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Group categories and their field theories

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Abstract A group–category is an additively semisimple category with a monoidal product structure in which the simple objects are invertible. For example in the category of representations of a group, 1–dimensional representations are the invertible simple objects. This paper gives a detailed exploration of "topological quantum field theories" for group–categories, in hopes of finding clues to a better understanding of the general situation. Group–categories are classified in several ways extending results of Frölich and Kerler. Topological field theories based on homology and cohomology are constructed, and these are shown to include theories obtained from group–categories by Reshetikhin–Turaev constructions. Braided–commutative categories most naturally give theories on 4–manifold thickenings of 2–complexes; the usual 3–manifold theories are obtained from these by normalizing them (using results of Kirby) to depend mostly on the boundary of the thickening. This is worked out for group–categories, and in particular we determine when the normalization is possible and when it is not.

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Dedicated to Rob Kirby, on the occasion of his 60^{th} *birthday*

1 Introduction

There is a close connection between monoidal categories and low-dimensional modular topological field theories. Specifically, symmetric monoidal categories correspond to field theories on 2–dimensional CW complexes [2, 17]; monoidal categories correspond to theories on 3–manifolds with boundary, and tortile (braided–commutative) categories correspond to theories on 4–dimensional thickenings of 2–complexes. These last can usually be normalized to give theories on extended 3–manifolds, and this is the most familiar context [19, 22, 11, 20, 23]. Particularly interesting braided categories are obtained from representations of "quantum groups" at roots of unity, cf [14, 10], and analogous symmetric mod p categories were defined by Gelfand and Kazhdan [9].

This subject has produced a voluminous literature but not a lot of new information. Presumably we do not yet understand the geometric significance, wider contexts, methods of computation, etc, well enough to effectively exploit these theories. This paper presents a class of examples in which everything can be worked out in detail, as a source of clues for the general case. Descriptions of the categories gives a connection to recent work on classifying spaces. The field theories turn out to be special cases of constructions using homology of CW complexes, or more generally cohomology of manifold thickenings of CW complexes. This clarifies the nature of the objects on which the fields are defined, and hints at higher-dimensional versions. The examples illuminate the normalization procedure used to pass to fields on extended 3–manifolds. Finally group–categories occur as tensor factors of the "quantum" categories (2.2.4), so understanding them is an essential ingredient of the general case.

Finite groups provide another class of examples that have been worked out in detail [7, 17, 25], but these have not been so helpful. Representations of the group give a (symmetric) monoidal category, and a field theory (on all finite CW complexes) defined in terms of homomorphisms of fundamental groups into the finite group. The restriction of the field theory to 2–complexes is the field theory corresponding to the representation category. However the restriction of the field theory to 3–manifolds corresponds to the *double* of the category [16], not the category itself. Constructions using a double are much easier but also much less informative than the general case, so this is a defect in this model.

A *group category* is a semisimple additive category with a product structure in which the simple objects are invertible. Isomorphism classes of simple objects then form a group, called the "underlying group" of the category. Section 2 begins with a slightly more precise definition (2.1) and some examples. The conjectural appearance of group–categories as tensor factors of quantum categories (2.2.4) is particularly curious. Three views of the classification of group–categories are then presented. The first and only novel view (2.3) uses recent work on classifying spaces of braided categories [6] to give a characterization in terms of spaces with two nonvanishing homotopy groups. Specifically, group–categories over a ring R with underlying group G correspond to spaces E with $\pi_d(E) = G$ and $\pi_{d+1}(E) = \mathrm{units}(R)$. The cases $d = 1, 2$, and $d \geq 3$ correspond to monoidal, braided–commutative, and symmetric categories respectively. The Postnikov decomposition gives an equivalence of this to k–invariants in group cohomology. The second approach (2.4) derives a category structure directly from group cohomology using cellular cochains in a model for the classifying space. This approach was developed by Frölich and Kerler [8]. The third approach (2.5) gives a "numerical presentation" for the category. This is a format developed for machine computation [3, 18], but in this case it gives an explicit and efficient low-level description.

Group cohomology in the context of topological field theories first appeared in Dijkgraaf–Witten [5] as lagrangians for fields with finite gauge group. Their lagrangians lie in $H^3(B_G)$, which we now see as classifying monoidal (no commutativity conditions) group–categories. The field theory they construct corresponds to the double of the category.

