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Coarse extrinsic geometry: a survey

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Abstract This paper is a survey of some of the developments in coarse extrinsic geometry since its inception in the work of Gromov. Distortion, as measured by comparing the diameter of balls relative to different metrics, can be regarded as one of the simplist extrinsic notions. Results and examples concerning distorted subgroups, especially in the context of hyperbolic groups and symmetric spaces, are exposed. Other topics considered are quasiconvexity of subgroups; behaviour at infinity, or more precisely continuous extensions of embedding maps to Gromov boundaries in the context of hyperbolic groups acting by isometries on hyperbolic metric spaces; and distortion as measured using various other filling invariants.

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To David Epstein on his sixtieth birthday

1 Introduction

Extrinsic geometry deals with the study of the geometry of subspaces relative to that of an ambient space. Given a Riemannian manifold M and a submanifold N, classical (differential) extrinsic geometry studies infinitesimal changes in the Riemannian metric on N induced from M. This involves an analysis of the second fundamental form or shape operator [35]. In coarse geometry local or infinitesimal machinery is absent. Thus it does not make sense to speak of tangent spaces or Riemannian metrics. However, the large scale notion of metric continues to make sense. Given a metric space X and a subspace Y one can still compare the intrinsic metric on Y to the metric inherited from X. This is especially useful for finitely generated subgroups of finitely generated groups. To formalize this, Gromov introduced the notion of distortion in his seminal paper [33].

Definition ([33],[22]) If $i\colon \Gamma_H \to \Gamma_G$ is an embedding of the Cayley graph of H into that of G, then the *distortion* function is given by

$$disto(R) = Diam_{\Gamma_H}(\Gamma_H \cap B(R)),$$

where $B(R)$ is the ball of radius R around $1 \in \Gamma_G$.

The definition above differs from the one in [33] by a linear factor and coincides with that in [22].

Note The above definition continues to make sense when Γ_G and Γ_H are replaced by graphs or (more generally) path-metric spaces (see below for definition) X and Y respectively.

Definition A *path-metric space* is a metric space (X, d) such that for all $x, y \in X$ there exists an isometric embedding $f\colon [0, d(x,y)] \to X$ with $f(0) = x$ and $f(d(x,y)) = y$.

If the distortion function is linear we say Γ_H (or Y) is *quasi-isometrically* (often abbreviated to *qi*) embedded in Γ_G (or X). This is equivalent to the following:

Definition A map f from one metric space (Y, d_Y) into another metric space (Z, d_Z) is said to be a (K,ϵ)*–quasi-isometric embedding* if

$$\frac{1}{K}(d_Y(y_1, y_2)) - \epsilon \leq d_Z(f(y_1), f(y_2)) \leq K d_Y(y_1, y_2) + \epsilon.$$

If f is a quasi-isometric embedding, and every point of Z lies at a uniformly bounded distance from some $f(y)$ then f is said to be a *quasi-isometry.* A (K,ϵ)–quasi-isometric embedding that is a quasi-isometry will be called a (K,ϵ)–quasi-isometry.

We collect here a few other closely related notions:

Definition A subset Z of X is said to be k*–quasiconvex* if any geodesic joining $a, b \in Z$ lies in a k–neighborhood of Z. A subset Z is *quasiconvex* if it is k–quasiconvex for some k.

A (K,ϵ)*–quasigeodesic* is a (K,ϵ)–quasi-isometric embedding of a closed interval in $\mathbb{R}$. A $(K,0)$–quasigeodesic will also be called a K–quasigeodesic.

For hyperbolic metric spaces (in the sense of Gromov [34]) the notions of quasiconvexity and qi embeddings coincide. This is because quasigeodesics lie close to geodesics in hyperbolic metric spaces [3], [31], [21].

Distortion can be regarded, in some sense, as the simplest extrinsic notion in coarse geometry. However a complete understanding of distortion is lacking even in special situations like subgroups of hyperbolic groups or discrete (infinite co-volume) subgroups of higher rank semi-simple Lie groups. One of the aims of this survey is to expose some of the issues involved. This is done in Section 2.

A characterisation of quasi-isometric embeddings in terms of group theory is another topic of extrinsic geometry that has received some attention of late. This will be dealt with in Section 3.

A different perspective of coarse extrinsic geometry comes from the asymptotic point of view. The issue here is behavior 'at infinity'. From this perspective it seems possible to introduce and study finer invariants involving distortion along specified directions. Section 4 deals with this in the special context of hyperbolic subgroups of hyperbolic groups.

Finally in Section 5, we discuss some other invariants of extrinsic geometry that have come up in different contexts.

It goes without saying that this survey reflects the author's bias and is far from comprehensive.

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2 Distortion

If a finitely generated subgroup H of a finitely generated group G is qi–embedded we shall refer to it as undistorted. Otherwise H will be said to be distorted. We shall also have occasion to replace the Cayley graph of G by a symmetric space (equipped with its invariant metric) or more generally a path metric space (X, d). In the latter case, distortion will be measured with respect to the metric d on X.

Distorted subgroups of hyperbolic groups or symmetric spaces are somewhat difficult to come by. This has resulted in a limited supply of examples. Brief accounts will be given of some of the known sources of examples.

An aspect that will not be treated in any detail is the connection to algorithmic problems, especially the Magnus problem. See [33] or (for a more detailed account) [22] for a treatment.

Subgroups of hyperbolic groups and $SL_2(\mathbb{C})$

One of the earliest classes of examples of distorted hyperbolic subgroups of hyperbolic groups came from Thurston's work on 3–manifolds fibering over the circle [62]. Let M be a closed hyperbolic 3–manifold fibering over the circle with fiber F. Then $\pi_1(F)$ is a hyperbolic subgroup of the hyperbolic group $\pi_1(M)$. The distortion is easily seen to be exponential.

