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Noncommutative localisation in algebraic K–theory I

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Abstract

This article establishes, for an appropriate localisation of associative rings, a long exact sequence in algebraic K–theory. The main result goes as follows. Let A be an associative ring and let $A \longrightarrow B$ be the localisation with respect to a set σ of maps between finitely generated projective A–modules. Suppose that $\mathrm{Tor}_n^A(B,B)$ vanishes for all $n > 0$. View each map in σ as a complex (of length 1, meaning one non-zero map between two non-zero objects) in the category of perfect complexes $D^{\mathrm{perf}}(A)$. Denote by $\langle\sigma\rangle$ the thick subcategory generated by these complexes. Then the canonical functor $D^{\mathrm{perf}}(A) \longrightarrow D^{\mathrm{perf}}(B)$ induces (up to direct factors) an equivalence $D^{\mathrm{perf}}(A)/\langle\sigma\rangle \longrightarrow D^{\mathrm{perf}}(B)$. As a consequence, one obtains a homotopy fibre sequence

$$K(A,\sigma) \longrightarrow K(A) \longrightarrow K(B)$$

(up to surjectivity of $K_0(A) \longrightarrow K_0(B)$) of Waldhausen K–theory spectra.

In subsequent articles [26, 27] we will present the K– and L–theoretic consequences of the main theorem in a form more suitable for the applications to surgery. For example if, in addition to the vanishing of $\mathrm{Tor}_n^A(B,B)$, we also assume that every map in σ is a monomorphism, then there is a description of the homotopy fiber of the map $K(A) \longrightarrow K(B)$ as the Quillen K–theory of a suitable exact category of torsion modules.

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Introduction

Probably the most useful technical tool in algebraic K–theory is localisation. Localisation tells us about certain long exact sequences in K–theory. Given a ring A, Quillen told us how to assign to it a K–theory spectrum $K(A)$. Given rings A and $\sigma^{-1}A$, where $\sigma^{-1}A$ is a localisation of A, there is a map of spectra $K(A) \longrightarrow K(\sigma^{-1}A)$. A localisation theorem expresses the homotopy fiber as $K(\mathbf{R})$, the K–theory of some suitable $\mathbf{R}$. In the early localisation theorems $K(\mathbf{R})$ was Quillen's K–theory of some exact category of torsion modules. But in more recent, more general theorems one allows $K(\mathbf{R})$ to be the Waldhausen K–theory of some suitable Waldhausen model category $\mathbf{R}$. In more concrete terms we get a long exact sequence

$$\cdots \longrightarrow K_1(A) \longrightarrow K_1(\sigma^{-1}A) \longrightarrow K_0(\mathbf{R}) \longrightarrow K_0(A) \longrightarrow K_0(\sigma^{-1}A) \longrightarrow 0.$$

There has been extensive literature over the years, proving localisation theorems in algebraic K–theory. Let us provide a brief sample of the existing literature. For commutative rings, or more generally for schemes, the reader can see Bass' [1], Quillen's [29], Grayson's [13], Levine's [20], Weibel's [39, 40] and Thomason's [35]. For non-commutative rings see Grayson's [14], Schofield's [33], Weibel and Yao's [41] and Yao's [42].

Some situations are very well understood. For example, let A be an associative ring with unit. Let $\sigma \subset A$ be a multiplicative set of elements in the center $Z(A)$ of the ring A. It is very classical to define $\sigma^{-1}A$ as the ring of fractions a/s, with $a \in A$ and $s \in \sigma$. The ring $\sigma^{-1}A$ is called the *commutative localisation* of A with respect to the multiplicative set σ. Note that the rings A and $\sigma^{-1}A$ are not assumed commutative; the only commutativity hypothesis is that $\sigma \subset Z(A)$. If every element of σ is a non-zero-divisor then the existence of a localisation exact sequence is classical.

Over the years people have found more general localisation theorems in the K–theory of non-commutative rings. What we want to do in this article is treat localisation in the generality that comes up in topology. In applications to higher dimensional knot theory and surgery, the ring A might be the group ring of the fundamental group of some manifold, and the set σ almost never lies in the center of A. We first need to remind the reader of the generality in which the localisation of rings arises in topology.

Let us agree to some notation first. Our rings are all associative and have units. Let A be a ring. When we say "A–module" without an adjective, we mean left A–module.

Let A be any non-commutative ring. Let σ be any set of maps of finitely generated, projective A–modules. In symbols

$$\sigma = \{s_i\colon P_i \longrightarrow Q_i \mid \text{where } P_i, Q_i \text{ are f.g. projective}\}.$$

Definition 0.1 A ring homomorphism $A \longrightarrow B$ is called *σ–inverting* if, for all $s_i\colon P_i \longrightarrow Q_i$ in σ, the map

$$B \otimes_A P_i \xrightarrow{1\otimes_A s_i} B \otimes_A Q_i$$

is an isomorphism.

The collection of σ–inverting homomorphisms $A \longrightarrow B$ is naturally a category. A morphism in this category is a commutative triangle of ring homomorphisms:

$$\begin{tikzcd} & B \arrow[dd] \\ A \arrow[ur] \arrow[dr] & \\ & B' \end{tikzcd}$$

There is an old observation, due to Cohn [7] and Schofield [33], which says that the category of σ–inverting homomorphisms $A \longrightarrow B$ has an initial object.

Definition 0.2 The initial object in the category of σ–inverting homomorphisms is called the *Cohn localisation* or the *universal localisation* of A with respect to σ. In this article it will be denoted $A \longrightarrow \sigma^{-1}A$.

See Vogel [36, 37], Farber and Vogel [10], Farber and Ranicki [9], and Ranicki [30, 31] for some of the applications of the algebraic K– and L–theory of Cohn localisations in topology.

In this article, a *perfect complex* of A–modules is a bounded complex of finitely generated projective modules. Let $C^{\mathrm{perf}}(A)$ be the Waldhausen category of all perfect complexes. That is, the objects are the perfect complexes, the morphisms are the chain maps, the weak equivalences are the homotopy equivalences, and the cofibrations are the degreewise split monomorphisms. Let $D^{\mathrm{perf}}(A)$ be the associated homotopy category; the objects are still the perfect complexes, but the morphisms are homotopy equivalence classes of chain maps.

Remark 0.3 Our convention is slightly different from the standard one. In the literature there is a distinction made between *perfect complexes* and *strictly perfect complexes.* What we call a perfect complex is what, elsewhere in the literature, is often referred to as a strictly perfect complex. We have almost no use for perfect complexes which are not strictly perfect. For this reason we let the adverb "strictly" be understood.

The set σ of maps $s_i\colon\ P_i \longrightarrow Q_i$ can be thought of as a set of objects in either $C^{\mathrm{perf}}(A)$ or $D^{\mathrm{perf}}(A)$ (the two categories share the same set of objects). We simply take the chain complexes

$$\cdots \longrightarrow 0 \longrightarrow P_i \xrightarrow{\ s_i\ } Q_i \longrightarrow 0 \longrightarrow \cdots$$

We define a *Waldhausen category* $\mathbf{R} \subset C^{\mathrm{perf}}(A)$ as follows:

Definition 0.4 The category $\mathbf{R}$ is the smallest subcategory of $C^{\mathrm{perf}}(A)$ which

(i) Contains all the complexes in σ, in the sense above.

(ii) Contains all acyclic complexes.

(iii) Is closed under the formation of mapping cones and suspensions.

(iv) Contains any direct summand of any of its objects.

The main K–theoretic result of the article becomes:

Theorem 0.5 *Suppose A is a ring, σ a set of maps of finitely generated, projective A–modules. Suppose $\mathrm{Tor}_n^A(\sigma^{-1}A, \sigma^{-1}A) = 0$ for all $n > 0$. Then the homotopy fiber of the map $K(A) \longrightarrow K(\sigma^{-1}A)$ is naturally identified, up to the failure of surjectivity of the map $K_0(A) \longrightarrow K_0(\sigma^{-1}A)$, with the spectrum $K(\mathbf{R})$. By $K(\mathbf{R})$ we mean the Waldhausen K–theory of the Waldhausen category $\mathbf{R}$ of Definition 0.4.*

Theorem 0.5 gives a precise version of what was stated, slightly less precisely, in the abstract. It turns out that Theorem 0.5 is a consequence of a statement about triangulated categories. Next we explain the triangulated categories results, and why the K–theoretic statement in Theorem 0.5 is a formal consequence.

Let $\mathcal{R}^c$ be the smallest triangulated subcategory of $D^{\mathrm{perf}}(A)$, containing σ and containing all direct summands of any of its objects. The category $\mathbf{R}$ of Definition 0.4 is simply a Waldhausen model for $\mathcal{R}^c$.

Remark 0.6 In the abstract we used the notation $\langle\sigma\rangle$ to denote what we now call $\mathcal{R}^c$. The new notation is to avoid ambiguity, clearly distinguishing the subcategory $\mathbf{R}$ generated by σ in $C^{\mathrm{perf}}(A)$ from the subcategory $\mathcal{R}^c$ generated by σ in $D^{\mathrm{perf}}(A)$.

Let $\mathcal{T}^c$ be the idempotent completion of the Verdier quotient $D^{\mathrm{perf}}(A)/\mathcal{R}^c$. That is, form the Verdier quotient, and in it split all idempotents. The main theorem, in its triangulated category incarnation, asserts:

Theorem 0.7 *Consider the natural functor $D^{\mathrm{perf}}(A) \longrightarrow D^{\mathrm{perf}}(\sigma^{-1}A)$, which takes a complex in $D^{\mathrm{perf}}(A)$ and tensors it with $\sigma^{-1}A$. Take the canonical factorisation*

$$D^{\mathrm{perf}}(A) \xrightarrow{\ \pi\ } \mathcal{T}^c \xrightarrow{\ T\ } D^{\mathrm{perf}}(\sigma^{-1}A).$$

Then the following are equivalent:

(i) *The functor $T\colon \mathcal{T}^c \longrightarrow D^{\mathrm{perf}}(\sigma^{-1}A)$ is an equivalence of categories.*

(ii) *For all $n \geq 1$ the group $\mathrm{Tor}_n^A(\sigma^{-1}A, \sigma^{-1}A) = 0$.*[1]

We call the localisation $A \longrightarrow \sigma^{-1}A$ stably flat[2] *if the equivalent conditions above hold.*

Next we will sketch how Theorem 0.5 follows from Theorem 0.7 (more detail will be given in later sections). We have a sequence

$$\mathcal{R}^c \longrightarrow D^{\mathrm{perf}}(A) \longrightarrow D^{\mathrm{perf}}(\sigma^{-1}A).$$

This sequence always has a lifting to Waldhausen models

$$\mathbf{R} \longrightarrow C^{\mathrm{perf}}(A) \longrightarrow C^{\mathrm{perf}}(\sigma^{-1}A).$$

Assume further that $\mathrm{Tor}_n^A(\sigma^{-1}A, \sigma^{-1}A) = 0$ for all $n \geq 1$. Theorem 0.7 tells us that the category $D^{\mathrm{perf}}(\sigma^{-1}A)$ is identified, up to splitting idempotents, as the Verdier quotient $D^{\mathrm{perf}}(A)/\mathcal{R}^c$. Waldhausen's Localisation Theorem (Theorem 2.3), coupled with Grayson cofinality (Theorem 2.4), tell us that

$$K(\mathbf{R}) \longrightarrow K\big(C^{\mathrm{perf}}(A)\big) \xrightarrow{\ \phi\ } K\big(C^{\mathrm{perf}}(\sigma^{-1}A)\big)$$

identifies $K(\mathbf{R})$ as the (-1)–connected cover of the homotopy fiber of the map ϕ above. By a theorem of Gillet, the Waldhausen K–theory $K(C^{\mathrm{perf}}(A))$ is

[1] The implication (i)$\Longrightarrow$(ii) may be found in Geigle and Lenzing [11]. The implication (ii)$\Longrightarrow$(i) is related to the telescope conjecture. The relation is studied in Krause [19]. Unfortunately the telescope conjecture is false in general; see Keller [17]. More precisely: there are hardly any non-commutative rings for which the telescope conjecture is known to hold. What we give here is a proof independent of the telescope conjecture.

[2] In Dicks' article [8, p. 565] the terminology for such a map is *a lifting,* while in Geigle and Lenzing's [11] it would be called a *homological epimorphism.* When we coined the term *stably flat* we were unaware of the earlier literature.

naturally isomorphic to Quillen's $K(A)$; in other words, we get a commutative square where the vertical maps are homotopy equivalences

$$\begin{array}{ccc} K(A) & \longrightarrow & K(\sigma^{-1}A) \\ \Big\downarrow\wr & & \Big\downarrow\wr \\ K\big(C^{\mathrm{perf}}(A)\big) & \xrightarrow{\ \phi\ } & K\big(C^{\mathrm{perf}}(\sigma^{-1}A)\big). \end{array}$$

Hence $K(\mathbf{R})$ may be identified with the (-1)–connected cover of the homotopy fiber of the map $K(A) \longrightarrow K(\sigma^{-1}A)$. In other words: Theorem 0.5 follows easily from Theorem 0.7, modulo well known results of Waldhausen, Grayson and Gillet.