Topological field theories based on homology with coefficients in a finite group are studied in section 3. Suppose G is a finite abelian group and R a ring. State spaces of the H_n theory are the free modules $R[H_n(Y;G)]$. Induced homomorphisms are defined by summing over H_{n+1}: if $X\supset Y_1,Y_2$ and $y\in H_1(Y_1;G)$ then

$$Z_X(y)=\Sigma_{\{x\in H_{n+1}(X;G)|\partial_1 x=-y\}}\partial_2 x.$$

We determine (3.1.3) exactly when this satisfies various field theory axioms. The H_1 theory is the one that connects with categories: on 2–complexes it corresponds to the standard (untwisted) group–category. On 3–manifolds it provides examples of field theories that are not modular. This illustrates the role of doubling or extended structures in obtaining modularity on 3–manifolds. The higher-dimensional versions are new, and suggest interesting connections with classical algebraic topology.

Probably the eventual proper setting for field theories will be covariant (homological), but the current constructions are too rigid. In section 4 we restrict to manifolds and consider the dual cohomology-based theories. Here we can build in a twisting by evaluating group cohomology classes on fundamental classes. Again we get examples for any n, and it is the $n=1$ cases that relate to group–categories. Again homological calculations determine when these satisfy field axioms. For $n=1$ state spaces are associated to manifold with the homotopy type of 1–complexes (we refer to these as "thickenings" of 1–complexes); induced homomorphisms come from thickenings of 2–complexes, and corners used in modular structures are thickenings of 0–complexes. The dimensions of these thickenings depend on the type of category. To establish notation we relate both fields and categories to spaces with two homotopy groups. Let E have $\pi_d(E)=G$ and $\pi_{d+1}(E)=\text{units}(R)$. Then E determines a category and a field theory:

d	category structure	fields on
1	associative	$(3,2,1)$–thickenings
2	braided–commutative	$(4,3,2)$–thickenings
≥ 3	symmetric	$(d+2,d+1,d)$–thickenings

We show (4.3) that the field theory is in fact the one obtained by a Reshetikhin–Turaev construction from the category.

Section 5 concerns field theories on 3–manifolds. The basic plan [23, 22] is to start with a theory on 4–dimensional thickenings of 2–complexes, associated to a braided–commutative category, and try to extract a theory that depends only on the boundary of the thickening. The geometric ingredient is the basis of the Kirby calculus [12]: a 3–manifold bounds a simply-connected 4–manifold, and this 4–manifold is well-defined up to connected sums with CP^2 and $\overline{CP}^2$. If we specify the index of the 4–manifold then it is well-defined up to sums with $CP^2\#\overline{CP}^2$. These connected sums change the induced homomorphisms by multiplication by an element in R. If the element associated to $CP^2\#\overline{CP}^2$ has an inverse square root then we can use it to nomalize the theory (tensor with an Euler characteristic theory) to be insensitive to such sums. This gives a theory defined on "extended" 3–manifolds: manifolds together with an integer specifying the index of the bounding 4–manifold. For group–categories we evaluate the effect of these connected sums in terms of structure constants of the category. When the underlying group is cyclic the conclusions are very explicit, and determine exactly when the field theory can be normalized. For instance over an algebraically closed field there are four categories with underlying group $Z/2Z$, distinguished by how the non-unit simple object commutes with itself. The possibilities are multiplication by ± 1 or $\pm i$, i a primitive fourth root of unity. The ± 1 cases are symmetric, $\pm i$ braided–symmetric. The canonical and braided cases can be normalized; the -1 case cannot.

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2 Group–categories

This section gives the formal definition and examples, then proceeds to classification. Classification is approached on three levels: modern homotopy theory gives a quick general description. Explicit CW models for classifying spaces give associativity and commutativity isomorphisms satisfying the standard axioms. Finally chosing bases for morphism sets gives a very explicit description in terms of sequences of units in the ring. Much of this material is essentially already known, so proofs are designed to clarify connections rather than nail down every detail. For instance the iterated bar construction is explained in detail in 2.4 because the connection with categories comes from the details, while the technically more powerful multi-simplicial construction behind 2.3 is not discussed. We do give a lot of detail, though, since new insights tend to be found in details.

2.1 Definition

A group–category is an additive category over a commutative ring R, with a product (monoidal structure) that distributes over addition, and in addition:

(1) it is additively semisimple in the sense that each object is a finite sum of certain specified "simple" objects;

(2) there are no nontrivial morphisms between distinct simple objects; and

(3) the simple objects are invertible.