It follows from work of Bonahon [8] and Thurston [61] that if H is a closed surface subgroup of the fundamental group $\pi_1(M)$ of a closed hyperbolic 3–manifold M then the distortion of H is either linear or exponential. This continues to be true if H is replaced by any freely indecomposable group. In fact exponential distortion of a freely indecomposable group corresponds precisely (up to passing to a finite cover of M) to the case of a hyperbolic 3–manifold fibering over the circle.

The situation is considerably less clear when we come to freely decomposable subgroups of hyperbolic 3–manifolds. The tameness conjecture (attributed to Marden [40]) asserts that the covering of a closed hyperbolic 3–manifold corresponding to a finitely generated subgroup of its fundamental group is topologically tame, ie is homeomorphic to the interior of a compact 3–manifold with boundary. If this conjecture were true, it would follow (using a Theorem of Canary [19]) that any finitely generated subgroup H of the fundamental group $\pi_1(M) = G$ is either quasiconvex in G or is exponentially distorted. Moreover, exponential distortion corresponds precisely (up to passing to a finite cover of M) to the case of a fiber of a hyperbolic 3–manifold fibering over the circle. Much of this theory can be extended to take parabolics into account.

This class of examples can be generalized in two directions. One can ask for distorted discrete subgroups of $SL_2(\mathbb{C})$ or for distorted hyperbolic subgroups of hyperbolic groups (in the sense of Gromov). We look first at discrete subgroups of $SL_2(\mathbb{C})$. A substantial class of examples comes from geometrically tame groups. In fact the simplest surface group, the fundamental group of a punctured torus (the puncture corresponds to a parabolic element), displays much of the exotic extrinsic geometry that may occur. These examples were studied in great detail by Minsky in [45]. The distortion function was calculated in [49].

Let S be a hyperbolic punctured torus so that the two shortest geodesics a and b are orthogonal and of equal length. Let S_0 denote S minus a neighborhood of the cusp. Let $N_\delta(a)$ and $N_\delta(b)$ be regular collar neighborhoods of a and b in S_0. For $n \in \mathbb{N}$, define $\gamma_n = a$ if n is even and equal to b if n is odd. Let T_n be the open solid torus neighborhood of $\gamma_n \times \{n + \frac{1}{2}\}$ in $S_0 \times [0, \infty)$ given by

$$T_n = N_\delta(\gamma_n)\times(n, n+1)$$

and let $M_0 = (S_0)\times[0,\infty)\backslash\bigcup_{n\in\mathbb{N}}T_n$.

Let $a(n)$ be a sequence of positive integers greater than one. Let $\hat{\gamma_n} = \gamma_n\times\{n\}$ and let μ_n be an oriented meridian for ∂T_n with a single positive intersection with $\hat{\gamma_n}$. Let M denote the result of gluing to each ∂T_n a solid torus $\hat{T_n}$, such that the curve $\hat{\gamma_n}^{a(n)}\mu_n$ is glued to a meridian. Let q_{nm} be the mapping class from S_0 to itself obtained by identifying S_0 to $S_0\times m$, pushing through M to $S_0\times n$ and back to S_0. Then $q_{n(n+1)}$ is given by $\Phi_n = D_{\gamma_n}^{a(n)}$, where D_c^k denotes Dehn twist along c, k times. Matrix representations of Φ_n are given by

$$\Phi_{2n} = \begin{pmatrix} 1 & a(2n) \\ 0 & 1 \end{pmatrix}$$

and

$$\Phi_{2n+1} = \begin{pmatrix} 1 & 0 \\ a(2n+1) & 1 \end{pmatrix}.$$

Recall that the metric on M_0 is the restriction of the product metric. The $\hat{T_n}$'s are given hyperbolic metrics such that their boundaries are uniformly quasi-isometric to $\partial T_n \subset M_0$. Then from [45], M is quasi-isometric to the complement of a rank one cusp in the convex core of a hyperbolic manifold $M_1 = \mathbb{H}^3/\Gamma$. Let σ_n denote the shortest path from $S_0\times 1$ to $S_0\times n$. Let $\overline{\sigma_n}$ denote σ_n with reversed orientation. Then $\tau_n = \sigma_n\gamma_n\overline{\sigma_n}$ is a closed path in M of length $2n+1$. Further τ_n is homotopic to a curve $\rho_n = \Phi_1\cdots\Phi_n(\gamma_n)$ on S_0. Then

$$\Pi_{i=1\cdots n}a(i) \le l(\rho_n) \le \Pi_{i=1\cdots n}(a(i)+2)$$

Hence

$$\Pi_{i=1\cdots n}a(i) \le (2n+1)disto(2n+1) \le \Pi_{i=1\cdots n}(a(i)+2)$$

Since M is quasi-isometric to the complement of the cusp of a hyperbolic manifold [45] and γ_n's lie in a complement of the cusp, the distortion function of Γ is of the same order as the distortion function above. In particular, functions of arbitrarily fast growth may be realized. This answers a question posed by Gromov in [33] page 66.

A closely related class of examples (the so called 'drill–holes' examples of which the punctured torus examples above may be regarded as special cases) appears in work of Thurston [62] and Bonahon and Otal [9].

Let us now turn to finitely generated subgroups of hyperbolic groups. If we restrict ourselves to hyperbolic subgroups there is a considerable paucity of examples. The chief ingredient for constructing distorted hyperbolic subgroups of hyperbolic groups is the celebrated combination theorem of Bestvina and Feighn [4]. This theorem was partly motivated by Thurston's hyperbolization theorem for Haken manifolds [43], [62] and continues to be an inevitable first step in constructing any distorted hyperbolic subgroups. The following Proposition summarizes these examples. The proof follows easily from normal forms.

Proposition 2.1 *Let G be a hyperbolic group acting cocompactly on a simplicial tree T such that all vertex and edge stabilizers are hyperbolic. Also suppose that every inclusion of an edge stabilizer in a vertex stabilizer is a quasi-isometric embedding. Let H be the stabilizer of a vertex or edge of T. Then the distortion of H is linear or exponential.*

Based on Bestvina and Feighn's combination theorem and work of Thurston's on stable and unstable foliations of surfaces [23], Mosher [53] constructed a class of examples of normal surface subgroups of hyperbolic groups where the quotient is free of rank strictly greater than one.

