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Correction to: Construction of 2–local finite groups of a type studied by Solomon and Benson

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Abstract A p–local finite group is an algebraic structure with a classifying space which has many of the properties of p–completed classifying spaces of finite groups. In our paper [2], we constructed a family of 2–local finite groups which are "exotic" in the following sense: they are based on certain fusion systems over the Sylow 2–subgroup of $\mathrm{Spin}_7(q)$ (q an odd prime power) shown by Solomon not to occur as the 2–fusion in any actual finite group. As predicted by Benson, the classifying spaces of these 2–local finite groups are very closely related to the Dwyer–Wilkerson space $BDI(4)$. An error in our paper [2] was pointed out to us by Andy Chermak, and we correct that error here.

Keywords Classifying space, p–completion, finite groups, fusion

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A *saturated fusion system* over a finite p–group S is a category whose objects are the subgroups of S, whose morphisms are all monomorphisms between the subgroups, and which satisfy certain axioms first formulated by Puig, and also described at the start of the first section in [2]. The main result of [2] is the construction of saturated fusion systems over certain 2–groups, motivated by a theorem of Solomon [3], which implies that these systems cannot be induced by fusion in any finite group. Recently, Andy Chermak has pointed out to us that the fusion systems actually constructed in [2] are *not* saturated (do not satisfy all of Puig's axioms). In this note, we describe how to modify that construction in a way so as to obtain saturated fusion systems of the desired type, and explain why all of the results in [2] (aside from [2, Lemma A.10]) are true under this new construction.

The following is the main theorem in [2]:

Theorem 1.1 [2, Theorem 2.1] *Let q be an odd prime power, and fix $S \in \mathrm{Syl}_2(\mathrm{Spin}_7(q))$. Let $z \in Z(\mathrm{Spin}_7(q))$ be the central element of order 2. Then there is a saturated fusion system $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$ which satisfies the following conditions:*

(a) *$C_{\mathcal{F}}(z) = \mathcal{F}_S(\mathrm{Spin}_7(q))$ as fusion systems over S.*

(b) *All involutions of S are $\mathcal{F}$–conjugate.*

Furthermore, there is a unique centric linking system $\mathcal{L} = \mathcal{L}^c_{\mathrm{Sol}}(q)$ associated to $\mathcal{F}$.

We have, in fact, found two errors in our proof of this theorem which we correct here. The more serious one is in [2, Lemma A.10], which is not true as stated: the last sentence in its proof is wrong. This has several implications on the rest of our construction, all of which are systematically treated here. There is also an error in the statement of [2, Lemma 2.8(b)] which is corrected below (Lemma 1.8).

We first state and prove here a corrected version of [2, Lemma A.10], and then state a modified version of the main technical proposition, [2, Proposition 1.2], used to prove saturation. Afterwards, we describe the changes which are needed in [2, Section 2] to prove the main theorem. In table 1, we list the correspondence between results and proofs in [2, Section 2] and those here. This is intended as a guide to the reader who is not yet familiar with [2], and who wants to read it simultaneously with this correction.

The only difference between [2, Lemma A.10] and the corrected version shown here is that in [2], we claimed that the "correction factor" Z must lie in a certain subgroup of order 2 in $SL_3(\mathbb{Z}/2^k)$, which is definitely not the case. Also, for convenience, we state this lemma here for matrices over the 2–adic integers $\widehat{\mathbb{Z}}_2$, instead of for matrices over the finite rings $\mathbb{Z}/2^k$.

Lemma 1.2 (Modified [2, Lemma A.10]) *Let T_1 and T_2 be the two maximal parabolic subgroups of $GL_3(2)$:*

$$T_1 = GL_2^1(\mathbb{Z}/2) = \{(a_{ij}) \in GL_3(2) \mid a_{21} = a_{31} = 0\}$$

Reference in [2]	**Reference here**	**Remarks**
Proposition 1.2	Proposition 1.3	more general statement
Theorem 2.1	Theorem 1.1	unchanged
Definition 2.2, Lemma 2.3	—	unchanged
Definition 2.4	—	omit definition of Γ_n
Prp. 2.5, Def. 2.6, Lem. 2.7	—	unchanged
—	Lem. 1.4, Prp. 1.5	added
—	Definition 1.6	new definition of Γ_n
—	Lemma 1.7	added
Lemma 2.8	Lemma 1.8	(b) restated, new proofs of (b), (e)
Proposition 2.9	Proposition 1.9	partly new proof
Lemma 2.10	Lemma 1.10	partly new proof
Proposition 2.11	Proposition 1.11	partly new proof

Table 1

and

$$T_2 = GL_1^2(\mathbb{Z}/2) = \{(a_{ij}) \in GL_3(2) \mid a_{31} = a_{32} = 0\}.$$

Set $T_0 = T_1 \cap T_2$: the group of upper triangular matrices in $GL_3(2)$. Assume, for some $k \geq 2$, that

$$\mu_i \colon T_i \longrightarrow SL_3(\widehat{\mathbb{Z}}_2)$$

are lifts of the inclusions $T_i \longrightarrow GL_3(2) = SL_3(2)$ (for $i = 1,2$) such that $\mu_1|_{T_0} = \mu_2|_{T_0}$. Then there is a homomorphism

$$\mu\colon GL_3(2) \longrightarrow SL_3(\widehat{\mathbb{Z}}_2),$$

and an element $Z \in C_{SL_3(\widehat{\mathbb{Z}}_2)}(\mu_1(T_0))$, such that $\mu|_{T_1} = \mu_1$, and $\mu|_{T_2} = c_Z \circ \mu_2$.

Proof By [1, Lemma 4.4], there is a lifting $\mu' \colon GL_3(2) \longrightarrow SL_3(\widehat{\mathbb{Z}}_2)$ of the identity on $GL_3(2)$; and any two liftings to $SL_3(\widehat{\mathbb{Z}}_2)$ of the inclusion of T_1 or of T_2 into $GL_3(2)$ differ by conjugation by an element of $SL_3(\widehat{\mathbb{Z}}_2)$. In particular, there are elements $Z_1, Z_2 \in SL_3(\widehat{\mathbb{Z}}_2)$ such that $\mu_i = c_{Z_i} \circ \mu'|_{T_i}$ (for $i = 1,2$). Set $\mu = c_{Z_1} \circ \mu'$, and $Z = Z_1 Z_2^{-1}$. Then $\mu_1 = \mu|_{T_i}$, and $\mu|_{T_2} = c_Z \circ \mu_2$. Since $\mu_1|_{T_0} = \mu_2|_{T_0}$, conjugation by Z is the identity on $\mu_1(T_0) = \mu_2(T_0)$, and thus $Z \in C_{SL_3(\widehat{\mathbb{Z}}_2)}(\mu_1(T_0))$. □

Whenever G is a finite group, $S \in \mathrm{Syl}_p(G)$, $S_0 \lhd S$, and $\Gamma \leq \mathrm{Aut}(S_0)$, then

$$\langle \mathcal{F}_S(G); \mathcal{F}_{S_0}(\Gamma) \rangle$$

denotes the smallest fusion system over S which contains all G–fusion, and which also contains all restrictions of automorphisms in Γ. In other words, if $\mathcal{F}$ denotes this fusion system, then for each $P,Q \le S$, $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ is the set of all composites

$$P = P_0 \xrightarrow{\varphi_1} P_1 \xrightarrow{\varphi_2} P_2 \longrightarrow \cdots \longrightarrow P_{k-1} \xrightarrow{\varphi_k} P_k = Q,$$

where for each i, $P_i \le S$, and either $\varphi_i \in \mathrm{Hom}_G(P_{i-1},P_i)$, or (if $P_{i-1},P_i \le S_0$) $\varphi_i = \psi_i|_{P_{i-1}}$ for some $\psi_i \in \Gamma$ such that $\psi_i(P_{i-1}) = P_i$.

Whenever G is a finite group and $S \in \mathrm{Syl}_p(G)$, an automorphism $\varphi \in \mathrm{Aut}(S)$ is said to *preserve G–fusion* if for each $P,Q \le S$ and each $\alpha \in \mathrm{Iso}(P,Q)$, $\alpha \in \mathrm{Iso}_G(P,Q)$ if and only if $\varphi\alpha\varphi^{-1} \in \mathrm{Iso}_G(\varphi(P),\varphi(Q))$. The proof of Theorem 1.1 is based on the following proposition.

Proposition 1.3 (Modified [2, Proposition 1.2]) *Fix a finite group G, a prime p dividing $|G|$, and a Sylow p–subgroup $S \in \mathrm{Syl}_p(G)$. Fix a normal subgroup $Z \lhd G$ of order p, an elementary abelian subgroup $U \lhd S$ of rank two containing Z such that $C_S(U) \in \mathrm{Syl}_p(C_G(U))$, and a group $\Gamma \le \mathrm{Aut}(C_S(U))$ of automorphisms which preserve all $C_G(U)$–fusion, and such that $\gamma(U) = U$ for all $\gamma \in \Gamma$. Set*

$$S_0 = C_S(U) \qquad \text{and} \qquad \mathcal{F} = \langle \mathcal{F}_S(G); \mathcal{F}_{S_0}(\Gamma)\rangle,$$

and assume the following hold.

(a) *All subgroups of order p in S different from Z are G–conjugate.*

(b) *Γ permutes transitively the subgroups of order p in U.*

(c) $\{\varphi \in \Gamma \mid \varphi(Z) = Z\} = \mathrm{Aut}_{N_G(U)}(C_S(U))$.