An object is invertible if there is another object so that the product of the two is isomorphic to the multiplicative unit. This is a very restrictive condition. In particular it follows that the product of any two simple objects is again simple, so isomorphism classes of simple objects form a group. This is called the "underlying group" of the group category. Condition (2) is usually automatic for simple objects because the category is usually assumed to be abelian (have kernels and cokernels, see [15]). We avoid this assumption to enable use of integers and other non-fields as coefficient rings. The extra generality is useful in the abstract theory and really vital in some numerical computations.

2.2 Examples

The canonical examples are analogs of group rings. Other examples come from representations of groups and Lie algebras.

2.2.1 Canonical examples

Suppose G is a group and R a commutative ring. Define $\mathcal{R}[G]$ to be the category with objects G–graded free R–modules of finite total dimension. Morphisms are R–homomorphisms that preserve the grading, with the usual composition. The product is the standard graded product: if a and b are G–graded modules then

$$(a \otimes b)_f = \oplus_{\{g,h:\ gh=f\}} a_g \otimes b_h.$$

Products of morphisms are defined similarly. This product is naturally associative with associating isomorphism the "identity"

$$\oplus_{\{g,h,i:\ (gh)i=f\}}(a_g \otimes b_h) \otimes c_i \quad = \quad \oplus_{\{g,h,i:\ g(hi)=f\}} a_g (\otimes b_h \otimes c_i).$$

The simple objects are the "delta functions" that take all but one group element to zero, and that one to a copy of R. In 2.3.3 and 2.4.1 we see that general group–categories are obtained (up to equivalence) by modifying the associativity and commutativity structures in this standard example.

2.2.2 Sub group–categories

If $\mathcal{C}$ is an additive category with a product then the subcategory generated by the invertible objects is a group–cateory. The following examples are of this type.

2.2.3 One–dimensional representations

If R is a commutative ring and G is a group then a representation of G over R is a finitely generated free R–module on which G acts. Equivalently, these are $R[G]$ modules that are finitely generated free as R–modules. Tensor product over R gives a monoidal structure on the category of finite dimensional representations. The invertible elements in this category are the one–dimensional representations. Therefore the subcategory with objects sums of 1–dimensional representations is a group category. In fact it is equivalent to the canonical group–category $\mathcal{R}[\hom(G, \mathrm{units}(R)]$. We briefly describe the equivalences between the two descriptions since they are models for several other constructions.

A homomorphism $\rho\colon G \to \mathrm{units}(R)$ determines a 1–dimensional representation $R^\rho)$, where elements g act by multiplication by $\rho(g)$.

An object in the group–category is a free $\hom(G, \mathrm{units}(R))$–graded R–module, so associates to each homomorphism ρ a free module a_ρ. Take such an object to the representation $\oplus_\rho(a_\rho \otimes_R R^\rho)$. This clearly extends to morphisms. The canonical identification $R^\rho \otimes R^\tau = R^{\rho\tau}$ makes this a monoidal functor from the group–category to representations.

To go the other way suppose V is a representation. Define a $\hom(G, \mathrm{units}(R))$–graded R–module by associating to each homomorphism ρ the space $\hom(R^\rho, V)$. To give an object in the group–category these must be finitely generated free modules. This process therefore defines a functor on the subcategory of representations with this property, and this certainly contains sums of 1–dimensional representations. Note this functor may not be monoidal on its entire domain: there may be indecomposable modules of dimension greater than 1 whose product has 1–dimensional summands. However it is monoidal on the subcategory of sums of 1–dimensional representations. It is also easy to see it gives an inverse equivalence for the functor defined above.

2.2.4 Quantum categories

Let G be a simple Lie algebra, or more precisely an algebraic Chevalley group over Z, and p a prime larger than the Coxeter number of G. Some of the categories are defined for non-prime p, but the prime case is simpler and computations are currently limited to primes. Let X be the weight lattice. The "quantum" categories are obtained by: consider either mod p representations (Gelfand and Kazhdan [9]), or deform the universal enveloping algebra and then specialize the deformation parameter to a p^{th} root of unity [14, 10]. Define $\mathcal{G}_p$ to be the additive category generated by highest weight representations whose weights lie in the standard alcove of the positive Weyl chamber in X. Define a product on $\mathcal{G}_p$ by: take the usual tensor product of representations and throw away all indecomposable summands that are not of the specified type. The miracle is that this operation is associative, and gives a tortile or symmetric monoidal category in the root of unity or mod p cases respectively.