This idea was used by Bestvina, Feighn and Handel in [5] to construct similar examples where the normal subgroup is free.

Thus one has examples of exact sequences

$$1 \to N \to G \to Q \to 1$$

of hyperbolic groups where N is a free group or a surface group. Owing to a general theorem of Mosher's regarding the existence of quasi-isometric sections of Q [54] the distortion of any normal hyperbolic subgroup N of infinite index in a hyperbolic group G is exponential.

Further, it follows from work of Rips and Sela [59], [57] that a torsion free normal hyperbolic subgroup of a hyperbolic group is a free product of free groups and surface groups. However, the only known restriction on Q is that it is hyperbolic [54]. It seems natural to wonder if there exist examples where the exact sequence does not split or at least where Q is not virtually free.

We now describe some examples exhibiting higher distortion [49]. Start with a hyperbolic group G such that $1 \to F \to G \to F \to 1$ is exact, where F is free of rank 3.

Let $F_1 \subset G$ denote the normal subgroup. Let $F_2 \subset G$ denote a section of the quotient group. Let $G_1, \cdots, G_n$ be n distinct copies of G. Let F_{i1} and F_{i2} denote copies of F_1 and F_2 respectively in G_i. Let

$$G = G_1 *_{H_1} G_2 * \cdots *_{H_{n-1}} G_n$$

where each H_i is a free group of rank 3, the image of H_i in G_i is F_{i2} and the image of H_i in G_{i+1} is $F_{(i+1)1}$. Then G is hyperbolic.

Let $H = F_{11} \subset G$. Then the distortion of H is superexponential for $n > 1$. In fact, it can be checked inductively that the distortion function is an iterated exponential of height n.

Starting from Bestvina, Feighn and Handel's examples above, one can construct examples with distortion a tower function. Let a_1, a_2, a_3 be generators of F_1 and b_1, b_2, b_3 be generators of F_2. Then

$$G = \{a_1, a_2, a_3, b_1, b_2, b_3 : b_i^{-1} a_j b_i = w_{ij}\}$$

where w_{ij} are words in a_i's. We add a letter c conjugating a_i's to 'sufficiently random' words in b_j's to get G_1. Thus,

$$G_1 = \{a_1, a_2, a_3, b_1, b_2, b_3, c : b_i^{-1} a_j b_i = w_{ij}, c^{-1} a_i c = v_i\},$$

where v_i's are words in b_j's satisfying a small-cancellation type condition to ensure that G_1 is hyperbolic. See [34], page 151 for details on addition of 'random' relations.

It can be checked that these examples have distortion function greater than any iterated exponential.

The above set of examples were motivated largely by examples of distorted cyclic subgroups in [33], page 67 and [28] (these examples will be discussed later in this paper).

So far, there is no satisfactory way of manufacturing examples of hyperbolic subgroups of hyperbolic groups exhibiting arbitrarily high distortion. It is easy to see that a subgroup of sub-exponential distortion is quasiconvex [33]. Not much else is known. One is thus led to the following question:

Question Given any increasing function $f\colon \mathbb{N} \to \mathbb{N}$ does there exist a hyperbolic subgroup H of a hyperbolic group G such that the distortion of H is of the order of $e^{f(n)}$?

Note that the above question has a positive answer if G is replaced by $SL_2(\mathbb{C})$.

If one does not restrict oneself to hyperbolic subgroups of hyperbolic groups, one has a large source of examples coming from finitely generated subgroups of small cancellation groups. These examples are due to Rips [56].

Let $Q = \{g_1, \cdots, g_n : r_1, \cdots, r_m\}$ be any finitely presented group. Construct a small cancellation ($C'(1/6)$) group G with presentation as follows:

$$G = \{g_1, \cdots, g_n, a_1, a_2 : g_i^{-1} a_j g_i = u_{ij}, g_i a_j g_i^{-1} = v_{ij}, r_k = w_k \\ \text{for } i = 1 \cdots n,\ j = 1,2 \text{ and } k = 1 \cdots m.\ \}$$

where u_{ij}, v_{ij}, w_k are words in a_1, a_2 satisfying $C'(1/6)$.

Then one has an exact sequence $1 \to H \to G \to Q \to 1$ where H is the subgroup of G generated by a_1, a_2 and Q is the given finitely presented group. The distortion of H can be made to vary by varying Q (one basically needs to vary the complexity of the word problem in Q). However the subgroups H are generally not finitely presented.

A remarkable example of a finitely presented normal subgroup H of a hyperbolic group G has recently been discovered by Brady [15]. This is the first example of a finitely presented non-hyperbolic subgroup of a hyperbolic group. The distortion in this example is exponential as the quotient group is infinite cyclic.

Distortion in symmetric spaces

Now let G be a semi-simple Lie group. Cyclic discrete subgroups generated by unipotent elements are exponentially distorted. This is because discrete subgroups of the nilpotent subgroup N in a KAN decomposition of G is distorted in this way. This is the most well known source of distortion.

Other known examples seem to have their origin in rank 1 phenomena. Given any Lie group G containing $F_2 \times F_2$ as a discrete subgroup one has distorted subgroups coming from a construction due to Mihailova [44], [33], [22] (see below). In some sense these examples are 'reducible'. Truly higher rank phenomena are hard to come by. One has the following basic question:

Question Are there examples of distorted finitely generated discrete subgroups H of irreducible lattices in higher rank semi-simple Lie groups G such that H has no unipotent element? (See [22] also).

