the same Alexander invariant as k; in particular, they have the same Alexander polynomial.

E. Kalfagianni showed in [9] that given any positive integer n, there exists a hyperbolic knot with trivial Alexander polynomial, trivial finite type invariants of orders $\leq n$ and volume greater than n. Our result can be seen as a partial generalization.

We are grateful to Abhijit Champanerkar, Tim Cochran and Danny Ruberman for helpful discussions. The first author is partially supported by NSF grant DMS-0304971.

Note added in proof Professor A. Kawauchi has informed the authors that many of the results of this paper can be found in [10] or [11].

2 Statement of results

We denote the group $\pi_1(S^3\backslash\mathrm{Int}(V),*)$ of a knot $k\subset S^3$ by πk. Here $V\cong k\times D^2$ is a tubular neighborhood of k, and $*$ is a basepoint chosen on the boundary $\partial V\cong k\times S^1$. An essential simple closed curve in ∂V that is contractible in V is called a *meridian*, and it is denoted by m. An essential simple closed curve $l\subset\partial V$ that is nullhomologous in $S^3\setminus\mathrm{Int}(V)$ is called a *longitude*. Once k is oriented, both m and l acquire induced orientations. The inclusion map $\partial V\hookrightarrow S^3\setminus\mathrm{Int}(V)$ induces an injection of fundamental groups. Its image is the subgroup $\langle m,l\rangle$ generated by m and l.

Let $k_i (i=1,2)$ be knots with meridian-longitude pairs m_i,l_i.

Definition 2.1 A homomorphism $\phi:\pi k_1\to\pi k_2$ *preserves peripheral structure* if the image of $\langle m_1,l_1\rangle$ is conjugate to a subgroup of $\langle m_2,l_2\rangle$. When ϕ is an epimorphism, we write $k_1\succeq k_2$.

The relation $\succeq$ is a partial order [23]. After an appropriate choice of orientation, we can assume that $\phi(m_1)=m_2l_2^p$ and $\phi(l_1)=m_2^ql_2^r$, for some integers p,q,r. Since $m_2^ql_2^r$ must be in $(\pi k_2)''\cap Z(m_2)$ [8], we have $q=0$. Furthermore, since the normal subgroup of πk_2 generated by $m_2l_2^p$ is all of πk_2, Corollary 2 of [3] implies that $p\in\{0,1,-1\}$; in fact the recent proof that every nontrivial knot satisfies Property P [13] implies that $p=0$. Hence $\phi(m_1)=m_2$ and $\phi(l_1)=l_2^r$. When $r=1$, we write $k_2\succeq_1 k_1$. In [23] we showed that $k_1\succeq_1 k_2$ implies $k_1\succeq k_2$ but not conversely.

A *ribbon concordance* from a knot k_1 to another knot k_0 is a smooth concordance $C\subset\mathbb{S}^3\times I$ with $C\cap\mathbb{S}^3\times\{i\}=k_i(i=0,1)$, and such that the restriction to C of the projection $\mathbb{S}^3\times I\to I$ is a Morse function with no local maxima. Visualizing such a concordance by cross-sections, we see a sequence of saddle points (called fusions) and local minima (the result of shrinking to points unknotted, unlinked components). We do not see any local maxima.

The notion of ribbon concordance was introduced by C. Gordon [5], who wrote $k_1\geq k_0$ if there is a ribbon concordance from k_1 to k_0. The term was motivated by the fact that a knot k is ribbon concordant to the trivial knot if it bounds an immersed disk in $\mathbb{S}^3$ with only ribbon singularities. Gordon conjectured that $\geq$ is a partial order. The conjecture remains open. It is immediate from [16] that ribbon concordance does not imply $\succeq$, nor does $\succeq$ imply ribbon concordance.

Theorem 2.2 *Let k be a knot. There exists a hypberbolic knot $\tilde{k}$ with the following properties.*

(i) $\tilde{k} \succeq_1 k$;

(ii) *The Alexander invariants of $\tilde{k}$ and k are isomorphic;*

(iii) *$\tilde{k}$ has arbitrarily large volume;*

(iv) *$\tilde{k}$ is ribbon concordant to k.*

The 4-*ball genus* of a knot $k \subset \mathbb{S}^3 = \partial B^4$ is the minimum genus of any properly embedded surface $F \subset B^4$ bounding k.

Corollary 2.3 *Every Alexander polynomial is realized by hyperbolic knots with arbitrarily large volume and arbitrarily large 4-ball genus.*

Corollary 2.3 is proven using results of J. Rasmussen [21] and C. Livingston [15]. The statement of Corollary 2.3 was shown earlier by A. Stoimenow using more combinatorial methods.

3 Proof of Theorem 2.2

The idea for the proof Theorem 2.2 was suggested by [16]. The rough idea is as follows. First, we invoke [23] so that we may assume without loss of generality that k is prime. Having chosen a diagram for k with a minimal number of crossings, we introduce a carefully devised unknot (called a "staple") into a small neighborhood of each crossing. The greater part of the proof is devoted to showing that the resulting link is hyperbolic. Finally, we perform $1/q$ surgery on each of the staples. Thurston's hyperbolic surgery theorem implies that the resulting knots $\tilde{k}$ will be hyperbolic provided that the values of q are sufficiently large. The special form of the staples ensures that $\tilde{k}$ has the same abelian invariants as k.

The main result of [23] implies that there exists a prime knot $\tilde{k}$ such that $\tilde{k} \succeq_1 k$. In fact there are infinitely many. Hence we can assume without any loss of generality that k is prime.

Take a regular projection of k with a minimal number m of crossings. We may assume that k lies in the projection plane except near the crossings. Number the crossings $i = 1, \ldots, m$, and for each i, let B_i be a 3-ball that meets k in two subarcs t_{i_1} and t_{i_2} that form the ith crossing. Thus each $(B_i, t_{i_1} \cup t_{i_2})$, abbreviated by (B_i, t_i), is either the tangle $+1$ or -1, depending on the crossing (Figure 1). We also assume that each B_i meets the projection plane in an equatorial disk, and that $B_i \cap B_j = \emptyset$ when $i \neq j$. We assume that the balls B_i are chosen so that $k \setminus t_1 \cup \cdots \cup t_m$ is in the projection plane.

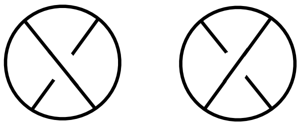

Figure 1: Tangle (B_i, t_i)

Next we insert an unknot γ_i in the interior of each $B_i \setminus k_i$, as in Figure 2. We refer to γ_i as a *staple*. We orient k in order to make the location of each staple specific. Note that (B_i, t_i, γ_i) is homeomorphic to (B_j, t_j, γ_j), for each i and j.