Now let $R \subset X$ denote the root lattice of the algebra. The quotient X/R is a finite abelian group, and each highest weight representation determines an element in X/R (the equivalence class of its weight). Subgroups of X/R correspond to Lie groups with algebra G, and representations of the group are those with weights in the given subgroup. In particular the "class 0" representations, ones with weights in the root lattice, form a monoidal subcategory.

Conjecture *The category $\mathcal{G}_p$ has a group subcategory with underlying group X/R, and $\mathcal{G}_p$ decomposes as a tensor product of this subcategory and the class 0 representations $\mathcal{G}_p^0$. Further $\mathcal{G}_p^0$ is "simple" in the sense that it has no proper subcategories closed under products and summands.*

This is true in the few dozen numerically computed examples, though the tensor product in the root-of-unity cases might be slightly twisted. In these examples the objects in the group subcategory have weights lying just below the upper wall of the alcove. The values of these weights are available through the "Category Comparison" software in [18] (see the Category Guide).

2.3 Homotopy classification of group–categories

Current homotopy-theory technology is used to obtain the classification in terms of spaces with two homotopy groups, or equivalently group cohomology. The result is essentially due to [8; section 7.5] where these are called "Θ–categories":

Proposition *Suppose R is a commutative ring and G is a group. Then*

(1) *monoidal group–categories over R with underlying group G correspond to $H^3(B_G; \mathrm{units}(R))$;*

(2) *tortile (ie, balanced braided–commutative monoidal) group–categories correspond to $H^4(B_G^2; \mathrm{units}(R))$; and*

(3) *symmetric monoidal group–categories to $H^{d+2}(B_G^d; \mathrm{units}(R))$, for $d > 2$.*

It has been known for a long time that the group completion of the nerve of a category with an associative monoidal structure is a loop space. It has been known almost as long that if the category is symmetric then the group completion is an infinite loop space. Recently this picture has been refined [6, 1] to include braided categories: the group completion of the nerve of a braided–commutative monoidal category is a 2–fold loop space. This can be applied to group–categories to obtain:

2.3.1 Lemma *Suppose R is a commutative ring and G a group (abelian in cases 2 and 3). Then*

(1) *equivalence classes of monoidal group–categories over R with group G correspond to homotopy classes of simple spaces with loop space $B_{\mathrm{units}(R)} \times G$;*

(2) *braided–commutative group categories correspond to spaces with second loop space $B_{\mathrm{units}(R)} \times G$; and*

(3) *symmetric group–categories correspond to spaces with d–fold loop $B_{\mathrm{units}(R)} \times G$, for $d > 2$.*

In practice this version seems to be more fundamental than the cohomology description of the Proposition.

Proof Consider the monoidal subcategory of simple objects and isomorphisms in the category. The nerve of a category is the simplicial set with vertices the objects, and n–simplices for $n > 0$ composable sequences of morphisms of length n. Condition 2.1(2) implies this is a disjoint union of components, one for each isomorphism class. Invertibility implies the components are all homotopy equivalent. Endomorphisms of the unit object in a category over a ring R are assumed to be canonically isomorphic to R, so the isomorphisms of each simple are given by units(R). This identifies the nerve of the whole category as $B_{\mathrm{units}(R)} \times G$.

The next step in applying the loop-space theory is group completion. Ordinarily π_0 of a category nerve is a monoid, and group completion converts this to a group. Here π_0 is already the group G so the nerve is equivalent to its group completion. Thus application of [6, 1] shows that the nerve is a 1–, 2– or $d > 2$–fold loop space when the category is monoidal, braided–commutative, and symmetric respectively.

A few refinements are needed:

(1) In the single loop case the delooping is X with $\pi_1(X) = G$ and $\pi_2 =$ units(R). In general π_1 acts on higher homotopy groups. Here the action

is trivial (the space is simple) because in the category G acts trivially on the coefficient ring.

(2) It is not necessary to be specific about which $d > 2$ in the symmetric case because in this particular setting a 3–fold delooping is automatically an infinite delooping. This follows from the cohomology description below.

(3) Generally the construction does not quite give a correspondence: monoidal structures give deloopings of the group completion of the nerve, while deloopings give monoidal structures on categories whose nerve is already the group completion. Here, however, the nerve is group–completed to begin with, so the inverse construction does give monoidal structures on categories equivalent to the original one.