Note that Thurston's construction of normal subgroups cannot possibly go through here on account of the following basic theorem of Kazhdan–Margulis:

Theorem 2.2 [41] *Let Γ be an irreducible lattice in a symmetric space of real rank greater than one. Then any normal subgroup Λ of Γ is either finite or the quotient Γ/Λ is finite.*

Another non-distortion theorem has recently been proven by Lubotzky–Moses–Raghunathan [39] answering a question of Kazhdan:

Theorem 2.3 *Any irreducible lattice in a symmetric space X of rank greater than one is undistorted in X.*

The above theorems indicate the difficulty in obtaining distorted subgroups of higher rank Lie groups.

Similar questions may be asked for rank one symmetric spaces also eg for complex hyperbolic, quaternionic hyperbolic and the Cayley hyperbolic planes. Here, too there is a dearth of examples.

In real hyperbolic spaces, the situation is slightly better owing to Thurston's examples of 3–manifolds fibering over the circle. Based partly on Thurston's examples, Bowditch and Mess [13] have described an example of a finitely generated subgroup of a uniform lattice in $SO(4,1)$ that is not finitely presented. Abresch and Schroeder [1] have given an arithmetic construction of this lattice, too. One wonders if this arithmetic description can be used to give similar examples in $SU(4,1)$ or $Sp(4,1)$.

Such infinitely presented subgroups are necessarily distorted. Related examples have also been discovered by Potyagailo and Kapovich [55], [37].

A natural question is whether Thurston's construction goes through in higher dimensions or not:

Question Does there exist a uniform lattice in a rank one symmetric (other than $\mathbb{H}^3$) space containing a finitely presented (or even finitely generated) infinite normal subgroup of infinite index?

One should note that any such normal subgroup cannot be hyperbolic (by [57]).

Distortion in finitely presented groups

There are certain special classes of distorted subgroups of finitely presented groups that do not fall into any of the above categories.

A basic class of examples comes from the Baumslag Solitar groups

$$BS(1,n) = \{a, t : tat^{-1} = a^n\}$$

where the cyclic group generated by a has exponential distortion for $n > 1$.

A class of examples with higher distortion have appeared in work of Gersten [28]. We briefly describe these.

Take $G = \{g_1, \cdots, g_n : g_{i-1}{}^{g_i} = g_{i-1}^2 \text{ for } i = 2 \cdots n\}$. Then the cyclic subgroup generated by g_1 has distortion an iterated exponential function of height n.

Next consider $G = \{a, b, c : a^b = a^2, a^c = b\}$. Then the cyclic group generated by a has distortion greater than any iterated exponential.

Another class of subgroups with distortion a fractional power occurs in work of Bridson [16]:

Let $G_c = \mathbb{Z}^c \rtimes_{\phi_c} \mathbb{Z}$ where $\phi_c \in GL_c\mathbb{Z}$ is the unipotent matrix with ones on the diagonal and superdiagonal and zeroes elsewhere. For $c > 1$, G_c has infinite cyclic center. Given two such groups G_a, G_b amalgamate them along their cyclic center $\langle z \rangle$ to get $G(a,b) = G_a *_{\langle z \rangle} G_b$. Then the distortion function of G_b in $G(a,b)$ is of the form $n^{\frac{a}{b}}$.

A large class of examples of distortion arise from subgroups of nilpotent and solvable groups [33].

Finally we describe a class of examples due to Mihailova [44] which give rise to non-recursive distortion (see also [33] [22]). Let $G = \{g_1, \cdots, g_n : r_1 \cdots r_m\}$ be any finitely presented group with defining presentation $f\colon F_n \to G$. Then $f \times f$ maps $F_n \times F_n$ to $G \times G$. The pull-back H under this map of the 'diagonal subgroup' $\{(g,g) : g \in G\}$ is generated by elements of the form (g_i, g_i), $i = 1 \cdots n$ and $(1, r_j)$, $j = 1 \cdots m$. If G has unsolvable word problem, then the distortion of H in $F_n \times F_n$ is non-recursive.

3 Characterization of quasiconvexity

It was seen in the previous section that construction of distorted subgroups usually involves some amount of work. In fact for subgroups of hyperbolic groups, Gromov [34] describes 'length–angle' relationships between generators that would ensure quasiconvexity of the subgroup. This can be taken as a genericity result. In another setting, one could ask for examples of groups all whose finitely generated subgroups are undistorted. This is known for free groups, surface groups and abelian groups.

However, a general group-theoretic characterization of quasiconvexity seems far off. Gersten has recently described a functional analytic approach to this problem. We briefly describe this. Later we shall discuss a more group-theoretic approach. We shall restrict ourselves to finitely generated subgroups of hyperbolic groups (in the sense of Gromov) in this section.

The following discussion appears in [27], [25], [2]. Let X' be a complex of type $K(G,1)$ with finite $(n+1)$ skeleton $X'^{(n+1)}$ and let X be the universal cover of X'. The vector space of cellular chains $C_i(X,\mathbb{R})$ is equipped with the l_1 norm for a basis of i–cells. Then the boundary maps $\delta_{i+1}\colon C_{i+1}(X,\mathbb{R}) \to C_i(X,\mathbb{R})$ are bounded linear and (owing to the finiteness of the $n+1$–skeleton) one gets quasi-isometry invariant homology groups $H_i^{(1)}(X,\mathbb{R})$ for $i \leq n$. Since these homology groups are quasi-isometry invariant it makes sense to define $H_i^{(1)}(G,\mathbb{R}) = H_i^{(1)}(X,\mathbb{R})$ for $i \leq n$ for any such X. The following Theorem of Gersten's occurs in [27].

Theorem 3.1 *The finitely presented group G is hyperbolic if and only if $H_1^{(1)}(G,\mathbb{R}) = 0$. Moreover, if H is a finitely generated subgroup of G then H is quasiconvex if and only if the map $H_1^{(1)}(H,\mathbb{R}) \to H_1^{(1)}(G,\mathbb{R})$ induced by inclusion is injective.*

Earlier results along these lines had been found in [26], [25], [2].

In a different direction, one would like a purely group-theoretic characterization of quasiconvexity. We start with some definitions.