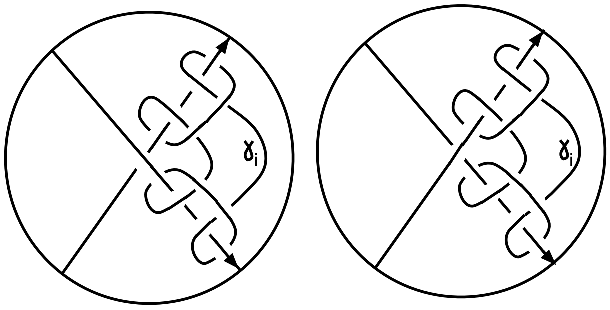

Figure 2: Tangle (B_i, t_i, γ_i)

The proof of Theorem 2.2 proceeds by a sequence of lemmas.

Lemma 3.1 *The link $L = k \cup \gamma_1 \cup \cdots \cup \gamma_m$ is unsplittable.*

Proof By construction, the sublink $\gamma_1 \cup \cdots \cup \gamma_m$ is trivial. It suffices to show that $k \cup \gamma_i$ is unsplittable, for each i.

It is convenient to have another view of (B_i, t_i, γ_i), obtained in the style of Montisenos by stretching ∂B_i into an "arc," as in Figure 3a. Figure 3b gives a view of the 2-fold cover of $B_i \setminus \gamma_i$ branched over t_i. It is a solid torus V_i minus the 2-component link $\tilde{\gamma}_i = \tilde{\gamma}_{i_1} \cup \tilde{\gamma}_{i_2}$. The program Snap shows that $\tilde{\gamma}_i$ is a hyperbolic link in V_i; that is, $\mathrm{Int}(V_i \setminus \tilde{\gamma}_i)$ is a hyperbolic 3-manifold.

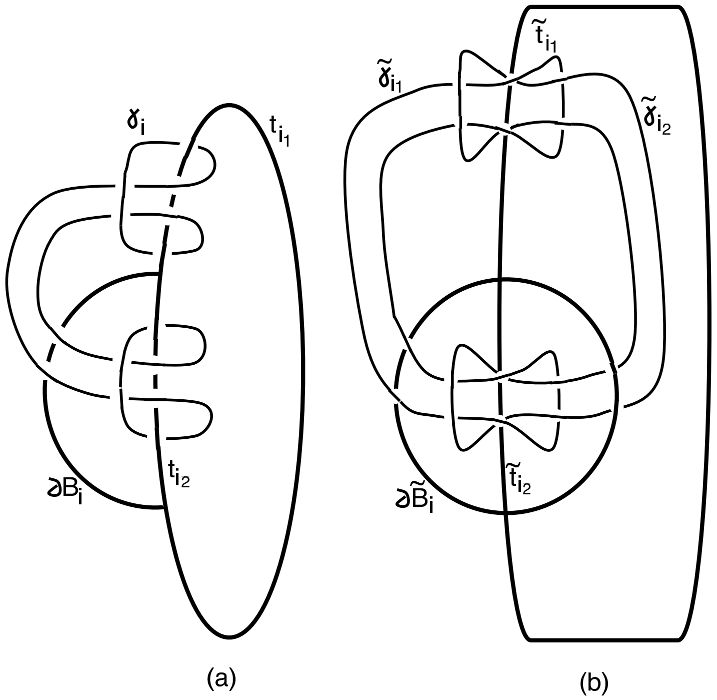

Figure 3: (a) Tangle (B_i, t_i, γ_i) (b) 2-Fold branched cover

If $k \cup \gamma_i$ is splittable, then there exists a 2-sphere S bounding a pair of 3-balls, one containing k, the other, which we call A, containing γ_i. Since each of B_i and A contains γ_i, their interiors intersect. Clearly B_i is not a subset of A, as B_i contains two subarcs of k. Therefore if A is not a subset of B_i, we can assume that $S \cap \partial B_i$ is a finite collection of pairwise disjoint simple closed curves. Let α be one of the curves that is innermost in S.

If α bounds a disk D in $S \cap \mathrm{cl}(S^3 \setminus B_i)$, then it also bounds a disk D' in ∂B_i that is in A, and since $D' \cap k = \emptyset$, the sphere $D \cup D'$ bounds a 3-ball not containing $k \cup \gamma_i$. Isotoping D through the ball, we can remove α without moving $k \cup \gamma_i$.

If, on the other hand, α bounds a disk $D \subset B_i$, then α also bounds a disk $D' \subset \partial B_i$ that contains no points of $t_i \cap \partial B_i$, since otherwise either $D \cap t_i \neq \emptyset$ or else D lifts to a pair of meridianal disks of V_i neither of which meets $\tilde{\gamma}_{i_1} \cup \tilde{\gamma}_{i_2}$. But $D \cap t_i = \emptyset$ by construction, and $\tilde{\gamma}_{i_1} \cup \tilde{\gamma}_{i_2}$ is essential in the 2-fold cover of B_i branched over t_i. Hence $D \cup D'$ bounds a 3-ball $A' \subset B_i \setminus t_i$. If $\gamma_i \subset A'$, then we push D' slightly into B_i and replace S by $D \cup D'$. If γ_i is not a subset of A', then we push D through A' into $\mathrm{cl}(S^3 \setminus B_i)$, and thereby eliminate α.

Inductively, we remove all curves of $S \cap \partial B_i$, and assume henceforth that S and hence A are contained in the interior of B_i. However, the lift of S to the 2-fold cover $V_i \setminus \tilde{\gamma}_i$ of $B_i \setminus \gamma_i$ branched over t_i is a pair of 2-spheres, each of which splits $\tilde{\gamma}_i = \tilde{\gamma}_{i_1} \cup \tilde{\gamma}_{i_2}$. Since $\mathrm{Int}(V_i \setminus \tilde{g}_i)$ is hyperbolic and hence irreducible, this is impossible. Therefore, $k \cup \gamma_i$ is unsplittable. □

Lemma 3.2 *The link $L = k \cup \gamma_1 \cup \cdots \cup \gamma_m$ is prime.*

Proof Let S be a 2-sphere that meets L transversely in exactly two points. The two points must belong to the same component of L. Suppose first that this component is a staple γ_i. Then S bounds a pair of 3-balls, one of which contains k. The other 3-ball, which we call A, contains an arc of γ_i, which must be unknotted as γ_i is trivial. It is not possible for A to contain another staple γ_j, $j \neq i$, since in that case S would split $k \cup \gamma_j$, thereby contradicting Lemma 3.1. Thus the ball A meets L in an unknotted spanning arc.

To complete the proof, we need to show that if the two points of $S \cap L$ belong to k, then S bounds a ball that intersects L in an unknotted spanning arc.

Suppose first that S is contained in the interior of some B_i. Then S bounds a 3-ball $A \subset B_i$ meeting t_i in a spanning arc of A. Since (B_i, t_i) is a trivial tangle, this spanning arc is unknotted. The lift of S to the 2-fold cover of $B_i \setminus \gamma_i$ branched over t_i is a 2-sphere bounding a 3-ball that projects to A, as $V_i \setminus \tilde{\gamma}_i$ is irreducible. Thus γ_i is not contained in A, and hence A meets L in an unknotted spanning arc.