At the level of homotopy groups this means we have an infinite long exact sequence

$$\cdots \to K_1(\mathbf{R}) \to K_1(A) \to K_1(\sigma^{-1}A) \to K_0(\mathbf{R}) \to K_0(A) \to K_0(\sigma^{-1}A)$$

which continues indefinitely to the left. We are, however, *not* asserting that the map $K_0(A) \longrightarrow K_0(\sigma^{-1}A)$ is surjective. The homotopy fiber F of the map $K(A) \longrightarrow K(\sigma^{-1}A)$ has in general a non-vanishing π_{-1}, and the map $K(\mathbf{R}) \longrightarrow F$ is an isomorphism in π_i for all $i \geq 0$, but $\pi_{-1}K(\mathbf{R})$ vanishes.

We have stated the main theorems mostly in the case where the localisation is stably flat (see Theorem 0.7 for the definition of stable flatness). There are examples of Cohn localisations which are not stably flat; see [28]. Even in the non-stably-flat case the study of the functor $T\colon \mathcal{T}^c \longrightarrow D^{\mathrm{perf}}(\sigma^{-1}A)$ is illuminating and has K–theoretic consequences. This article is devoted to studying the functor T.

This article contains the proof of the theorems above, and some other formal, triangulated category facts about the functor T. The applications will appear separately. See [26] for K–theoretic consequences, [27] for consequences in L–theory, as well as Krause's beautiful article [19] which further develops some of our results.

In presenting the proofs we tried to keep in mind that the reader might not be an expert in derived categories. This is a paper of interest in topology and surgery theory. We therefore try to give a survey of the results, from the literature on K–theory and on triangulated categories, which we need to appeal to. We give clear statements and careful references. We also try to break down the proofs into a series of very easy steps.

The result is that the paper is much longer than necessary to communicate the results to the experts; we ask the experts for patience. The other drawback, of presenting the proof in many easy steps, is that the key issues can become disguised. We will address this soon, in the discussion of the proof.

It is never clear how much the introduction ought to say about the details of the proofs. Let us confine ourselves to the following. It is easy to produce the functors

$$D^{\mathrm{perf}}(A) \xrightarrow{\ \pi\ } \mathcal{T}^c \xrightarrow{\ T\ } D^{\mathrm{perf}}(\sigma^{-1}A).$$

For any integer $n \in \mathbb{Z}$, they induce maps of abelian groups

$$\begin{array}{ccc} \mathrm{Hom}_{D^{\mathrm{perf}}(A)}(A, \Sigma^n A) & & \\ \downarrow & & \\ \mathrm{Hom}_{\mathcal{T}^c}(\pi A, \Sigma^n \pi A) & \xrightarrow{\ \varphi_n\ } & \mathrm{Hom}_{D^{\mathrm{perf}}(\sigma^{-1}A)}(T\pi A, \Sigma^n T\pi A). \end{array}$$

If T is an equivalence of categories, then the map φ_n above must be an isomorphism. A minor variant of a theorem of Rickard's tells us that the converse also holds. To prove that T is an equivalence of categories, it suffices to show that the map φ_n is an isomorphism for every $n \in \mathbb{Z}$.

The construction of T gives that $T\pi A = \sigma^{-1}A$. This means that the abelian group $\mathrm{Hom}_{D^{\mathrm{perf}}(\sigma^{-1}A)}(T\pi A, \Sigma^n T\pi A)$ is just $\mathrm{Ext}^n_{\sigma^{-1}A}(\sigma^{-1}A, \sigma^{-1}A)$. It vanishes when $n \neq 0$. For $n = 0$, the endomorphisms of $\sigma^{-1}A$, viewed as a left $\sigma^{-1}A$–module, are right multiplication by elements of $\sigma^{-1}A$. Therefore we are reduced to showing

$$\mathrm{Hom}_{\mathcal{T}^c}(\pi A, \Sigma^n \pi A) = \begin{cases} 0 & \text{if } n \neq 0 \\ \{\sigma^{-1}A\}^{\mathrm{op}} & \text{if } n = 0. \end{cases}$$

In other words, the proof reduces to computing the groups $\mathrm{Hom}_{\mathcal{T}^c}(\pi A, \Sigma^n \pi A)$.

It happens to be very useful to turn the problem into one about unbounded complexes. Although Theorem 0.7 deals only with perfect complexes, the proof looks at $D(A)$, the unbounded derived category. It is possible to embed the category $\mathcal{T}^c$ in a larger category $\mathcal{T}$, and extend the map $\pi\colon D^{\mathrm{perf}}(A) \longrightarrow \mathcal{T}^c$ to a map $\pi\colon D(A) \longrightarrow \mathcal{T}$. What makes this useful is that the extended functor π has a right adjoint $G\colon \mathcal{T} \longrightarrow D(A)$. By adjunction

$$\mathrm{Hom}_{\mathcal{T}^c}(\pi A, \Sigma^n \pi A) \quad = \quad \mathrm{Hom}_{D(A)}(A, \Sigma^n G\pi A) \quad = \quad H^n(G\pi A),$$

and we are reduced to computing $H^n(G\pi A)$.

Remark 0.8 It turns out that, for $n \geq 0$, there is no need to assume the vanishing of $\mathrm{Tor}^A_n(\sigma^{-1}A, \sigma^{-1}A)$. Without any hypotheses we get $H^n(G\pi A) = 0$ if $n > 0$, while

$$H^0(G\pi A) \quad = \quad \mathrm{Hom}_{\mathcal{T}^c}(\pi A, \pi A) \quad = \quad \{\sigma^{-1}A\}^{\mathrm{op}}.$$

The key lemma, which underpins everything we prove, is Lemma 6.3. The lemma looks like a trivial little fact. It asserts that, for the standard t–structure, the truncations of any object of the form $G\pi x$ are also of the form $G\pi y$. This is the one point where we use the fact that we are dealing with a Cohn localisation, not just a general localisation in a triangulated category. The lemma crucially depends on the complexes σ, which generate the subcategory $\mathcal{R}^c$, being of length ≤ 1. In the case where A is a commutative noetherian ring, [21] tells us all the localisations of the derived category. It is easy to see that, without the hypothesis that the complexes generating $\mathcal{R}^c$ be of length ≤ 1, essentially all our theorems fail. The proof amounts to following the consequences of Lemma 6.3. We play around with some spectral sequences when necessary, the argument is a little tricky at points, but none of this changes the fact that Lemma 6.3 is the foundation for everything we prove.

This is our second attempt to expose the results; the first may be found in [25]. All but the experts in triangulated categories found the first exposition difficult to read. As we have already explained, this is our attempt to make the article readable. We begin with a survey of the main results we need from the literature. Then follows a sequence of easy steps, reducing the proof of Theorem 0.7 to the computation of $H^n(G\pi A)$. The computations, which are the hard core of the article, come only at the end, in sections 6, 7 and 8.

Since we want this article to be easy to read, we try not to assume that the reader is very familiar with triangulated categories. We have therefore gone to some trouble to keep our references to the literature focused. In order to read the article, the triangulated category background that is needed is:

(i) The standard t–structure on $D(A)$; [2, Chapter 1].

(ii) Homotopy limits and colimits; [4, Sections 1–3].

(iii) The generalisation of Thomason's localisation theorem; [22, Sections 1,2]

The reader will note that all the needed information is contained near the beginning of the papers cited. We make a serious effort not to refer any place else. But we feel free to quote any of the results in the brief literature given above.

The fact that we cite only the three papers above leads to historical inaccuracies. For example, the existence of the right adjoint G to the functor $\pi\colon\ D(A) \longrightarrow \mathcal{T}$ was first proved by Bousfield [5, 6]. The many people who have done excellent work in triangulated categories do not receive the credit they deserve: see for example Keller's articles [16, 17] or Krause's [18]. Also, there is a sense in which our main theorems are descended from Thomason's [35]. In the survey article [24] we try to correct at least one of the historical inaccuracies, indicating the crucial role of Thomason's work.

To keep the length from mushrooming to infinity we have separated off the applications, which now appear in [26, 27].

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1 Notation, and a reminder of t–structures

All our rings in this article will be associative rings with units. Let A be a ring. Unless otherwise specified, all modules are left A–modules. The derived category $D(A)$ means the unbounded derived category of all complexes of A–modules. An object x is a complex

$$\cdots \longrightarrow x^{n-2} \longrightarrow x^{n-1} \xrightarrow{\partial^{n-1}} x^n \xrightarrow{\partial^n} x^{n+1} \longrightarrow x^{n+2} \longrightarrow \cdots$$

As the reader has undoubtedly noticed, we write our complexes cohomologically. Since one gets tired of adding a "co" to every word, let it be understood. What we call chain maps is what in the literature is usually called cochain maps. What we call chain complexes is usually called cochain complexes.

The nth homology of the complex x above (which is what most people refer to as the cohomology of the cochain complex) will be denoted $H^n(X)$.

When it is clear which category we are dealing with, we write the Hom–sets as $\mathrm{Hom}(x,y)$. When there are several categories around, we freely use the notation $\mathcal{T}(x,y)$ for $\mathrm{Hom}_{\mathcal{T}}(x,y)$.

In this article, t–structures on triangulated categories play a key role in many of our proofs. For an excellent exposition of this topic see Chapter 1 of [2]. We give here the bare essentials. Let $\mathcal{S} = D(A)$ as above. The only t–structure we use in this article is the standard one on $\mathcal{S}$. We remind the reader.

For any integer $n \in \mathbb{Z}$ there are two full subcategories of $\mathcal{S}$. The objects are given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{S}^{\leq n}) &= \{X \in \mathrm{Ob}(\mathcal{S}) \mid H^r(X) = 0 \text{ for all } r > n\}, \\ \mathrm{Ob}(\mathcal{S}^{\geq n}) &= \{X \in \mathrm{Ob}(\mathcal{S}) \mid H^r(X) = 0 \text{ for all } r < n\}. \end{aligned}$$

The properties they satisfy are

(i) $\mathcal{S}^{\leq n} \subset \mathcal{S}^{\leq n+1}$.

(ii) $\mathcal{S}^{\geq n} \subset \mathcal{S}^{\geq n-1}$.

(iii) $\Sigma\mathcal{S}^{\geq n} = \mathcal{S}^{\geq n-1}$, and $\Sigma\mathcal{S}^{\leq n} = \mathcal{S}^{\leq n-1}$.

(iv) If $x \in \mathcal{S}^{\leq -1}$ and $y \in \mathcal{S}^{\geq 0}$, then $\mathrm{Hom}(x,y) = 0$.

(v) For every object $x \in \mathcal{S}$ there is a unique, canonical distinguished triangle

$$x^{\leq n-1} \longrightarrow x \longrightarrow x^{\geq n} \longrightarrow \Sigma x^{\leq n-1}$$

with $x^{\leq n-1} \in \mathcal{S}^{\leq n-1}$ and $x^{\geq n} \in \mathcal{S}^{\geq n}$.

Remark 1.1 If x is the complex

$$\cdots \longrightarrow x^{n-2} \longrightarrow x^{n-1} \xrightarrow{\partial^{n-1}} x^n \xrightarrow{\partial^n} x^{n+1} \longrightarrow x^{n+2} \longrightarrow \cdots$$

then the maps

$$x^{\leq n-1} \longrightarrow x \longrightarrow x^{\geq n}$$

are concretely given by the chain maps

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & x^{n-2} & \longrightarrow & \mathrm{Ker}(\partial^{n-1}) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & x^{n-2} & \longrightarrow & x^{n-1} & \xrightarrow{\partial^{n-1}} & x^n & \longrightarrow & x^{n+1} & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Coker}(\partial^{n-1}) & \longrightarrow & x^{n+1} & \longrightarrow \cdots
\end{array}$$

2 Preliminaries, based on Waldhausen's work

We begin with a brief review of Waldhausen's foundational work. The reader can find much more thorough treatments in Waldhausen's article [38], or in Section 1 of Thomason's [35].

Let $\mathbf{S}$ be a small category with cofibrations and weak equivalences. Out of $\mathbf{S}$ Waldhausen constructs a spectrum, denoted $K(\mathbf{S})$. In Thomason's [35] the category $\mathbf{S}$ is assumed to be a full subcategory of the category of chain complexes over some abelian category, the cofibrations are maps of complexes which are split monomorphisms in each degree, and the weak equivalences contain the quasi-isomorphisms. We will call such categories *permissible Waldhausen categories.* In this article, we may assume that all categories with cofibrations and weak equivalences are permissible Waldhausen categories.

Remark 2.1 Thomason's term for them is *complicial biWaldhausen categories.*

Given a small, permissible Waldhausen category $\mathbf{S}$, one can form its derived category; just invert the weak equivalences. We denote this derived category by $D(\mathbf{S})$. We have two major theorems here, both of which are special cases of more general theorems of Waldhausen. The first theorem may be found in Thomason's [35, Theorem 1.9.8]:

Theorem 2.2 (Waldhausen's Approximation Theorem) *Let $F\colon \mathbf{S} \longrightarrow \mathbf{T}$ be an exact functor of small, permissible Waldhausen categories (categories of chain complexes, as above). Suppose that the induced map of derived categories*

$$D(F)\colon\ D(\mathbf{S}) \longrightarrow D(\mathbf{T})$$

is an equivalence of categories. Then the induced map of spectra

$$K(F)\colon\ K(\mathbf{S}) \longrightarrow K(\mathbf{T})$$

is a homotopy equivalence.

In this sense, Waldhausen's K–theory is almost an invariant of the derived categories. To construct it one needs to have a great deal more structure. One must begin with a permissible category with cofibrations and weak equivalences. But the Approximation Theorem asserts that the dependence on the added structure is not strong.