(4) Since the original group–category is additively semisimple, monoidal structures on the simple objects extend linearly, and uniquely up to equivalence, to products on the whole category that distribute over sums. This shows that classification of structures on the subcategory of simples does classify the group–category. □

The final step in the classification is to relate this to group cohomology.

2.3.2 Lemma *Connected spaces with $\pi_d = G$, $\pi_{d+1} = \mathrm{units}(R)$, and all other homotopy trivial (and simple if $d = 1$) are classified up to homotopy equivalence by elements of $H^{d+2}(B^d_G; \mathrm{units}(R))$.*

Proof This is an almost trivial instance of Postnikov systems [24; chapter IX]. Suppose E is the space with only two non-vanishing homotopy groups. There is a map $E \to B^d_G$ (obtained, for instance, by killing π_{d+1}), and up to homotopy this gives a fibration

$$B^{d+1}_{\mathrm{units}(R)} \to E \to B^d_G.$$

The point of Postnikov systems is that this extends to the right: there is a map $k\colon B^d_G \to B^{d+2}_{\mathrm{units}(R)}$ well-defined up to homotopy, so that

$$E \to B^d_G \xrightarrow{k} B^{d+2}_{\mathrm{units}(R)}$$

is a fibration up to homotopy. This determines E, again up to homotopy. Homotopy classes of such maps k are exactly $H^{d+2}(B^d_G; \mathrm{units}(R))$, so the spaces E correspond to cohomology classes. □

Putting 2.3.1 and 2.3.2 together gives the classification theorem.

2.3.3 Monoidal categories from spaces with two homotopy groups

In many ways the delooping of 2.3.1 is more fundamental than its k–invariant of 2.3.2. We finish this section by showing how to recover the category from the space. A description directly in terms of the k–invariant is given in 2.4. Suppose E is a space with $\pi_1(E) = G$, $\pi_2(E) = \text{units}(R)$, and π_1 acts trivially on π_2. This data specifies (up to monoidal equivalence) a group–category over R with underlying group G. Here we show how to describe a category in the equivalence class. In 2.3.4 this is extended to braided–monoidal and symmetric categories.

Begin with the canonical category $\mathcal{R}[G]$ of 2.2.1. $\mathcal{G}$ has the same underlying additive category over R and the same product functor, but we change the associativity isomorphisms. Specifically we find $\alpha(f,g,h)$ so that the isomorphism $(a_f \otimes b_g) \otimes c_h \to a_f \otimes (b_g \otimes c_h)$ obtained by multiplying the standard isomorphism by α gives an associativity. The key property is the pentagon axiom.

The definition of α depends on lots of choices. For each $g \in \pi_1(E)$ choose a map $\hat{g}\colon I/\partial I \to E$ in the homotopy class. For each $g, h \in G$ choose a homotopy $m_{g,h}\colon \hat{g}\hat{h} \sim \widehat{gh}$. Here $\hat{g}\hat{h}$ indicates composition of paths. The only restrictions are that the identity element of the group lifts to the constant path, and $m_{1,g}$ and $m_{g,1}$ are constant homotopies.

Now define $\alpha(f,g,h)$ as follows: use these standard homotopies to construct a homotopy $\widehat{fgh} \sim \widehat{fg}\hat{h} \sim \hat{f}\hat{g}\hat{h} \sim \hat{f}\widehat{gh} \sim \widehat{fgh}$. Since this is a homotopy of a loop to itself the ends can be identified to give a map $I \times S^1/(\partial I \times S^1) \to E$. Think of $I \times S^1/(\{0\} \times S^1)$ as D^2, then this defines an element in $\pi_2(E) = \text{units}(R)$. Define $\alpha(a,b,c)$ to be this element of R.

We explain why the pentagon axiom holds. A huge diagram goes with this explanation, but the reader may find it easier to reconstruct the diagram than to make sense of a printed version. Thus we stick with words. It is sufficient to verify the axiom for simple objects, and we write g for the G–graded R–module that takes g to R and all other elements to 0. The pentagon has various associations of a 4–fold product $efgh$ at the five corners, and connects them with reassociation isomorphisms α. The routine for constructing the isomorphisms can be described as follows. Put the loop $\widehat{efgh}$ at each corner, and put the composite loop $\hat{e}\hat{f}\hat{g}\hat{h}$ in the center. Along each radius from a corner to the center put the concatenation of homotopies $m_{*,*}$ corresponding to the way of associating the product at that corner. The α for an edge comes from the homotopy of $\widehat{efgh}$ to itself obtained by going from one corner radially in to the center and then back out to the other corner. Going all the way around the pentagon corresponds to going in and out five times. But going out and back in along a single radius gives the composition of a homotopy with its inverse, so cancels, up to homotopy. Therefore the homotopy obtained from the full circuit is homotopic to the constant homotopy of $\widehat{efgh}$ to itself. In $\pi_2 E = \mathrm{units}(R)$ this is the statement that the product of the α terms associated to the edges is the identity, so the diagram commutes.

