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Transversal torus knots

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Abstract

We classify positive transversal torus knots in tight contact structures up to transversal isotopy.

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1 Introduction

The study of special knots in contact three manifolds provided great insight into the geometry and topology of three manifolds. In particular, the study of Legendrian knots (ones tangent to the contact planes) has been useful in distinguishing homotopic contact structures on T^3 [12] and homology spheres [2]. Moreover, Rudolph [15] has shown that invariants of Legendrian knots can be useful in understanding slicing properties of knots. The first example of the use of knot theory in contact topology was in the work of Bennequin. In [3] Bennequin used transversal knots (ones transversal to the contact planes) to show that $\mathbb{R}^3$ has exotic contact structures. This was the genesis of Eliashberg's insightful tight versus overtwisted dichotomy in three dimensional contact geometry.

In addition to its importance in the understanding of contact geometry, the study of transversal and Legendrian knots is quite interesting in its own right. Questions concerning transversal and Legendrian knots have most prominently appeared in [6] and Kirby's problem list [13]. Currently there are very few general theorems concerning the classification of these knots. In [6], Eliashberg classified transversal unknots in terms of their self-linking number. In [7], Legendrian unknots were similarly classified. In this paper we will extend this classification to positive transversal torus knots[1]. In particular we prove:

Theorem *Positive transversal torus knots are transversely isotopic if and only if they have the same topological knot type and the same self-linking number.*

In the process of proving this result we will examine *transversal stabilization*. This is a simple method for creating one transversal knot from another. By showing that all positive transversal torus knots whose self-linking number is less than maximal come from this stabilization process we are able to reduce the above theorem to the classification of positive transversal torus knots with maximal self-linking number. Stabilization also provides a general way to approach the classification problem for other knot types. For example, we can reprove Eliashberg's classification of transversal unknots using stabilization ideas and basic contact topology.

It is widely believed that the self-linking number is not a complete invariant for transversal knots. However, as of the writing of this paper, there is no known

[1]By "positive transversal torus knot" we mean a positive (right handed) torus knot that is transversal to a contact structure.

knot type whose transversal realizations are not determined by their self-linking number. For Legendrian knots, in contrast, Eliashberg and Hofer (currently unpublished) and Chekanov [4] have produced examples of Legendrian knots that are not determined by their corresponding invariants.

In Section 2 we review some standard facts concerning contact geometry on three manifolds. In Section 3 we prove our main theorem modulo some details concerning the characteristic foliations on tori which are proved in Section 4 and some results on stabilizations proved in Section 5. In the last section we discuss some open questions.

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2 Contact structures in three dimensions

We begin by recalling some basic facts from contact topology. For a more detailed introduction, see [1, 11]. Recall an orientable plane field ξ is a *contact structure* on a three manifold if $\xi = \ker\ \alpha$ where α is a nondegenerate 1–form for which $\alpha \wedge d\alpha \neq 0$. Note $d\alpha$ induces an orientation on ξ. Two contact structures are called *contactomorphic* if there is a diffeomorphism taking one of the plane fields to the other. A contact structure ξ induces a singular foliation on a surface Σ by integrating the singular line field $\xi \cap T\Sigma$. This is called the *characteristic foliation* and is denoted Σ_ξ. Generically, the singularities are elliptic (if local degree is 1) or hyperbolic (if the local degree is -1). If Σ is oriented then the singularities also have a sign. A singularity is positive (respectively negative) if the orientations on ξ and $T\Sigma$ agree (respectively disagree) at the singularity.

Lemma 2.1 (Elimination Lemma [10]) *Let Σ be a surface in a contact 3–manifold (M,ξ). Assume that p is an elliptic and q is a hyperbolic singular point in Σ_ξ, they both have the same sign and there is a leaf γ in the characteristic foliation Σ_ξ that connects p to q. Then there is a C^0–small isotopy $\phi\colon\ \Sigma \times [0,1] \to M$ such that ϕ_0 is the inclusion map, ϕ_t is fixed on γ and outside any (arbitrarily small) pre-assigned neighborhood U of γ and $\Sigma' = \phi_1(\Sigma)$ has no singularities inside U.*

