Geometry & Topology Monographs
Volume 1: The Epstein Birthday Schrift
Pages 335–340

Complex projective structures on Kleinian groups

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Abstract Let M^3 be a compact, oriented, irreducible, and boundary incompressible 3–manifold. Assume that its fundamental group is without rank two abelian subgroups and $\partial M^3 \neq \emptyset$. We will show that every homomorphism $\theta\colon \pi_1(M^3) \to PSL(2,\mathbf{C})$ which is not "boundary elementary" is induced by a possibly branched complex projective structure on the boundary of a hyperbolic manifold homeomorphic to M^3.

AMS Classification 30F50; 30F45, 30F60, 30F99, 30C99

Keywords Projective structures on Riemann surfaces, hyperbolic 3–manifolds

1 Introduction

Let M^3 be a compact, oriented, irreducible, and boundary incompressible 3–manifold such that its fundamental group $\pi_1(M^3)$ is without rank two abelian subgroups. Assume that $\partial M^3 = R_1 \cup \ldots \cup R_n$ has $n \geq 1$ components, each a surface necessarily of genus exceeding one.

We will study homomorphisms

$$\theta\colon \pi_1(M^3) \to G \subset PSL(2,\mathbf{C})$$

onto groups G of Möbius transformations. Such a homomorphism is called *elementary* if its image G fixes a point or pair of points in its action on $\mathbf{H}^3 \cup \partial \mathbf{H}^3$, ie on hyperbolic 3–space and its "sphere at infinity". More particularly, the homomorphism θ is called *boundary elementary* if the image $\theta(\pi_1(R_k))$ of some boundary subgroup is an elementary group. (This definition is independent of how the inclusion $\pi_1(R_k) \hookrightarrow \pi_1(M^3)$ is taken as the images of different inclusions of the same boundary group are conjugate in G).

The purpose of this note is to prove:

Theorem 1 *Every homomorphism $\theta\colon \pi_1(M^3) \to PSL(2,\mathbf{C})$ which is not boundary elementary is induced by a possibly branched complex projective structure on the boundary of some Kleinian manifold $\mathbf{H}^3 \cup \Omega(\Gamma)/\Gamma \cong M^3$.*

This result is based on, and generalizes:

Theorem A (Gallo–Kapovich–Marden [1]) *Let R be a compact, oriented surface of genus exceeding one. Every homomorphism $\pi_1(R) \to PSL(2,\mathbf{C})$ which is not elementary is induced by a possibly branched complex projective structure on $\mathbf{H}^2/\Gamma \cong R$ for some Fuchsian group Γ.*

Theorem 1 is related to Theorem A as simultaneous uniformization is related to uniformization. Its application to quasifuchsian manifolds could be called simultaneous projectivization. For Theorem A finds a single surface on which the structure is determined whereas Theorem 1 finds a structure simultaneously on the pair of surfaces arising from some quasifuchsian group.

2 Kleinian groups

Thurston's hyperbolization theorem [3] implies that M^3 has a hyperbolic structure: there is a Kleinian group $\Gamma_0 \cong \pi_1(M^3)$ with regular set $\Omega(\Gamma_0) \subset \partial\mathbf{H}^3$ such that $\mathcal{M}(\Gamma_0) = \mathbf{H}^3 \cup \Omega(\Gamma_0)/\Gamma_0$ is homeomorphic to M^3. The group Γ_0 is not uniquely determined by M^3, rather M^3 determines the deformation space $\mathcal{D}(\Gamma_0)$ (taking a fixed Γ_0 as its origin).

We define $\mathcal{D}^*(\Gamma_0)$ as the set of those isomorphisms $\phi\colon \Gamma_0 \to \Gamma \subset PSL(2,\mathbf{C})$ onto Kleinian groups Γ which are induced by orientation preserving homeomorphisms $\mathcal{M}(\Gamma_0) \to \mathcal{M}(\Gamma)$. Then $\mathcal{D}(\Gamma_0)$ is defined as $\mathcal{D}^*(\Gamma_0)/PSL(2,\mathbf{C})$, since we do not distinguish between elements of a conjugacy class.

Let $\mathcal{V}(\Gamma_0)$ denote the representation space $\mathcal{V}^*(\Gamma_0)/PSL(2,\mathbf{C})$ where $\mathcal{V}^*(\Gamma_0)$ is the space of boundary nonelementary homomorphisms $\theta\colon \Gamma_0 \to PSL(2,\mathbf{C})$.

By Marden [2], $\mathcal{D}(\Gamma_0)$ is a complex manifold of dimension $\sum[3(\text{genus } R_k) - 3]$ and an open subset of the representation variety $\mathcal{V}(\Gamma_0)$. If M^3 is acylindrical, $\mathcal{D}(\Gamma_0)$ is relatively compact in $\mathcal{V}(\Gamma_0)$ (Thurston [4]).