If S is not in the interior of any 3-ball B_i, then we can assume that $S \cap (\partial B_1 \cup \cdots \cup \partial B_m)$ is a finite collection of pairwise disjoint simple closed curves in which S meets $\partial B_1 \cup \cdots \cup \partial B_m$ transversely. Our immediate goal is to show that we can move S without disturbing L setwise so that either S is contained in some B_i or else $S \cap (\partial B_1 \cup \cdots \cup \partial B_m) = \emptyset$.

Let α be a component of $S \cap (\partial B_1 \cup \cdots \cup \partial B_m)$ that is innermost in S. We can assume that $\alpha \subset \partial B_i$ and that α bounds a disk $D \subset S$ such that $D \cap B_j = \emptyset$, for $j \neq i$, and either $D \cap k = \emptyset$ or else $D \cap k$ is one of the two points of $S \cap k$. If $D \cap k = \emptyset$, then either $D \subset B_i$ or $D \subset \mathrm{cl}(S^3 \setminus B_i)$. In the first case, D can be moved off B_i, as L is not splittable and t_{i_1} and t_{i_2} are not separated by D in $B_i \setminus \gamma_i$. In the second case, α also bounds a disk $D' \subset \partial B_i$ such that the cardinality $|D' \cap k|$ is $0, 1$ or 2. If $|D' \cap k| = 0$, then the sphere $D \cup D'$

bounds a 3-ball A such that $A \cap L = \emptyset$, since L is unsplittable or equivalently $\mathbb{S}^3 \setminus L$ is irreducible, and we can therefore push D into B_i and thereby remove α without moving L. The case $|D' \cap k| = 1$ cannot occur, since $D \cap k = \emptyset$. If $|D' \cap k| = 2$, then $D \cup D'$ bounds a 3-ball outside $\mathrm{Int}(B_i)$ containing an arc of k and perhaps some of the balls B_j. This implies, however, that the crossing of k in B_i is nugatory, contradicting minimality of the projection of k. Hence $|D' \cap k| = 2$ also cannot occur.

Assume now that $D \cap k$ is one point, and recall that $\partial D = \alpha \subset \partial B_i$. Then α bounds a disk $D' \subset \partial B_i$ meeting k in one point.

If $D \subset B_i$, then $D \cup D'$ bounds a 3-ball $A \subset B_i$ meeting k in a spanning arc. Since (B_i, t_i) is a trivial tangle, the arc is unknotted. The irreducibility of the 2-fold cover of $B_i \setminus \gamma_i$ branched over t_i implies that γ_i is not a subset of A. Hence we can isotop D through A to remove α while keeping L setwise fixed.

If $D \subset \mathrm{cl}(\mathbb{S}^3 \setminus B_i)$, then the fact that D is an innermost disk (with $\partial D = \alpha$) in S implies that $D \cap B_j = \emptyset$, for all $j \neq i$, and hence $D \cap k$ is a point in the projection plane. Let A dnote the 3-ball in $\mathbb{S}^3$ with $\partial A = D \cup D'$ and $\mathrm{Int}(B_i)$ not a subset of A. If A contains any B_j, $j \neq i$, then we can move D' slightly off B_i while keeping k setwise fixed to obtain a 2-sphere $D \cup D'$ such that $(D \cup D') \cap (\partial B_1 \cup \cdots \cup \partial B_m) = \emptyset$ and such that $D \cup D'$ bounds two 3-balls each of which contains at least one of the balls $B_1, \ldots, B_m$. As we will see shortly, this cannot occur, and so $D \cap k$ is a point in one of the four planar arcs of k protruding from B_i. These arcs are unknotted by construction, and no staple γ_j or ball B_j is now in A. Hence we can push D back into B_i and remove α, again while keeping L setwise fixed.

We can, therefore, assume that either S is contained in some B_i or $S \cap (B_1 \cup \cdots \cup B_m) = \emptyset$. As we have seen, if S is in some B_i, then S bounds a 3-ball in B_i meeting L in an unknotted spanning arc. So assume that $S \cap (B_1 \cup \cdots \cup B_m) = \emptyset$. Let A_1 and A_2 be the two 3-balls bounded by S. Since k is prime, one of A_1 and A_2, say A_2, meets k in an unknotted spanning arc b of A_2.

Assume that S is in general position with respect to the projection plane P of L. Since the general position isotopy of S can be chosen to fix the two points x_1 and x_2 of $S \cap k$, we can assume that S meets P in a simple closed curve containing x_1, x_2 together with a collection of simple closed curves bounding disks in S. Since we can also assume that S meets a tubular neighborhood N of k (see proof of Lemma 3.3) in two disks, the disks in S bounded by the latter curves belong to the handlebody $\mathrm{cl}(\mathbb{S}^3 \setminus \mathrm{cl}[(\cup_{i=1}^m B_i) \cup N])$, and thus the curves themselves can be removed by cut and paste arguments. Hence there is a simple arc $\beta \subset P \cap S$ with $\partial\beta = \{x_1, x_2\}$ and a subarc α of k

such that $k = (\alpha \cup \beta)\sharp(\beta \cup b)$, where $\beta \cup b$ is an unknot, and k is ambient isotopic to $\alpha \cup \beta$. Since the projection of k in P has a minimal number of crossings m (equal to the crossing number of k), so does $\alpha \cup \beta$, and so $A_1 \supset B_1 \cup \cdots \cup B_m \supset \gamma_1 \cup \cdots \cup \gamma_m$. Therefore, $b \subset P$ and $A_2 \cap L = b$. $\square$

Lemma 3.3 *The link $L = k \cup \gamma_1 \cup \cdots \cup \gamma_m$ is hyperbolic.*

Proof Let N be a tubular neighborhood of k in $S^3 \setminus (\gamma_1 \cup \cdots \cup \gamma_m)$, and let N_i be a tubular neighborhood of γ_i, $i = 1, \ldots, m$, such that $N, N_1, \ldots, N_m$ are pairwise disjoint and $N_i \subset \text{Int}(B_i)$, for each i. We also assume that $N \cap \partial B_i$ is a collection of four meridianal disks of N, for each i. Set $\text{Ext}(L) = \text{cl}(\mathbb{S}^3 \setminus (N \cup N_1 \cup \cdots \cup N_m))$. With $\tilde{\gamma}_i = \tilde{\gamma}_{i_1} \cup \tilde{\gamma}_{i_2}$ $(i = 1, \ldots, m)$, the trivial link $\gamma_1 \cup \cdots \cup \gamma_m$ lifts to a $2m$-component link in the 2-fold cover M_2 of k, and each N_i lifts to a pair of tubular neighborhoods, $\tilde{N}_{i_1}$ and $\tilde{N}_{i_2}$, of $\tilde{\gamma}_{i_1}$ and $\tilde{\gamma}_{i_2}$, respectively, in M_2. Clearly, $\tilde{N}_{i_1} \cap \tilde{N}_{i_2} = \emptyset$ and $\tilde{N}_{i_1} \cup \tilde{N}_{i_2}$ is contained in the 2-fold cover of B_i branched over t_i, which is in M_2. We set $M = \text{Ext}(\tilde{\gamma}_1 \cup \cdots \cup \tilde{\gamma}_m) = \text{cl}(M_2 \setminus \cup_{i=1}^m (\tilde{N}_{i_1} \cup \tilde{N}_{i_2}))$, which can be shown to be irreducible by a straightforward application of Lemma 3.2 and the $\mathbb{Z}_2$ sphere theorem [12]. Since each of $\text{Ext}(L)$ and M is an irreducible (in fact, a Haken) 3-manifold that has torus boundary components and is not a solid torus, it is a standard fact that each of them has incompressible boundary.