Next we state Waldhausen's Localisation Theorem. The statement we give is an easy consequence of Theorem 2.2, coupled with Waldhausen's [38, 1.6.4] or Thomason's [35, 1.8.2]:

Theorem 2.3 (Waldhausen's Localisation Theorem) *Let $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{T}$ be small, permissible Waldhausen categories. Suppose*

$$\mathbf{R} \longrightarrow \mathbf{S} \longrightarrow \mathbf{T}$$

are exact functors of permissible Waldhausen categories. Suppose further that

(i) *The induced triangulated functors of derived categories*

$$D(\mathbf{R}) \longrightarrow D(\mathbf{S}) \longrightarrow D(\mathbf{T})$$

compose to zero.

(ii) *The functor $\varphi\colon\ D(\mathbf{R}) \longrightarrow D(\mathbf{S})$ is fully faithful.*

(iii) *If x and x' are objects of $D(\mathbf{S})$, and the direct sum $x \oplus x'$ is isomorphic in $D(\mathbf{S})$ to $\varphi(z)$ for some $z \in D(\mathbf{R})$, then x, x' are isomorphic to $\varphi(y), \varphi(y')$ for some $y, y' \in D(\mathbf{R})$.*

(iv) *The natural map*

$$D(\mathbf{S})/D(\mathbf{R}) \longrightarrow D(\mathbf{T})$$

is an equivalence of categories.

Then the sequence of spectra

$$K(\mathbf{R}) \longrightarrow K(\mathbf{S}) \longrightarrow K(\mathbf{T})$$

is a homotopy fibration.

We need one more general theorem, this one due to Grayson [15].

Theorem 2.4 (Grayson's Cofinality Theorem) *Let $\varphi\colon \mathbf{T} \longrightarrow \mathbf{T}'$ be an exact functor of permissible Waldhausen categories. Suppose the induced map $D(\varphi)\colon D(\mathbf{T}) \longrightarrow D(\mathbf{T}')$ is an idempotent completion. That is, the functor $D(\varphi)$ is fully faithful, and for every object $x \in D(\mathbf{T}')$ there exists an object $x' \in D(\mathbf{T}')$ and an isomorphism $x \oplus x' \simeq D(\varphi)(y)$, with $y \in D(\mathbf{T})$.*

Then the map of spectra $K(\varphi)\colon K(\mathbf{T}) \longrightarrow K(\mathbf{T}')$ satisfies

(i) $K_i(\mathbf{T}) \longrightarrow K_i(\mathbf{T}')$ *is an isomorphism if $i \geq 1$.*

(ii) $K_0(\mathbf{T}) \longrightarrow K_0(\mathbf{T}')$ *is injective.*

In the article we will apply the results of this section. None of the results is very sensitive to changes in Waldhausen models. The additivity theorem, which we did not discuss in this section, is sensitive to changes of permissible Waldhausen categories. In this article, and the two subsequent ones [26, 27], we never once use the additivity theorem. We can afford to confine ourselves to proving the existence of one way to make the choice of models. Of course it is possible that, in the future, someone will want to apply the results of the articles in conjunction with the additivity theorem. Such a person will have to pay more attention to the choice of Waldhausen categories.

Let us discuss one cheap way to produce models.

Lemma 2.5 *Let $\mathcal{S}^c$ be a small triangulated category, $\mathcal{R}^c \subset \mathcal{S}^c$ a triangulated subcategory containing all direct summands of its objects. Suppose we are given a permissible Waldhausen category $\mathbf{S}$ and an equivalence of triangulated categories $\varphi\colon D(\mathbf{S}) \longrightarrow \mathcal{S}^c$. Define $\mathbf{R}$ to be the full Waldhausen subcategory* *of all objects $x \in \mathbf{S}$ so that $\varphi(x)$ is isomorphic in $\mathcal{S}^c$ to an object in $\mathcal{R}^c \subset \mathcal{S}^c$. Define the permissible Waldhausen category $\mathbf{S}_{\mathbf{R}}$ so that the objects, morphisms and cofibrations are as in $\mathbf{S}$, but the weak equivalences in $\mathbf{S}_{\mathbf{R}}$ are the maps in $\mathbf{S}$ whose mapping cones lie in $\mathbf{R}$.*

Then there is a commutative diagram of triangulated functors, where the vertical maps are equivalences

$$\begin{array}{ccccc} D(\mathbf{R}) & \longrightarrow & D(\mathbf{S}) & \longrightarrow & D(\mathbf{S}_{\mathbf{R}}) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \mathcal{R}^c & \longrightarrow & \mathcal{S}^c & \longrightarrow & \mathcal{S}^c/\mathcal{R}^c . \end{array}$$

Idea of the Proof The axioms of permissible Waldhausen categories guarantee that the calculus of fractions, in the passage from a permissible Waldhausen category $\mathbf{S}$ to its derived category $D(\mathbf{S})$, is quite simple. Every morphism $x \longrightarrow y$ in $D(\mathbf{S})$ can be written as $\beta\alpha^{-1}$, for maps in $\mathbf{S}$

$$x \xleftarrow{\ \alpha\ } x' \xrightarrow{\ \beta\ } y$$

with α a weak equivalence. If x and y are in $\mathbf{R}$ then, since $\mathbf{R}$ contains all the isomorphs in $D(\mathbf{S})$ of any of its objects, $\mathbf{R}$ must contain x'. Hence any morphism in $D(\mathbf{S})$ between objects in the image of $D(\mathbf{R})$ lifts to $D(\mathbf{R})$. The equivalence relation between pairs (α,β) as above is slightly more complicated, but also only involves objects isomorphic in $\mathcal{S}^c$ to x. Hence $\beta\alpha^{-1}$ will equal $\beta'\{\alpha'\}^{-1}$ in $D(\mathbf{S})$ if and only if they are equal in $D(\mathbf{R})$. Thus the functor $D(\mathbf{R}) \longrightarrow D(\mathbf{S})$ is fully faithful. The objects in $D(\mathbf{R})$ are, by definition of $\mathbf{R}$, precisely the ones isomorphic to objects in $\mathcal{R}^c$.

The fact that $D(\mathbf{S}_{\mathbf{R}}) \simeq \mathcal{S}^c/\mathcal{R}^c$ is obvious from Verdier's construction of the quotient $\mathcal{S}^c/\mathcal{R}^c$. $\square$

3 The machine to produce examples

In order to apply the theorems of the last section, we will produce triangulated categories $\mathcal{R}^c \subset \mathcal{S}^c$ and a triangulated functor $\mathcal{S}^c/\mathcal{R}^c \longrightarrow \mathcal{T}^c$ which is an idempotent completion (as in Theorem 2.4). There is a general machine which constructs examples. It is based on a theorem by the first author. In this section we will set up the notation, state the theorem and explain how it is applied.

Definition 3.1 Let $\mathcal{S}$ be a triangulated category, containing all small coproducts of its objects. An object $c \in \mathcal{S}$ is called *compact* if every map from c to any coproduct factors through a finite part of the coproduct. That is, any map

$$c \longrightarrow \coprod_{\lambda \in \Lambda} t_\lambda$$

factors as

$$c \longrightarrow \coprod_{i=1}^{n} t_{\lambda_i} \longrightarrow \coprod_{\lambda \in \Lambda} t_\lambda .$$

Equivalently, c is compact if and only if

$$\bigoplus_{\lambda \in \Lambda} \operatorname{Hom}(c, t_\lambda) \quad = \quad \operatorname{Hom}\left(c\,, \coprod_{\lambda \in \Lambda} t_\lambda\right).$$

Example 3.2 Let A be a ring, and let $\mathcal{S} = D(A)$ be the unbounded derived category of A. The category $\mathcal{S}$ contains all small coproducts of its objects; we can form direct sums of unbounded complexes. Let $A \in D(A) = \mathcal{S}$ be the chain complex which is A in degree 0, and vanishes in all other degrees. For any $X \in \mathcal{S}$ he have $\operatorname{Hom}(A, X) = H^0(X)$, and hence

$$\bigoplus_{\lambda \in \Lambda} \operatorname{Hom}(A, t_\lambda) = \bigoplus_{\lambda \in \Lambda} H^0(t_\lambda) = H^0\left(\coprod_{\lambda \in \Lambda} t_\lambda\right) = \operatorname{Hom}\left(A\,, \coprod_{\lambda \in \Lambda} t_\lambda\right).$$

Thus A is a compact object of $\mathcal{S}$.

Definition 3.3 The full subcategory $\mathcal{S}^c \subset \mathcal{S}$ has for its objects all the compact objects of $\mathcal{S}$.

Remark 3.4 It is easy to show that the full subcategory $\mathcal{S}^c \subset \mathcal{S}$ is closed under triangles and direct summands.

Example 3.5 In the special case of the category $\mathcal{S} = D(A)$ of Example 3.2, we know that A is compact. Any finite direct sum of compact objects is compact, and any direct summand of a compact object is compact. We conclude that all finitely generated, projective A–modules are compact. The subcategory $\mathcal{S}^c \subset \mathcal{S}$ is triangulated, and hence we conclude that all bounded chain complexes of finitely generated, projective modules are compact. In Corollary 4.4 we will see that every compact object in $\mathcal{S}$ is isomorphic to a bounded complex of finitely generated, projective modules.

Notation 3.6 Next we set up the notation for the main theorem. Let $\mathcal{S}$ be a triangulated category containing all small coproducts. Let $\mathcal{R} \subset \mathcal{S}$ be a full triangulated subcategory, closed under the formation of the coproducts in $\mathcal{S}$ of any set of its objects. Form the category $\mathcal{T} = \mathcal{S}/\mathcal{R}$. It is easy to show that the category $\mathcal{T}$ contains all small coproducts, and that the natural map $\mathcal{S} \longrightarrow \mathcal{T}$ respects coproducts.

We have $\mathcal{R} \subset \mathcal{S}$, with $\mathcal{T} = \mathcal{S}/\mathcal{R}$. The reader might imagine trying to apply Waldhausen's localisation theorem directly to the triple $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$. There are two problems with this:

(i) Since $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ contain all small coproducts of their objects they tend to be huge categories. The classes of objects are not small sets. This means that any permissible Waldhausen model would not be small, and there are set theoretic difficulties even in defining the Waldhausen K–theory $K(\mathcal{R})$, $K(\mathcal{S})$ and $K(\mathcal{T})$.

(ii) Even if we are willing to enlarge the universe and define $K(\mathcal{R})$ in this enlarged universe, we still get nonsense. The fact that $\mathcal{R}$ contains countable coproducts of its objects permits us to do the Eilenberg swindle, and show that $K(\mathcal{R})$ is contractible. Similarly for $K(\mathcal{S})$ and $K(\mathcal{T})$.

The useful way to produce a non-trivial, interesting example is by passing to compact objects. The categories $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ each has a subcategory of compact objects. The main theorem tells us:

Theorem 3.7 *Let the notation be as in Notation 3.6. Assume further that there exist:*

(i) *A set of objects $S \subset \mathcal{S}^c$, so that any subcategory of $\mathcal{S}$ containing S and closed under triangles and coproducts is all of $\mathcal{S}$.*

(ii) *A set of objects $R \subset \mathcal{R} \cap \mathcal{S}^c$, so that any subcategory of $\mathcal{R}$ containing R and closed under triangles and coproducts is all of $\mathcal{R}$.*

Then the natural map $\mathcal{R} \longrightarrow \mathcal{S}$ takes compact objects to compact objects, and so does the natural map $\mathcal{S} \longrightarrow \mathcal{T}$. In other words, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{R}^c & \longrightarrow & \mathcal{S}^c & \longrightarrow & \mathcal{T}^c \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{R} & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{T}. \end{array}$$

Of course the composite $\mathcal{R}^c \longrightarrow \mathcal{S}^c \longrightarrow \mathcal{T}^c$ *must vanish, since it is just the restriction to* $\mathcal{R}^c$ *of a vanishing functor on* $\mathcal{R}$*. We therefore have a factorisation of* $\mathcal{S}^c \longrightarrow \mathcal{T}^c$ *as*

$$\mathcal{S}^c \longrightarrow \mathcal{S}^c/\mathcal{R}^c \xrightarrow{\ i\ } \mathcal{T}^c.$$

The functor $i\colon \mathcal{S}^c/\mathcal{R}^c \longrightarrow \mathcal{T}^c$ *is fully faithful, and every object of* $\mathcal{T}^c$ *is a direct summand of an isomorph of an object in the image of* i*.*

Remark 3.8 Thomason proved this theorem in the special case where $\mathcal{S}$ and $\mathcal{T}$ are the derived categories of quasi-coherent sheaves on a scheme X (respectively, on an open subset $U \subset X$). In the generality above, the theorem may be found in the first author's [22, Theorem 2.1]. In a recent book [23] the first author generalises the theorem even further, to deal with the large cardinal case. There are now two proofs of Theorem 3.7. The proof presented in the old paper [22], and the more general proof in the book [23]. These two proofs are quite different from each other.

Remark 3.9 In the situation of Theorem 3.7 the map $\mathcal{R} \longrightarrow \mathcal{S}$ is fully faithful, and hence so is its restriction to $\mathcal{R}^c \longrightarrow \mathcal{S}^c$. Furthermore, every idempotent in the category $\mathcal{R}$ splits, because $\mathcal{R}$ is closed under coproducts; see [4, Proposition 3.2 and Remark 3.3]. Since every direct summand in $\mathcal{R}$ of a compact object is obviously compact, every idempotent in $\mathcal{R}^c$ splits. It follows that $\mathcal{R}^c \subset \mathcal{S}^c$ is closed under direct summands.