(d) *For each $E \le S$ which is elementary abelian of rank three, contains U, and is fully centralized in $\mathcal{F}_S(G)$,*

$$\{\alpha \in \mathrm{Aut}_{\mathcal{F}}(C_S(E)) \mid \alpha(Z) = Z\} = \mathrm{Aut}_G(C_S(E)).$$

(e) *For all $E,E' \le S$ which are elementary abelian of rank three and contain U, if E and E' are Γ–conjugate, then they are G–conjugate.*

Then $\mathcal{F}$ is a saturated fusion system over S. Also, for any $P \le S$ such that $Z \le P$,

$$\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S) \mid \varphi(Z) = Z\} = \mathrm{Hom}_G(P,S). \tag{1}$$

This proposition is slightly more general than [2, Proposition 1.2], in that Γ is assumed only to be a group of automorphisms of $C_S(U)$ which preserves $C_G(U)$–fusion, and not a group of automorphisms of $C_G(U)$ itself. This extra generality is necessary when proving that the fusion systems $\mathcal{F}_{\mathrm{Sol}}(q)$, under our modified construction, are saturated. The changes needed to prove this more general version of [2, Proposition 1.2] are described in Section 2.

Proposition 1.3 is applied with $G = \mathrm{Spin}_7(q)$, $Z = Z(G)$, and $S \in \mathrm{Syl}_2(G)$, $U \leq S$, and $\Gamma \leq \mathrm{Aut}(C_G(U))$ to be chosen shortly. The error in [2] arose in the choice of Γ, as will be explained in detail below.

We first recall some of the definitions and notation used in [2]. Throughout, we fix an odd prime power q, let $\mathbb{F}_q$ be a field with q elements, and let $\bar{\mathbb{F}}_q$ be its algebraic closure. We write $SL_2(q^\infty) = SL_2(\bar{\mathbb{F}}_q)$, $\mathrm{Spin}_7(q^\infty) = \mathrm{Spin}_7(\bar{\mathbb{F}}_q)$, etc., for short. For each n, ψ^{q^n} denotes the automorphism of $\mathrm{Spin}_7(q^\infty)$ or of $SL_2(q^\infty)$ induced by the field isomorphism $(x \mapsto x^{q^n})$. We then fix elements $z, z_1 \in \mathrm{Spin}_7(q)$ of order 2, where $\langle z\rangle = Z(\mathrm{Spin}_7(q))$, set $U = \langle z, z_1\rangle$, and construct an explicit homomorphism

$$\omega\colon SL_2(q^\infty)^3 \longrightarrow \mathrm{Spin}_7(q^\infty)$$

such that

$$\mathrm{Im}(\omega) = C_{\mathrm{Spin}_7(q^\infty)}(U) \qquad \text{and} \qquad \mathrm{Ker}(\omega) = \langle(-I,-I,-I)\rangle.$$

Write

$$H(q^\infty) \stackrel{\text{def}}{=} \omega(SL_2(q^\infty)^3) = C_{\mathrm{Spin}_7(q^\infty)}(U) \quad \text{and} \quad [\![X_1,X_2,X_3]\!] = \omega(X_1,X_2,X_3)$$

for short. In particular,

$$z = [\![I,I,-I]\!] \qquad \text{and} \qquad z_1 = [\![-I,I,I]\!],$$

and thus

$$U = \{[\![\pm I, \pm I, \pm I]\!]\}$$

(with all combinations of signs). By [2, Lemma 2.3 & Proposition 2.5], there is an element $\tau \in N_{\mathrm{Spin}_7(q)}(U)$ of order 2 such that

$$\tau{\cdot}[\![X_1,X_2,X_3]\!]{\cdot}\tau^{-1} = [\![X_2,X_1,X_3]\!]$$

for all $X_1,X_2,X_3 \in SL_2(q^\infty)$, and such that

$$N_{\mathrm{Spin}_7(q^\infty)}(U) = H(q^\infty){\cdot}\langle\tau\rangle.$$

We next fix elements $A, B \in SL_2(q)$ of order 4, such that $\langle A,B\rangle \cong Q_8$ (a quaternion group of order 8). Most of the following notation is taken from [2, Definition 2.6]. We set

$$\widehat{A} = [\![A,A,A]\!] \qquad \text{and} \qquad \widehat{B} = [\![B,B,B]\!];$$

$$C(q^\infty) = \{X \in C_{SL_2(q^\infty)}(A) \mid X^{2^k} = I, \text{ some } k\} \cong \mathbb{Z}/2^\infty;$$

and

$$Q(q^\infty) = \langle C(q^\infty), B \rangle.$$

Here, $\mathbb{Z}/2^\infty$ means a union of cyclic 2–groups $\mathbb{Z}/2^n$ for all n; equivalently, the group $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$. We then define

$$\begin{aligned} A(q^\infty) &= \omega(C(q^\infty)^3) \cong (\mathbb{Z}/2^\infty)^3, \\ S_0(q^\infty) &= \omega(Q(q^\infty)^3) \le H(q^\infty) \\ S(q^\infty) &= S_0(q^\infty)\cdot\langle\tau\rangle \le H(q^\infty)\cdot\langle\tau\rangle \le \mathrm{Spin}_7(q^\infty). \end{aligned}$$

In all cases, whenever a subgroup $\Theta(q^\infty) \le \mathrm{Spin}_7(q^\infty)$ has been defined, we set

$$\Theta(q^n) = \Theta(q^\infty) \cap \mathrm{Spin}_7(q^n).$$

Since $\mathrm{Spin}_7(q^n)$ is the fixed subgroup of ψ^{q^n} acting on $\mathrm{Spin}_7(q^\infty)$ (cf. [2, Lemma A.3]), $H(q^n) \le H(q^\infty)$ is the subgroup of all elements of the form $[\![X_1, X_2, X_3]\!]$, where either $X_i \in SL_2(q^n)$ for each i, or $\psi^{q^n}(X_i) = -X_i$ for each i. By [2, Lemma 2.7], for all n,

$$S(q^n) \in \mathrm{Syl}_2(\mathrm{Spin}_7(q^n)).$$

The following lemma is what is needed to tell us how to choose a subgroup $\Gamma_n \le \mathrm{Aut}(S_0(q^n))$ so that the fusion system $\langle \mathcal{F}_{S(q^n)}(\mathrm{Spin}_7(q^n)); \mathcal{F}_{S_0(q^n)}(\Gamma_n)\rangle$ is saturated. Note that since each element of $C(q^\infty)$ has 2–power order, it makes sense to write $X^u \in C(q^\infty)$ for $X \in C(q^\infty)$ and $u \in \widehat{\mathbb{Z}}_2$.

Lemma 1.4 *Assume $\alpha \in \mathrm{Aut}(A(q^\infty))$ centralizes $\mathrm{Aut}_{S(q^\infty)}(A(q^\infty))$. Then α has the form*

$$\alpha([\![X_1, X_2, X_3]\!]) = [\![X_1^v, X_2^v, X_3^u]\!]$$

for some $u, v \in (\widehat{\mathbb{Z}}_2)^*$.

Proof Set

$$\Delta_0 = \mathrm{Aut}_{S(q^\infty)}(A(q^\infty)) = \langle c_{[\![B,I,I]\!]}, c_{[\![I,B,I]\!]}, c_{[\![I,I,B]\!]}, c_\tau\rangle$$

for short. The second equality follows since $S(q^\infty)$ is by definition generated by $A(q^\infty)$ and the four elements listed. Set

$$A_1 \stackrel{\text{def}}{=} \langle z, z_1, \widehat{A}\rangle \cong C_2^3,$$

the 2–torsion subgroup in $A(q^\infty)$. The image of $\Delta_0 \le \mathrm{Aut}(A(q^\infty))$ in the group $\mathrm{Aut}(A_1) \cong GL_3(2)$ (the image under restriction) is the group of all automorphisms which leave $\langle z\rangle$ and $U=\langle z,z_1\rangle$ invariant (ie, the group of upper triangular matrices with respect to the ordered basis $\{z,z_1,\widehat{A}\}$).

By assumption, $[\alpha,\Delta_0]=1$, and in particular, $\alpha\Delta_0\alpha^{-1}=\Delta_0$. Since each element of Δ_0 sends U to itself, this means that each element of Δ_0 also sends $\alpha(U)$ to itself. Also, U is the only subgroup of rank 2 left invariant by all elements of Δ_0 (since Δ_0 contains all automorphisms which leave $\langle z\rangle$ and U invariant), and hence $\alpha(U)=U$.