Definition Let H be a subgroup of a group G. We say that the elements $\{g_i | 1 \leq i \leq n\}$ of G are essentially distinct if $Hg_i \neq Hg_j$ for $i \neq j$. Conjugates of H by essentially distinct elements are called essentially distinct conjugates.

Note that we are abusing notation slightly here, as a conjugate of H by an element belonging to the normalizer of H but not belonging to H is still essentially distinct from H. Thus in this context a conjugate of H records (implicitly) the conjugating element.

Definition We say that the height of an infinite subgroup H in G is n if there exists a collection of n essentially distinct conjugates of H such that the intersection of all the elements of the collection is infinite and n is maximal possible. We define the height of a finite subgroup to be 0.

The main theorem of [32] states:

Theorem 3.2 *If H is a quasiconvex subgroup of a hyperbolic group G, then H has finite height.*

The following question of Swarup was prompted partly by this result:

Question (Swarup) *Suppose H is a finitely presented subgroup of a hyperbolic group G. If H has finite height is H quasiconvex in G?*

So far only some partial answers have been obtained. The first result is due to Scott and Swarup:

Theorem 3.3 [58] *Let $1 \to H \to G \to \mathbb{Z} \to 1$ be an exact sequence of hyperbolic groups induced by a pseudo Anosov diffeomorphism of a closed surface with fundamental group H. Let H_1 be a finitely generated subgroup of infinite index in H. Then H_1 is quasiconvex in G.*

In [51] an analogous result for free groups was derived. The methods also provide a different proof of Scott and Swarup's theorem above:

Theorem 3.4 [51] *Let $1 \to H \to G \to \mathbb{Z} \to 1$ be an exact sequence of hyperbolic groups induced by a hyperbolic automorphism ϕ of the free group H. Let $H_1 (\subset H)$ be a finitely generated distorted subgroup of G. Then there exist $N > 0$ and a free factor K of H such that the conjugacy class of K is preserved by ϕ^N and H_1 contains a finite index subgroup of a conjugate of K.*

Another special case where one has a positive answer is the following:

Theorem 3.5 [50] *Let G be a hyperbolic group splitting over H (ie $G = G_1 {*_H} G_2$ or $G = G_1 {*_H}$) with hyperbolic vertex and edge groups. Further, assume the two inclusions of H are quasi-isometric embeddings. Then H is of finite height in G if and only if it is quasiconvex in G.*

Swarup's question is therefore still open in the following special case, which can be regarded as a next step following the Theorems of [51] and [50] above.

Question Suppose G splits over H satisfying the hypothesis of Theorem 3.5 above and H_1 is a quasiconvex subgroup of H. If H_1 has finite height in G is it quasiconvex in G? More generally, if H_1 is an edge group in a hyperbolic graph of hyperbolic groups satisfying the qi–embedded condition, is H quasiconvex in G if and only if it has finite height in G?

A closely related problem can be formulated in more geometric terms:

Question Let X_G be a finite 2 complex with fundamental group G. Let X_H be a cover of X_G corresponding to the finitely presented subgroup H. Let $I(x)$ be the injectivity radius of X_H at x.

Does $I(x) \to \infty$ as $x \to \infty$ imply that H is quasi-isometrically embedded in G?

A positive answer to this question for G hyperbolic would provide a positive answer to Swarup's question.

The answer to this question is negative if one allows G to be only finitely generated instead of finitely presented as the following example shows:

Example Let $F = \{a, b, c, d\}$ denote the free group on four generators. Let $u_i = a^i b^i$ and $v_i = c^{f(i)} d^{f(i)}$ for some function $f\colon \mathbb{N} \to \mathbb{N}$. Introducing a stable letter t conjugating u_i to v_i one has a finitely generated HNN extension G. The free subgroup generated by a, b provides a negative answer to the question above for suitable choice of f. In fact one only requires that f grows faster than any linear function.

If f is recursive one can embed the resultant G in a finitely presented group by Higman's Embedding Theorem. But then one might lose malnormality of the free subgroup generated by a, b. If one can have some control over the embedding in a finitely presented group, one might look for a counterexample. A closely related example was shown to the author by Steve Gersten.

So far the following question (attributed to Bestvina and Brady) remains open:

Question Let G be a finitely presented group with a finite $K(G,1)$. Suppose moreover that G does not contain any subgroup isomorphic to $BS(m,n)$. Is G hyperbolic?

A malnormal counterexample to Swarup's question would provide a counterexample for the above question (observed independently by M. Sageev).

4 Boundary extensions

The purpose of this section is to take an asymptotic rather than a coarse point of view and expose some of the problems from this perspective. Since virtually all the work in this area involves actions of hyperbolic groups on hyperbolic metric spaces we restrict ourselves mostly to this.

Roughly speaking, one would like to know what happens 'at infinity'. We put this in the more general context of a hyperbolic group H acting freely and properly discontinuously by isometries on a proper hyperbolic metric space X. Then there is a natural map $i\colon \Gamma_H \to X$, sending the vertex set of Γ_H to the orbit of a point under H, and connecting images of adjacent vertices in Γ_H by geodesics in X. Let $\widehat{X}$ denote the Gromov compactification of X.

The basic question discussed in this section is the following:

Question Does the continuous proper map $i\colon \Gamma_H \to X$ extend to a continuous map $\hat{\imath}\colon \widehat{\Gamma_H} \to \widehat{X}$?

A measure–theoretic version of this question was asked by Bonahon in [7]. A positive answer to the above would imply a positive answer to Bonahon's question. Related questions in the context of Kleinian groups have been studied by Cannon and Thurston [20], Bonahon [8], Floyd [24] and Minsky [47].