It is important to note that after the above cancellation there is a curve in the characteristic foliation on which the singularities had previously sat. In the case of positive singularities this curve will consist of the (closure of the) stable manifolds of the hyperbolic point and *any* arc leaving the elliptic point (see [7, 8]), and similarly for the negative singularity case. One may also reverse this process and add a canceling pair of singularities along a leaf in the characteristic foliation. It is also important to note:

Lemma 2.2 *The germ of the contact structure ξ along a surface Σ is determined by Σ_ξ.*

Now recall that a contact structure ξ on M is called *tight* if no disk embedded in M contains a limit cycle in its characteristic foliation, otherwise it is called *overtwisted*. The standard contact structure on S^3, induced from the complex tangencies to $S^3 = \partial B^4$ where B^4 is the unit 4–ball in $\mathbb{C}^2$, is tight.

A closed curve $\gamma\colon S^1 \to M$ in a contact manifold (M, ξ) is called *transversal* if $\gamma'(t)$ is transverse to $\xi_{\gamma(t)}$ for all $t \in S^1$. Notice a transversal curve can be *positive* or *negative* according as $\gamma'(t)$ agrees with the co-orientation of ξ or not. We will restrict our attention to positive transversal knots (thus in this paper "transversal" means "positive transversal"). It can be shown that any curve can be made transversal by a C^0 small isotopy. It will be useful to note:

Lemma 2.3 (See [6]) *If $\psi_t\colon S^1 \to M$ is a transversal isotopy, then there is a contact isotopy $f_t\colon M \to M$ such that $f_t \circ \psi_0 = \psi_t$.*

Given a transverse knot γ in (M, ξ) that bounds a surface Σ we define the *self-linking number*, $l(\gamma)$, of γ as follows: take a nonvanishing vector field v in $\xi|_\gamma$ that extends to a nonvanishing vector field in $\xi|_\Sigma$ and let γ' be γ slightly pushed along v. Define

$$l(\gamma, \Sigma) = I(\gamma', \Sigma),$$

where $I(\,\cdot\,,\,\cdot\,)$ is the oriented intersection number. There is a nice relationship between $l(\gamma, \Sigma)$ and the singularities of the characteristic foliation of Σ. Let $d_\pm = e_\pm - h_\pm$ where $e_\pm$ and $h_\pm$ are the number of $\pm$ elliptic and hyperbolic points in the characteristic foliation Σ_ξ of Σ, respectively. In [3] it was shown that

$$l = d_- - d_+. \tag{1}$$

When ξ is a *tight* contact structure and Σ is a *disk*, Eliashberg [5] has shown, using the elimination lemma, how to eliminate all the positive hyperbolic and negative elliptic points from Σ_ξ. Thus in a tight contact structure when γ is an unknot $l(\gamma, \Sigma)$ is always negative. More generally one can show (see [3, 5]) that

$$l(\gamma) \leq -\chi(\Sigma), \tag{2}$$

where Σ is a Seifert surface for γ and $\chi(\Sigma)$ is its Euler number.

Any odd negative integer can be realized as the self-linking number for some transversal unknot. The first general result concerning the classification of transversal knots was the following:

Theorem 2.4 (Eliashberg [6]) *Two transversal unknots are transversely isotopic if and only if they have the same self-linking number.*

Let $\mathcal{T}$ be the transversal isotopy classes of transversal knots in S^3 with its unique tight contact structure. Let $\mathcal{K}$ be the isotopy classes of knots in S^3. Given a transversal knot $\gamma \in \mathcal{T}$ we have two pieces of information: its knot type $[\gamma] \in \mathcal{K}$ and its self-linking number $l(\gamma) \in \mathbb{Z}$. Define

$$\phi\colon \mathcal{T} \to \mathcal{K} \times \mathbb{Z} : \gamma \mapsto ([\gamma], l(\gamma)). \tag{3}$$

The main questions concerning transversal knots can be phrased in terms of the image of this map and preimages of points. In particular the above results say that ϕ is onto

$$U = [\text{unknot}] \times \{\text{negative odd integers}\}$$

and ϕ is one-to-one on $\phi^{-1}(U)$.