It follows that α induces an automorphism α' of $A(q^\infty)/U=(C(q^\infty)/\langle -I\rangle)^3$. Also, α' commutes with the following automorphisms of $(C(q^\infty)/\langle -I\rangle)^3$:

$$(Y_1,Y_2,Y_3)\mapsto(Y_1^{\pm1},Y_2^{\pm1},Y_3^{\pm1})\qquad\text{and}\qquad(Y_1,Y_2,Y_3)\mapsto(Y_2,Y_1,Y_3)$$

since these are induced by automorphisms in Δ_0. Hence α' has the form $\alpha'(Y_1,Y_2,Y_3)=(Y_1^v,Y_2^v,Y_3^u)$ for some $u,v\in(\widehat{\mathbb{Z}}_2)^*$. Thus for all $X_1,X_2,X_3\in C(q^\infty)$,

$$\alpha([\![X_1,X_2,X_3]\!])=[\![\pm X_1^v,\pm X_2^v,\pm X_3^u]\!].$$

Since all elements of $C(q^\infty)$ are squares, these signs must all be positive, and α has the form $\alpha([\![X_1,X_2,X_3]\!]=[\![X_1^v,X_2^v,X_3^u]\!]$. $\square$

For each $u\in(\widehat{\mathbb{Z}}_2)^*$, let $\delta_u\in\mathrm{Aut}(A(q^\infty))$ be the automorphism

$$\delta_u([\![X_1,X_2,X_3]\!])=[X_1,X_2,X_3^u].$$

Define $\gamma,\gamma_u\in\mathrm{Aut}(A(q^\infty))$ by setting

$$\gamma([\![X_1,X_2,X_3]\!])=[\![X_3,X_1,X_2]\!]\qquad\text{and}\qquad\gamma_u=\delta_u\gamma\delta_u^{-1}.$$

Proposition 1.5 *There is an element $u\in\widehat{\mathbb{Z}}_2$ such that $u\equiv1 \pmod 4$, and such that the subgroup $\Omega_u\le\mathrm{Aut}(A(q^\infty))$ given by*

$$\Omega_u\overset{\text{def}}{=}\langle\,\mathrm{Aut}_{\mathrm{Spin}_7(q^\infty)}(A(q^\infty)),\gamma_u\rangle$$

is isomorphic to $C_2\times GL_3(2)$. Furthermore, the following hold:

(a) *The subgroup of elements of Ω_u which act via the identity on all 2–torsion in $A(q^\infty)$ has order 2, and contains only the identity and the automorphism $(g\mapsto g^{-1})$.*

(b) *For each $n\ge1$,*

$$\langle\,\mathrm{Aut}_{\mathrm{Spin}_7(q^n)}(A(q^n)),\gamma_u\rangle\cong C_2\times GL_3(2).$$

Proof For each $k \geq 1$, let $A_k \leq A(q^\infty)$ denote the 2^k–torsion subgroup. In particular,

$$A_1 = \langle z, z_1, \widehat{A} \rangle \cong C_2^3, \qquad \text{where} \qquad \widehat{A} = [\![A, A, A]\!].$$

Let $R_1, R_2, \cdots \in C(q^\infty) \cong \mathbb{Z}/2^\infty$ be elements such that $R_1 = -I$, $R_2 = A$, and $(R_i)^2 = R_{i-1}$ for all $k \geq 2$. Thus, $|R_i| = 2^i$ for all i. For each $k \geq 1$, let $\{\mathbf{r}_1^{(k)}, \mathbf{r}_2^{(k)}, \mathbf{r}_3^{(k)}\}$ be the basis of A_k defined by

$$\mathbf{r}_1^{(k)} = [\![I, I, R_k]\!], \qquad \mathbf{r}_2^{(k)} = [\![R_k, I, I]\!], \qquad \text{and} \qquad \mathbf{r}_3^{(k)} = [\![R_{k+1}, R_{k+1}, R_{k+1}]\!].$$

In particular, $\mathbf{r}_1^{(1)} = z$, $\mathbf{r}_2^{(1)} = z_1$, and $\mathbf{r}_3^{(1)} = \widehat{A}$. Using these bases, we identify $\mathrm{Aut}(A_k) = GL_3(\mathbb{Z}/2^k)$ and $\mathrm{Aut}(A(q^\infty)) = GL_3(\widehat{\mathbb{Z}}_2)$.

Set

$$\Delta_0 = \mathrm{Aut}_{S(q^\infty)}(A(q^\infty)), \quad \Delta_1 = \mathrm{Aut}_{\mathrm{Spin}_7(q^\infty)}(A(q^\infty)), \quad \text{and} \quad \Delta_2 = \langle \Delta_0, \gamma \rangle.$$

In particular, Δ_2 is the group of all signed permutations

$$[\![X_1, X_2, X_3]\!] \mapsto [\![X_{\sigma(1)}^{\pm 1}, X_{\sigma(2)}^{\pm 1}, X_{\sigma(3)}^{\pm 1}]\!]$$

for $\sigma \in \Sigma_3$.

For each $i = 0, 1, 2$ and each $k \geq 1$, let $\Delta_i^{(k)} \leq \mathrm{Aut}(A_k)$ be the image of Δ_i under restriction. By [2, Proposition A.8], $\Delta_1^{(1)} = \mathrm{Aut}_{\mathrm{Spin}}(A_1)$ is the group of all elements of $\mathrm{Aut}(A_1) \cong GL_3(\mathbb{Z}/2)$ which send z to itself. Also, $\Delta_0^{(1)}$ was seen in the proof of Lemma 1.4 to be the group of all automorphisms of A_1 which leaves both z and $U = \langle z, z_1 \rangle$ invariant; and a similar argument shows that $\Delta_2^{(1)}$ is the group of all automorphisms of A_1 which leaves U invariant. Hence, with respect to the ordered basis $\{z, z_1, \widehat{A}\}$ of A_1, each group $\Delta_i^{(1)} \leq \mathrm{Aut}(A_1)$ $(i = 0, 1, 2)$ can be identified with the subgroup $T_i \leq GL_3(\mathbb{Z}/2)$ of Lemma 1.2.

By [2, Proposition 2.5],

$$C_{\mathrm{Spin}_7(q^\infty)}(U) = H(q^\infty) \cong (SL_2(q^\infty)^3)/\langle(-I, -I, -I)\rangle.$$

An element $[\![X_1, X_2, X_3]\!] \in H(q^\infty)$ $(X_i \in SL_2(q^\infty))$ centralizes $\widehat{A} \overset{\mathrm{def}}{=} [\![A, A, A]\!]$ if and only if $[X_i, A] = 1$ for each i, or $X_i A X_i^{-1} = -A = A^{-1}$ for each i. Set $\overline{C} = C_{SL_2(q^\infty)}(A)$. This is an abelian group (the union of the finite cyclic groups $C_{SL_2(q^n)}(A)$), and $N_{SL_2(q^\infty)}(A) = \overline{C} \cdot \langle B \rangle$. Hence, since $A_1 = \langle U, \widehat{A} \rangle$ and $\widehat{B} = [\![B, B, B]\!]$, we have

$$C_{\mathrm{Spin}_7(q^\infty)}(A_1) = C_{H(q^\infty)}(\widehat{A}) = \omega(\overline{C}^3) \cdot \langle \widehat{B} \rangle. \tag{1}$$

Since $\omega(\bar{C}^3)$ is abelian (thus centralizes $A(q^\infty)$), this shows that the kernel of each of the projection maps $\Delta_i \twoheadrightarrow \Delta_i^{(1)}$ is generated by conjugation by $\widehat{B}$; ie, by the automorphism $(g \mapsto g^{-1})$.

Since each Δ_i is finite, their elements all have determinant of finite order in $(\widehat{\mathbb{Z}}_2)^*$, hence are ± 1 in all cases. Also, for each i, Δ_i surjects onto $\Delta_i^{(1)} = T_i$ with kernel generated by the automorphism $(g \mapsto g^{-1})$ of determinant (-1). Hence the elements of determinant one in Δ_i are sent isomorphically to T_i, and define a lift

$$\mu_i \colon T_i \longrightarrow SL_3(\widehat{\mathbb{Z}}_2)$$

with respect to the given bases. In particular, $\mu_1|_{T_0} = \mu_2|_{T_0} = \mu_0$.

For all $i, j, k \in (\widehat{\mathbb{Z}}_2)^*$, we define $\psi_{i,j,k} \in \mathrm{Aut}(A(q^\infty))$ by setting

$$\psi_{i,j,k}(\llbracket X_1, X_2, X_3 \rrbracket) = \llbracket X_1^i, X_2^j, X_3^k \rrbracket$$

for all $\llbracket X_1, X_2, X_3 \rrbracket \in A(q^\infty)$. Recall that $c_\tau(\llbracket X_1, X_2, X_3 \rrbracket) = \llbracket X_2, X_1, X_3 \rrbracket$ (by choice of τ). Since $\Delta_0 = \mathrm{Aut}_{S(q^\infty)}(A(q^\infty))$ is generated by $c_{\llbracket B,I,I \rrbracket}$, $c_{\llbracket I,B,I \rrbracket}$, $c_{\llbracket I,I,B \rrbracket}$, and c_τ (corresponding to generators of $S(q^\infty)/A(q^\infty)$), μ_0 has image

$$\mathrm{Im}(\mu_0) = \mu_1(T_0) = \langle \psi_{-1,-1,1}, \psi_{1,-1,-1}, c_\tau \rangle,$$

where $\psi_{-1,-1,1} = c_{\llbracket B,B,I \rrbracket}$ and $\psi_{1,-1,-1} = c_{\llbracket I,B,B \rrbracket}$.

By Lemma 1.2, there is a homomorphism

$$\mu \colon GL_3(2) \longrightarrow SL_3(\widehat{\mathbb{Z}}_2),$$

and an element $Z \in \mathrm{Aut}(A(q^\infty)) \cong GL_3(\widehat{\mathbb{Z}}_2)$ which commutes with all elements of $\mu_1(T_0)$, such that $\mu|_{T_1} = \mu_1$ and $\mu|_{T_2} = c_Z \circ \mu_2$. By Lemma 1.4, $Z = \psi_{v,v,u}$ (using the above notation) for some $u, v \in (\widehat{\mathbb{Z}}_2)^*$. Since $\psi_{v,v,v}$ lies in the center of $\mathrm{Aut}(A(q^\infty))$ (it sends every element to its v-th power), we can assume that $v = 1$ (without changing c_Z), and thus that $Z = \psi_{1,1,u} = \delta_u$. Finally, since $\delta_{-1} = \psi_{1,1,-1} \in \Delta_0$, we can replace δ_u by δ_{-u} if necessary, and assume that $u \equiv 1 \pmod 4$.