Much of the work around this problem was inspired by a seminal (unpublished) paper of Cannon and Thurston [20]. The main theorem of [20] states:

Theorem 4.1 [20] *Let M be a closed hyperbolic 3–manifold fibering over the circle with fiber F. Let $\widetilde{F}$ and $\widetilde{M}$ denote the universal covers of F and M respectively. Then $\widetilde{F}$ and $\widetilde{M}$ are quasi-isometric to $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively. Let $\mathbb{D}^2 = \mathbb{H}^2 \cup \mathbb{S}^1_\infty$ and $\mathbb{D}^3 = \mathbb{H}^3 \cup \mathbb{S}^2_\infty$ denote the standard compactifications. Then the usual inclusion of $\widetilde{F}$ into $\widetilde{M}$ extends to a continuous map from $\mathbb{D}^2$ to $\mathbb{D}^3$.*

The proof of the above theorem involved the construction of a local 'Sol-like' metric using affine structures on surfaces coming from stable and unstable foliations. Coupled with Thurston's hyperbolization of 3–manifolds fibering over the circle one has a very explicit description of the boundary extension.

Using these (local) methods Minsky [47] generalized this theorem to the following:

Theorem 4.2 [47] *Let Γ be a Kleinian group isomorphic (as a group) to the fundamental group of a closed surface, such that $\mathbb{H}^3/\Gamma = M$ has injectivity radius uniformly bounded below by some $\epsilon > 0$. Then there exists a continuous map from the Gromov boundary of Γ (regarded as an abstract group) to the limit set of Γ in $\mathbb{S}^2_\infty$.*

Finally Klarreich [38] generalized the above theorem to the case of freely indecomposable Kleinian groups. A different proof was given by the author [49] (see below).

Theorem 4.3 ([38],[49]) *Let Γ be a freely indecomposable Kleinian group, such that $\mathbb{H}^3/\Gamma = M$ has injectivity radius uniformly bounded below by some $\epsilon > 0$. Then there exists a continuous map from the Gromov boundary of Γ (regarded as an abstract group) to the limit set of Γ in $\mathbb{S}^2_\infty$.*

Klarreich proved Theorem 4.3 by combining her Theorem 4.4 below with Theorem 4.2 above.

Theorem 4.4 [38] *Let X and Y be proper, geodesic Gromov–hyperbolic spaces, H_α a collection of closed, disjoint path-connected subsets of X, and $h\colon X \to Y$ a quasi–Lipschitz map such that for every H_α, h restricted to H_α extends continuously to the boundary at infinity. Suppose that the following hold:*

(1) *The complement in X of the sets H_α is open and path-connected as also the complement of $h(H_\alpha)$ in Y.*
(2) *There is some real number $k > 0$ such that the sets H_α are all k–quasiconvex in X and $h(H_\alpha)$'s are k–quasiconvex in Y.*
(3) *There is a real number $c > 0$ such that $d(H_\alpha, H_\beta) > c$ and such that $d(h(H_\alpha), h(H_\beta)) > 0$ for all α and β.*

Then if the map h induced on the electric spaces is a quasi-isometry, h extends continuously to a continuous map from the boundary of X to the boundary of Y. Here the electric spaces are the spaces obtained from X and Y by collapsing each space H_α (or $h(H_\alpha)$) to points: they inherit path metrics from X and Y.

One should note that since Cannon and Thurston's Theorem 4.1 deals with asymptotic behavior it might well be regarded as a theorem in coarse geometry. The above Theorems are all of this form. But the proof techniques in [20], [47] are local as they rely on Thurston's theory of singular foliations of surfaces. In [48] and [49] a different approach was described using purely large-scale techniques giving generalized versions of Theorems 4.1 4.2 and 4.3.

Theorem 4.5 [48] *Let G be a hyperbolic group and let H be a hyperbolic subgroup that is normal in G. Let $i\colon \Gamma_H \to \Gamma_G$ be the continuous proper embedding of Γ_H in Γ_G described above. Then i extends to a continuous map $\hat{i}$ from $\widehat{\Gamma_H}$ to $\widehat{\Gamma_G}$.*

A more useful generalization of Theorem 4.1 is:

Theorem 4.6 [49] *Let (X,d) be a tree (T) of hyperbolic metric spaces satisfying the quasi-isometrically embedded condition. Let v be a vertex of T. Let (X_v, d_v) denote the hyperbolic metric space corresponding to v. If X is hyperbolic then the inclusion of X_v in X extends continuously to the boundary.*

A direct consequence of Theorem 4.6 above is the following:

Corollary 4.7 *Let G be a hyperbolic group acting cocompactly on a simplicial tree T such that all vertex and edge stabilizers are hyperbolic. Also suppose that every inclusion of an edge stabilizer in a vertex stabilizer is a quasi-isometric embedding. Let H be the stabilizer of a vertex or edge of T. Then an inclusion of the Cayley graph of H into that of G extends continuously to the boundary.*

In [4], Bestvina and Feighn give sufficient conditions for a graph of hyperbolic groups to be hyperbolic. Vertex and edge subgroups are thus natural examples of hyperbolic subgroups of hyperbolic groups. These examples are covered by the above corollary.

Using Thurston's pleated surfaces technology one then gives a 'coarse' proof of Theorem 4.3. With some further work and using a theorem of Minsky [46], one can give [49] a 'partly coarse' proof of another result of Minsky [47]: Thurston's Ending Lamination Conjecture for geometrically tame manifolds with freely indecomposable fundamental group and a uniform lower bound on injectivity radius.

Theorem 4.8 [47] *Let N_1 and N_2 be homeomorphic hyperbolic 3–manifolds with freely indecomposable fundamental group. Suppose there exists a uniform lower bound $\epsilon > 0$ on the injectivity radii of N_1 and N_2. If the end invariants of corresponding ends of N_1 and N_2 are equal, then N_1 and N_2 are isometric.*

One should note here that the coarse techniques referred to circumvent only the building of a 'model manifold' — a local construction in [47]. It might be worthwhile to obtain a coarse proof of the main theorem of [46]. A positive answer to the following coarse question will do the job (as can be seen from [49]):

Question Let $\sigma\colon \mathbb{N} \to Teich(S)$ be a map. For l a closed curve on S, let l_i denote the length of the shortest curve freely homotopic to l on $\sigma(i)$.