We will also need to consider Legendrian knots. A knot γ is a *Legendrian knot* if it is tangent to ξ. The contact structure ξ defines a canonical framing on a Legendrian knot γ. If γ is null homologous we may associate a number to this framing which we call the *Thurston–Bennequin invariant* of γ and denote it $\text{tb}(\gamma)$. If we let Σ be the surface exhibiting the null homology of γ then we may trivialize ξ over Σ and use this trivialization to measure the rotation of $\gamma'(t)$ around γ. This number $r(\gamma)$ is called the *rotation number* of γ. Note that the rotation number depends on an orientation on γ. From an oriented Legendrian knot γ one can obtain canonical positive and negative transversal knots $\gamma_\pm$ by pushing γ by vector fields tangent to ξ but transverse to $\gamma'(t)$. One may compute

$$l(\gamma_\pm) = \text{tb}(\gamma) \mp r(\gamma). \tag{4}$$

This observation combined with Equation (2) implies

$$\text{tb}(\gamma) + |r(\gamma)| \leq -\chi(\Sigma). \tag{5}$$

Consider an oriented (nonsingular) foliation $\mathcal{F}$ on a torus T. The foliation is said to have a *Reeb component* if two oppositely oriented periodic orbits cobound an annulus containing no other periodic orbits.

Lemma 2.5 *Consider a torus T in a contact three manifold (M,ξ). If the characteristic foliation on T is nonsingular and contains no Reeb components then any closed curve on T may be isotoped to be transversal to T_ξ or into a leaf of T_ξ. Moreover there is at most one homology class in $H_1(T)$ that can be realized by a leaf of T_ξ.*

Now let ξ be a tight contact structure on a solid torus S with nonsingular characteristic foliation on it boundary $T = \partial S$. It is easy to arrange for T_ξ to have no Reeb components [14]. Since ξ is tight the lemma above implies the meridian μ can be made transversal to T_ξ. We say S has self-linking number l if $l = l(\mu)$ (ie, the self-linking number of S is the self-linking number of its meridian).

Theorem 2.6 (Makar–Limanov [14]) *Any two tight contact structures on S which induce the same nonsingular foliation on the boundary and have self-linking number -1 are contactomorphic.*

3 Positive transversal torus knots

Let U be an unknot in a 3–manifold M, D an embedded disk that it bounds and V a tubular neighborhood of U. The boundary T of V is an embedded torus in M, we call such a torus a *standardly embedded torus.* Let μ be the unique curve on T that bounds a disk in V and $\lambda = D \cap V$. Orient μ arbitrarily and then orient λ so that μ, λ form a positive basis for $H_1(T)$ where T is oriented as the boundary of V. Up to homotopy any curve in T can be written as $p\mu + q\lambda$, we shall denote this curve by $K_{(p,q)}$. If p and q are relatively prime then $K_{(p,q)}$ is called a (p,q)*–torus knot.* If $pq > 0$ we say $K(p,q)$ is a *positive* torus knot otherwise we call it *negative.* One may easily compute that the Seifert surface of minimal genus for $K_{(p,q)}$ has Euler number $|p| + |q| - |pq|$. Thus for a transversal torus knot Equation 2 implies

$$l(K_{(p,q)}) \leq -|p| - |q| + |pq|. \tag{6}$$

In fact, if $\bar{l}_{(p,q)}$ denotes the maximal self-linking number for a transversal $K_{(p,q)}$ then one may easily check that

$$\bar{l}_{(p,q)} = -p - q + pq, \tag{7}$$

if $p, q > 0$, ie, for a positive torus knot. (Note: for a positive transversal torus knot Lemma 3.6 says we have $p, q > 0$ not just $pq > 0$.) From the symmetries involved in the definition of a torus knot we may assume that $p > q$, which we do throughout the rest of the paper. We now state our main theorem.

Theorem 3.1 *Positive transversal torus knots in a tight contact structure are determined up to transversal isotopy by their knot type and their self-linking number.*

Remark 3.2 We may restate this theorem by saying the map ϕ defined in equation 3 is one-to-one when restricted to

$$(\mathrm{pr} \circ \phi)^{-1}(\text{positive torus knots})$$

(here $\mathrm{pr}\colon \mathcal{K} \times \mathbb{Z} \to \mathcal{K}$ is projection). Moreover, the image of ϕ restricted to the above set is $G = \cup_{(p,q)} K_{(p,q)} \times N(p,q)$ where the union is taken over relatively prime positive p and q, and $N(p,q)$ is the set of odd integers less than or equal to $-p - q + pq$.