Under the identification $GL_3(\widehat{\mathbb{Z}}_2) = \mathrm{Aut}(A(q^\infty))$, we now have

$$\mathrm{Aut}_{\mathrm{Spin}_7(q^\infty)}(A(q^\infty)) = (g \mapsto g^{-1}) \times \mu(T_1)$$

and

$$\gamma_u = \delta_u \gamma \delta_u^{-1} = c_Z(\gamma) = \mu\left(\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)\right),$$

where the matrix is that of $\gamma|_{A_1}$ with respect to the basis $\{\mathbf{r}_1^{(1)}, \mathbf{r}_2^{(1)}, \mathbf{r}_3^{(1)}\}$. Also, T_1 is a maximal subgroup of $GL_3(2)$ — the subgroup of invertible matrices which send $\mathbf{r}_1^{(1)}$ to itself — and so T_1 together with this matrix generate $GL_3(2)$. Thus

$$\Omega_u = \langle\, \mathrm{Aut}_{\mathrm{Spin}_7(q^\infty)}(A(q^\infty)), \gamma_u \rangle = \langle g \mapsto g^{-1} \rangle \times \mu(GL_3(2)) \cong C_2 \times GL_3(2). \quad (2)$$

This proves the first claim in the proposition. Point (a) follows by construction, since each nonidentity element of $\mu(GL_3(2))$ acts nontrivially on A_1. Point (b) follows from (2), once we know that each element of $\mathrm{Spin}_7(q^n)$ which normalizes A_1 (hence which normalizes $A(q^n)$) also normalizes $A(q^\infty)$ — and this follows from (1). □

We are now in a position to define the fusion systems we want. Roughly, they are generated by the fusion systems of $\mathrm{Spin}_7(q^n)$ together with one extra automorphism: the cyclic permutation $[\![X_1, X_2, X_3]\!] \mapsto [\![X_3, X_1, X_2]\!]$ "twisted" by the automorphism δ_u of the last proposition. By comparison, the construction in [2] was similar but without the twisting (ie, done with $u = 1$), and the resulting fusion system is, in fact, not saturated.

We regard $Q(q^\infty) = C(q^\infty) \rtimes \langle B \rangle$ as an infinite quaternion group: $BA'B^{-1} = A'^{-1}$ for each $A' \in C(q^\infty)$, and each element of the coset $C(q^\infty){\cdot}B$ has order 4. Hence any automorphism of $C(q^\infty)$ extends to a unique automorphism of $Q(q^\infty)$ which sends B to itself.

Definition 1.6 *Let $u \in (\widehat{\mathbb{Z}}_2)^*$ be as in Proposition 1.5. Let*

$$\widehat{\gamma}, \widehat{\delta}_u \in \mathrm{Aut}(S_0(q^\infty))$$

be the automorphisms

$$\widehat{\gamma}([\![X_1, X_2, X_3]\!]) = [\![X_3, X_1, X_2]\!]$$

and

$$\widehat{\delta}_u([\![X_1, X_2, A'B^j]\!]) = [\![X_1, X_2, (A')^u B^j]\!]$$

for all $X_i \in Q(q^\infty)$, $A' \in C(q^\infty)$, and $i, j \in \mathbb{Z}$; and set $\widehat{\gamma}_u = \widehat{\delta}_u \widehat{\gamma} \widehat{\delta}_u^{-1} \in \mathrm{Aut}(S_0(q^\infty))$. For each $n \geq 1$, set

$$\Gamma_n = \langle\, \mathrm{Inn}(S_0(q^n)), c_\tau, \widehat{\gamma}_u \rangle \leq \mathrm{Aut}(S_0(q^n));$$

and set

$$\mathcal{F}_n = \mathcal{F}_{\mathrm{Sol}}(q^n) = \langle \mathcal{F}_{S(q^n)}(\mathrm{Spin}_7(q^n)), \mathcal{F}_{S_0(q^n)}(\Gamma_n) \rangle.$$

In order to be able to apply Proposition 1.3, it is important to know that Γ_n is fusion preserving. This follows immediately from the following lemma.

Lemma 1.7 *For each $n \geq 1$, the automorphisms c_τ, $\widehat{\gamma}$, $\widehat{\delta}_u$, and $\widehat{\gamma}_u$ all preserve $H(q^n)$–fusion in $S_0(q^n)$.*

Proof The fusion in $SL_2(q^n)$ is generated by inner automorphisms of its Sylow subgroup $Q(q^n)$, together with the groups $\mathrm{Aut}(P)$ for subgroups $P \leq Q(q^n)$ isomorphic to Q_8. This follows from Alperin's fusion theorem, since these are the only subgroups whose automorphism group is not a 2–group. Thus any automorphism of $Q(q^n)$ — in particular, the automorphism φ_u defined by $\varphi_u(A') = A'^u$, $\varphi_u(B) = B$ for $A' \in C(q^n)$ — preserves fusion.

Set $H_0(q^n) = \{[\![X_1, X_2, X_3]\!] \mid X_i \in SL_2(q^n),\ \forall i\}$. Fix a generator Y of $C(q^{2n})$; then $C(q^n) = \langle Y^2 \rangle$, and $\psi^{q^n}(Y) = -Y$. For each $g = [\![X_1, X_2, X_3]\!] \in H(q^n)$, $[\![\psi^{q^n}(X_1), \psi^{q^n}(X_2), \psi^{q^n}(X_3)]\!] = [\![X_1, X_2, X_3]\!]$; and hence there is some fixed $\epsilon = \pm 1$ for which $\psi^{q^n}(X_i) = \epsilon \cdot X_i$ for $i = 1, 2, 3$. When $\epsilon = 1$, this means that $X_i \in SL_2(q^n)$ for each i; while if $\epsilon = -1$ it means that $X_i \in SL_2(q^n) \cdot Y$ for each i. So every element of $H(q^n)$ either lies in $H_0(q^n)$, or has the form $g \cdot [\![Y, Y, Y]\!]$ for some $g \in H_0(q^n)$.

Assume $g \in H(q^n)$ and $P, Q \leq S_0(q^n)$ are such that $gPg^{-1} = Q$. Clearly, $P \leq H_0(q^n)$ if and only if $Q \leq H_0(q^n)$ ($H_0(q^n)$ is normal in $H(q^n)$). We claim that there is $h \in H(q^n)$ such that

$$h\widehat{\delta}_u(P)h^{-1} = \widehat{\delta}_u(Q) \qquad \text{and} \qquad c_h \circ \widehat{\delta}_u|_P = \widehat{\delta}_u \circ c_g|_P\,. \tag{1}$$

Let $P_i, Q_i \leq SL_2(q^{2n})$ be the projections of P and Q to the i-th factor ($i = 1, 2, 3$), and write $g = [\![X_1, X_2, X_3]\!]$ (thus $X_i P_i X_i^{-1} = Q_i$). Consider the following cases.

(a) Assume $g \in H_0(q^n)$ and $P, Q \leq H_0(q^n)$. Since φ_u preserves fusion in $SL_2(q^n)$, there is $Y_3 \in SL_2(q^n)$ such that $Y_3 \varphi_u(P_3) Y_3^{-1} = \varphi_u(Q_3)$ and $c_{Y_3} \circ \varphi_u|_{P_3} = \varphi_u \circ c_{X_3}|_{P_3}$. Then $h \overset{\text{def}}{=} [\![X_1, X_2, Y_3]\!]$ satisfies (1).

(b) Assume $g \notin H_0(q^n)$ and $P, Q \leq H_0(q^n)$. Write $g = g' \cdot [\![Y, Y, Y]\!]$, where $g' \in H_0(q^n)$. Choose h' as in (a), so that (1) is satisfied with g, h replaced by g', h'. Then the element $h = h' \cdot [\![Y, Y, Y^u]\!]$ satisfies (1).

(c) Finally, assume that $P, Q \not\leq H_0(q^n)$. Then none of the subgroups $P_i, Q_i \leq SL_2(q^{2n})$ is contained in $SL_2(q^n)$. By the same procedure as was used in (a), we can find $h \in H_0(q^{2n})$ which satisfies (1); the problem is to do this so that $h \in H(q^n)$.

    As noted above, fusion in $SL_2(q^{2n})$ is generated by inner automorphisms of $Q(q^{2n})$ and automorphisms of subgroups isomorphic to Q_8. Hence if P_i

is not isomorphic to Q_8 or one of its subgroups, then there is $X_i' \in Q(q^{2n})$ such that $c_{X_i}|_{P_i} = c_{X_i'}|_{P_i}$. If, on the other hand, P_i (and hence Q_i) is isomorphic to Q_8 or C_4, then $P_i \le \langle A, Y^r B\rangle$ and $Q_i \le \langle A, Y^s B\rangle$ for some odd $r,s \in \mathbb{Z}$, and we can choose $k \in \mathbb{Z}$ such that $X_i' \stackrel{\text{def}}{=} Y^k \in Q(q^{2n})$ has the same conjugation action as X_i. Thus in all cases, we can write $X_i = X_i'X_i''$ for some $X_i' \in Q(q^{2n})$ and some $X_i'' \in C_{SL_2(q^{2n})\cdot\langle Y\rangle}(P_i)$.