4 The $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ to which we apply Theorem 3.7

In sections 2 and 3 we reviewed the general K–theoretic and triangulated category results we will be using. Now it is time to explain how we will apply them. We want to use the general theorems to deduce a K–theory localisation theorem for the Cohn localisation.

The Cohn localisation begins with a ring A and a set σ of morphisms $s_i\colon P_i \longrightarrow Q_i$, as in Definitions 0.1 and 0.2. To apply the results of Section 3, we need to choose suitable triangulated categories $\mathcal{R} \subset \mathcal{S}$, and $\mathcal{T} = \mathcal{S}/\mathcal{R}$. Our choices are:

Definition 4.1 Let A be a ring, σ a set of maps of finitely generated, projective A–modules. We define the triangulated categories

(i) $\mathcal{S} = D(A)$ is the unbounded derived category of complexes of A–modules.

(ii) We are given a set of maps $\sigma = \{s_i\colon P_i \longrightarrow Q_i\}$. We can view these as objects in $\mathcal{S} = D(A)$ just by turning them into complexes

$$\cdots \longrightarrow 0 \longrightarrow P_i \xrightarrow{s_i} Q_i \longrightarrow 0 \longrightarrow \cdots$$

The category $\mathcal{R} \subset \mathcal{S}$ is defined to be the smallest triangulated subcategory of $\mathcal{S} = D(A)$, which contains σ and is closed in $\mathcal{S}$ under the formation of arbitrary coproducts of its objects.

(iii) $\mathcal{T}$ is defined to be $\mathcal{S}/\mathcal{R}$.

Remark 4.2 The categories $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ depend on A and on σ. In most of this article we can view A and σ as fixed. For this reason our notation makes no explicit mention of this dependence.

Next we prove that our choices of $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ satisfy the technical hypotheses of Theorem 3.7. We need a little lemma:

Lemma 4.3 *Let A be a ring, and let $\mathcal{S} = D(A)$. The object $A \in \mathcal{S}$ is the complex which is A in degree 0, and vanishes in all other degrees. If $\mathcal{B} \subset \mathcal{S}$ is a triangulated subcategory which contains A and is closed under coproducts, then $\mathcal{B} = \mathcal{S}$.*

Sketch of proof This lemma is well-known and there are several proofs. We include a sketch of one just for completeness.

Since $A \in \mathcal{B}$ and $\mathcal{B}$ is closed under direct sums, $\mathcal{B}$ must contain all free A modules. Since $\mathcal{B}$ is triangulated, it must contain all bounded complexes of free A–modules. If $X \in \mathcal{B}$ is a bounded-above complex, then X is quasi-isomorphic to a bounded above complex F of free modules. But F is a direct limit of its stupid truncations, all of which are bounded complexes of free modules. The stupid truncations lie in $\mathcal{B}$, and by [4, Remark 2.2] so does the direct limit $X \cong F$.

Now let Y be an arbitrary (unbounded) object in $\mathcal{B}$. Then Y is the direct limit of its (bounded above) t–structure truncations $Y^{\leq i}$, all of which lie in $\mathcal{B}$ by the above. Using [4, Remark 2.2] again, we conclude that $Y \in \mathcal{B}$. □

In passing we mention the following corollary of Lemma 4.3:

Corollary 4.4 *As in Lemma 4.3, let A be a ring and $\mathcal{S} = D(A)$. An object $c \in \mathcal{S}$ is compact if and only if it is isomorphic to a perfect complex; that is, if and only if c is isomorphic in $\mathcal{S}$ to a bounded chain complex of finitely generated, projective modules.*

Proof In Example 3.5 we saw that every perfect complex is compact in $\mathcal{S}$. We need to show that, up to isomorphism in $D(A)$, these are the only compact objects.

It is well known that the natural functor $D^{\mathrm{perf}}(A) \longrightarrow D(A)$ is fully faithful; $D^{\mathrm{perf}}(A)$ may be viewed as a full subcategory of $\mathcal{S} = D(A)$. Let $\widetilde{\mathcal{R}}$ be the subcategory of $D(A)$ containing all objects isomorphic to objects of $D^{\mathrm{perf}}(A) \subset D(A)$. The subcategory $\widetilde{\mathcal{R}}$ is triangulated, and from [4, Proposition 3.4] any direct summand of an object in $\widetilde{\mathcal{R}}$ lies in $\widetilde{\mathcal{R}}$. The subcategory $\widetilde{\mathcal{R}}$ also contains the object $A \in \mathcal{S}$, which generates $\mathcal{S}$ by Lemma 4.3. By [22, Lemma 2.2] it follows that $\widetilde{\mathcal{R}}$ contains all the compact objects. $\square$

Proposition 4.5 *Let the triangulated categories $\mathcal{R} \subset \mathcal{S}$, $\mathcal{T} = \mathcal{S}/\mathcal{R}$ be as in Definition 4.1. Then $\mathcal{R} \subset \mathcal{S}$, $\mathcal{T} = \mathcal{S}/\mathcal{R}$ satisfy the hypotheses of Theorem 3.7.*

Proof The category $\mathcal{S} = D(A)$ clearly contains coproducts of its objects, and by its definition $\mathcal{R} \subset \mathcal{S}$ is closed in $\mathcal{S}$ under coproducts. That is, the notation is as in Notation 3.6.

It remains to verify the hypotheses 3.7(i) and (ii). For $S \subset \mathcal{S}^c$ take the set $\{A\}$. Lemma 4.3 tells us that 3.7(i) holds. For the set $R \subset \mathcal{R}$ of 3.7(ii) we take σ. The definition of $\mathcal{R}$ is as the smallest triangulated subcategory of $\mathcal{S}$, closed under coproducts and containing σ. To prove 3.7(ii), it suffices to establish that $\sigma \subset \mathcal{S}^c$.

But every object of σ is a chain complex

$$\cdots \longrightarrow 0 \longrightarrow P_i \xrightarrow{s_i} Q_i \longrightarrow 0 \longrightarrow \cdots$$

with P_i and Q_i finitely generated and projective. By Corollary 4.4 (or even by Example 3.5) it follows that every object in σ is compact in $\mathcal{S} = D(A)$. $\square$

Since the hypotheses of Theorem 3.7 hold, so does its conclusion. We deduce a diagram of triangulated categories:

$$\begin{tikzcd}
\mathcal{R}^c \arrow[r] & \mathcal{S}^c \arrow[rr, "\pi"] \arrow[dr] & & \mathcal{T}^c \\
& & \mathcal{S}^c/\mathcal{R}^c \arrow[ur, "i"'] &
\end{tikzcd}$$

Remark 4.6 Now we make our choices of permissible Waldhausen categories. We let $\mathbf{S} = C^{\mathrm{perf}}(A)$ be the Waldhausen category of all perfect chain complexes of A–modules, with morphisms the chain maps, weak equivalences the homotopy equivalences, and cofibrations the degreewise split monomorphisms. Clearly $\mathbf{S} = C^{\mathrm{perf}}(A)$ is a model for $D^{\mathrm{perf}}(A)$. As in Lemma 2.5 we produce $\mathbf{R}$ and $\mathbf{S_R}$. The category $\mathbf{R}$ is the full subcategory of all objects in $\mathbf{S} = C^{\mathrm{perf}}(A)$ which become isomorphic in $D(A)$ to objects in $\mathcal{R}^c$. The category $\mathbf{S_R}$ has the same objects, morphisms and cofibrations as $\mathbf{S}$, but the weak equivalences are any morphisms whose mapping cones lie in $\mathbf{R}$.

Slightly more delicate is our choice for $\mathbf{T}$. For this we need:

Definition 4.7 For any ring B, the category $C(B,\aleph_0)$ will be defined as follows. The objects are certain chain complexes of projective A–modules, to be specified below. The morphisms are the chain maps, the weak equivalences are the quasi-isomorphisms, and the cofibrations are the degreewise split monomorphisms. The restrictions on the objects are given by specifying that $C(B,\aleph_0)$ is the smallest category which:

(i) Contains the perfect complexes.

(ii) Is closed under countable direct sums.

(iii) Is closed under the formation of mapping cones.

If A is not just any ring, but comes with a set σ of maps of finitely generated projective A–modules, then $C(A,\sigma,\aleph_0)$ has the same objects, morphisms and cofibrations as $C(A,\aleph_0)$. Only the weak equivalences change. A morphism in $C(A,\sigma,\aleph_0)$ is a weak equivalence if its mapping cone maps to $\mathcal{R} \subset \mathcal{S}$ under the functor $C(A,\aleph_0) \longrightarrow D(A) = \mathcal{S}$.

Our choice for $\mathbf{T}$ is to take the full Waldhausen subcategory of $C(A,\sigma,\aleph_0)$ whose objects are isomorphic in $\mathcal{T} = D(A)/\mathcal{R}$ to compact objects.[3]

Lemma 4.8 *If we take the diagram of permissible Waldhausen categories*

$$\begin{tikzcd}
\mathbf{R} \arrow[r] & \mathbf{S} \arrow[rr, "\pi"] \arrow[dr] & & \mathbf{T} \\
& & \mathbf{S_R} \arrow[ur, "i"'] &
\end{tikzcd}$$

and pass to derived categories, we obtain (up to equivalence):

$$\begin{tikzcd}
\mathcal{R}^c \arrow[r] & \mathcal{S}^c \arrow[rr, "\pi"] \arrow[dr] & & \mathcal{T}^c \\
& & \mathcal{S}^c/\mathcal{R}^c \arrow[ur, "i"'] &
\end{tikzcd}$$

[3]The reason for this definition is that $\mathbf{T}$ must be essentially small to have a K–theory. Hence we only include countable coproducts.

Proof By Corollary 4.4, there is an equivalence of categories $D^{\mathrm{perf}}(A) \longrightarrow \mathcal{S}^c$. This makes $\mathbf{S} = C^{\mathrm{perf}}(A)$ a model for $\mathcal{S}^c$. The definitions of the categories $\mathbf{R}$ and $\mathbf{S}_{\mathbf{R}}$, together with Lemma 2.5, make the lemma immediate for the part of the diagram

$$\begin{tikzcd} \mathbf{R} \arrow[r] & \mathbf{S} \arrow[dr] & \\ & & \mathbf{S}_{\mathbf{R}} \end{tikzcd}$$

The slight subtlety comes from $\mathbf{T}$. The key point is that the derived category $D(A,\sigma,\aleph_0)$ of $C(A,\sigma,\aleph_0)$ maps fully faithfully to $\mathcal{T} = \mathcal{S}/\mathcal{R}$. This is not entirely trivial;[4] it may be found in [23, Proposition 4.4.1]. Since $C(A,\sigma,\aleph_0)$ is closed under countable direct sums so is $D(A,\sigma,\aleph_0)$. From [4, Remark 3.3] we conclude that any idempotent in $D(A,\sigma,\aleph_0)$ splits. But $D(A,\sigma,\aleph_0)$ is equivalent to a full subcategory of $\mathcal{T}$ containing the image of $D^{\mathrm{perf}}(A)$ and closed under direct summands. It follows that $\mathcal{T}^c \cap D(A,\sigma,\aleph_0)$ is equivalent to $\mathcal{T}^c$. Since $\mathbf{T}$ is defined to be the preimage in $C(A,\sigma,\aleph_0)$ of $\mathcal{T}^c \cap D(A,\sigma,\aleph_0)$, its derived category must be equivalent to $\mathcal{T}^c$ (again by Lemma 2.5). □

Corollary 4.9 *In the sequence*

$$K(\mathbf{R}) \longrightarrow K(\mathbf{S}) \xrightarrow{K(\pi)} K(\mathbf{T})$$

the spectrum $K(\mathbf{R})$ is the (-1)–connected cover of the homotopy fiber of $K(\pi)$. Furthermore, $K(\mathbf{S})$ agrees with $K(A)$, the Quillen K–theory of A.

Proof The statement about the fiber is immediate from Theorems 2.3 and 2.4. By definition $\mathbf{S} = C^{\mathrm{perf}}(A)$, and hence $K(\mathbf{S}) = K\big(C^{\mathrm{perf}}(A)\big)$. The assertion $K\big(C^{\mathrm{perf}}(A)\big) = K(A)$ may be found in Gillet's [12, 6.2]. □

5 The map $\mathcal{T} \longrightarrow D(\sigma^{-1}A)$

Let A be an associative ring, and let σ be a set of maps of finitely generated, projective A–modules. Let the categories $\mathcal{S} = D(A)$, $\mathcal{R} \subset \mathcal{S}$ and $\mathcal{T} = \mathcal{S}/\mathcal{R}$ be as in Definition 4.1.

In the previous section we showed that in the sequence

$$K(\mathbf{R}) \longrightarrow K(A) \xrightarrow{K(\pi)} K(\mathbf{T})$$

[4]This is the only point in the article where we appeal to a theorem about triangulated categories which cannot be found in the three basic references [2, 4, 22].

$K(\mathbf{R})$ is the (-1)–connected cover of the homotopy fiber of the map $K(\pi)$. Next we want necessary and sufficient conditions for $\mathcal{T}^c$ to be $D^{\mathrm{perf}}(\sigma^{-1}A)$. Under these conditions

$$K(\mathbf{T}) = K\big(C^{\mathrm{perf}}(\sigma^{-1}A)\big) = K(\sigma^{-1}A)$$

where the last equality is by Gillet's [12, 6.2]. The sequence above becomes, up to some nonsense in the (-1)–homotopy groups, a homotopy fibration

$$K(\mathbf{R}) \longrightarrow K(A) \longrightarrow K(\sigma^{-1}A)$$

and this is what we are after.