To see that L is hyperbolic, we need to show that $S^3 \setminus L$ is not a Seifert fibered space and that every incompressible torus is $\text{Ext}(L)$ is boundary parallel [26]. That $\mathbb{S}^3 \setminus L$ is not Seifert fibered follows from [2], which yields a geometric description of the unsplittable links in $\mathbb{S}^3$ with Seifert fibered complements. Each component of such a link can be chosen to be a fiber of some Seifert fibration of $\mathbb{S}^3$. In particular, our link L has four or more components, so if $\mathbb{S}^3 \setminus L$ is Seifert fibered, then either (1) each component of L is unknotted; or (2) one or two components are unknotted and each of the remaining components is a nontrivial torus knot (of a given fixed type (α, β)); or (3) all components are nontrivial torus knots of the same type. Since L has exactly one knotted component but three or more unknotted components, it follows that $\mathbb{S}^3$ is not Seifert fibered.

We show now that $\text{Ext}(L)$ is atoroidal, by which we mean that every incompressible torus in $\text{Ext}(L)$ is boundary parallel. (Our argument was suggested by that of Case 3 in the proof of Theorem 2 of [6].) Suppose first that a torus $T \subset \text{Ext}(L)$ is incompressible but not boundary parallel and that $T \subset \text{Int}[B_i \setminus (t_i \cup \gamma_i)]$, for some i. Then the lift $\tilde{T}$ of T to $V_i \setminus \tilde{\gamma}_i$ is either one or two tori. Since $V_i \setminus \tilde{\gamma}_i$ is hyperbolic (and thus atoroidal), there is a compressing disk $\tilde{D}$ for $\tilde{T}$ in $V_i \setminus \tilde{\gamma}_i$ such that $g(\tilde{D}) \cap \tilde{D} = \emptyset$, or $g(\tilde{D}) = \tilde{D}$ and $\tilde{D}$ meets the fixed point set $\tilde{t}_i$ of the involution g transversely in a single point [12] (see also Theorem 3 of [6]). Let D denote the image of $\tilde{D}$ under the projection map $V_i \setminus \tilde{\gamma}_i \to B_i \setminus \gamma_i$. If $g(\tilde{D}) \cap \tilde{D} = \emptyset$, then the disk D compresses T in $\mathrm{Int}[B_i \setminus (t_i \cup \gamma_i)]$, which is a contradiction. If, however, $g(\tilde{D}) = \tilde{D}$, then the disk D meets t_{i_1} or t_{i_2} – say t_{i_1} – transversely in a single point. We then split T along D to obtain a 2-sphere S meeting t_{i_1} in two points. As was shown in the proof of Lemma 3.2, S bounds a 3-ball A in $B_i \setminus \gamma_i$ meeting t_{i_1} in a spanning arc of A. It is now clear that T itself must bound the exterior of a nontrivial knot in $B_i \setminus \gamma_i$, since T is incompressible. This, however, implies that t_{i_1} is a knotted arc, which is a contradiction. Hence T is not contained in $\mathrm{Int}[B_i \setminus (t_i \cup \gamma_i)]$, for any i.

On the other hand, the incompressible torus $T \subset \mathrm{Ext}(L)$ is also not in $\mathrm{cl}[\mathbb{S}^3 \setminus (N \cup \cup_{i=1}^m B_i]$, as this is clearly a handlebody ($\neq \mathbb{S}^1 \times D^2$).

Thus we can assume that $T \cap (\partial B_1 \cup \cdots \cup \partial B_m)$ is a finite collection of disjoint simple closed curves along which T and $\partial B_1 \cup \cdots \cup \partial B_m$ meet transversely. Let α be one of these curves, on B_i say.

If α is homotopically trivial on T, then it bounds a disk $D \subset T$, and we can assume that α is innermost on T in the sense that there is no curve α' in $T \cap (\partial B_1 \cup \cdots \cup \partial B_m)$ such that $\alpha' \subset \mathrm{Int}(D)$. Note that $\alpha \cap \partial(t_{i_1} \cup t_{i_2}) = \emptyset$ and that D is properly imbedded in $B_i \setminus (t_{i_1} \cup t_{i_2} \cup \gamma_i)$ or in $\mathrm{cl}(\mathbb{S}^3 \setminus B_i)$.

Case 1 $D \subset B_i \setminus (t_{i_1} \cup t_{i_2} \cup \gamma_i)$ In this case, the disk D lifts to a pair of disks $\tilde{D}_1$ and $\tilde{D}_2$ in $V_i \setminus \tilde{\gamma}_i$, each of which is properly imbedded with $\partial \tilde{D}_j \subset \partial V_i$ and $\tilde{D}_j \cap (\tilde{\gamma}_i \cup \tilde{t}_i) = \emptyset$ ($j = 1, 2$ and i fixed). Since moreover ∂V_i is incompressible in $V_i \setminus \tilde{\gamma}_i$, it follows that $\partial \tilde{D}_1$ and $\partial \tilde{D}_2$ (the lifts of α) bound disks $\tilde{D}'_1$ and $\tilde{D}'_2$, respectively, in ∂V_i such that $\tilde{D}'_j \cap \partial \tilde{t}_i = \emptyset$ ($j = 1, 2$). The projection of $\tilde{D}'_1 \cup \tilde{D}'_2$ is a disk $D' \subset \partial B_i$ such that $D' \cap \partial(t_{i_1} \cup t_{i_2}) = \emptyset$ and $\partial D' = \alpha$, and so $D \cup D'$ bounds a 3-ball A in B_i such that $A \cap L = \emptyset$, since $\mathbb{S}^3 \setminus L$ is irreducible. Thus we can isotop T to remove α.