Suppose there exists $\lambda > 1$ such that for all closed curves l on S one has

$$\lambda l_i \leq max(l_{i-1}, l_{i+1}) \text{ for all } i \in \mathbb{N}.$$

Then does σ lie in a bounded neighborhood of a Teichmuller geodesic?

The above question was motivated in part by the 'hallways flare' condition of [4] and a recent relative hyperbolicity result of Masur–Minsky [42].

Since a continuous image of a compact locally connected set is locally connected [36] Theorem 4.3 also shows that the limit sets of freely indecomposable Kleinian groups with a uniform lower bound on the injectivity radius are locally connected. The issue of local connectivity has received a lot of attention lately due to some recent foundational work of Bowditch and Swarup [10], [11], [14], [12], [60] following earlier work by Bestvina and Mess [6].

Theorem 4.9 ([10], [60]) *Let H be a one-ended hyperbolic group. Then its boundary is locally connected. Next assume H does not split over any two-ended group and acts on a proper hyperbolic metric space X with limit set $\Lambda \subset \partial X$. Then Λ is locally connected.*

The existence of continuous boundary extensions in general would thus imply (using Theorem 4.9) local connectivity of limit sets of hyperbolic groups acting on proper hyperbolic metric spaces. One wonders if some kind of a converse exists.

Such speculations are prompted on the one hand by Theorem 4.9 and by the following observation. Let Γ be a simply degenerate Kleinian group isomorphic to a surface group. Further assume Γ has no parabolics. Let Λ be the limit set of Γ, Ω its domain of discontinuity and X the boundary of the convex hull of Λ. Then 'nearest point projections' give a natural homeomorphism between Ω and X. From this it is easy to conclude that a continuous boundary extension exists if and only if a neighborhood of Λ in S^2_∞ deformation retracts onto Λ. In this special case therefore local connectivity is equivalent to continuous boundary extensions.

Before concluding this section it is worth pointing out that one needs finer invariants than distortion to understand asymptotic extrinsic geometry. One way of approaching the problem is to consider extrinsic geometry of rays (starting at $1 \in \Gamma_H$) and describe those which are not quasigeodesics in the ambient space

X. If one looks at bi-infinite geodesics instead of rays one gets 'ending laminations'. For 3–manifolds fibering over the circle with fiber F and monodromy ϕ one can think of these as the stable and unstable foliations of ϕ. Motivated by this, the author gave a more group theoretic description in [52] in the special case of a hyperbolic normal subgroup of a hyperbolic group.

Recall that for a hyperbolic 3–manifold M fibering over the circle with fiber F Cannon and Thurston show in [20] that the usual inclusion of $\widetilde{F}$ into $\widetilde{M}$ extends to a continuous map from $\mathbb{D}^2$ to $\mathbb{D}^3$. An explicit description of this map was also described in [20] in terms of 'ending laminations' [See [61] for definitions]. The explicit description depends on Thurston's theory of stable and unstable laminations for pseudo-anosov diffeomorphisms of surfaces [23]. In the case of normal hyperbolic subgroups of hyperbolic groups, though existence of a continuous extension $\hat{i}\colon \widehat{\Gamma_H} \to \widehat{\Gamma_G}$ was proven in [48], an explicit description was missing. In [52] some parts of Thurston's theory of ending laminations were generalized to the context of normal hyperbolic subgroups of hyperbolic groups. Using this an explicit description of the continuous boundary extension $\hat{i}\colon \widehat{\Gamma_H} \to \widehat{\Gamma_G}$ was given for H a normal hyperbolic subgroup of a hyperbolic group G.

In general, if

$$1 \to H \to G \to Q \to 1$$

is an exact sequence of finitely presented groups where H, G and hence Q (from [54]) are hyperbolic, one has ending laminations naturally parametrized by points in the boundary $\partial\Gamma_Q$ of the quotient group Q.

Corresponding to every element $g \in G$ there exists an automorphism of H taking h to $g^{-1}hg$ for $h \in H$. Such an automorphism induces a bijection ϕ_g of the vertices of Γ_H. This gives rise to a map from Γ_H to itself, sending an edge $[a,b]$ linearly to a shortest edge-path joining $\phi_g(a)$ to $\phi_g(b)$.

Fixing $z \in \partial\Gamma_Q$ for the time being (for notational convenience) we shall define the set of ending laminations corresponding to z.

Let $[1,z)$ be a semi-infinite geodesic ray in Γ_Q starting at the identity 1 and converging to $z \in \partial\Gamma_Q$. Let σ be a single-valued quasi-isometric section of Q into G. Let z_n be the vertex on $[1,z)$ such that $d_Q(1,z_n) = n$ and let $g_n = \sigma(z_n)$.

Given $h \in H$ let Σ_n^h be the (H–invariant) collection of all free homotopy representatives (or shortest representatives in the same conjugacy class) of $\phi_{g_n^{-1}}(h)$

in Γ_H. Identifying equivalent geodesics in Σ_n^h one obtains a subset S_n^h of (unordered) pairs of points in $\widehat{\Gamma_H}$. The intersection with $\partial^2\Gamma_H$ of the union of all subsequential limits (in the Chabauty topology) of $\{S_n^h\}$ will be denoted by Λ_z^h.

Definition The set of *ending laminations corresponding to* $z \in \partial\Gamma_Q$ is given by

$$\Lambda_z = \bigcup_{h\in H} \Lambda_z^h.$$

Definition The set Λ of all *ending laminations* is defined by

$$\Lambda = \bigcup_{z\in\partial\Gamma_Q} \Lambda_z.$$

It was shown in [52] that the continuous boundary extension $\hat{\imath}$ identifies end-points of leaves of the ending lamination. Further if $\hat{\imath}$ identifies a pair of points in $\partial\Gamma_H$, then a bi-infinite geodesic having these points as its end-points is a leaf of the ending lamination.