The first step is to find a functor comparing $\mathcal{T}^c$ and $D^{\mathrm{perf}}(\sigma^{-1}A)$. We define it at the level of unbounded complexes. Let us remind the reader first of the tensor product of unbounded complexes.

Reminder 5.1 Let B be any $(A-A)$–bimodule. The *derived tensor product with* B is a triangulated, coproduct-preserving functor $D(A) \longrightarrow D(A)$. We will denote it

$$X \mapsto B{}^L\!\otimes_A X.$$

If B is an A–algebra, we can view this as a functor $D(A) \longrightarrow D(B)$. The existence of this functor was first proved by Spaltenstein [34]. A very short proof of the existence may be found in [4, Theorem 2.14]. Very concretely, to form $B{}^L\!\otimes_A X$ we take a K–projective resolution $P \longrightarrow X$, and define

$$B{}^L\!\otimes_A X \quad = \quad B \otimes_A P.$$

Recall that a map $P \longrightarrow X$ is a K–projective resolution if

(i) $P \longrightarrow X$ is a quasi-isomorphism.

(ii) Any chain map $P \longrightarrow Y$, with Y acyclic, is null homotopic.

The references above prove the existence of K–projective resolutions[5].

In this article, we consider tensor products both in the category of modules and in the derived category. We try to be careful to distinguish them in the notation.

It will be helpful to note that, for the categories $\mathbf{T} \subset C(A, \sigma, \aleph_0)$ of Remark 4.6 and Definition 4.7, the ordinary tensor product agrees with the derived tensor product.

[5]The terminology is a little misleading. Note that a K–projective resolution is not necessarily a projective resolution in the ordinary sense of the word. Any null homotopic complex N is a K–projective resolution of the zero complex. But N is not necessarily a complex of projectives.

Lemma 5.2 *The objects $P \in C(A, \aleph_0)$ are all K–projective.*

Proof The perfect complexes are clearly K–projective. Furthermore any coproduct of K–projectives is K–projective, and any mapping cone on a map of K–projectives is K–projective. □

Lemma 5.3 *Let $A \longrightarrow B$ be any σ–inverting ring homomorphism (Definition 0.1). By Reminder 5.1 there is a functor $D(A) \longrightarrow D(B)$, taking $X \in D(A)$ to $B^L\otimes_A X \in D(B)$. We assert that this functor factors uniquely as*

$$D(A) = \mathcal{S} \xrightarrow{\ \pi\ } \mathcal{T} \xrightarrow{\ T\ } D(B)$$

where $\pi\colon \mathcal{S} \longrightarrow \mathcal{T} = \mathcal{S}/\mathcal{R}$ is as in Definition 4.1, and T respects coproducts. Furthermore, the functor $T\colon \mathcal{T} \longrightarrow D(B)$ takes compact objects to compact objects.

Proof Let $s_i\colon P_i \longrightarrow Q_i$ be a map in σ. Tensoring with B takes it to an isomorphism. Hence tensoring with B takes the chain complex

$$\cdots \longrightarrow 0 \longrightarrow P_i \xrightarrow{\ s_i\ } Q_i \longrightarrow 0 \longrightarrow \cdots$$

to an acyclic complex. Therefore the functor $X \mapsto B^L\otimes_A X\colon D(A) \longrightarrow D(B)$ kills all the objects in σ. Since derived tensor product preserves triangles and coproducts, the subcategory of $\mathcal{S}$ annihilated by $X \mapsto B^L\otimes_A X$ must be closed under triangles and coproducts, and therefore contains all of $\mathcal{R}$. By the universal property of the Verdier quotient $\mathcal{T} = \mathcal{S}/\mathcal{R}$, there is a unique factorisation

$$D(A) = \mathcal{S} \xrightarrow{\ \pi\ } \mathcal{T} \xrightarrow{\ T\ } D(B)$$

and since $T\pi$ and π respect coproducts, so does T. It remains to show that T takes $\mathcal{T}^c \subset \mathcal{T}$ to $\{D(B)\}^c \subset D(B)$.

It is clear that the map $T\pi\colon \mathcal{S} \longrightarrow D(B)$ takes a bounded complex of finitely generated projective A–modules to a bounded complex of finitely generated projective B–modules; the map just tensors with B. By Corollary 4.4, this says that the functor $T\pi$ takes $\mathcal{S}^c$ to $\{D(B)\}^c \subset D(B)$. The last statement of Theorem 3.7 says that every object $t \in \mathcal{T}^c$ is a direct summand of $\pi(s)$, with $s \in \mathcal{S}^c$. Hence $T(t)$ is a direct summand of the compact $T\pi(s)$, and must therefore be compact. □

Summary 5.4 For the special $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ of Definition 4.1, Lemma 5.3 allows us to extend the diagram of Theorem 3.7 to:

$$\begin{tikzcd}
\mathcal{R}^c \arrow[r] \arrow[dd] & D^{\mathrm{perf}}(A) \arrow[rr, "\pi"] \arrow[dr] \arrow[dd] & & \mathcal{T}^c \arrow[r, "T"] \arrow[dd] & D^{\mathrm{perf}}(B) \arrow[dd] \\
& & D^{\mathrm{perf}}(A)/\mathcal{R}^c \arrow[ur, "i"'] & & \\
\mathcal{R} \arrow[r] & D(A) \arrow[rr, "\pi"'] & & \mathcal{T} \arrow[r, "T"'] & D(B)
\end{tikzcd} \qquad (**)$$

Now we construct $\mathbf{U}$, a permissible Waldhausen model for $D^{\mathrm{perf}}(B)$. First consider the category $C(B, \aleph_0)$ of Definition 4.7. The category $\mathbf{U}$ is defined to be the full subcategory of $C(B, \aleph_0)$ whose objects are quasi-isomorphic to perfect complexes. The map $X \mapsto B \otimes_A X$ defines an exact functor $C(A, \sigma, \aleph_0) \longrightarrow C(B, \aleph_0)$, which takes $\mathbf{T} \subset C(A, \sigma, \aleph_0)$ to $\mathbf{U} \subset C(B, \aleph_0)$. The diagram of permissible Waldhausen categories

$$\begin{tikzcd}
\mathbf{R} \arrow[r] & \mathbf{S} \arrow[rr, "\pi"] \arrow[dr] & & \mathbf{T} \arrow[r, "T"] & \mathbf{U} \\
& & \mathbf{S}_{\mathbf{R}} \arrow[ur, "i"'] & &
\end{tikzcd}$$

gives, when we pass to derived categories, precisely the top row of $(**)$.

Before applying Waldhausen's K–theory to this diagram, it is helpful to extend it a little bit. Put $\mathbf{D} = C^{\mathrm{perf}}(B)$. Remember that tensor product with B takes perfect complexes to perfect complexes. Therefore we have a commutative square, where the vertical maps are induced by tensor product with B:

$$\begin{array}{ccccccc}
C^{\mathrm{perf}}(A) & = & \mathbf{S} & \longrightarrow & \mathbf{T} & \subset & C(A, \sigma, \aleph_0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C^{\mathrm{perf}}(B) & = & \mathbf{D} & \longrightarrow & \mathbf{U} & \subset & C(B, \aleph_0)
\end{array}$$

The map from $C^{\mathrm{perf}}(A) = \mathbf{S}$ to $C^{\mathrm{perf}}(B) = \mathbf{D}$ clearly factors through $\mathbf{S}_{\mathbf{R}}$; after all, the only change from $\mathbf{S}$ to $\mathbf{S}_{\mathbf{R}}$ is in the weak equivalences, and the larger class of weak equivalences in $\mathbf{S}_{\mathbf{R}}$ maps to weak equivalences in $\mathbf{D}$. We therefore have a commutative diagram:

$$\begin{tikzcd}
\mathbf{R} \arrow[r] & \mathbf{S} \arrow[rr, "\pi"] \arrow[dr] & & \mathbf{T} \arrow[rr, "T"] & & \mathbf{U} \\
& & \mathbf{S}_{\mathbf{R}} \arrow[ur, "i"'] \arrow[rr] & & \mathbf{D} \arrow[ur] &
\end{tikzcd} \qquad (***)$$

In this diagram, the map $\mathbf{D} \longrightarrow \mathbf{U}$ induces an equivalence of derived categories; both $\mathbf{D}$ and $\mathbf{U}$ are models for $D^{\mathrm{perf}}(B)$. By Theorem 2.2 the map $K(\mathbf{D}) \longrightarrow K(\mathbf{U})$ is a homotopy equivalence. Gillet's [12, 6.2] tells us that

$$K(\mathbf{S}) = K(C^{\mathrm{perf}}(A)) = K(A), \qquad \text{and} \qquad K(\mathbf{D}) = K(C^{\mathrm{perf}}(B)) = K(B).$$

In K–theory, $(***)$ yields:

$$\begin{tikzcd}
K(\mathbf{R}) \arrow[r] & K(A) \arrow[rr, "K(\pi)"] \arrow[dr] & & K(\mathbf{T}) \arrow[rr, "K(T)"] & & K(\mathbf{U}) \\
& & K(\mathbf{S_R}) \arrow[ur, "K(i)"'] \arrow[rr] & & K(B) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

In the remainder of this section we will analyse necessary and sufficient conditions for the functor $T\colon \mathcal{T}^c \longrightarrow D^{\mathrm{perf}}(B)$ to be an equivalence of categories. When it is, it follows from Theorem 2.2 that $K(T)\colon K(\mathbf{T}) \longrightarrow K(\mathbf{U})$ is a homotopy equivalence. From the diagram and our previous discussion it then follows that, up to nonsense in degree (-1), $K(\mathbf{R})$ is the homotopy fiber of the natural map $K(A) \longrightarrow K(B)$.

Lemma 5.5 *Let the notation be as in Lemma 5.3. If the functor $T\colon \mathcal{T}^c \longrightarrow D^{\mathrm{perf}}(B)$ is an equivalence, then*

(i) *The functor $\pi\colon D(A) = \mathcal{S} \longrightarrow \mathcal{T}$ induces a homomorphism*

$$\mathrm{Hom}_{D(A)}(A, A) \longrightarrow \mathcal{T}(\pi A, \pi A)$$

which can be naturally identified with $A^{\mathrm{op}} \longrightarrow B^{\mathrm{op}}$.

(ii) *For all $n \neq 0$, $\mathcal{T}(\pi A, \Sigma^n \pi A) = 0$.*

Proof If T is an equivalence, then for all $s, t \in \mathcal{T}^c \subset \mathcal{T}$ we must have

$$\mathcal{T}(s, t) = \mathrm{Hom}_{D(B)}(Ts, Tt).$$

Put $s = \pi A$ and $t = \Sigma^n \pi A$. Then $Ts = T\pi A = B \otimes_A A = B$, and $Tt = \Sigma^n B$. This gives

$$\mathcal{T}(\pi A, \Sigma^n \pi A) = \mathrm{Hom}_{D(B)}(B, \Sigma^n B)$$

and the right hand side is B^{op} in degree zero, and vanishes if $n \neq 0$. Moreover, the induced map $A^{\mathrm{op}} \longrightarrow B^{\mathrm{op}}$ is the natural homomorphism. □

The most interesting case is $B = \sigma^{-1} A$. The proof of the next Proposition is a small modification of ideas that may be found in Rickard's [32].

Proposition 5.6 *Let the notation be as in Lemma 5.3, but with $B = \sigma^{-1}A$. The following are equivalent:*

(i) *The functor $T\colon \mathcal{T} \longrightarrow D(\sigma^{-1}A)$ is an equivalence.*

(ii) *The restriction to compact objects, that is $T\colon \mathcal{T}^c \longrightarrow D^{\mathrm{perf}}(\sigma^{-1}A)$, is an equivalence.*

(iii) $\mathcal{T}(\pi A, \Sigma^n \pi A) = \begin{cases} \{\sigma^{-1}A\}^{\mathrm{op}} & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$
where the isomorphism $\mathcal{T}(\pi A, \pi A) = \{\sigma^{-1}A\}^{\mathrm{op}}$ is as A^{op}–algebras.

Proof (i)$\Longrightarrow$(ii) is clear; if T is an equivalence, then it restricts to an equivalence on compact objects. (ii)$\Longrightarrow$(iii) was proved in Lemma 5.5. It remains to prove (iii)$\Longrightarrow$(i).