Changing the choices gives an isomorphic category. Specifically suppose $m'_{f,g}$ are different homotopies between compositions. They differ from the original m by elements of $\pi_2(E)$, so by units $\tau_{f,g} \in R$. Regard this as defining a natural isomorphism from the product functor to itself: $f \otimes g \to f \otimes g$ by multiplication by $\tau_{f,g}$. Then the identity functor $\mathcal{R}[G] \to \mathcal{R}[G]$ together with this transformation is a monoidal isomorphism, ie, associativity defined using m in the domain commutes with associativity using m' in the range. We revisit this construction in the context of group cohomology in 2.4.2, and make it more explicit using special choices in 2.5.

2.3.4 Braided group–categories from spaces with two homotopy groups

Suppose E has $\pi_2 E = G$ and $\pi_3 E = \mathrm{units}(R)$. According to 2.3.1 this corresponds to an equivalence class of braided–commutative group–categories with underlying group G. Here we show how to extract one such category from this data, extending the monoidal case of 2.3.3.

The loop space ΩE has $\pi_1 E = G$ and $\pi_2 E = \mathrm{units}(R)$, so specifies an associativity structure for the standard product on $\mathcal{R}[G]$. Let $\{\hat{g}\}\colon I \to \Omega E$ and $m_{f,g}\colon I^2 \to \Omega E$ be the choices used in 2.3.3 to make this explicit. Let $\widetilde{g}\colon I^2 \to E$ and $\widetilde{m}_{f,g}\colon I^3 \to E$ denote the adjoints. An element $\sigma(f,g) \in \mathrm{units}(R)$ is obtained as follows: define a homotopy $\widetilde{fg} \sim \widetilde{f}\widetilde{g} \sim \widetilde{g}\widetilde{f} \sim \widetilde{gf}$ by the reverse of $\widetilde{m}_{f,g}$, the clockwise standard commuting homotopy in π_2, and $\widetilde{m}_{g,f}$. Since G is abelian $gf = fg$, and this is a self-homotopy. Glueing the ends gives a map on $I^2 \times S^1/(\partial I^2 \times S^1)$. Regard this as a neighborhood of $S^1 \subset D^3$, and extend the map to D^3 by taking the complement to the basepoint. This gives an element of $\pi_3(E) = \mathrm{units}(R)$. Define this to be $\sigma(f,g)$. Define a commutativity natural transformation $f \otimes g \to g \otimes f$ by multiplying the natural identification by $\sigma(f,g)$.

We explain why this and the associativity from 2.3.3 satisfy the hexagon axiom. Again we omit the huge diagram. The hexagon has various associations of permutations of fgh at the corners, and reassociating and commuting isomorphisms alternate going around the edges. Imagine a triangle inside the hexagon, with two hexagon corners joined to each triangle corner. Put $\widetilde{fgh}$

at each hexagon corner, and the three permutations of $\widetilde{f}\widetilde{g}\widetilde{h}$ on the triangle corners. On the edges joining the triangle to the hexagon put compositions of homotopies $\widetilde{m}$ corresponding to different associations of the terms. On the edges of the triangle put clockwise commuting homotopies in π_2. The homotopies used to define associating or commuting units on the hexagon edges are obtained by going in to the triangle and either directly back out (for associations) or along a triangle edge and back out (for commutes). Going around the whole hexagon composes all these. The trips from the triangle out and back cancel, to give a homotopy of the big composition to the composition of the triangle edges. This composition is trivial (it gives the analog of the hexagon axiom for $\pi_2(E)$). Thus the composition of homotopies corresponding to the full circuit of the hexagon gives the trivial element in $\pi_3(E)$, and the diagram itself commutes.

Finally suppose E has $\pi_d(E) = G$ and $\pi_{d+1}(E) = \text{units}(R)$ for some $d > 2$. Then the same arguments as above apply except the representatives $\widetilde{g}$ are now defined on D^d, and there is only a single standard commuting homotopy, up to homotopy. This implies $\sigma(g,h)\sigma(h,g) = 1$, so the group–category is symmetric.