The subgroups of $Q(q^{2n})$ which are centralized by elements in the coset $SL_2(q^n)\cdot Y$ are precisely the cyclic subgroups. (The quaternion subgroups of order ≥ 8 are all centric in $SL_2(q^{2n})$.) Hence we can choose elements Y_i'' as follows: $Y_i'' = 1$ if $X_i'' \in SL_2(q^n)$, and $Y_i'' \in SL_2(q^n)\cdot Y$ and centralizes $\varphi_u(P_i)$ if $X_i'' \in SL_2(q^n)\cdot Y$. We now define

$$h = [\![X_1', X_2', \varphi_u(X_3')]\!]\cdot[\![Y_1'', Y_2'', Y_3'']\!];$$

then $h \in H(q^n)$ and satisfies (1).

This shows that $\widehat{\delta}_u$ preserves $H(q^n)$–fusion as an automorphism of $S_0(q^n)$. Also, $\widehat{\gamma}$ and c_τ preserve fusion, since both extend to automorphisms of $H(q^n)$; and hence $\widehat{\gamma}_u = \widehat{\delta}_u\widehat{\gamma}\widehat{\delta}_u^{-1}$ also preserves fusion. □

Let $\psi = \psi^{q^n} \in \mathrm{Aut}(\mathrm{Spin}_7(q^\infty))$ be induced by the field automorphism $x \mapsto x^{q^n}$. By [2, Proposition A.9(a)], if $E \le \mathrm{Spin}_7(q^n)$ is an arbitrary elementary abelian 2–subgroup of rank 4, then there is an element $a \in \mathrm{Spin}_7(q^\infty)$ such that $aEa^{-1} = E_*$, and we define

$$x_{\mathcal{C}}(E) = a^{-1}\psi(a).$$

Then $x_{\mathcal{C}}(E) \in E$, and is independent of the choice of a.

In the following lemma, we correct the statement and proof of points (b) and (e). The proof of (e) is affected by both the changes in the statement of (b) and those in the definition of Γ_n.

Lemma 1.8 [2, Lemma 2.8] *Fix $n \ge 1$, set $E_* = \langle z, z_1, \widehat{A}, \widehat{B}\rangle \le S(q^n)$, and let $\mathcal{C}$ be the $\mathrm{Spin}_7(q^n)$–conjugacy class of E_*. Let $\mathcal{E}_4^U$ be the set of all elementary abelian subgroups $E \le S(q^n)$ of rank 4 which contain $U = \langle z, z_1\rangle$. Fix a generator $X \in C(q^n)$ (the 2–power torsion in $C_{SL_2(q^n)}(A)$), and choose $Y \in C(q^{2n})$ such that $Y^2 = X$. Then the following hold.*

(a) *E_* has type I.*

(b) *Each subgroup in $\mathcal{E}_4^U$ which contains $\widehat{A}$ is of the form*

$$E_{ijk} = \langle z, z_1, \widehat{A}, [\![X^iB, X^jB, X^kB]\!]\rangle \quad \text{or} \quad E'_{ijk} = \langle z, z_1, \widehat{A}, [\![X^iYB, X^jYB, X^kYB]\!]\rangle.$$

Each subgroup in $\mathcal{E}_4^U$ is $H(q^n)$–conjugate to one of these subgroups E_{ijk} or E'_{ijk} for some $i,j,k \in \mathbb{Z}$.

(c) $x_{\mathcal{C}}(E_{ijk}) = [\![(-I)^i, (-I)^j, (-I)^k]\!]$ *and* $x_{\mathcal{C}}(E'_{ijk}) = [\![(-I)^i, (-I)^j, (-I)^k]\!]\cdot\widehat{A}$.

(d) *All of the subgroups E'_{ijk} have type II. The subgroup E_{ijk} has type I if and only if $i \equiv j \pmod 2$, and lies in $\mathcal{C}$ (is conjugate to E_*) if and only if $i \equiv j \equiv k \pmod 2$. The subgroups E_{000}, E_{001}, and E_{100} thus represent the three conjugacy classes of rank four elementary abelian subgroups of* $\mathrm{Spin}_7(q^n)$ *(and $E_* = E_{000}$).*

(e) *For any $\varphi \in \Gamma_n \le \mathrm{Aut}(S_0(q^n))$ and any $E \in \mathcal{E}_4^U$, $\varphi(x_{\mathcal{C}}(E)) = x_{\mathcal{C}}(\varphi(E))$.*

Proof We prove only points (b) and (e) here, and refer to [2] for the proofs of the other points.

(b) Assume first that $\widehat{A} \in E$; ie, that $E \ge A_1 = \langle z, z_1, \widehat{A}\rangle$. By definition, $S(q^n)$ is generated by $A(q^n)$, whose elements clearly centralize A_1; and elements $[\![B^i, B^j, B^k]\!]$ for $i,j,k \in \{0,1\}$. Since an element of this form centralizes $\widehat{A}$ only if $i=j=k$, this shows that

$$E \le C_{S(q^n)}(\langle z, z_1, \widehat{A}\rangle) = A(q^n)\cdot\langle\widehat{B}\rangle.$$

Since $A(q^n)$ is a finite abelian 2–group of rank 3, we have $E = \langle z, z_1, \widehat{A}, g\widehat{B}\rangle$ for some $g \in A(q^n)$. Also,

$$A(q^n) = \omega(C(q^\infty)^3) \cap \mathrm{Spin}_7(q^n) = \{[\![X^i, X^j, X^k]\!], [\![X^iY, X^jY, X^kY]\!] \mid i,j,k \in \mathbb{Z}\},$$

and hence $E = E_{ijk}$ or E'_{ijk} for some $i,j,k \in \mathbb{Z}$.

Now let $E \in \mathcal{E}_4^U$ be arbitrary. Each element of E has the form $[\![X_1, X_2, X_3]\!]$, where either $X_i \in SL_2(q^n)$ for all i, or $X_i \in SL_2(q^n)\cdot Y$ for all i — and the elements of the first type ($X_i \in SL_2(q^n)$) form a subgroup of index at most 2. Since U has index 4 in E, this means that there is some $g = [\![X_1, X_2, X_3]\!] \in E \smallsetminus U$ for which $X_i \in SL_2(q^n)$ for all i. Also, $|X_i| = 4$ for all i, since $g \notin U = \{[\![\pm I, \pm I, \pm I]\!]\}$. Since all elements of order 4 in $SL_2(q^n)$ are conjugate (cf. [4, 3.6.23]), this implies that g is $H(q^n)$–conjugate to $\widehat{A}$, and hence that E is $H(q^n)$–conjugate to one of the above subgroups E_{ijk} or E'_{ijk}.

(e) By construction, $x_{\mathcal{C}}(-)$ is preserved under conjugation by elements of the group $\mathrm{Spin}_7(q^n)$. Since Γ_n is generated by $\widehat{\gamma}_u$ and conjugation by elements of $\mathrm{Spin}_7(q^n)$, it suffices to prove the result when $\varphi = \widehat{\gamma}_u$. Since $u \equiv 1 \pmod 4$ by Proposition 1.5, $\widehat{\gamma}_u(\widehat{A}) = \widehat{A}$, $\widehat{\gamma}_u(E_{ijk}) = E_{k',i',j'}$, and $\widehat{\gamma}_u(E'_{ijk}) = E'_{k'',i'',j''}$ for some $i' \equiv i'' \equiv i \pmod 2$, and similarly for the other indices. Hence by (c), $\widehat{\gamma}_u(x_{\mathcal{C}}(E)) = x_{\mathcal{C}}(\widehat{\gamma}_u(E))$ whenever $E = E_{ijk}$ or $E = E'_{ijk}$ for some $i,j,k \in \mathbb{Z}$.

Now assume $E \in \mathcal{E}_4^U$ is not one of the subgroups E_{ijk} or E'_{ijk}. By (b), there is $g \in H(q^n)$ such that $E' \overset{\text{def}}{=} gEg^{-1}$ is of this form. Since $\widehat{\gamma}_u$ preserves $H(q^n)$–fusion by Lemma 1.7, there is $h \in H(q^n)$ such that the following square commutes:

$$\begin{tikzcd} E \arrow[r, "\widehat{\gamma}_u"] \arrow[d, "c_g"'] & \widehat{\gamma}_u(E) \arrow[d, "c_h"'] \\ E' \arrow[r, "\widehat{\gamma}_u"] & \widehat{\gamma}_u(E')\,. \end{tikzcd}$$

We have seen that $\widehat{\gamma}_u(x_{\mathcal{C}}(E')) = x_{\mathcal{C}}(\widehat{\gamma}_u(E'))$; and also that c_g and c_h preserve $x_{\mathcal{C}}(-)$. Hence $\widehat{\gamma}_u(x_{\mathcal{C}}(E)) = x_{\mathcal{C}}(\widehat{\gamma}_u(E))$ by the commutativity of the square. □

The following is the crucial result needed to apply Proposition 1.3. The statement is exactly the same as that in [2], but the proof has to be modified slightly due to the changed definition of Γ_n (hence of $\mathcal{F}_n$).