Assume now that (iii) holds. Among other things, we know that there is some isomorphism $\mathcal{T}(\pi A, \Sigma^n \pi A) = \{\sigma^{-1}A\}^{\mathrm{op}}$ of A^{op}–algebras. We first want to show how it follows that the functor T induces an isomorphism

$$\mathcal{T}(\pi A, \pi A) \xrightarrow{\quad T \quad} \mathrm{Hom}_{D(\sigma^{-1}A)}(\sigma^{-1}A, \sigma^{-1}A) = \sigma^{-1}A.$$

We have ring homomorphisms

$$\mathrm{Hom}_{D(A)}(A, A) \xrightarrow{\quad \pi \quad} \mathcal{T}(\pi A, \pi A) \xrightarrow{\quad T \quad} \mathrm{Hom}_{D(\sigma^{-1}A)}(\sigma^{-1}A, \sigma^{-1}A).$$

That is,

$$A^{\mathrm{op}} \xrightarrow{\quad \pi \quad} \mathcal{T}(\pi A, \pi A) \xrightarrow{\quad T \quad} \{\sigma^{-1}A\}^{\mathrm{op}}.$$

The composite $T\pi\colon A^{\mathrm{op}} \longrightarrow \{\sigma^{-1}A\}^{\mathrm{op}}$ is easily computed to be the natural map. By hypothesis (iii), there is an isomorphism $\mathcal{T}(\pi A, \pi A) \cong \{\sigma^{-1}A\}^{\mathrm{op}}$, as A^{op}–algebras. But then both $\pi\colon A \longrightarrow \mathcal{T}(\pi A, \pi A)^{\mathrm{op}}$ and $T\pi : A \longrightarrow \sigma^{-1}A$ are initial in the category of σ–inverting ring homomorphisms. Hence the map $T\colon \mathcal{T}(\pi A, \pi A) \longrightarrow \{\sigma^{-1}A\}^{\mathrm{op}}$ must be an isomorphism.

Let $\mathcal{C} \subset \mathcal{T}$ be the full subcategory with objects

$$\mathrm{Ob}(\mathcal{C}) = \left\{ x \in \mathrm{Ob}(\mathcal{T}) \;\middle|\; \begin{array}{c} \forall n \in \mathbb{Z} \text{ the map induced by } T \\ \mathcal{T}(\pi A, \Sigma^n x) \longrightarrow \mathrm{Hom}_{D(\sigma^{-1}A)}(T\pi A, \Sigma^n T x) \\ \text{is an isomorphism} \end{array} \right\}.$$

Since $\pi A \in \mathcal{T}$ and $T\pi A = \sigma^{-1}A \in D(\sigma^{-1}A)$ are both compact, the category $\mathcal{C}$ is closed under coproducts. It is clearly closed under triangles, and by (iii) and the above it contains πA. Its inverse image under the projection map

$$\pi\colon D(A) = \mathcal{S} \longrightarrow \mathcal{S}/\mathcal{R} = \mathcal{T}$$

is a triangulated subcategory, closed under coproducts and containing A. Now Lemma 4.3 tells us that $\pi^{-1}\mathcal{C} = \mathcal{S}$, and hence $\mathcal{C} = \mathcal{T}$.

Next let $\mathcal{D} \subset \mathcal{T}$ be the full subcategory with objects

$$\mathrm{Ob}(\mathcal{D}) = \left\{ x \in \mathrm{Ob}(\mathcal{T}) \;\middle|\; \begin{array}{c} \forall n \in \mathbb{Z} \text{ and } \forall y \in \mathcal{T} \text{ the map} \\ \mathcal{T}(x, \Sigma^n y) \longrightarrow \mathrm{Hom}_{D(\sigma^{-1}A)}(Tx, \Sigma^n Ty) \\ \text{is an isomorphism} \end{array} \right\}.$$

By the above, $\mathcal{D}$ contains πA. It is clear that $\mathcal{D}$ is closed under triangles and coproducts. As above, it follows that $\pi^{-1}\mathcal{D} = \mathcal{S}$, and hence $\mathcal{D} = \mathcal{T}$.

This proves that T is fully faithful. It embeds $\mathcal{T}$ as a full, triangulated subcategory of $D(\sigma^{-1}A)$, closed under coproducts and containing $T\pi A = \sigma^{-1}A$. Applying Lemma 4.3 to the inclusion $\mathcal{T} \subset D(\sigma^{-1}A)$, we conclude it must be an equivalence. □

6 The case $n > 0$

Let the notation be as in Proposition 5.6. That is, A is a ring, σ is a set of maps of finitely generated, projective A–modules, $\sigma^{-1}A$ is the Cohn localisation, $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ are the triangulated categories of Definition 4.1, and $T\colon \mathcal{T} \longrightarrow D(\sigma^{-1}A)$ is the functor of Lemma 5.3. Proposition 5.6 tells us that everything is reduced to computing the groups $\mathcal{T}(\pi A, \Sigma^n \pi A)$. In the next three sections we will prove

(i) If $n > 0$, then $\mathcal{T}(\pi A, \Sigma^n \pi A) = 0$. [This section].

(ii) If $n = 0$, then the ring homomorphism $A^{\mathrm{op}} \longrightarrow \mathcal{T}(\pi A, \Sigma^n \pi A)$ agrees with $A^{\mathrm{op}} \longrightarrow \{\sigma^{-1}A\}^{\mathrm{op}}$. [Section 7].

(iii) The groups $\mathcal{T}(\pi A, \Sigma^n \pi A)$ vanish for all $n < 0$ if and only if the groups $\mathrm{Tor}_n^A(\sigma^{-1}A, \sigma^{-1}A)$ vanish for all $n > 0$. [Section 8].

In every paper there comes a time for hard work. The day of reckoning has come in this paper. We now have to prove something. The key tool in the proofs is:

Proposition 6.1 *The functor $\pi\colon \mathcal{S} \longrightarrow \mathcal{T}$ has a right adjoint $G\colon \mathcal{T} \longrightarrow \mathcal{S}$. The unit of adjunction $\eta_x\colon x \longrightarrow G\pi x$ can be completed to a distinguished triangle*

$$k \longrightarrow x \xrightarrow{\eta_x} G\pi x \longrightarrow \Sigma k.$$

In this triangle, the object k lies in $\mathcal{R}$.

Proof See [22, Lemma 1.7]. The notation there is slightly different; the functor we have been calling $\pi\colon \mathcal{S} \longrightarrow \mathcal{T}$ is called j^* there, and the adjoint we call G goes by the name j_* there. $\square$

Reminder 6.2 An object $y \in \mathcal{S}$ is called *σ–local* (or just *local* if σ is understood) if for all $r \in \mathcal{R}$ we have $\mathcal{S}(r, y) = 0$. If x is any object of $\mathcal{S}$, then the object $G\pi x$ is local; after all

$$\begin{aligned} \mathcal{S}(r, G\pi x) &= \mathcal{T}(\pi r, \pi x) && \text{by adjunction} \\ &= \mathcal{T}(0, \pi x) && \text{since } r \in \mathcal{R}, \text{ hence } \pi r = 0 \\ &= 0. \end{aligned}$$

Lemma 6.3 *If $x \in \mathcal{S}$ is a local object, then so are its t–structure truncations $x^{\leq n}$ and $x^{\geq n}$.*

Proof Pick a σ–local object x and an integer $n \in \mathbb{Z}$; we will show first that $x^{\geq n}$ is σ–local. Without loss of generality we may assume $n = 0$. To prove that $x^{\geq 0}$ is σ–local, take any $s \in \sigma$. We will show that $\mathcal{S}(s, x^{\geq 0}) = 0$. Assume for a second that we have shown this, for all $s \in \sigma$. This will mean that the full subcategory $\mathcal{C} \subset \mathcal{S}$ whose objects are

$$\mathrm{Ob}(\mathcal{C}) = \{c \in \mathcal{S} \mid \forall n \in \mathbb{Z},\ \mathcal{S}(\Sigma^n c, x^{\geq 0}) = 0\}$$

contains σ. But $\mathcal{C}$ is clearly triangulated and closed under coproducts. Hence $\mathcal{R} \subset \mathcal{C}$, which means that $x^{\geq 0}$ is local.

Hence it needs to be shown that, for any $s \in \sigma$ and any local x, $\mathcal{S}(s, x^{\geq 0}) = 0$. Let s be the chain complex

$$\cdots \longrightarrow 0 \longrightarrow P_i \xrightarrow{s_i} Q_i \longrightarrow 0 \longrightarrow \cdots$$

Now P_i is in some degree m and Q_i is in degree $m+1$. There are two cases:

Case 1 If $m \leq -2$, then s is a complex concentrated in degrees ≤ -1; that is, $s \in \mathcal{S}^{\leq -1}$. But $x^{\geq 0}$ is in $\mathcal{S}^{\geq 0}$. Hence all maps $s \longrightarrow x^{\geq 0}$ vanish.

Case 2 Suppose $m \geq -1$. The t–structure gives a distinguished triangle

$$x^{\leq -1} \longrightarrow x \longrightarrow x^{\geq 0} \xrightarrow{w} \Sigma x^{\leq -1}.$$

For any map $s \longrightarrow x^{\geq 0}$, the composite

$$s \longrightarrow x^{\geq 0} \xrightarrow{w} \Sigma x^{\leq -1}$$

is a map from a bounded above complex of projectives s to some object in $\mathcal{S} = D(A)$, and hence it is represented by a chain map. But the chain complex

s lives in degrees m and $m+1$, both of which are ≥ -1, while the complex $\Sigma x^{\leq -1}$ lies in $\mathcal{S}^{\leq -2}$. Hence the map vanishes. From the triangle we deduce that the map $s \longrightarrow x^{\geq 0}$ must factor as

$$s \longrightarrow x \longrightarrow x^{\geq 0}.$$

Now x is σ–local by hypothesis, and hence any map $s \longrightarrow x$ vanishes.

This proves that $x^{\geq 0}$ is σ–local. We have a triangle

$$x^{\leq -1} \longrightarrow x \longrightarrow x^{\geq 0} \xrightarrow{w} \Sigma x^{\leq -1} .$$

The long exact sequence for $\mathcal{S}(r,-)$, with $r \in \mathcal{R}$, allows us to deduce that $x^{\leq -1}$ is also σ–local. Shifting by powers of Σ, we deduce that $x^{\leq n}$ is σ–local for any $n \in \mathbb{Z}$. □

Lemma 6.4 *Let the notation be as above. If $x \in \mathcal{S}^{\leq n}$, then so is $G\pi x$.*

Proof We may assume without loss that $n = 0$. Pick any $x \in \mathcal{S}^{\leq 0}$. By Reminder 6.2, $G\pi x$ is local. By Lemma 6.3 so is $\{G\pi x\}^{\leq 0}$.

Now the unit of adjunction $\eta_x\colon x \longrightarrow G\pi x$ is a map from an object $x \in \mathcal{S}^{\leq 0}$, and therefore factors (uniquely) as

$$x \xrightarrow{\alpha} \{G\pi x\}^{\leq 0} \xrightarrow{f} G\pi x .$$

On the other hand, we have a triangle

$$k \longrightarrow x \xrightarrow{\eta_x} G\pi x \longrightarrow \Sigma k$$

with $k \in \mathcal{R}$. The composite

$$k \longrightarrow x \xrightarrow{\alpha} \{G\pi x\}^{\leq 0}$$

must vanish, since $k \in \mathcal{R}$ and $\{G\pi x\}^{\leq 0}$ is local. It follows that α factors as

$$x \xrightarrow{\eta_x} G\pi x \xrightarrow{g} \{G\pi x\}^{\leq 0}.$$

The composite

$$x \xrightarrow{\eta_x} G\pi x \xrightarrow{g} \{G\pi x\}^{\leq 0} \xrightarrow{f} G\pi x$$

is η_x, by construction of f and g. It follows that

$$x \xrightarrow{\eta_x} G\pi x \xrightarrow{1-fg} G\pi x$$

vanishes. From the triangle

$$k \longrightarrow x \xrightarrow{\eta_x} G\pi x \longrightarrow \Sigma k$$

we have that $1 - fg$ must factor through a map $\Sigma k \longrightarrow G\pi x$. But as $\Sigma k \in \mathcal{R}$ and $G\pi x$ is local, $1 - fg$ must vanish. In other words, $fg = 1$.

But then the identity on $G\pi x$ factors through an object in $\mathcal{S}^{\leq 0}$. Therefore $1\colon H^n(G\pi x) \longrightarrow H^n(G\pi x)$ vanishes for all $n > 0$. This means that for $n > 0$ we have $H^n(G\pi x) = 0$. In other words, $G\pi x \in \mathcal{S}^{\leq 0}$. □

Corollary 6.5 *Let the notation be as above. For any $n > 0$ we have*

$$\mathcal{T}(\pi A, \Sigma^n \pi A) = 0.$$

Proof We know $A \in \mathcal{S}^{\leq 0}$, and from Lemma 6.4 we deduce $G\pi A \in \mathcal{S}^{\leq 0}$. Now we compute

$$\begin{array}{rcll} \mathcal{T}(\pi A, \Sigma^n \pi A) & = & \mathcal{S}(A, \Sigma^n G\pi A) & \text{by adjunction} \\ & = & H^n(G\pi A) & \text{because } \mathcal{S}(A, \Sigma^n X) = H^n(X) \\ & = & 0 & \text{because } G\pi A \in \mathcal{S}^{\leq 0}. \end{array}$$

□

7 The case $n = 0$

In this section we will show that $\mathcal{T}(\pi A, \pi A) = \{\sigma^{-1}A\}^{\mathrm{op}}$. We know that $\mathcal{T}(\pi A, \pi A)$ is a ring, and comes with a natural ring homomorphism

$$A^{\mathrm{op}} = \mathcal{S}(A, A) \longrightarrow \mathcal{T}(\pi A, \pi A) = B^{\mathrm{op}}.$$

What we prove is that the ring homomorphism $A \longrightarrow B$ above is the initial σ–inverting homomorphism.

Lemma 7.1 *The unit of adjunction gives us a map $\eta_A\colon A \longrightarrow G\pi A$. Applying the functor H^0 gives a map*

$$A = H^0(A) \xrightarrow{H^0(\eta_A)} H^0(G\pi A).$$

We assert that this agrees with the natural homomorphism

$$A \longrightarrow \mathcal{T}(\pi A, \pi A)^{\mathrm{op}} = B.$$

That is, there is an isomorphism of left A–modules $B \longrightarrow H^0(G\pi A)$, commuting with the inclusion of A.