We first prove the following auxiliary result:

Proposition 3.3 *Two positive transversal (p,q)–torus knots K and K' in a tight contact structure with maximal self-linking number (ie, $l(K) = l(K') = \bar{l}_{(p,q)}$) are transversally isotopic.*

Proof Let T and T' be tori standardly embedded in M on which K and K', respectively, sit.

Lemma 3.4 *If the self-linking number of K is maximal then T may be isotoped relative to K so that the characteristic foliation on T is nonsingular.*

This lemma and the next are proved in the following section.

Lemma 3.5 *Two transversal knots on a torus T with nonsingular characteristic foliation that are homologous are transversally isotopic, except possibly when there is a closed leaf in the foliation isotopic to the transversal knots.*

Our strategy is to isotop T onto T', keeping K and K' transverse to ξ, so that K and K' are homologous, and thus transversally isotopic. We now show that T can be isotoped into a standard form keeping K transverse (and similarly for K' and T' without further mention). Let V be the solid torus that T bounds (recall we are choosing V so that $p > q$). Let D_μ and D_λ be the disk that μ and λ respective bound. Now observe:

Lemma 3.6 *We may take μ and λ to be positive transversal curves and with this orientation μ, λ form a positive basis for $T = \partial V$.*

Proof Clearly we may take μ and λ to be positive transversal knots, for if we could not then Lemma 2.5 implies that we may isotop one of them to a closed leaf in T_ξ contradicting the tightness of ξ. Thus we are left to see that μ, λ is a positive basis. Assume this is not the case. By isotoping T slightly we may assume that T_ξ has closed leaf (indeed if T_ξ does not already have a closed leaf then the isotopy will give an intervals worth of rotation numbers, and hence some rational rotation numbers, for the return map induced on μ by T_ξ). Let C be one of these closed leaves and let $n = \lambda \cdot C$ and $m = \mu \cdot C$. Note n and m are both positive since μ and λ are positive transversal knots. Since μ, λ is not a positive basis C is an (n, m)–torus knot. In particular C is a positive torus knot. Moreover, the framing on C induced by ξ is the same as the framing induced by T. Thus $\mathrm{tb}(C) = mn$ contradicting Equation (5). So μ, λ must be a positive basis for T. □

Now let $m = l(\mu)$ and $l = l(\lambda)$ and recall $m, l \leq -1$.

Lemma 3.7 *If γ is a transversal (p, q) knot on T (with nonsingular characteristic foliation) then*

$$l(\gamma) = pm + ql + pq. \tag{8}$$

Proof Let v be a section of ξ over an open 3–ball containing T and its meridional and longitudinal disks. If C is a curve on T then define $f(C)$ to be the framing of ξ over C induced by v relative to the framing of ξ over C induced by T. Note f descends to a map on $H_1(T)$ and $f(A + B) = f(A) + f(B)$ where $A, B \in H_1(T)$. One easily computes $f(\mu) = m$ and $f(\lambda) = l$. Thus $f(p\mu + q\lambda) = pm + ql$. Now for a transversal curve C on T the normal bundle to C can be identified with ξ thus $f(C)$ differs from $l(C)$ by the framing induced on C by T relative to the framing induced on C by its Seifert surface. So $l(C) = f(C) + pq = pm + ql + pq$. □

Thus since K has maximal self-linking number we must have $m = l = -1$. Now by Theorem 2.6 we may find a contactomorphism from V to $S_f = \{(r, \theta, \phi) \in \mathbb{R}^2 \times S^1 | r \leq f(\theta, \phi)\}$ for some positive function $f \colon T^2 \to \mathbb{R}$, with the standard tight contact structure $\ker(d\phi + r^2\, d\theta)$.