Similar descriptions of laminations have been used by Bestvina, Feighn and Handel for free groups [5]. Using these two descriptions in conjunction gives further information eg about subgroup structure [51].

5 Other invariants in extrinsic geometry

To fix notions consider a finitely generated group H acting on a path-metric space X. As mentioned in the introduction distortion arises out of comparing the intrinsic metric on Γ_H to the metric inherited from the ambient space X. Alternately this can be regarded as arising out of comparing filling functions, where one fills a copy of S^0 in Γ_H and X and compares the sizes of the chains required.

In Chapter 5 of [33] Gromov defines several filling invariants of spaces. Each of these gives rise to a relative version and corresponding distortion functions. Recall some of these from [33].

Given a simplicial n–cycle S in a homotopically (or homologically) n–connected simplicial complex X one constructs fillings of S by $(n+1)$ chains in X.

Definition Filling volume, denoted $FillVol_n(S,X)$ is the infimal simplicial volume of $(n+1)$ chains filling S.

Definition Filling radius, denoted $FillRad_n(S,X)$ is the minimal R such that S bounds in an R–neighborhood $U_R(S) \subset X$.

A host of other filling invariants are defined in [33] but we focus on these two.

We will define relative versions of the above two notions. Since the definitions of these invariants require n–connectedness of the spaces we shall assume that whenever these invariants are defined, the spaces in question are quasi-isometric to (or admit thickenings that are) n–connected. It will be clear that one gets quasi-isometry invariants in the process. Reference to an explicit quasi-isometry may at times be suppressed.

Distortion of $Fillvol_n$ and $FillRad_n$ can be defined in a somewhat more general context. Fix classes $\mathcal{S}_n(X)$ and $\mathcal{S}_n(Y)$ of n–cycles in X, Y respectively (eg one might restrict to connected cycles or images of spheres) such that $\mathcal{S}_n(X) \subset \mathcal{S}_n(Y)$. Let f_n be one be one of the functions $FillVol_n$ or $FillRad_n$. Define

$$\mathcal{S}_n(f_n, m, X) = \{S \in \mathcal{S}_n(X) : f_n(S) \le m\}.$$

Finally define

$$Disto(f, X, Y, m) = sup(f_n(S, Y))$$
$$\text{where the } sup \text{ is taken over } S \in \mathcal{S}_n(f_n, m, X) \cap \mathcal{S}_n(Y).$$

For $n = 0$, $\mathcal{S}-$ the set of maps of the 0–sphere S^0 and $f_0 = FillVol_0$ or $FillRad_0$ we get back the original distortion function. Note that $FillRad_0$ is approximately half of $FillVol_0$.

For $n = 1$, $\mathcal{S}$ the set of maps of S^1 and $f_0 = FillVol_1$ we get *area distortion* in the sense of Gersten [29]. Distortion has been surveyed in Section 2. We give a brief sketch of Gersten's results on area distortion.

Definition An automorphism of a finitely presented group is *tame* if it lifts to an automorphism of the free group on its generators, preserving the normal subgroup generated by relators.

Theorem 5.1 *Let ϕ be a tame automorphism of a one-relator group G. Then area is undistorted for $G \subset G \rtimes_\phi \mathbb{Z}$.*

In [29] Gersten shows that in extensions of $\mathbb{Z}$ by finitely presented groups G area distortion of G is at most an exponential of an isoperimetric function for the extension. Moreover, he describes examples of undistorted (in the usual sense of length) subgroups that exhibit area distortion. He observes further that for torus bundles over the circle with Sol geometry, area in the fiber subgroup is undistorted whereas length is exponentially distorted.

Gersten showed further that area is undistorted for finitely presented subgroups of finitely presented groups of cohomological dimension 2. From this it follows that finitely presented subgroups H of hyperbolic groups G are finitely presented provided G has cohomological dimension 2 or G is a hyperbolic small cancellation group [30].

The remaining distortion functions are yet to be studied systematically. The first class of examples where $Disto(FillVol_n, X, Y, m)$ seem interesting and tractable are examples coming from extensions of $\mathbb{Z}$ by $\mathbb{Z}^n$, ie for $G = \mathbb{Z}^n \rtimes_\phi \mathbb{Z}$ where $\phi \in GL_n\mathbb{Z}$. Such examples have been studied by Bridson [17] and Bridson and Gersten [18].

Much less is known about $Disto(FillRad_n, X, Y, m)$. These functions are related to topology of balls in groups (Chapter 4 of [33]). For a group Γ admitting a uniformly k–connected thickening X (see [33] for definitions) Gromov defines $\overline{R_k}(r)$ to be the infimal radius $R \geq r$ such that the inclusion of balls $B(r) \subset B(R)$ is k–connected.

The following observations are straightforward generalizations of corresponding statements (for $n = 0$) on pages 74–76 of [33]. Fix a group Γ' and a subgroup Γ.

Proposition 5.2 *If $Disto(FillRad_n, \Gamma', \Gamma, m)$ is superexponential in m then the function $\overline{R_k}(m)$ for $(\Gamma, dist_{\Gamma'}|\Gamma)$ grows faster than any linear function Cm.*

Proposition 5.3 *Take two copies of $(\Gamma', \Gamma \subset \Gamma')$ and let $\Gamma_1 = \Gamma' *_\Gamma \Gamma'$ be the double. Then the function $\overline{R_k}(m)$ for Γ_1 is minorized by $\overline{R_{k-1}}(m)$ for $(\Gamma, dist_{\Gamma'}|\Gamma)$.*

This leads to the following

Question Do there exist pairs of groups $H \subset G$ (with n–connected inclusions of thickenings of the Cayley Graph) such that $Disto(FillRad_n, \Gamma_G, \Gamma_H, m)$ is superexponential in m?

A positive answer will furnish (via Proposition 5.3) examples of groups with fast growing $\overline{R_k}(m)$ for $k \geq 2$ (page 80 of [33]). No such example has been found yet.

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