Proof It is clear how to give an isomorphism of sets $H^0(G\pi A) \cong \mathcal{T}(\pi A, \pi A)$. We have

$$\begin{array}{rcll} \mathcal{T}(\pi A, \pi A) & = & \mathcal{S}(A, G\pi A) & \text{by adjunction} \\ & = & H^0(G\pi A). & \end{array}$$

As sets, we have an equality $\mathcal{T}(\pi A, \pi A) = \mathcal{T}(\pi A, \pi A)^{\mathrm{op}}$. Let us call the isomorphism of sets given above $\varphi\colon \mathcal{T}(\pi A, \pi A)^{\mathrm{op}} \longrightarrow H^0(G\pi A)$. We have a triangle

$$\begin{tikzcd} & \mathcal{T}(\pi A, \pi A)^{\mathrm{op}} \arrow[d, "\varphi"] \\ A \arrow[ur] \arrow[r, "H^0(\eta_A)"'] & H^0(G\pi A). \end{tikzcd}$$

We need to show that the triangle commutes, and that φ is a map of left A–modules. For a second let us suppose we know that φ is a homomorphism of left A–modules. Then all three maps are A–module homomorphisms, and the commutativity can be checked by evaluating the maps on the single element $1 \in A$. We leave this to the reader.

It remains to show that φ is a homomorphism of left A–modules. We must show that φ takes right multiplication by $a \in A$ in the ring $\mathcal{T}(\pi A, \pi A)$ to left multiplication by a in $H^0(G\pi A)$.

Therefore we let x be any element of $\mathcal{T}(\pi A, \pi A)$, and let $a \in A$. Then xa is an element of the ring $\mathcal{T}(\pi A, \pi A)$; it is the composite

$$\pi A \xrightarrow{\pi\rho_a} \pi A \xrightarrow{x} \pi A,$$

where $\rho_a\colon A \longrightarrow A$ is the map induced by right multiplication by a. The naturality of $\eta\colon 1 \longrightarrow G\pi$ gives a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\rho_a"] \arrow[d, "\eta_A"'] & A \arrow[d, "\eta_A"] & \\ G\pi A \arrow[r, "G\pi\rho_a"'] & G\pi A \arrow[r, "Gx"'] & G\pi A. \end{tikzcd}$$

Consider the image of $1 \in A$. We have

$$\begin{array}{rcll} \varphi(xa) & = & G(x\ \pi\rho_a)\,\eta_A(1) & \text{adjunction formula} \\ & = & Gx\ G\pi\rho_a\ \eta_A(1) & G \text{ respects composition} \\ & = & Gx\ \eta_A\ \rho_a(1) & \text{by commutative diagram above} \\ & = & Gx\ \eta_A(a) & \text{since } \rho_a(1) = a \\ & = & a\,Gx\ \eta_A(1) & \text{since } Gx\,\eta_A \text{ is a homomorphism} \\ & & & \text{of left } A\text{–modules} \\ & = & a\,\varphi(x). & \end{array}$$

□

Lemma 7.2 *The ring homomorphism $A \longrightarrow \mathcal{T}(\pi A, \pi A)^{\mathrm{op}}$ is σ–inverting.*

Proof Let $s_i\colon P_i \longrightarrow Q_i$ be any element of σ. Because $G\pi A$ is local we know that, for any $n \in \mathbb{Z}$, $\mathcal{S}(\Sigma^n s_i, G\pi A) = 0$. From the distinguished triangle

$$P_i \longrightarrow Q_i \longrightarrow s_i \longrightarrow \Sigma P_i$$

we conclude that the natural map

$$\mathcal{S}(Q_i, G\pi A) \longrightarrow \mathcal{S}(P_i, G\pi A)$$

is an isomorphism. But both P_i and Q_i are projective A–modules, viewed as complexes concentrated in degree 0. Therefore the isomorphism above is the natural map

$$\mathrm{Hom}_A\big(Q_i, H^0(G\pi A)\big) \longrightarrow \mathrm{Hom}_A\big(P_i, H^0(G\pi A)\big).$$

Put $B = \mathcal{T}(\pi A, \pi A)^{\mathrm{op}}$. As left A–modules, we have $B \cong H^0(G\pi A)$. By the above we know that

$$\mathrm{Hom}_A(Q_i, B) \longrightarrow \mathrm{Hom}_A(P_i, B)$$

is an isomorphism of right B–modules. Applying the functor $\mathrm{Hom}_B(-, B)$ to it, and recalling that

$$\mathrm{Hom}_B\big(\mathrm{Hom}_A(P, B), B\big) = B \otimes_A P,$$

we deduce that

$$B \otimes_A P_i \longrightarrow B \otimes_A Q_i$$

is also an isomorphism. □

Lemma 7.3 *Any σ–inverting ring homomorphism $A \longrightarrow C$ factors through the natural map $A \longrightarrow \mathcal{T}(\pi A, \pi A)^{\mathrm{op}}$.*

Proof By Lemma 5.3 the functor $D(A) \longrightarrow D(C)$, taking X to $C^L\otimes_A X$, factors as

$$D(A) \xrightarrow{\ \pi\ } \mathcal{T} \xrightarrow{\ T\ } D(C).$$

Hence we have ring homomorphisms

$$\mathrm{Hom}_{D(A)}(A, A) \longrightarrow \mathcal{T}(\pi A, \pi A) \longrightarrow \mathrm{Hom}_{D(C)}(T\pi A, T\pi A).$$

Now $T\pi A = C \otimes A = C$. Taking opposed rings, we have

$$A \longrightarrow \mathcal{T}(\pi A, \pi A)^{\mathrm{op}} \longrightarrow C,$$

and the composite is easily seen to be the given map $A \longrightarrow C$. □

Theorem 7.4 *The natural map $A \longrightarrow \mathcal{T}(\pi A, \pi A)^{\mathrm{op}}$ is the initial object in the category of σ–inverting homomorphisms.*

Proof Lemma 7.2 tells us that the map is σ–inverting, while Lemma 7.3 tells us any σ–inverting map factors through it. We need to prove the factorisation unique. We will prove the uniqueness even as maps of left A–modules.

Assume therefore that we are given a σ–inverting ring homomorphism $A \longrightarrow C$. Suppose we have a factorisation, as maps of left A–modules,

$$A \longrightarrow \mathcal{T}(\pi A, \pi A)^{\mathrm{op}} \longrightarrow C.$$

We wish to show it unique.

By Lemma 7.2, the map $A \longrightarrow \mathcal{T}(\pi A, \pi A)^{\mathrm{op}}$ can be identified with $A \longrightarrow H^0(G\pi A)$. By Lemma 6.4, $G\pi A \in \mathcal{S}^{\leq 0}$. This allows us to identify, in the derived category $\mathcal{S} = D(A)$, the objects $H^0(G\pi A)$ and $\{G\pi A\}^{\geq 0}$. In the derived category $D(A)$, our factorisation of $A \longrightarrow C$ becomes

$$A \longrightarrow \{G\pi A\}^{\geq 0} \longrightarrow C.$$

We can factor this further as

$$A \longrightarrow G\pi A \longrightarrow \{G\pi A\}^{\geq 0} \longrightarrow C.$$

In the distinguished triangle

$$\{G\pi A\}^{\leq -1} \longrightarrow G\pi A \longrightarrow \{G\pi A\}^{\geq 0} \longrightarrow \Sigma\{G\pi A\}^{\leq -1}$$

we have that both $\{G\pi A\}^{\leq -1}$ and $\Sigma\{G\pi A\}^{\leq -1}$ lie in $\mathcal{S}^{\leq -1}$. Since $C \in \mathcal{S}^{\geq 0}$, we conclude that the map

$$\mathcal{S}(\{G\pi A\}^{\geq 0}, C) \longrightarrow \mathcal{S}(G\pi A, C)$$

is an isomorphism. The factorisation

$$A \longrightarrow G\pi A \longrightarrow \{G\pi A\}^{\geq 0} \longrightarrow C$$

is completely determined by

$$A \longrightarrow G\pi A \longrightarrow C.$$

Now consider the distinguished triangle

$$k \longrightarrow A \longrightarrow G\pi A \longrightarrow \Sigma k.$$

We know that $k \in \mathcal{R}$. But then, for every $n \in \mathbb{Z}$,

$$\begin{aligned} \mathrm{Hom}_{D(A)}(\Sigma^n k, C) &= \mathrm{Hom}_{D(C)}(\Sigma^n C^L\!\otimes_A k\ , C) \\ &= 0. \end{aligned}$$

The last equality is because $C^L\!\otimes_A k = 0$, by Lemma 5.3. From the distinguished triangle we conclude that

$$\mathcal{S}(G\pi A, C) \longrightarrow \mathcal{S}(A, C)$$

is an isomorphism. The factorisation

$$A \longrightarrow G\pi A \longrightarrow C$$

is unique. □

8 The case $n < 0$

In this section we will study what happens to the groups $\mathcal{T}(\pi A, \Sigma^n \pi A) = H^n(G\pi A)$ when $n < 0$. We will prove that they vanish if and only if $\sigma^{-1}A$ is stably flat; that is, if and only if $\mathrm{Tor}_n^A(\sigma^{-1}A, \sigma^{-1}A) = 0$ for all $n > 0$. We even prove more. We prove that the first non-vanishing $\mathrm{Tor}_n^A(\sigma^{-1}A, \sigma^{-1}A)$ is isomorphic (up to changing n to $n-1$) with the first non-vanishing $H^{-n}(G\pi A)$. For the precise statement see Theorem 8.7.

It might help to give a sketch of the argument. Lemmas 8.1 and 8.2 prove that all the homology groups $H^n(G\pi A)$ are naturally modules over $\sigma^{-1}A$. Lemma 8.3 is a technical observation: let $A \longrightarrow B$ be a ring homomorphism, and let M be a B module. Under some hypotheses one can say something about $\mathrm{Tor}_i^A(B, M)$. The idea is to apply these observations in the case where $B = \sigma^{-1}A$ and the $\sigma^{-1}A$–modules in question are $H^n(G\pi A)$.

Lemma 8.4 and Remark 8.5 are the crucial part of the argument. They introduce the spectral sequence which does the work. Lemma 8.6 tells us that the technical conditions of Lemma 8.3 are satisfied in the case of the ring homomorphism $A \longrightarrow \sigma^{-1}A$. And then Theorem 8.7 clinches the computation.

Lemma 8.1 *Let M be any A–module. There is an isomorphism of left A–modules*

$$H^0(G\pi M) \cong \{\sigma^{-1}A\} \otimes_A M.$$

The A–module structure on $H^0(G\pi M)$ therefore extends (uniquely) to an $\sigma^{-1}A$–module structure.

Proof Put $B = \sigma^{-1}A$, and $\theta\colon A \longrightarrow B$ the initial σ–inverting homomorphism. Let $x \in D(A)$ be any object. The map

$$A{}^L\!\otimes_A x \xrightarrow{\theta\otimes 1} B{}^L\!\otimes_A x$$

gives a natural transformation from the identity functor to $B{}^L\!\otimes_A(-)$. In the distinguished triangle

$$k \longrightarrow x \xrightarrow{\eta_x} G\pi x \longrightarrow \Sigma k$$

the object k lies in $\mathcal{R}$. Applying the functor $\mathrm{Hom}_{D(A)}\big(- \,,\, B^L\!\otimes_A x\big)$ to this triangle, we deduce an exact sequence

$$\begin{tikzcd}
\mathrm{Hom}_{D(A)}\big(\Sigma k \,,\, B^L\!\otimes_A x\big) \arrow[d] & \\
\mathrm{Hom}_{D(A)}\big(G\pi x \,,\, B^L\!\otimes_A x\big) \arrow[r] & \mathrm{Hom}_{D(A)}\big(x \,,\, B^L\!\otimes_A x\big) \arrow[d] \\
& \mathrm{Hom}_{D(A)}\big(k \,,\, B^L\!\otimes_A x\big).
\end{tikzcd}$$

But for every $n \in \mathbb{Z}$ there is an isomorphism

$$\mathrm{Hom}_{D(A)}\big(\Sigma^n k \,,\, B^L\!\otimes_A x\big) = \mathrm{Hom}_{D(B)}\big(\Sigma^n B^L\!\otimes_A k \,,\, B^L\!\otimes_A x\big).$$

This vanishes since, by Lemma 5.3, $B^L\!\otimes_A k = 0$. From the exact sequence we conclude that the map

$$\mathrm{Hom}_{D(A)}\big(G\pi x \,,\, B^L\!\otimes_A x\big) \longrightarrow \mathrm{Hom}_{D(A)}\big(x \,,\, B^L\!\otimes_A x\big)$$

is an isomorphism. Our map $\theta \otimes 1\colon\ A^L\!\otimes_A x \longrightarrow B^L\!\otimes_A x$ factors uniquely as

$$x \xrightarrow{\ \eta_x\ } G\pi x \xrightarrow{\ \varphi_x\ } B^L\!\otimes_A x.$$

The uniqueness allows us to easily show that the φ_x assemble to a natural transformation $\varphi\colon\ G\pi(-) \Longrightarrow B^L\!\otimes_A(-)$. Applying the functor H^0, we have a natural transformation

$$H^0(\varphi)\colon\ H^0\big(G\pi(-)\big) \Longrightarrow H^0\big(B^L\!\otimes_A(-)\big).$$

What we will show is that,when x is a chain complex concentrated in degree 0 (ie x is just a module), then $H^0(\varphi)$ is an isomorphism. Observe that, when x is just a module concentrated in degree 0, then $H^0\big(B^L\!\otimes_A(-)\big)$ simplifies to $B \otimes_A x$.