Clearly $T = \partial S_f$ may be isotoped to $S_\epsilon = \{(r, \theta, \phi) \in \mathbb{R}^2 \times S^1 | r < \epsilon\}$ for arbitrarily small $\epsilon > 0$. We now show this isotopy may be done keeping our knot K transverse to the characteristic foliation. To a foliation on ∂S_f we may associate a real valued rotation number $r(S_f)$ for the return map on μ induced by $(\partial S_f)_\xi$ (see [14]). For a standardly embedded torus this number must be negative since if not then some nearby torus would have a positive (r, s) torus knot as a closed leaf in its characteristic foliation violating the Bennequin inequality (as in the proof of Lemma 3.6). So as we isotop ∂S_f to ∂S_ϵ we may keep our positive torus knot transverse to the characteristic foliation by Lemma 2.5 (since closed leaves in $(\partial S_f)_\xi$ have slope $r(S_f)$ and K has positive slope). Thus we assume that the solid torus V is contactomorphic to S_ϵ. If C is the core of $V(=S_\epsilon)$ then it is a transversal unknot with self-linking $l(\lambda) = -1$.

Finally, let V and V' be the solid tori associated to the torus knots K and K' and let C and C' be the cores of V and V'. Now since C and C' are unknots with the same self-linking number they are transversely isotopic. Thus we may think of V and V' as neighborhoods of the same transverse curve $C = C'$. From above, V and V' may both be shrunk to be arbitrarily small neighborhoods of C keeping K and K' transverse to ξ. Hence we may assume that V and V' both sit in a neighborhood of C which is contactomorphic to, say, S_c (using the notation from the previous paragraph). By shrinking V and V' further we may assume they are the tori S_ϵ and $S_{\epsilon'}$ inside S_c for some ϵ and ϵ'. Note that this is not immediately obvious but follows from the fact that a contactomorphism from the standard model S_f for, say, V to $V \subset S_c$ may be constructed to take a neighborhood of the core of S_f to a neighborhood of the core of S_c. This allows us to finally conclude that we may isotop V so that $V = V'$. Now since K and K' represent the same homology class on ∂V and they are both transverse to the foliation we may use Lemma 3.5 to transversely isotop K to K'. □

A transversal knot K is called a *stabilization* of a transversal knot C if $K = \alpha \cup A$, $C = \alpha \cup A'$ and $A \cup A'$ cobound a disk with only positive elliptic and negative hyperbolic singularities (eg Figure 1). We say K is obtained from C by a *single stabilization* if K is a stabilization of C and $l(K) = l(C) - 2$ (ie, the disk that $A \cup A'$ cobound is the one shown in Figure 1). The key observation concerning stabilizations is the following:

Theorem 3.8 *If the transversal knots K and K' are single stabilizations of transversal knots C and C' then K is transversely isotopic to K' if C is transversely isotopic to C'.*

This theorem will be proved in Section 5. The proof of Theorem 3.1 is completed by an inductive argument using the following observation.

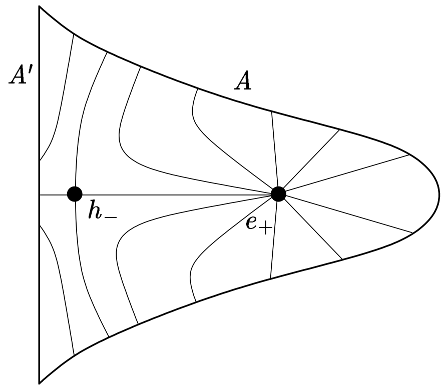

Figure 1: Stabilization disk

Lemma 3.9 *If K is a positive transversal (p,q)–torus knot and $l(K) < \bar{l}_{(p,q)}$ then K is a single stabilization of a (p,q)–torus knot with larger self-linking number.*

The proof of this lemma will be given in the next section following the proof of Lemma 3.4.

4 Characteristic foliations on tori

In this section we prove various results stated in Section 3 related to foliations on tori. Let T be a standardly embedded torus in M^3 and K a positive (p,q)–torus knot on T that is transverse to a tight contact structure ξ. We are now ready to prove:

Lemma 3.4 *If the self-linking number of K is maximal then T may be isotoped relative to K so that the characteristic foliation on T is nonsingular.*

Proof Begin by isotoping T relative to K so that the number of singularities in T_ξ is minimal. Any singularities that are left must occur in pairs: a positive (negative) hyperbolic h and elliptic e point connected by a stable (unstable) manifold c. Moreover, since h and e cannot be canceled without moving K we must have $c \cap K \neq \emptyset$.