Case 2 $D \subset \mathrm{cl}(\mathbb{S}^3 \setminus B_i)$ The curve α bounds two disks $D_1, D_2 \subset \partial B_i$ such that $D_1 \cap D_2 = \alpha$ and $D_1 \cup D_2 = \partial B_i$. If each of $\mathrm{Int}(D_1)$ and $\mathrm{Int}(D_2)$ contains a point of $\partial(t_{i_1} \cup t_{i_2})$, then the minimal number of points in either disk is one or two. Since D contains no points of k, however, this minimal number clearly must be two, and since $|\partial(t_{i_1} \cup t_{i_2})| = 4$, each of $\mathrm{Int}(D_1)$ and $\mathrm{Int}(D_2)$ must therefore contain two points of k. Using D_1, say, it follows that $D \cup D_1$ is a 2-sphere meeting L in two points of k. Since L is prime, $D \cup D_1$ bounds a

3-ball meeting L in an unknotted arc b, a subarc of k. Considering B_i, this implies that either k consists of two components or the crossing of t_{i_1} and t_{i_2} in B_i is nugatory. Since neither of these is possible, one of D_1 and D_2 must miss $\partial(t_{i_1} \cup t_{i_2})$, say D_1. By irreducibility of $\mathbb{S}^3 \setminus L$, it follows that $D \cup D_1$ bounds a 3-ball $A \subset \mathrm{cl}(\mathbb{S}^3 \setminus B_i)$ such that $A \cap L = \emptyset$, and we can move T to eliminate α.

Application of Cases 1 and 2 can be used to remove all other curves in $T \cap (\partial B_1 \cup \cdots \cup \partial B_m)$ that are homotopically trivial in T without disturbing the remaining curves. We therefore assume now that $T \cap (\partial B_1 \cup \cdots \cup \partial B_m)$ is a collection of homotopically nontrivial curves in T, which must of course be parallel. If this collection is empty, then T is either in some B_i or else T is in the handlebody $\mathrm{cl}[\mathbb{S}^3 \setminus (N \cup \cup_{i=1}^m B_i)]$. Clearly then, a pair of curves, α_1 and α_2, in $T \cap (\partial B_1 \cup \cdots \cup \partial B_m)$ must bound an annulus F in T with $F \subset B_i \setminus (t_i \cup \gamma_i)$, for some i, and no α' in $T \cap (\partial B_1 \cup \cdots \cup \partial B_m)$ is contained in $\mathrm{Int}(F)$. We now show that either F bounds a tubular neighborhood of t_{i_1} or t_{i_2} in $B_i \setminus \gamma_i$ or else F can be slightly isotoped off B_i.

The curves α_1 and α_2 bound disjoint disks D_1 and D_2, respectively, in ∂B_i, and $\mathrm{Int}(D_j) \cap \partial(t_{i_1} \cup t_{i_2}) \neq \emptyset$ $(j = 1, 2)$. Since $|\mathrm{lk}(k, \alpha_1)| = |\mathrm{lk}(k, \alpha_2)|$ and $|\partial B_i \cap \partial(t_{i_1} \cup t_{i_2})| = 4$, there are three possible cases, two of which we combine into Case (b).

Case (a) $|\mathrm{Int}(D_j) \cap \partial(t_{i_1} \cup t_{i_2})| = 1$ $(j = 1, 2)$ Since $D_1 \cup F \cup D_2$ is a 2-sphere S, it is clear that each of $\mathrm{Int}(D_1)$ and $\mathrm{Int}(D_2)$ contains an endpoint of the same arc t_{i_1}, say. Isotoping S into B_i, it follows that S bounds a 3-ball A in $B_i \setminus \gamma_i$ meeting t_{i_1} in an unknotted spanning arc of A (as in the proof of Lemma 3.2.) Isotoping S back to its original position, it follows that F is boundary parallel. (Recall that we began with the original assumption that $T \subset \mathrm{Ext}(L)$.)

Case (b) Either $|\mathrm{Int}(D_j) \cap \partial(t_{i_1} \cup t_{i_2})| = 2$ $(j = 1, 2)$, or $|\mathrm{Int}(D_1) \cap \partial(t_{i_1} \cup t_{i_2})| = 1$ and $|\mathrm{Int}(D_2) \cap \partial(t_{i_1} \cup t_{i_2})| = 3$. (In the second possiblity, the disks' numbering can be switched.)

Let F' denote the annulus $\mathrm{cl}[\partial B_i \setminus (D_1 \cup D_2)]$, and isotop the torus $F \cup F'$ slightly into $\mathrm{Int}[B_i \setminus (t_i \cup \gamma_i)]$ without moving L setwise. As we have seen, the image torus must be compressible in $\mathrm{Int}[B_i \setminus (t_i \cup \gamma_i)]$. Now there exist knot exteriors A_1 and A_2 (at least one of which is a solid torus) such that $\mathbb{S}^3 = A_1 \cup A_2$ with $A_1 \cap A_2 = F \cup F'$. One of A_1 and A_2 (say A_1) is in $\mathrm{Int}[B_i \setminus (t_i \cup \gamma_i)]$; suppose that A_1 is the exterior of a nontrivial knot k', that is, suppose that A_1 is not a solid torus. Then the compressing disk D of $F \cup F'$ in

$\mathrm{Int}[B_i \setminus (t_i \cup \gamma_i)]$ is properly imbedded in A_2. The boundary ∂D is not parallel to α_1 (or to α_2) in $F \cup F'$, since each of α_1 and α_2 represents a nontrivial element of πL (see Case 2). If $(\partial D, \ell')$ is a meridian-longitude pair for k' (with $\{\partial D, \ell'\} \subset \partial A_1 = F \cup F'$), it follows that α_1 represents an element of $\pi k' (= \pi A_1)$ of the form $(\partial D)^p (\ell')^q$, where $p, q \in \mathbb{Z}$ with $q \neq 0$. This means, however, that as a simple closed curve in $\mathbb{S}^3$, α_1 must be knotted. But α_1 bounds a compressing disk for T in $\mathbb{S}^3$, and we have a contradiction. Hence A_1 is a solid torus. Moving $F \cup F'$ back to its original position, we can thus isotop F through A_1 off B_i without disturbing L, since α_1 and α_2 are unknotted in $\mathbb{S}^3$.

Applying Cases (a) and (b) to $T \cup (B_1 \cup \cdots \cup B_m)$, we can assume that $T \cup (B_1 \cup \cdots \cup B_m)$ is empty except when Case (a) holds for some collection $B_{i_1}, \ldots, B_{i_r}$ $(1 \leq r \leq m)$. If $T \cup (B_1 \cup \cdots \cup B_m) = \emptyset$, then T is in the handlebody $\mathrm{cl}[\mathbb{S}^3 \setminus (N \cup \cup_{i=1}^m B_i)]$, which is a contradiction, since T is incompressible in $\mathrm{Ext}(L)$. Thus we assume that, for some i, T meets B_i in an annulus F that is boundary parallel (in B_i) to ∂N. The following proposition will enable us to conclude the proof of the lemma.