Let x be A, viewed as an object in $D(A)$ concentrated in degree 0. We have maps

$$A \xrightarrow{\ \eta_A\ } G\pi A \xrightarrow{\ \varphi_A\ } B^L\!\otimes_A A = B.$$

Applying the functor H^0, this becomes

$$A \xrightarrow{\ H^0(\eta_A)\ } H^0(G\pi A) \xrightarrow{\ H^0(\varphi_A)\ } B;$$

By Lemma 7.1 and Theorem 7.4 there is an isomorphism of left A–modules $\rho\colon\ H^0(G\pi A) \longrightarrow B$, so that the composite

$$A \xrightarrow{\ H^0(\eta_A)\ } H^0(G\pi A) \xrightarrow{\ \rho\ } B$$

equals $\theta\colon A \longrightarrow B$. But in the proof of Theorem 7.4 we saw that any such factorisation (as maps of left A–modules) is unique. Hence $\rho = H^0(\varphi_A)$, and $H^0(\varphi_A)$ must be an isomorphism. Because both $H^0\big(G\pi(-)\big)$ and $B \otimes_A (-)$ commute with direct sums, $H^0(\varphi_x)$ must be an isomorphism for any free A–module x.

Let M be any A–module. Choose a free module F surjecting onto M. We have a short exact sequence of A–modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0.$$

We deduce a commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^0(G\pi K) & \longrightarrow & H^0(G\pi F) & \longrightarrow & H^0(G\pi M) & \longrightarrow & H^1(G\pi K) \\
{\scriptstyle H^0(\varphi_K)}\downarrow & & {\scriptstyle H^0(\varphi_F)}\downarrow & & {\scriptstyle H^0(\varphi_M)}\downarrow & & \downarrow \\
B \otimes_A K & \longrightarrow & B \otimes_A F & \longrightarrow & B \otimes_A M & \longrightarrow & 0.
\end{array}$$

By Lemma 6.4 we know that $G\pi K \in D(A)^{\leq 0}$, hence $H^1(G\pi K) = 0$. By the above we know that $H^0(\varphi_F)$ is an isomorphism. This allows us to conclude first that $H^0(\varphi_M)$ is surjective. This being true for every A–module M, it must be true for K. This means $H^0(\varphi_K)$ is surjective, and hence $H^0(\varphi_M)$ must be an isomorphism.

We have proved that $H^0(G\pi M)$ is isomorphic to $B \otimes_A M$, which is obviously a module over $B = \sigma^{-1}A$. The fact that the B–module structure is unique is easy: To say that X is an A–module is to give a ring homomorphism

$$A \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X, X).$$

To say this extends to a B–module structure is to give a factorisation of the ring homomorphism through $\theta\colon A \longrightarrow B$. The fact that θ is initial says that any such factorisation is unique. □

Lemma 8.2 *For any local object $x \in D(A)$ and any integer $n \in \mathbb{Z}$, the A–module structure on $H^n(x)$ extends (uniquely) to a $\sigma^{-1}A$–module structure.*

Proof Replacing x by $\Sigma^n x$, we may assume $n = 0$. The object x is local. Lemma 6.3 tells us that so is $\{x^{\leq 0}\}^{\geq 0}$. That is the module $M = H^0(x)$, viewed as a complex concentrated in degree 0, is a local object. We need to prove that M is a module over $\sigma^{-1}A$.

Consider the distinguished triangle

$$k \xrightarrow{\ \alpha\ } M \longrightarrow G\pi M \longrightarrow \Sigma k.$$

Since M is local and $k \in \mathcal{R}$, it follows that $\alpha\colon\ k \longrightarrow M$ must vanish. Therefore the map $G\pi M \longrightarrow \Sigma k$ admits a splitting; there is a split inclusion $\beta\colon\ \Sigma k \longrightarrow G\pi M$. By Reminder 6.2 the object $G\pi M$ is local, while $\Sigma k \in \mathcal{R}$. It follows that $\beta\colon\ \Sigma k \longrightarrow G\pi M$ must vanish. Hence $k = 0$, and M is isomorphic to $G\pi M$. But then

$$M = H^0(M) = H^0(G\pi M),$$

and by Lemma 8.1 $H^0(G\pi M)$ is naturally a $\sigma^{-1}A$–module. □

The next lemmas are based on studying two hyperTor spectral sequences. The key one, of Remark 8.5, has for its E_2 term $\mathrm{Tor}^A_{-i}\big(\sigma^{-1}A\,,\,H^j(G\pi A)\big)$. Lemma 8.2 tells us that $H^j(G\pi A)$ is naturally a $\sigma^{-1}A$–module. Thus we are interested in general lemmas that apply to $\mathrm{Tor}^A_n(B,M)$, where $A \longrightarrow B$ is a ring homomorphism and M is a B–module. We do this by means of another hyperTor spectral sequence.

Lemma 8.3 *Suppose $A \longrightarrow B$ is a ring homomorphism such that the multiplication map $\mu\colon\ B \otimes_A B \longrightarrow B$ is an isomorphism. Suppose also that for some $n \geq 1$*

$$\mathrm{Tor}^A_i(B,B) \ = \ 0 \ (1 \leq i \leq n).$$

Then for every B–module M we have:

(i) *The multiplication map $B \otimes_A M \longrightarrow M$ is an isomorphism.*

(ii) *$\mathrm{Tor}^A_i(B,M) = 0$ for all $1 \leq i \leq n$.*

Proof Choose a resolution of M by free (left) B–modules

$$\cdots \longrightarrow Q^{-2} \longrightarrow Q^{-1} \longrightarrow Q^0 \longrightarrow M \longrightarrow 0,$$

and choose a resolution of B by free right A–modules

$$\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow B \longrightarrow 0.$$

The tensor product $P \otimes_A Q$ gives a double complex whose cohomology computes $\mathrm{Tor}^A_{-i-j}(B,M)$. But there is a spectral sequence for it, whose E_1 term is

$$E_1^{i,j} \ = \ \mathrm{Tor}^A_{-j}(B,Q^i).$$

Now $E_1^{i,0} = B \otimes_A Q^i = Q^i$, since Q^i is free and, by hypothesis, $B \otimes_A B \longrightarrow B$ is an isomorphism. In E_2, we have

$$E_2^{i,0} \ = \ \begin{cases} M & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

By hypothesis, we also have $\mathrm{Tor}^A_{-j}(B,B)=0$ for all $1\le -j\le n$. Since Q^i are free, this gives $\mathrm{Tor}^A_{-j}(B,Q^i)=0$, for all i and for all $1\le -j\le n$. In other words, $E_1^{i,j}=0$ if $1\le -j\le n$, and hence $E_2^{i,j}=0$ if either $j=0$, $i\neq 0$, or if $1\le -j\le n$. The assertions of the lemma immediately follow. □

Lemma 8.4 *The map*

$$\{\sigma^{-1}A\}\otimes_A A \xrightarrow{1^L\otimes_A\eta_A} \{\sigma^{-1}A\}{}^L\!\otimes_A G\pi A$$

is an isomorphism.

Proof Consider the distinguished triangle

$$k \longrightarrow A \xrightarrow{\eta_A} G\pi A \longrightarrow \Sigma k.$$

We know that $k\in\mathcal{R}$, and Lemma 5.3 tells us that $\{\sigma^{-1}A\}{}^L\!\otimes_A k=0$. Tensoring the triangle with $\sigma^{-1}A$, the lemma follows. □

Remark 8.5 Lemma 8.4 produced an isomorphism $\sigma^{-1}A=\{\sigma^{-1}A\}{}^L\!\otimes_A G\pi A$ in the derived category $D(A)$. Of course, there is a spectral sequence which computes the cohomology of $\{\sigma^{-1}A\}{}^L\!\otimes_A G\pi A$. The E_2 term is given by

$$E_2^{i,j}=\mathrm{Tor}^A_{-i}\big(\sigma^{-1}A\,,\,H^j(G\pi A)\big).$$

Lemma 8.4 can be viewed as telling us the limit of this spectral sequence. In the rest of the section we will study the consequences.

Lemma 8.6 *We have:*

(i) *The multiplication map* $\mu\colon\ \{\sigma^{-1}A\}\otimes_A\{\sigma^{-1}A\}\longrightarrow\{\sigma^{-1}A\}$ *is an isomorphism.*

(ii) $\mathrm{Tor}^A_1(\sigma^{-1}A,\sigma^{-1}A)=0$.

Proof The results of Lemma 8.6 are not new. They first appeared in an article by Bergman and Dicks [3]. The proof that μ is an isomorphism may be found in [3, (4) on page 298], combined with the remark in Construction 2.2 on page 300. The vanishing of $\mathrm{Tor}^A_1(\sigma^{-1}A,\sigma^{-1}A)$ is in [3, (95) on page 326]. See also Schofield [33, page 58]. Even though the results are known, both statements are essentially immediate from the spectral sequence of Remark 8.5. Hence we include the proof.

By Lemma 6.4 we know that $G\pi A\in D(A)^{\le 0}$. This means $H^j(G\pi A)=0$ when $j>0$. For $\mathrm{Tor}^A_{-i}(-,-)$, we know it vanishes whenever $i>0$. In the spectral sequence of Remark 8.5 we therefore have $E_2^{i,j} = 0$ unless $i \le 0$ and $j \le 0$. The spectral sequence is third quadrant.

This immediately means that $E_2^{0,0} = E_\infty^{0,0}$ and $E_2^{-1,0} = E_\infty^{-1,0}$. Lemma 8.4 tells us precisely that the map

$$1^L \otimes_A \eta_A \colon \{\sigma^{-1}A\} \otimes_A A \longrightarrow \{\sigma^{-1}A\}^L \otimes_A G\pi A$$

is an isomorphism. Evaluating H^{-1} of this isomorphism, we have that $E_\infty^{-1,0}$ is a quotient of $H^{-1}(\sigma^{-1}A) = 0$, and hence

$$E_2^{-1,0} = \mathrm{Tor}_1^A(\sigma^{-1}A, \sigma^{-1}A) = 0.$$

Evaluating H^0 of the isomorphism of Lemma 8.4 and recalling that $E_2^{0,0} = E_\infty^{0,0}$, we have an isomorphism

$$1 \otimes_A H^0(\eta_A) \colon \{\sigma^{-1}A\} \otimes_A A \longrightarrow \{\sigma^{-1}A\} \otimes_A H^0(G\pi A).$$

By Lemma 7.1 we know that the map $H^0(\eta_A) \colon A \longrightarrow H^0(G\pi A)$ can be identified with the natural homomorphism $\theta \colon A \longrightarrow \sigma^{-1}A$. It follows that

$$1 \otimes_A \theta \colon \{\sigma^{-1}A\} \otimes_A A \longrightarrow \{\sigma^{-1}A\} \otimes_A \{\sigma^{-1}A\}$$

is an isomorphism. But the composite

$$\{\sigma^{-1}A\} \otimes_A A \xrightarrow{1 \otimes_A \theta} \{\sigma^{-1}A\} \otimes_A \{\sigma^{-1}A\} \xrightarrow{\mu} \sigma^{-1}A$$

is clearly the identity. This makes μ left-inverse to the invertible map $1 \otimes_A \theta$. Hence μ must be the two-sided inverse, and is invertible. □

Theorem 8.7 *Suppose* $\mathrm{Tor}_i^A(\sigma^{-1}A, \sigma^{-1}A) = 0$, *for all* $1 \le i \le n$. *Then for all* $1 \le i \le n-1$ *we have* $H^{-i}(G\pi A) = 0$, *and*

$$\mathrm{Tor}_{n+1}^A(\sigma^{-1}A, \sigma^{-1}A) \;=\; H^{-n}(G\pi A).$$

Proof The proof is a slightly more sophisticated computation with the spectral sequence of Remark 8.5. Recall that we have a spectral sequence whose E_2 term is

$$E_2^{i,j} \;=\; \mathrm{Tor}_{-i}^A\big(\sigma^{-1}A, H^j(G\pi A)\big),$$

which converges to $H^{i+j}(\sigma^{-1}A)$. By Reminder 6.2 the object $G\pi A$ is local. By Lemma 8.2 the homology groups $H^j(G\pi A)$ are all modules over $\sigma^{-1}A$. By Lemma 8.6 we know that the multiplication map $\mu \colon \{\sigma^{-1}A\} \otimes_A \{\sigma^{-1}A\} \longrightarrow \sigma^{-1}A$ is an isomorphism. Lemma 8.3 now applies, and we deduce that if $1 \le -i \le n$ then $E_2^{i,j} = 0$. This forces the differential

$$E_2^{-i-1,0} \longrightarrow E_2^{0,-i}$$

to be an isomorphism, for all $1 \le i \le n$. For $1 \le i \le n-1$ we read off that $H^{-i}(G\pi A) = 0$. For $i = n$, we deduce that

$$\mathrm{Tor}_{n+1}^A(\sigma^{-1}A, \sigma^{-1}A) \;=\; H^{-n}(G\pi A). \qquad \square$$

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