The fact that $\mathcal{D}(\Gamma_0)$ is a manifold depends on a uniqueness theorem (Marden [2]). Namely two isomorphisms $\phi_i\colon \Gamma_0 \to \Gamma_i$, $i = 1, 2$, are conjugate if and only if $\phi_2\phi_1^{-1}\colon \Gamma_1 \to \Gamma_2$ is induced by a homeomorphism $\mathcal{M}(\Gamma_1) \to \mathcal{M}(\Gamma_2)$ which is homotopic to a conformal map.

3 Complex projective structures

For the purposes of this note we will use the following definition (cf [1]). A *complex projective structure* for the Kleinian group Γ is a locally univalent meromorphic function f on $\Omega(\Gamma)$ with the property that

$$f(\gamma z) = \theta(\gamma) f(z), \ z \in \Omega(\Gamma), \ \gamma \in \Gamma,$$

for some homomorphism $\theta\colon \Gamma \to PSL(2, \mathbf{C})$. We are free to replace f by a conjugate AfA^{-1}, for example to normalize f on one component of $\Omega(\Gamma)$.

Such a function f solves a Schwarzian equation

$$S_f(z) = q(z), \quad q(\gamma z)\gamma'(z)^2 = q(z); \ \gamma \in \Gamma, \ z \in \Omega(\Gamma),$$

where $q(z)$ is the lift to $\Omega(\Gamma)$ of a holomorphic quadratic differential defined on each component of $\partial\mathcal{M}(\Gamma)$. Conversely, solutions of the Schwarzian,

$$S_g(z) = q(z), \ z \in \Omega(\Gamma),$$

are determined on each component of $\Omega(\Gamma)$ only up to post composition by any Möbius transformation. The function f has the property that it not only is a solution on each component, but that its restrictions to the various components fit together to determine a homomorphism $\Gamma \to PSL(2, \mathbf{C})$. Automatically (cf [1]), the homomorphism θ induced by f is boundary nonelementary.

When *branched* complex projective structures for a Kleinian group are required, it suffices to work with the simplest ones: $f(z)$ is meromorphic on $\Omega(\Gamma)$, induces a homomorphism $\theta\colon \Gamma \to PSL(2, \mathbf{C})$ (which is automatically boundary nonelementary), and is locally univalent except at most for one point, modulo $\mathrm{Stab}(\Omega_0)$, on each component Ω_0 of $\Omega(\Gamma)$. At an exceptional point, say $z = 0$,

$$f(z) = \alpha z^2(1 + o(z)), \ \alpha \neq 0.$$

Such f are characterized by Schwarzians with local behavior

$$S_f(z) = q(z) = -3/2z^2 + b/z + \Sigma a_i z^i, \qquad b^2 + 2a_0 = 0.$$

At any designated point on a component R_k of $\partial\mathcal{M}(\Gamma)$, there is a quadratic differential with leading term $-3/2z^2$. To be admissible, a differential must be the sum of this and any element of the $(3g_k - 2)$–dimensional space of quadratic differentials with at most a simple pole at the designated point. In addition it must satisfy the relation $b^2 + 2a_0 = 0$. That is, the admissible differentials are parametrized by an algebraic variety of dimension $3g_k - 3$. For details, see [1].

If a branch point needs to be introduced on a component R_k of $\partial\mathcal{M}(\Gamma)$, it is done during a construction. According to [1], a branch point needs to be introduced if and only if the restriction

$$\theta\colon \pi_1(R_k) \to PSL(2, \mathbf{C})$$

does *not* lift to a homomorphism

$$\theta^*\colon \pi_1(R_k) \to SL(2, \mathbf{C}).$$

4 Dimension count

The vector bundle of holomorphic quadratic differentials over the Teichmüller space of the component R_k of $\partial\mathcal{M}(\Gamma_0)$ has dimension $6g_k - 6$. All together these form the vector bundle $\mathcal{Q}(\Gamma_0)$ of quadratic differentials over the Kleinian deformation space $\mathcal{D}(\Gamma_0)$. That is, $\mathcal{Q}(\Gamma_0)$ has *twice* the dimension of $\mathcal{V}(\Gamma_0)$. The count remains the same if there is a branching at a designated point.

For example, if Γ_0 is a quasifuchsian group of genus g, $\mathcal{Q}(\Gamma_0)$ has dimension $12g - 12$ whereas $\mathcal{V}(\Gamma_0)$ has dimension $6g - 6$. Corresponding to each nonelementary homomorphism $\theta\colon \Gamma_0 \to PSL(2, \mathbf{C})$ that lifts to $SL(2, \mathbf{C})$ is a group Γ in $\mathcal{D}(\Gamma_0)$ and a quadratic differential on the designated component of $\Omega(\Gamma)$. This in turn determines a differential on the other component. There is a solution of the associated Schwarzian equation $S_g(z) = q(z)$ satisfying

$$f(\gamma z) = \theta(\gamma) f(z), \ z \in \Omega(\Gamma), \ \gamma \in \Gamma.$$

Theorem 1 implies that $\mathcal{V}(\Gamma_0)$ has at most 2^n components. For this is the number of combinations of $(+,-)$ that can be assigned to the n–components of $\partial\mathcal{M}(\Gamma_0)$ representing whether or not a given homomorphism lifts. For a quasifuchsian group Γ_0, $\mathcal{V}(\Gamma_0)$ has two components (see [1]).