Proposition 3.4 *Let $\beta = \beta_1 \cup \cdots \cup \beta_n$ be a prime link in $\mathbb{S}^3$ of n components, and let T be a torus imbedded in $\mathbb{S}^3 \setminus \beta$. Suppose that D is a compressing disk for T (in $\mathbb{S}^3$) meeting β transversely in a single point. Then either β is contained in one component of $\mathbb{S}^3 \setminus T$ or else T bounds a tubular neighborhood of β_i, for some i.*

Proof Assume that $D \cap \beta = D \cap \beta_1$ is the single point of transverse intersection. Assume also that β is not contained in one component of $\mathbb{S}^3 \setminus T$. If some of $\beta_2, \ldots, \beta_n$ are contained in each component, then we surger T along D to obtain a splitting 2-sphere S for β (Figure 4), contradicting primality.

Assume now that $\beta_2 \cup \cdots \cup \beta_n$ lies in the component of $\mathbb{S}^3 \setminus T$ not containing β_1 (Figure 5(a)). As in the previous case, surger T along D to obtain a 2-sphere S (Figure 5(b)). Let B be the 3-ball with boundary S that does not contain $\beta_2 \cup \cdots \cup \beta_n$. By primality of β, the 1-tangle $(B, B \cup \beta_1)$ must be trivial. Regard the neighborhood of D removed in surgery as a 1-handle h with core equal to the part of β_1 not contained in B. It is easy to arrange for h to miss $\beta_2 \cup \cdots \cup \beta_n$, since the disk D does not intersect it. Now $B \cup h$ is a solid torus V bounded by T. Moreover, the product structure on h extends over B so that β_1 is the core of V (Figure 6). □

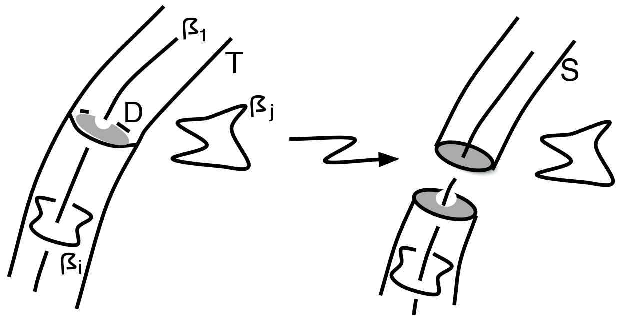

Figure 4: Splitting 2-sphere S

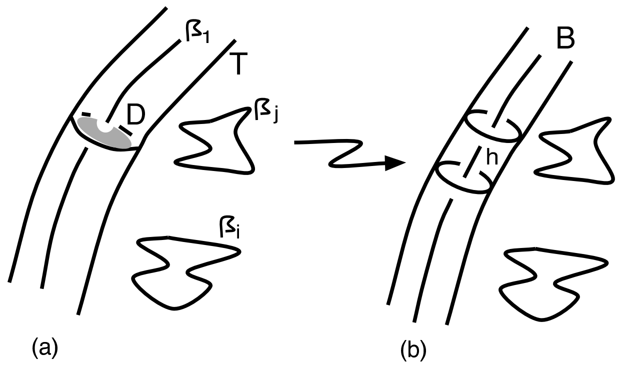

Figure 5: Surgery on T

Continuing with the proof of Lemma 3.3, we have $T \cap B_i = F$, which is boundary parallel to the tubular neighborhood N of k. The boundary ∂F is a pair of unknotted curves, α_1 and α_2, bounding disks D_1 and D_2 in ∂B_i, which are compressing disks for T, each meeting k transversely in one point. If $T \cap B_j = \emptyset$, for some $j \neq i$, then B_j is contained in a component U_1 of $\mathbb{S}^3 \setminus T$. Hence $k \cup \gamma_j \subset U_1$, and by Proposition 3.4, $L \subset U_1$. But if U_2 denotes the other component of $\mathbb{S}^3 \setminus T$, it is clear that $B_i \cap U_2 \neq \emptyset$ and, moreover, that $\gamma_i \subset U_2$. Thus $T \cap B_j \neq \emptyset$, for all j, and T is boundary parallel. Therefore $\mathrm{Ext}(L)$ is atoroidal, and the proof of Lemma 3.3 is complete. □

Since γ_1 is unknotted in $\mathbb{S}^3$ and represents the trivial element in πk, a $1/q_1$-surgery on γ_1 changes k into a knot k_1 such that $k_1 \succeq_1 k$. Now, $\gamma_2 \subset B_2$, and

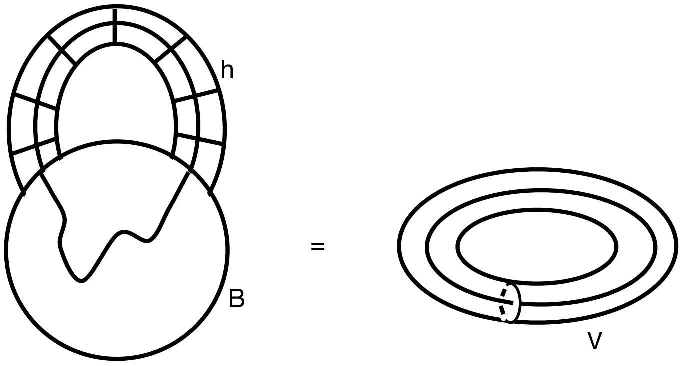

Figure 6: $B \cup h$ seen as solid torus

the $1/q_1$-surgery on γ_1 can be regarded as a $(-q_1)$-twist on a disk $D_1 \subset B_1$ that is transverse to k such that $\partial D_1 = \gamma_1$ and $D_1 \cap k$ is a set of four points. Thus since $B_1 \cap B_2 = \emptyset$, it follows that γ_2 represents the trivial element of πk_1, and hence that a $1/q_2$-surgery on γ_2 changes k_1 into a knot k_2 such that $k_2 \succeq_1 k_1$. Continuing this process, we arrive at the mth stage, in which we do $1/q_m$-surgery on γ_m. This changes k_{m-1} into a knot k_m such that $k_m \succeq_1 k_{m-1}$. Thus

$$k_m \succeq_1 k_{m-1} \succeq_1 \cdots \succeq_1 k_1 \succeq_1 k,$$

and so $k_m \succeq_1 k$. By Thurston's hyperbolic surgery theorem [25], excluding all but a finite number of possible values of $q_i \in \mathbb{Z}$ for each i assures that k_m is hyperbolic. Hence statement (i) of Theorem 2.2 is proved.

In order to prove statement (ii) we observe that the staples γ_i bound pairwise disjoint ribbon disks in the complement of k (Figure 7). The disks can be lifted to the infinite cyclic cover of k, and since any two lifts meet only in ribbon singularities, it follows that each γ_i represents an element of the second commutator subgroup of πk. Hence $1/q$-surgery on γ_i will not change the Alexander invariant (see Lemma 2 of [18]).