Proposition 1.9 [2, Proposition 2.9] *Fix $n \geq 1$. Let $E \leq S(q^n)$ be an elementary abelian subgroup of rank 3 which contains U, and such that $C_{S(q^n)}(E) \in \mathrm{Syl}_2(C_{\mathrm{Spin}_7(q^n)}(E))$. Then*

$$\{\varphi \in \mathrm{Aut}_{\mathcal{F}_n}(C_{S(q^n)}(E)) \mid \varphi(z) = z\} = \mathrm{Aut}_{\mathrm{Spin}_7(q^n)}(C_{S(q^n)}(E)). \tag{1}$$

Proof Set

$$\mathrm{Spin} = \mathrm{Spin}_7(q^n), \qquad S = S(q^n), \qquad \Gamma = \Gamma_n, \qquad \text{and} \qquad \mathcal{F} = \mathcal{F}_n$$

for short, and consider the subgroups

$$R_0 = R_0(q^n) \overset{\text{def}}{=} A(q^n) \qquad \text{and} \qquad R_1 = R_1(q^n) \overset{\text{def}}{=} C_S(\langle U, \widehat{A}\rangle) = \langle R_0, \widehat{B}\rangle.$$

Then

$$R_0 \cong (C_{2^k})^3 \qquad \text{and} \qquad R_1 = R_0 \rtimes \langle \widehat{B}\rangle,$$

where 2^k is the largest power which divides $q^n \pm 1$, and where $\widehat{B} = [\![B,B,B]\!]$ has order 2 and acts on R_0 via $(g \mapsto g^{-1})$. Also,

$$\langle U, \widehat{A}\rangle = \langle [\![\pm I, \pm I, \pm I]\!], [\![A,A,A]\!]\rangle \cong C_2^3$$

is the 2–torsion subgroup of R_0. It was shown in the proof of [2, Proposition 2.9] that

$$R_0 \text{ is the only subgroup of } S \text{ isomorphic to } (C_{2^k})^3. \tag{2}$$

Let $E \le S$ be an elementary abelian subgroup of rank 3 which contains U, and such that $C_S(E) \in \mathrm{Syl}_2(C_{\mathrm{Spin}}(E))$. There are two cases to consider: that where $E \le R_0$ and that where $E \not\le R_0$.

Case 1 Assume $E \le R_0$. Since R_0 is abelian of rank 3, we must have $E = \langle U, \widehat{A}\rangle$, the 2–torsion subgroup of R_0, and $C_S(E) = R_1$. Also, by (2), neither R_0 nor R_1 is isomorphic to any other subgroup of S; and hence

$$\mathrm{Aut}_{\mathcal{F}}(R_i) = \langle \mathrm{Aut}_{\mathrm{Spin}}(R_i), \mathrm{Aut}_{\Gamma}(R_i)\rangle = \langle \mathrm{Aut}_{\mathrm{Spin}}(R_i), \gamma_u|_{R_i}\rangle \qquad \text{for } i = 0, 1. \tag{3}$$

(Recall that $\mathrm{Aut}_{\Gamma}(R_i)$ is generated by $\mathrm{Aut}_{S_0(q^n)}(R_i)$ and the restrictions of c_τ and γ_u.) Hence by Proposition 1.5(b),

$$\mathrm{Aut}_{\mathcal{F}}(R_0) = \langle \mathrm{Aut}_{\mathrm{Spin}}(R_0), \gamma_u|_{R_0}\rangle \cong C_2 \times GL_3(2).$$

In other words, if we let $A_1 = \langle z, z_1, \widehat{A}\rangle$ denote the 2–torsion subgroup of R_0, then restriction to A_1 sends $\mathrm{Aut}_{\mathcal{F}}(R_0)$ onto $\mathrm{Aut}(A_1)$ with kernel $\langle c_{\widehat{B}}\rangle$ of order 2. Since $\mathrm{Aut}_{\mathrm{Spin}}(A_1)$ is the group of all automorphisms of A_1 which send z to itself [2, Proposition A.8], this shows that

$$\mathrm{Aut}_{\mathrm{Spin}}(R_0) = \{\varphi \in \mathrm{Aut}_{\mathcal{F}}(R_0) \mid \varphi(z) = z\}. \tag{4}$$

In the proof of [2, Proposition 2.9] (see formula (5) in that proof), we show the first of the following two equalities:

$$\begin{aligned}\mathrm{Aut}_{\mathrm{Spin}}(R_1) &= \mathrm{Inn}(R_1)\cdot\{\varphi \in \mathrm{Aut}_{\mathrm{Spin}}(R_1) \mid \varphi(\widehat{B}) = \widehat{B}\}\\ &= \mathrm{Inn}(R_1)\cdot\{\varphi \in \mathrm{Aut}(R_1) \mid \varphi(\widehat{B}) = \widehat{B},\ \varphi|_{R_0} \in \mathrm{Aut}_{\mathrm{Spin}}(R_0)\}.\end{aligned} \tag{5}$$

The second equality holds since $R_1 = \langle R_0, \widehat{B}\rangle$, and since R_0 is the unique abelian subgroup of R_1 of index 2. Since $\widehat{\gamma}_u(R_0) = R_0$ and $\widehat{\gamma}_u(\widehat{B}) = \widehat{B}$, this, together with (3), shows that

$$\mathrm{Aut}_{\mathcal{F}}(R_1) = \mathrm{Inn}(R_1)\cdot\{\varphi \in \mathrm{Aut}(R_1) \mid \varphi(\widehat{B}) = \widehat{B},\ \varphi|_{R_0} \in \mathrm{Aut}_{\mathcal{F}}(R_0)\},$$

and hence that

$$\begin{aligned}&\{\varphi \in \mathrm{Aut}_{\mathcal{F}}(R_1) \mid \varphi(z) = z\}\\ &\quad = \mathrm{Inn}(R_1)\cdot\{\varphi \in \mathrm{Aut}(R_1) \mid \varphi(\widehat{B}) = \widehat{B},\ \varphi(z) = z,\ \varphi|_{R_0} \in \mathrm{Aut}_{\mathcal{F}}(R_0)\}\\ &\quad = \mathrm{Inn}(R_1)\cdot\{\varphi \in \mathrm{Aut}(R_1) \mid \varphi(\widehat{B}) = \widehat{B},\ \varphi|_{R_0} \in \mathrm{Aut}_{\mathrm{Spin}}(R_0)\},\end{aligned} \tag{6}$$

where the second equality follows from (4). If $\varphi \in \mathrm{Aut}(R_1)$ is such that $\varphi(\widehat{B}) = \widehat{B}$ and $\varphi|_{R_0} = c_x|_{R_0}$ for some $x \in N_{\mathrm{Spin}}(R_0)$, then x normalizes R_1 (the centralizer of the 2–torsion in R_0), $c_x(\widehat{B}) = c_y(\widehat{B})$ for some $y \in R_1 = R_0 {\cdot} \langle \widehat{B} \rangle$ by (5), so we can assume $y \in R_0$. Since $R_0 = A(q^\infty)$ is abelian (so $[y, R_0] = 1$), this implies $c_{y^{-1}x}|_{R_1} = \varphi$, and hence $\varphi \in \mathrm{Aut}_{\mathrm{Spin}}(R_1)$. So $\{\varphi \in \mathrm{Aut}_{\mathcal{F}}(R_1) \,|\, \varphi(z) = z\} \le \mathrm{Aut}_{\mathrm{Spin}}(R_1)$ by (6), and the opposite inclusion is clear.

Case 2 Now assume that $E \not\le R_0$. By assumption, $U \le E$ (hence $E \le C_S(E) \le C_S(U)$), and $C_S(E)$ is a Sylow subgroup of $C_{\mathrm{Spin}}(E)$. Also, E contains an element of the form $g {\cdot} [\![B^i, B^j, B^k]\!]$ for $g \in R_0 = A(q^n)$ and some i, j, k not all even, and hence $A(q^n) \not\le C_S(E)$. Hence by (2), $C_S(E)$ is not isomorphic to $R_1 = C_S(\langle z, z_1, \widehat{A} \rangle)$, and this shows that E is not Spin–conjugate to $\langle z, z_1, \widehat{A} \rangle$. By [2, Proposition A.8], Spin contains exactly two conjugacy classes of rank 3 subgroups containing z, and thus E must have type II. So by [2, Proposition A.8(d)], $C_S(E)$ is elementary abelian of rank 4, and also has type II.

Let $\mathcal{C}$ be the $\mathrm{Spin}_7(q^n)$–conjugacy class of the subgroup $E_* = \langle U, \widehat{A}, \widehat{B} \rangle \cong C_2^4$, which by Lemma 1.8(a) has type I. Let $\mathcal{E}'$ be the set of all subgroups of S which are elementary abelian of rank 4, contain U, and are not in $\mathcal{C}$. By Lemma 1.8(e), for any $\varphi \in \mathrm{Iso}_\Gamma(E', E'')$ and any $E' \in \mathcal{E}'$, $E'' \overset{\mathrm{def}}{=} \varphi(E') \in \mathcal{E}'$, and φ sends $x_{\mathcal{C}}(E')$ to $x_{\mathcal{C}}(E'')$. The same holds for $\varphi \in \mathrm{Iso}_{\mathrm{Spin}}(E', E'')$ by definition of the elements $x_{\mathcal{C}}(-)$ ([2, Proposition A.9]). Since $C_S(E) \in \mathcal{E}'$, this shows that all elements of $\mathrm{Aut}_{\mathcal{F}}(C_S(E))$ send the element $x_{\mathcal{C}}(C_S(E))$ to itself. By [2, Proposition A.9(c)], $\mathrm{Aut}_{\mathrm{Spin}}(C_S(E))$ is the group of automorphisms which are the identity on the rank two subgroup $\langle x_{\mathcal{C}}(C_S(E)), z \rangle$; and (1) now follows. □

The proof of the following lemma is essentially unchanged.