5 Proof of Theorem 1

We will describe how the construction introduced in [1] also serves in the more general setting here.

By hypothesis, each component Ω_k of $\Omega(\Gamma_0)$ is simply connected and covers a component R_k of $\partial\mathcal{M}(\Gamma_0)$. In addition, the restriction

$$\theta\colon \pi_1(R_k) \cong Stab(\Omega_k) \to G_k \subset PSL(2, \mathbf{C})$$

is a homomorphism to the nonelementary group G_k.

The construction of [1] yields a simply connected Riemann surface $\mathcal{J}_k$ lying over S^2, called a pants configuration, such that:

(i) There is a conformal group Γ_k acting freely in $\mathcal{J}_k$ such that $\mathcal{J}_k/\Gamma_k$ is homeomorphic to R_k.

(ii) The holomorphic projection $\pi\colon \mathcal{J}_k \to S^2$ is locally univalent if θ lifts to a homomorphism $\theta^*\colon \pi_1(R_k) \to SL(2, \mathbf{C})$. Otherwise π is locally univalent except for one branch point of order two, modulo Γ_k.

(iii) There is a quasiconformal map $h_k\colon \Omega_k \to \mathcal{J}_k$ such that

$$\pi h_k(\gamma_z) = \theta(\gamma)\pi h_k(z),\ \gamma \in Stab(\Omega_k),\ z \in \Omega_k.$$

Once h_k is determined for a representative Ω_k for each component R_k of $\partial\mathcal{M}(\Gamma_0)$, we bring in the action of Γ_0 on the components of $\Omega(\Gamma_0)$ and the corresponding action of $\theta(\Gamma_0)$ on the range. By means of this action a quasiconformal map h is determined on all $\Omega(\Gamma_0)$ which satisfies

$$\pi h(\gamma z) = \theta(\gamma)\pi h(z),\ \gamma \in \Gamma_0,\ z \in \Omega(\Gamma_0).$$

The Beltrami differential $\mu(z) = (\pi h)_{\bar z}/(\pi h)_z$ satisfies

$$\mu(\gamma z)\bar\gamma'(z)/\gamma'(z) = \mu(z),\ \gamma \in \Gamma_0,\ z \in \Omega(\Gamma_0).$$

It may equally be regarded as a form on $\partial\mathcal{M}(\Gamma_0)$. Using the fact that the limit set of Γ_0 has zero area, we can solve the Beltrami equation $g_{\bar z} = \mu g_z$ on S^2. It has a solution which is a quasiconformal mapping g and is uniquely determined up to post composition with a Möbius transformation. Furthermore g uniquely determines, up to conjugacy, an isomorphism $\varphi\colon \Gamma_0 \to \Gamma$ to a group Γ in $\mathcal{D}(\Gamma_0)$.

The composition $\pi h g^{-1}$ is a meromorphic function on each component of $\Omega(\Gamma)$. It satisfies

$$(\pi h g^{-1})(\gamma z) = \theta\varphi^{-1}(\gamma)\pi h g^{-1}(z), \quad \gamma \in \Gamma,\ z \in \Omega(\Gamma).$$

The composition is locally univalent except for at most one point on each component of $\Omega(\Gamma)$, modulo its stabilizer in Γ. That is, $\pi \circ h \circ g^{-1}$ is a complex projective structure on Γ that induces the given homomorphism θ, via the identification φ.

6 Open questions

Presumably, a nonelementary homomorphism $\theta\colon \Gamma_0 \to PSL(2, \mathbf{C})$ can be elementary for one, or all, of the $n \geq 1$ components of $\partial\mathcal{M}(\Gamma_0)$. Presumably too, the restrictions to $\partial\mathcal{M}(\Gamma_0)$ of a boundary nonelementary homomorphism can lift to a homomorphism into $SL(2, \mathbf{C})$ without the homomorphism $\Gamma_0 \to PSL(2, \mathbf{C})$ itself lifting. However we have no examples of these phenomena.

According to Theorem 1, there is a subset $\mathcal{P}(\Gamma_0)$ of the vector bundle $\mathcal{Q}(\Gamma_0)$ consisting of those homomorphic differentials giving rise to, say, unbranched complex projective structures on the groups in $\mathcal{D}(\Gamma_0)$. What is the analytic structure of $\mathcal{P}(\Gamma_0)$; is it a nonsingular, properly embedded, analytic subvariety?

When does a given Schwarzian equation $S_f(z) = q(z)$ on $\Omega(\Gamma)$ have a solution which induces a homomorphism of Γ?

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Received: 1 June 1998