Next we prove statement (iii). Let k_0 be a hyperbolic knot with trivial Alexander polynomial. Consider the connected sum $k' = k\sharp k_0 \sharp \cdots \sharp k_0$ of k with N copies of k_0, where N is an arbitrary positive number. By [23] there exists a prime knot k'' such that $k'' \succeq_1 k'$. A proper degree-1 map can be constructed from $\mathrm{Ext}(k'')$ to $\mathrm{Ext}(k')$, and hence by [7] the simplicial volume of k'' is no less than the simplicial volume of k'. However, the simplicial volume of k' is

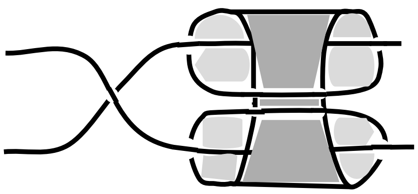

Figure 7: Ribbon disk bounded by staple

at least N times that of k_0, which is greater than zero. Consequently, the simplicial volume of k'' can be made arbitrarily large by choosing N sufficiently large. By part (i) of Theorem 2.2, we can find a hyperbolic knot $\tilde{k}$ such that $\tilde{k} \succeq_1 k''$. As before, the simplicial volume of $\tilde{k}$ is at least as large as that of k'', and hence the hyperbolic volume of $\tilde{k}$ can be made arbitrarily large.

By [23] and part (ii) of Theorem 2.2, the knots k', k'' and $\tilde{k}$ have the same Alexander invariants. Since k and k' have isomorphic Alexander invariants, so do $\tilde{k}$ and k.

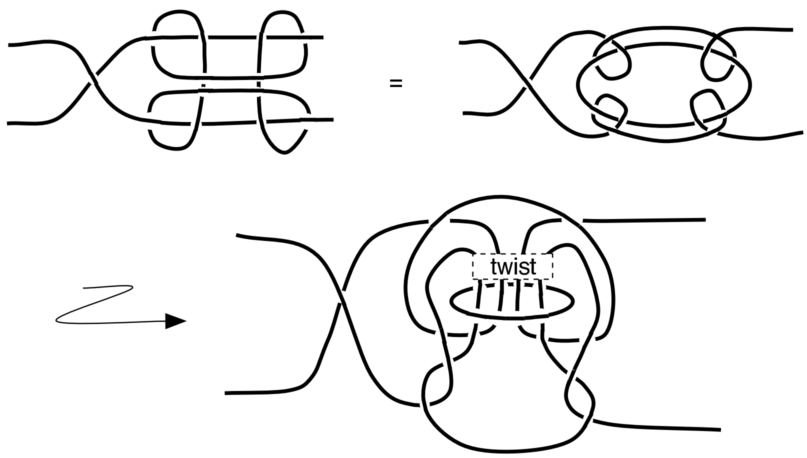

Figure 8: Twisting about the staple

Finally we prove statement (iv). The key idea is that $1/q$-surgery on any staple γ converts any knot k to a knot that is ribbon concordant to k. This is immediately seen in Figures 8 and 9. In Figure 8, we see the staple redrawn so that it bounds an obvious 2-disk. We perform $1/q$-surgery by cutting, twisting $-q$ full times and reconnecting the strands of k that pass through the disk.

Figure 9 shows how a pair of fusions produces two unknotted, unlinked circles that can be shrunk to points. Hence the knot produced from k by surgery is ribbon concordant to k.

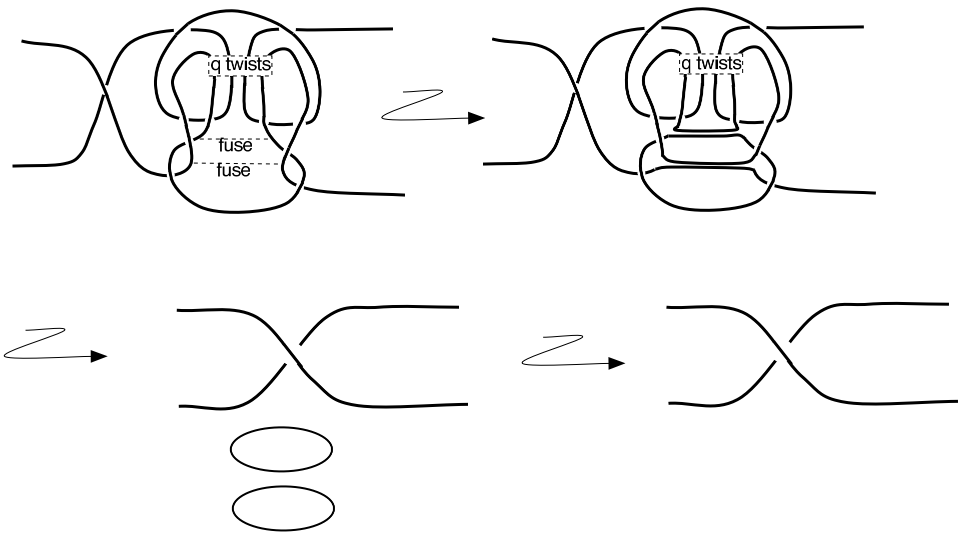

Figure 9: Ribbon fusions recovering k

Recall that we began the proof of Theorem 2.2 by appealing to the main result of [23]. There we began with any knot k, and produced a prime knot by surgery on an unknot C that is not a staple. We complete the proof of Theorem 2.2 (iv) by showing that in fact C can be taken to be a staple.

According to Proposition 2.5 of [4], we can consider k as the numerator closure T^N of a tangle T that is either prime or rational. Form the 2-component link $L = k \cup \gamma$ (Figure 10).

Let (B, t, γ) be any tangle, where B is a 3-ball, t is a finite collection of disjoint, properly embedded spanning arcs of B, and γ is a finite collection of disjoint simple closed curves in $\mathrm{Int}(B \setminus t)$ such that $t \neq \emptyset$. Following [20] and [1], we will say that (B, t, γ) is *prime* if it has the following properties.

(i) (No connected summand) Each 2-sphere in B intersecting $t \cup \gamma$ transversely in two points bounds a 3-ball in B that meets $t \cup \gamma$ in an unknotted spanning arc.

(ii) (Disk inseparable) No properly embedded disk in $B \setminus (t \cup \gamma)$ separates $t \cup \gamma$.

(iii) (Indivisible) Any properly embedded disk D in B such that $D \cap \gamma = \emptyset$ and such that D meets exactly one component of t transversely in a single

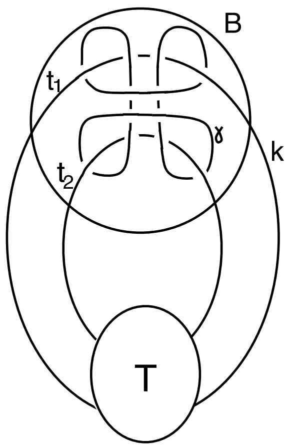

Figure 10: 2-component link $L = k \cup \gamma$

point divides (B, t, γ) into two tangles $(B_1, t', \emptyset)$ and (B_2, t'', γ) such that t' has only one component and that component is unknotted.

Lemma 3.5 *The tangle (B, t, γ) in Figure 10 is prime, where $t = t_1 \cup t_2$.*

Proof Form the denominator closure B^D. According to the program Snap, a computer program developed at Melbourne University for studying arithmetic invariants of hyperbolic 3-manifolds (http://www.ms.unimelb.edu.au/ snap/), B^D is a hyperbolic link. Hence (B, t, γ) has no connected summand since otherwise B^D would have a connected summand.