Lemma 1.10 [2, Lemma 2.10] *Fix $n \ge 1$, and let $E, E' \le S(q^n)$ be two elementary abelian subgroups of rank three which contain U, and which are Γ_n–conjugate. Then E and E' are $\mathrm{Spin}_7(q^n)$–conjugate.*

Proof Consider the sets

$$\mathcal{J}_1 = \{X \in SL_2(q^n) \,|\, X^2 = -I\}$$

and

$$\mathcal{J}_2 = \{X \in SL_2(q^{2n}) \,|\, \psi^{q^n}(X) = -X,\ X^2 = -I\}.$$

Here, as usual, ψ^{q^n} is induced by the field automorphism $(x \mapsto x^{q^n})$. It was shown in the proof of [2, Lemma 2.10] that all elements in each of these sets are $SL_2(q)$–conjugate to each other.

Since E and E contain U, $E, E' \le C_{\mathrm{Spin}_7(q^n)}(U)$. By [2, Proposition 2.5(a)],

$$C_{\mathrm{Spin}_7(q^n)}(U) = H(q^n) \stackrel{\text{def}}{=} \omega(SL_2(q^\infty)^3) \cap \mathrm{Spin}_7(q^n).$$

Thus

$$E = \langle z, z_1, [\![X_1, X_2, X_3]\!]\rangle \qquad \text{and} \qquad E' = \langle z, z_1, [\![X_1', X_2', X_3']\!]\rangle,$$

where the X_i are all in $\mathcal{J}_1$ or all in $\mathcal{J}_2$, and similarly for the X_i'. Also, E and E' are Γ_n–conjugate, and each element of Γ_n leaves $U = \langle z, z_1\rangle$ and $\omega(SL_2(q^n)^3)$ invariant. Hence either E and E' are both contained in $\omega(SL_2(q^n)^3)$, in which case the X_i and X_i' are all in $\mathcal{J}_1$; or neither is contained in $\omega(SL_2(q^n)^3)$, in which case the X_i and X_i' are all in $\mathcal{J}_2$. This shows that the X_i and X_i' are all $SL_2(q^n)$–conjugate, and so E and E' are $\mathrm{Spin}_7(q^n)$–conjugate. □

We are now ready to prove:

Proposition 1.11 [2, Proposition 2.11] *For a fixed odd prime power q, let $S(q^n) \le S(q^\infty) \le \mathrm{Spin}_7(q^\infty)$ be as defined above. Let $z \in Z(\mathrm{Spin}_7(q^\infty))$ be the central element of order 2. Then for each n, $\mathcal{F}_n = \mathcal{F}_{\mathrm{Sol}}(q^n)$ is saturated as a fusion system over $S(q^n)$, and satisfies the following conditions:*

(a) *For all $P, Q \le S(q^n)$ which contain z, if $\alpha \in \mathrm{Hom}(P,Q)$ is such that $\alpha(z) = z$, then $\alpha \in \mathrm{Hom}_{\mathcal{F}_n}(P,Q)$ if and only if $\alpha \in \mathrm{Hom}_{\mathrm{Spin}_7(q^n)}(P,Q)$.*

(b) *$C_{\mathcal{F}_n}(z) = \mathcal{F}_{S(q^n)}(\mathrm{Spin}_7(q^n))$ as fusion systems over $S(q^n)$.*

(c) *All involutions of $S(q^n)$ are $\mathcal{F}_n$–conjugate.*

Furthermore, $\mathcal{F}_m \subseteq \mathcal{F}_n$ for $m|n$. The union of the $\mathcal{F}_n$ is thus a category $\mathcal{F}_{\mathrm{Sol}}(q^\infty)$ whose objects are the finite subgroups of $S(q^\infty)$.

Proof We apply Proposition 1.3, where $p = 2$, $G = \mathrm{Spin}_7(q^n)$, $S = S(q^n)$, $Z = \langle z\rangle = Z(G)$, $U = \langle z, z_1\rangle$, $C_G(U) = H(q^n)$, $S_0 = C_S(U) = S_0(q^n)$, and $\Gamma = \Gamma_n \le \mathrm{Aut}(S_0)$. By Lemma 1.7, $\widehat{\gamma}_u$ preserves $H(q^n)$–fusion in S_0. Since Γ is generated by $\widehat{\gamma}_u$ and certain automorphisms of $H(q^n)$, this shows that all automorphisms in Γ preserve $H(q^n)$–fusion.

Condition (a) in Proposition 1.3 (all noncentral involutions in G are conjugate) holds since all subgroups in $\mathcal{E}_2$ are conjugate ([2, Proposition A.8]), and condition (b) holds by definition of Γ. Condition (c) holds since

$$\{\gamma \in \Gamma \mid \gamma(z) = z\} = \mathrm{Inn}(S_0(q^n))\cdot\langle c_\tau\rangle = \mathrm{Aut}_{N_G(U)}(S_0(q^n))$$

by [2, Proposition 2.5(b)]. Condition (d) was shown in Proposition 1.9, and condition (e) in Lemma 1.10. So by Proposition 1.3, $\mathcal{F}_n$ is a saturated fusion system, and $C_{\mathcal{F}_n}(Z) = \mathcal{F}_{S(q^n)}(\mathrm{Spin}_7(q^n))$.

The proofs of the other statements remain unchanged. □

In Section 3 of [2], these corrections affect only the proof of Lemma 3.1. In that proof, the groups E_{100} and E_{001} are not Γ_1–conjugate under the new definitions; instead E_{100} is Γ_1–conjugate to a subgroup which is $\mathrm{Spin}_7(q)$–conjugate to E_{001} (and hence the two are $\mathcal{F}$–conjugate). Also, when showing that $\mathrm{Aut}_{\mathcal{F}}(E_{001})$ is the group of all automorphisms which fix $z = x_{\mathcal{C}}(E_{001})$, it is important to know that all Γ_1–isomorphisms between subgroups in that conjugacy class preserve the elements $x_{\mathcal{C}}(-)$ (as shown in Lemma 1.8(e)), and not just that Γ_1–automorphisms of E_{001} do so.

The changes do not affect the later sections. Just note that we are able to consider $\mathcal{F}_{\mathrm{Sol}}(q^n)$ as a subcategory of $\mathcal{F}_{\mathrm{Sol}}(q^{kn})$ for $k > 1$, because they were both chosen using the same "correction factor" $u \in (\widehat{\mathbb{Z}}_2)^*$ in the definitions of Γ_n and Γ_{kn}.

2 Proof of Proposition 1.3

Proposition 1.3 follows from Lemmas 1.3, 1.4, and 1.5 in [2], once they are restated to assume the hypotheses of this new proposition, and not those of [2, Proposition 1.2]. The only one of these lemmas whose proof is affected by the change in hypotheses is Lemma 1.4, and so we restate and reprove it here.

Lemma 2.1 *Assume the hypotheses of Proposition 1.3, and let*

$$\mathcal{F} = \langle \mathcal{F}_S(G); \mathcal{F}_{S_0}(\Gamma)\rangle$$

be the fusion system generated by G and Γ. Then for all $P, P' \le S$ which contain Z,

$$\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, P') \mid \varphi(Z) = Z\} = \mathrm{Hom}_G(P, P').$$

Proof Upon replacing P' by $\varphi(P) \le P'$, we can assume that φ is an isomorphism, and thus that it factors as a composite of isomorphisms

$$P = P_0 \xrightarrow[\cong]{\varphi_1} P_1 \xrightarrow[\cong]{\varphi_2} P_2 \xrightarrow[\cong]{\varphi_3} \cdots \xrightarrow[\cong]{\varphi_{k-1}} P_{k-1} \xrightarrow[\cong]{\varphi_k} P_k = P',$$

where for each i, $\varphi_i \in \mathrm{Hom}_G(P_{i-1}, P_i)$ or $\varphi_i \in \mathrm{Hom}_\Gamma(P_{i-1}, P_i)$. Let $Z_i \le Z(P_i)$ be the subgroups of order p such that $Z_0 = Z_k = Z$ and $Z_i = \varphi_i(Z_{i-1})$.

To simplify the discussion, we say that a morphism in $\mathcal{F}$ is of type (G) if it is given by conjugation by an element of G, and of type (Γ) if it is the restriction of an automorphism in Γ. More generally, we say that a morphism is of type (G,Γ) if it is the composite of a morphism of type (G) followed by one of type (Γ), etc. We regard Id_P, for all $P \le S$, to be of both types, even if $P \not\le S_0$. By definition, if any nonidentity isomorphism is of type (Γ), then its source and image are both contained in $S_0 = C_S(U)$. In particular, if $P_i \not\le S_0$ for any $0 < i < k$, then φ_i and φ_{i+1} are both of type (G), and we can remove it and replace the two morphisms by their composite. We can thus assume that $[P_i, U] = 1$ for all $0 < i < k$, and hence that $Z_i \le Z(P_i U)$ for all i.

For each i, using [2, Lemma 1.3], choose some $\psi_i \in \mathrm{Hom}_{\mathcal{F}}(P_i U, S)$ such that $\psi_i(Z_i) = Z$. More precisely, using points (1) and (2) in [2, Lemma 1.3], we can choose ψ_i to be of type (Γ) if $Z_i \le U$ (the inclusion if $Z_i = Z$), and to be of type (G,Γ) if $Z_i \not\le U$. Set $P_i' = \psi_i(P_i)$. To keep track of the effect of morphisms on the subgroups Z_i, we write them as morphisms between pairs, as shown below. Thus, φ factors as a composite of isomorphisms

$$(P_{i-1}', Z) \xrightarrow{\psi_{i-1}^{-1}} (P_{i-1}, Z_{i-1}) \xrightarrow{\varphi_i} (P_i, Z_i) \xrightarrow{\psi_i} (P_i', Z).$$

If φ_i is of type (G), then this composite (after replacing adjacent morphisms of the same type by their composite) is of type (Γ, G, Γ). If φ_i is of type (Γ), then the composite is again of type (Γ, G, Γ) if either $Z_{i-1} \le U$ or $Z_i \le U$, and is of type $(\Gamma, G, \Gamma, G, \Gamma)$ if neither Z_{i-1} nor Z_i is contained in U. So we are reduced to assuming that φ is of one of these two forms.