2.4 Models for classifying spaces

Here we use explicit CW models for classifying spaces B^n_G to connect group cohomology to descriptions of categories using functorial isomorphisms. This is done in detail for $n = 1, 2$, and outlined for $n = 3$. The basis for the connection is a comparison between general group–categories and the standard example 2.2.1.

2.4.1 Lemma *Suppose $\mathcal{C}$ is a group category over R with underlying group G. Then there is an equivalence of categories $\mathcal{C} \to \mathcal{R}[G]$ and a natural transformation between the given product in $\mathcal{C}$ and the standard product in $\mathcal{R}[G]$.*

Note that this functor usually not monoidal since it usually will not commute with associativity morphisms. If a “lax” description of associativity is used then it can be transferred through such a categorical equivalence. The classification of group–categories then corresponds to classification of different associativity and commutativity structures for the standard product on $\mathcal{R}[G]$.

Proof By hypothesis G is identified with the set of equivalence classes of simple objects in $\mathcal{C}$, so we can choose a simple object s_g in each equivalence class g. Further we can choose isomorphisms $m_{g,h} \colon s_{gh} \to s_g \diamond s_h$.

Now define the functor $\hom_s\colon \mathcal{C} \to \mathcal{R}[G]$ by: an object X goes to the function that takes $g \in G$ to $\hom_{\mathcal{C}}(s_g, X)$. Comparison of products in the two categories involves the diagram

$$\begin{array}{ccc} \mathcal{C}\times\mathcal{C} & \xrightarrow{\hom_s\times\hom_s} & \mathcal{R}[G]\times\mathcal{R}[G] \\ \big\downarrow\diamond & & \big\downarrow\otimes \\ \mathcal{C} & \xrightarrow{\hom_s} & \mathcal{R}[G] \end{array}$$

A natural transformation $\otimes(\hom_s \times \hom_s) \to \hom_s\diamond$ consists of: for X, Y in $\mathcal{C}$ and $g \in G$ a natural homomorphism $\oplus_h \hom(s_h, X) \otimes \hom(s_{h^{-1}g}, Y) \to \hom(s_g, X \diamond Y)$. Define this by taking $(a, b) \in \hom(s_h, X) \otimes \hom(s_{h^{-1}g}, Y)$ to $(a \diamond b)m_{h,h^{-1}g}$. It is simple to check this has the required naturality properties. Note the lack of any coherence among the isomorphisms $m_{g,h}$ prevents any conclusions about associativity. □

Associativity structures for a product on a category are defined using natural isomorphisms satisfying the "pentagon axiom" [15]. These can be connected directly to group cohomology via the cellular chains of a particular model for the classifying space.

2.4.2 Lemma *Suppose G is a group and R a commutative ring.*

(1) *Cellular 3–cocycles for the bar construction B_G are natural associativity isomorphisms for the product on $\mathcal{R}[G]$, and coboundaries of 2–cochains correspond to compositions with natural endomorphisms.*

(2) *If G is abelian, cellular 4–cocycles for the iterated bar construction B_G^2 give braided–commutative monoidal structures for the product on $\mathcal{R}[G]$, and coboundaries of 3–cochains correspond to natural endomorphisms.*

(3) *If G is abelian, cellular 5–cocycles on B_G^3 give symmetric monoidal structures.*

Lemmas 2.4.1 and 2.4.2 together give the equivalences between group–categories and cohomology, except for "balance" in the braided case. This is addressed in 2.4.3. The analysis in the symmetric case is only sketched.

Proof Suppose G is a discrete group. The "bar construction" gives the following model for the classifying space B_G: n–cells are indexed by n–tuples $(g_1, \ldots, g_n)$ of elements in the group, so we denote the set of n–tuples by $B_G^{(n)}$. Note that there is a single 0–cell, the 0–tuple $(\ )$. There are $n+1$ boundary functions from n–tuples to $(n-1)$–tuples: ∂_0 omits the first element;

∂_n omits the last; and for $0 < i < n$, ∂_i multiplies the i and $i+1$ entries: $\partial_i(g_1, \dots, g_n) = (g_1, \dots, g_i g_{i+1}, \dots, g_n)$.