Furthermore, (B, t, γ) is disk inseparable since the 2-fold cover $V \setminus \tilde{\gamma}$ of $B \setminus \gamma$ branched over t is hyperbolic. A properly embedded disk in $B \setminus (t \cup \gamma)$ lifts to two disks in $V \setminus (\tilde{t} \cup \tilde{\gamma})$, each of which forms a 2-sphere with a corresponding disk in ∂V that bounds a 3-ball in V missing $\tilde{t} \cup \tilde{\gamma}$. Each of these balls projects to the same 3-ball in $B \setminus (t \cup \gamma)$.

According to Proposition 1.5 of [19], any tangle that has no connected summand, is disk inseparable, and has at most two spanning arcs is prime. Hence (B, t, γ) is prime. □

Lemma 3.6 *The link $L = k \cup \gamma$ is prime.*

Proof Since (B, t, γ) is prime, this follows immediately from Theorem 1.10 of [19] if T is a prime tangle. If T is rational, then we can replace it with a prime tangle T_1 such that $T_1^N = k$. The tangle T_1 is obtained as a partial sum of T with the prime tangle T_2 as shown in Figure 11. It follows from Theorem 3 of [14] that T_1 is prime, since T_2 is prime. Hence again L is prime. □

The remaining argument of [23] applies now, completing the proof of Theorem 2.2 (iv).

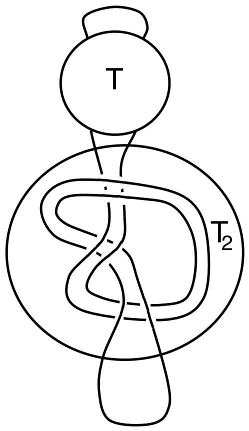

Figure 11: The knot k as the numerator closure of T_1

Proof of Corollary 2.3 Let k_0 be the untwisted double of a trefoil. Corollary 5 and Theorem 1 of [15] together imply that the 4-ball genus of the connected sum $k\sharp k_0 \sharp \cdots \sharp k_0$ can be made arbitrarily large by increasing the number of summands k_0. (The results of [15] are convenient for us, but earlier work of Rudolph [22] could be used instead.) We replace k by $k\sharp k_0 \sharp \cdots \sharp k_0$, which has the same Alexander invariant, and apply Theorem 2.2. Since the resulting knot $\tilde{k}$ is (ribbon) concordant, the two knots have the same 4-ball genus. □

References

[1] **S A Bleiler**, *Knots prime on many strings*, Trans. Amer. Math. Soc. 282 (1984) 385–401

[2] **G Burde**, **K Murasugi**, *Links and Seifert fiber spaces*, Duke Math. J. 37 (1970) 89–93

[3] **M Culler**, **C M Gordon**, **J Luecke**, **P B Shalen**, *Dehn surgery on knots*, Ann. of Math. (2) 125 (1987) 237–300

[4] **S Eliahou**, **L H Kauffman**, **M B Thistlethwaite**, *Infinite families of links with trivial Jones polynomial*, Topology 42 (2003) 155–169

[5] **C M Gordon**, *Ribbon concordance of knots in the 3-sphere*, Math. Ann. 257 (1981) 157–170

[6] **C M Gordon**, **R A Litherland**, *Incompressible surfaces in branched coverings*, from: "The Smith conjecture (New York, 1979)", Pure Appl. Math. 112, Academic Press, Orlando, FL (1984) 139–152

[7] **M Gromov**, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. (1982) 5–99 (1983)

[8] **D Johnson**, **C Livingston**, *Peripherally specified homomorphs of knot groups*, Trans. Amer. Math. Soc. 311 (1989) 135–146

[9] **E Kalfagianni**, *Alexander polynomial, finite type invariants and volume of hyperbolic knots*, Algebr. Geom. Topol. 4 (2004) 1111–1123

[10] **A Kawauchi**, *Almost identical link imitations and the skein polynomial*, from: "Knots 90 (Osaka, 1990)", Walter de Gruyter and Co. Berlin (1992) 465–476

[11] **A Kawauchi**, *Almost identical imitations of (3,1)-dimensional manifold pairs*, Osaka J. Math 26 (1989) 743–758

[12] **P K Kim**, **J L Tollefson**, *Splitting the PL involutions of nonprime 3-manifolds*, Michigan Math. J. 27 (1980) 259–274

[13] **P B Kronheimer**, **T S Mrowka**, *Witten's conjecture and Property P*, Geom. Topol. 8 (2004) 295–310

[14] **W B R Lickorish**, *Prime knots and tangles*, Trans. Amer. Math. Soc. 267 (1981) 321–332

[15] **C Livingston**, *Computations of the Ozsváth-Szabó knot concordance invariant*, Geom. Topol. 8 (2004) 735–742

[16] **K Miyazaki**, *Ribbon concordance does not imply a degree one map*, Proc. Amer. Math. Soc. 108 (1990) 1055–1058

[17] **R Myers**, *Simple knots in compact, orientable 3-manifolds*, Trans. Amer. Math. Soc. 273 (1982) 75–91

[18] **Y Nakanishi**, *Prime links, concordance and Alexander invariants*, Math. Sem. Notes Kobe Univ. 8 (1980) 561–568

[19] **Y Nakanishi**, *Primeness of links*, Math. Sem. Notes Kobe Univ. 9 (1981) 415–440

[20] **Y Nakanishi**, *Prime and simple links*, Math. Sem. Notes Kobe Univ. 11 (1983) 249–256

[21] **J Rasmussen**, *Khovanov homology and the slice genus*, e-print (2004) `arXiv:math.GT/0402131`

[22] **L Rudolph**, *Quasipositivity as an obstruction to sliceness*, Bull. Amer. Math. Soc. (N.S.) 29 (1993) 51–59

[23] **D S Silver**, **W Whitten**, *Knot group epimorphisms*, J. Knot Theory Ramifications, to appear

[24] **A Stoimenow**, *On polynomials and surfaces of variously positive links*, J. Eur. Math. Soc. (JEMS) 7 (2005) 477–509

[25] **W P Thurston**, *Three-dimensional geometry and topology, Vol 1*, (Silvio Levy, editor), Princeton Mathematical Series 35 Princeton University Press, Princeton, NJ (1997)

[26] **W P Thurston**, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. (N.S.) 6 (1982) 357–381

[27] **S Wang**, *Non-zero degree maps between 3-manifolds*, from: "Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)", Higher Ed. Press, Beijing (2002) 457–468

Department of Mathematics, University of South Alabama
Mobile AL 36688, USA
and
1620 Cottontown Road, Forest VA 24551, USA

Email: `silver@jaguar1.usouthal.edu, bjwcw@aol.com`

Received: 25 March 2005 Revised: 4 August 2005