Now $T \setminus K$ is an annulus A with the characteristic foliation flowing out of one boundary component and flowing in the other. Let c' be the component of c

connected to h in A. We can have no periodic orbits in A since such an orbit would be a Legendrian (p,q)–torus knot with Thurston–Bennequin invariant pq contradicting Equation (5). Thus the other stable (unstable) manifold c'' of h will have to enter (exit) A through the same boundary component. The manifolds c' and c'' separate off a disk D from A. We may use $D \subset T$ to push the arc $K \cap D$ across D to obtain another transverse (p,q)–torus knot K'. It is not hard to show that K is a stabilization of K'. In particular $l(K') > l(K)$, contradicting the maximality of $l(K)$. Thus we could have not have had any singularities left after our initial isotopy. □

The above proof provides some insight into Lemma 3.9. Recall:

Lemma 3.9 *If K is a positive transversal (p,q)–torus knot with and $l(K) < \bar{l}_{(p,q)}$ then K is a single stabilization of a (p,q)–torus knot with larger self-linking number.*

Proof We begin by noting that if K is a stabilization of another transversal knot then it is also a single stabilization of some transversal knot. Thus we just demonstrate that K is a stabilization of some transversal knot.

From the above proof it is clear that if we cannot eliminate all the singularities in the characteristic foliation of the torus T on which K sits then there is a disk on the torus which exhibits K as a stabilization.

If we can remove all the singularities from T then by Lemma 3.7 we know that the self-linking number of, say, the meridian μ is less than -1. Thus μ bounds a disk D_μ containing only positive elliptic and at least one negative hyperbolic singularity. To form a positive transversal torus knot K'' we can take p copies of the meridian μ and q copies of the longitude λ and "add" them (ie, resolve all the intersection points keeping the curve transverse to the characteristic foliation). This will produce a transversal knot on T isotopic to K thus transversely isotopic. Moreover, we may use the graph of singularities on D_μ to show that K'', and hence K, is a stabilization. □

We end this section by establishing (a more general version of) Lemma 3.5.

Lemma 4.1 *Suppose that $\mathcal{F}$ is a nonsingular foliation on a torus T and γ and γ' are two simple closed curves on T. If γ and γ' are homologous and transverse to $\mathcal{F}$ then they are isotopic through simple closed curves transverse to $\mathcal{F}$, except possibly if $\mathcal{F}$ has a closed leaf isotopic to γ.*

Proof We first note that if γ and γ' are disjoint and there are not closed leaves isotopic to them then the annulus that they cobound will provide the desired transverse isotopy. Thus we are left to show that we can make γ and γ' disjoint. We begin by isotoping them so they intersect transversely. Now assume we have transversely isotoped them so that the number of their intersection points is minimal. We wish to show this number is zero. Suppose not, then there are an even number of intersection points (since homologically their intersection is zero).

Using a standard innermost arc argument we may find a disk $D \subset T$ such that ∂D consists of two arcs, one a subarc of γ the other a subarc of γ'. We can use the disk D to guide a transverse isotopy of γ' that will decrease the number of intersections of γ and γ' contradicting our assumption of minimality. To see this, note that the local orientability of the foliation implies that we can define a winding number of $\mathcal{F}$ around ∂D. Moreover since ∂D is contractible and the foliation is nonsingular this winding number must be zero. Thus the foliation on D must be diffeomorphic to the one shown in Figure 2 where the desired isotopy is apparent. □

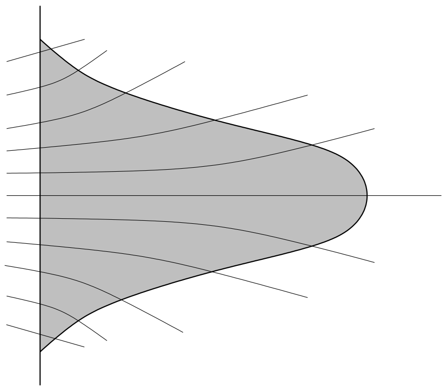

Figure 2: Foliation on D

5 Stabilizations of transversal knots

The main goal of this section is to prove Theorem 3.8:

Theorem 3.8 *If the transversal knots K and K' are single stabilizations of transversal knots C and C' then K is transversely isotopic to K' if C is transversely isotopic to C'.*

Proof Since C and C' are transversely isotopic we can assume that $C = C'$. Let D and D' be the disks that exhibit K and K' as stabilizations of C. Let e, h and e', h' be the elliptic/hyperbolic pairs on D and D'. Finally, let α and α' be the Legendrian arcs formed by the (closure of the) union of stable manifolds of h and h'. Using the characteristic foliation on D we may transversely isotop $K \setminus C$ to lie arbitrarily close to α (and similarly for K' and α'). We are thus done by the following simple lemmas.