Case 1 Assume first that φ is of type (Γ, G, Γ); ie, a composite of isomorphisms of the form

$$(P_0, Z) \xrightarrow[(\Gamma)]{\varphi_1} (P_1, Z_1) \xrightarrow[(G)]{\varphi_2} (P_2, Z_2) \xrightarrow[(\Gamma)]{\varphi_3} (P_3, Z).$$

Then $Z_1 = Z$ if and only if $Z_2 = Z$ because φ_2 is of type (G). If $Z_1 = Z_2 = Z$, then φ_1 and φ_3 are of type (G) by Proposition 1.3(c), and the result follows.

If $Z_1 \ne Z \ne Z_2$, then $U = ZZ_1 = ZZ_2$, and thus $\varphi_2(U) = U$. Neither φ_1 nor φ_3 can be the identity, so $P_i \le S_0 = C_S(U)$ for all i by definition of

$\mathrm{Hom}_\Gamma(-,-)$. Let $\gamma_1,\gamma_3\in\Gamma$ be such that $\varphi_i=\gamma_i|_{P_{i-1}}$, and let $g\in N_G(U)$ be such that $\varphi_2=c_g$. Then $gC_S(U)g^{-1}\in\mathrm{Syl}_p(C_G(U))$, so there is $h\in C_G(U)$ such that $hg\in N(C_S(U))$. Then $c_{hg}\in\Gamma$ by Proposition 1.3(c).

Since γ_3 preserves $C_G(U)$–fusion among subgroups of S_0, there is $g'\in C_G(U)$ such that $c_{g'}=\gamma_3\circ c_h^{-1}\circ\gamma_3^{-1}$. Thus φ is the composite

$$P_0 \xrightarrow[(\Gamma)]{\varphi_1} P_1 \xrightarrow[(\Gamma)]{c_{hg}} hP_2h^{-1} \xrightarrow[(\Gamma)]{\gamma_3} \gamma_3(hP_2h^{-1}) \xrightarrow[(G)]{c_{g'}} P_3.$$

The composite of the first three isomorphisms is of type (Γ) and sends Z to $g'^{-1}Zg'=Z$, hence is of type (G) by Proposition 1.3(c) again, and so φ is also of type (G).

Case 2 Assume now that φ is of type $(\Gamma,G,\Gamma,G,\Gamma)$; more precisely, that it is a composite of the form

$$(P_0,Z) \xrightarrow[(\Gamma)]{\varphi_1} (P_1,Z_1) \xrightarrow[(G)]{\varphi_2} (P_2,Z_2) \xrightarrow[(\Gamma)]{\varphi_3} (P_3,Z_3)$$
$$\xrightarrow[(G)]{\varphi_4} (P_4,Z_4) \xrightarrow[(\Gamma)]{\varphi_5} (P_5,Z),$$

where $Z_2,Z_3\not\leq U$. Then $Z_1,Z_4\leq U$ and are distinct from Z, and the groups P_0,P_1,P_4,P_5 all contain U since φ_1 and φ_5 (being of type (Γ)) leave U invariant. In particular, P_2 and P_3 contain Z, since P_1 and P_4 do and φ_2,φ_4 are of type (G). We can also assume that $U\leq P_2,P_3$, since otherwise $P_2\cap U=Z$ or $P_3\cap U=Z$, $\varphi_3(Z)=Z$, and hence φ_3 is of type (G) by Proposition 1.3(c) again. Finally, we assume that $P_2,P_3\leq S_0=C_S(U)$, since otherwise $\varphi_3=\mathrm{Id}$.

Let $E_i\leq P_i$ be the rank three elementary abelian subgroups defined by the requirements that $E_2=UZ_2$, $E_3=UZ_3$, and $\varphi_i(E_{i-1})=E_i$. In particular, $E_i\leq Z(P_i)$ for $i=2,3$ (since $Z_i\leq Z(P_i)$, and $U\leq Z(P_i)$ by the above remarks); and hence $E_i\leq Z(P_i)$ for all i. Also, $U=ZZ_4\leq\varphi_4(E_3)=E_4$ since $\varphi_4(Z)=Z$, and thus $U=\varphi_5(U)\leq E_5$. Via similar considerations for E_0 and E_1, we see that $U\leq E_i$ for all i.

Set $H=C_G(U)$ for short. Let $\mathcal{E}_3$ be the set of all elementary abelian subgroups $E\leq S$ of rank three which contain U, and let $\mathcal{E}_3^*\subseteq\mathcal{E}_3$ be the set of all $E\in\mathcal{E}_3$ such that $C_S(E)\in\mathrm{Syl}_p(C_G(E))$. Then for all $E\in\mathcal{E}_3$,

$$C_S(E)=C_{S_0}(E) \qquad \text{and} \qquad C_G(E)=C_H(E)$$

since $E\geq U$ (and $S_0=C_S(U)$). Thus $\mathcal{E}_3^*$ is the set of all subgroups $E\in\mathcal{E}_3$ which are fully centralized in the fusion system $\mathcal{F}_{S_0}(H)$, and so each subgroup in $\mathcal{E}_3$ is H–conjugate to a subgroup in $\mathcal{E}_3^*$.

Let $\gamma_1, \gamma_3, \gamma_5 \in \Gamma$ be such that $\varphi_i = \gamma_i|_{P_{i-1}}$ for odd i. Let $g_2, g_4 \in G$ be such that φ_i is conjugation by g_i for $i = 2, 4$. We will construct a commutative diagram of the following form

$$\begin{array}{ccccccccccc}
(P_0,E_0) & \xrightarrow{\gamma_1} & (P_1,E_1) & \xrightarrow{c_{g_2}} & (P_2,E_2) & \xrightarrow{\gamma_3} & (P_3,E_3) & \xrightarrow{c_{g_4}} & (P_4,E_4) & \xrightarrow{\gamma_5} & (P_5,E_5) \\
{\scriptstyle c_{a_0}}\big\downarrow & & {\scriptstyle c_{a_1}}\big\downarrow & & {\scriptstyle c_{a_2}}\big\downarrow & & {\scriptstyle c_{a_3}}\big\downarrow & & {\scriptstyle c_{a_4}}\big\downarrow & & {\scriptstyle c_{a_5}}\big\downarrow \\
(C'_0,E'_0) & \xrightarrow{\gamma_1} & (C'_1,E'_1) & \xrightarrow{c_{h_2}} & (C'_2,E'_2) & \xrightarrow{\gamma_3} & (C'_3,E'_3) & \xrightarrow{c_{h_4}} & (C'_4,E'_4) & \xrightarrow{\gamma_5} & (C'_5,E'_5)
\end{array}$$

where $E'_i \in \mathcal{E}_3^*$, $C'_i = C_S(E'_i)$, $h_2, h_4 \in G$, and $a_i \in H = C_G(U)$. To do this, first choose $a'_0, a'_2, a'_4 \in H$ such that $E'_i \overset{\text{def}}{=} a'_i E_i {a'_i}^{-1} \in \mathcal{E}_3^*$ for $i = 0, 2, 4$, and set $E'_i = \gamma_i(E'_{i-1})$ for $i = 1, 3, 5$. Then $C_S(E'_i) = \gamma_i(C_S(E'_{i-1}))$ for $i = 1, 3, 5$, so $C_S(E'_i) \in \mathrm{Syl}_p(C_H(E'_i))$ and $E'_i \in \mathcal{E}_3^*$ for all i. So we can choose $x_0 \in C_H(E'_0)$ such that $x_0(a'_0 P_0 {a'_0}^{-1})x_0^{-1} \le C_S(E'_0)$, and set $a_0 = x_0 a'_0 \in H$. Since $\gamma_1 \in \mathrm{Aut}(S_0)$ preserves H–fusion, there is $a_1 \in H$ which makes the first square in the above diagram commute. Now choose $x_2 \in C_H(E'_2)$ such that

$$x_2((a'_2 g_2 a_1^{-1}) C_S(E'_1)(a'_2 g_2 a_1^{-1})^{-1}) x_2^{-1} \le C_S(E'_2),$$

and set $a_2 = x_2 a'_2$ and $h_2 = a_2 g_2 a_1^{-1}$. Upon continuing this procedure we obtain the above diagram.

Let $\widehat{\varphi} \in \mathrm{Iso}_{\mathcal{F}}(C_S(E'_0), C_S(E'_5))$ be the composite of the morphisms in the bottom row of the above diagram. Then $\widehat{\varphi}(Z) = Z$, since φ and the c_{a_i} all send Z to itself. By Proposition 1.3(e), the rank three subgroups E'_i are all G–conjugate to each other. Choose $g \in G$ such that $gE'_5g^{-1} = E'_0$. Then $g{\cdot}C_S(E'_5){\cdot}g^{-1}$ and $C_S(E'_0)$ are both Sylow p–subgroups of $C_G(E'_0)$, so there is $h \in C_G(E'_0)$ such that $(hg)C_S(E'_5)(hg)^{-1} = C_S(E'_0)$. By Proposition 1.3(d), $c_{hg} \circ \widehat{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(C_S(E'_0))$ is of type (G), so $\widehat{\varphi}$ is of type (G), and hence $\varphi = c_{a_5}^{-1} \circ \widehat{\varphi} \circ c_{a_0}$ is also of type (G). □

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