Lemma 5.1 *There is a contact isotopy preserving C taking $\alpha \cap C$ to $\alpha' \cap C$.*

Working in a standard model for a transverse curve this lemma is quite simple to establish. Thus we may assume that α and α' both touch C at the same point.

Lemma 5.2 *There is a contact isotopy preserving C taking α to α'.*

Once again one can use a Darboux chart to check this lemma (for some details see [7]).

Lemma 5.3 *Any two single stabilizations of C along a fixed Legendrian arc are transversely isotopic.*

With this lemma our proof of Theorem 3.8 is complete. □

We now observe that using Theorem 3.8 we may reprove Eliashberg's result concerning transversal unknots. The reader should note that this "new proof" is largely just a reordering/rewording of Eliashberg's proof.

Theorem 5.4 *Two transversal unknots are transversally isotopic if and only if they have the same self-linking number.*

Proof Using Theorem 3.8 we only need to prove that two transversal unknots with self-linking number -1 are transversally isotopic, since by looking at the characteristic foliation on a Seifert disk it is clear that a transversal unknot with self-linking number less than -1 is a single stabilization of another unknot. But given a transversal unknot with self-linking number -1 we may find a disk that it bounds with precisely one positive elliptic singularity in its characteristic foliation. Using the characteristic foliation on the disk the unknot may be transversely isotoped into an arbitrarily small neighborhood of the elliptic point. Thus given two such knots we may now find a contact isotopy of taking the elliptic point on one of the Seifert disks to the elliptic point on the other. Since the Seifert disks are tangent at their respective elliptic points we may arrange that they agree in a neighborhood of the elliptic points. Now by shrinking the Seifert disks more we may assume that both unknots sit on the same disk. It is now a simple matter to transversely isotop one unknot to the other. □

6 Concluding remarks and questions

We would like to note that many of the techniques in this paper work for negative torus knots as well (though the proofs above do not always indicate this). There are two places where we cannot make the above proofs work for negative torus knots, they are:

- From Equation 8 we cannot conclude that the self-linking numbers of μ and λ are -1 when $l(K_{(p,q)})$ is maximal as we could for positive torus knots.
- We cannot always conclude that a negative torus knot with self-linking less than maximal is a stabilization.

Despite these difficulties we conjecture that negative torus knots are also determined by their self-linking number.

Let $S = S^1 \times D^2$ and let K be a (p,q)–curve on the boundary of S. Now if C is a null homologous knot in a three manifold M then let $f\colon S \to N$ be a diffeomorphism from S to a neighborhood N of C in M taking $S^1 \times \{\text{point}\}$ to a longitude for C. We now define the (p,q)*–cable of* C to be the knot $f(K)$.

Question 1 *If $\mathcal{C}$ is the class of topological knots whose transversal realizations are determined up to transversal isotopy by their self-linking number, then is $\mathcal{C}$ closed under cablings?*

Eliashberg's Theorem 2.4 says that the unknot U is in $\mathcal{C}$. Our main Theorem 3.1 says that any positive cable of the unknot is in $\mathcal{C}$. This provides the first bit of evidence that the answer to the question might be YES, at least for "suitably positive" cablings.

Given a knot type one might hope, using the observation on stabilizations in this paper, to prove that transversal knots in this knot type are determined by their self-linking number as follows: First establishing that there is a unique transversal knot in this knot type with maximal self-linking number. Then showing that any transversal knot in this knot type that does not have maximal self-linking number is a stabilization. The second part of this program is of independent interest so we ask the following question:

Question 2 *Are all transversal knots not realizing the maximal self-linking number of their knot type stabilizations of other transversal knots?*

It would be somewhat surprising if the answer to this question is YES in complete generality but understanding when the answer is YES and when and why it is NO should provide insight into the structure of transversal knots.

We end by mentioning that the techniques in this paper also seem to shed light on Legendrian torus knots. It seems quite likely that their isotopy class may be determined by their Thurston–Bennequin invariant and rotation number. We hope to return to this question in a future paper.

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