We get a space by geometrically realizing these formal cells:

$$B_G = \left(\cup_n B_G^{(n)} \times \Delta^n\right) / \simeq .$$

Here Δ^n is the standard n–dimensional simplex, and $\simeq$ is the equivalence relation that for each n–tuple τ identifies $\tau \times \partial_i \Delta^n$ with $\partial_i \tau \times \Delta^n$.

The cellular chains of this CW structure gives a model for the chain complex of the space. Specifically, $C_n^c(B_G)$ is the free abelian group generated by the formal n–cells $B_G^{(n)}$, and the boundary homomorphism $\partial \colon C_n^c(B_G) \to C_{n-1}^c(B_G)$ takes an n–tuple τ to the class representing the boundary $\partial\tau$. The boundary of the standard n–simplex Σ^n is the union of the faces $\partial_i \Sigma^n$, but the ones with odd i have the wrong orientation. Using the equivalence relation in B_G therefore gives $\partial\tau = \Sigma_{i=0}^n (-1)^i \partial_i \tau$.

Now suppose H is an abelian group. The model for chains of B_G gives a description for the cohomology $H^3(G; H)$. A 3–cocycle is a function $\alpha \colon B_G^3 \to H$ with composition $\alpha\partial$ is trivial. $\partial(a,b,c,d) = (b,c,d) - (ab,c,d) + (a,bc,d) - (a,b,cd) + (a,b,c)$, so the cocycle condition is

$$\alpha(b,c,d) + \alpha(a,bc,d) + \alpha(a,b,c) = \alpha(ab,c,d) + \alpha(a,b,cd).$$

In the application the coefficient group is units(R), with multiplication as group structure. Rewriting the cocycle condition multiplicatively gives exactly the pentagon axiom for associativity, so this gives a monoidal category.

Now we consider uniqueness. A 2–cochain is a function on the 2–cells, so $\mu(a,b)$ defined for all $a, b \in G$. The coboundary of this is the 3–cochain obtained by composing with the total boundary homomorphism. Written multiplicatively (in units(R)) this is

$$(\delta\mu)(a,b,c) = \mu(b,c)\mu(ab,c)^{-1}\mu(a,bc)\mu(a,b)^{-1}.$$

Thus a 3–cocycle α' differs from α by a coboundary if

$$\alpha'(a,b,c) = \mu(a,b)^{-1}\mu(ab,c)^{-1}\alpha(a,b,c)\mu(b,c)\mu(a,bc).$$

Interpreting this as commutativity in the diagram

$$\begin{tikzcd}
(ab)c \arrow[r, "\alpha(a,b,c)"] \arrow[d, "\mu(a,b)"] & (a(bc) \arrow[d, "\mu(b,c)"] \\
(ab)c \arrow[d, "\mu(ab,c)"] & (a(bc) \arrow[d, "\mu(a,bc)"] \\
(ab)c \arrow[r, "\alpha'(a,b,c)"] & (a(bc)
\end{tikzcd}$$

shows we can think of μ as a natural transformation from the standard product to itself, and then α' is obtained from α by composition with this transformation. This gives an isomorphism between categories where the associativity cocycles differ by a coboundary.

The braided case uses the iterated bar construction. If G is abelian then B_G is again a group, this time simplicial or topological rather than discrete. The same construction gives a simplicial (or Δ) space $B(B_G)$ whose realization is B^2_G. The first step in describing this is a description of the multiplication on B_G.

Cells in the product $B_G \times B_G$ are modeled on products $\Delta^i \times \Delta^j$. The map $B_G \times B_G \to B_G$ is defined by subdividing these products into simplices, and describing where in B_G to send these simplices. The standard subdivision of a product of simplices is obtained as follows: the vertices of Δ^i are numbered $0, 1, \ldots, i$. Suppose $((r_0, s_0), \ldots, (r_{i+j}, s_{i+j}))$ is a sequence of pairs of these, ie, vertices of $\Delta^i \times \Delta^j$, then the function of vertices $k \mapsto (r_k, s_k)$ extends to a linear map to the convex hull $\Delta^{i+j} \to \Delta^i \times \Delta^j$. Restrict the sequences to ones for which one coordinate of (r_{k+1}, s_{k+1}) is the same as in (r_k, s_k), and the other coordinate increases by exactly one. Then this gives a collection of embeddings with disjoint interiors, whose union is the whole product.

We relate this subdivision to the indexing of simplices by sequences in G. Think of a sequence $(a_1, \ldots, a_i)$ as labeling edges in Δ^i, specifically think of a_k as labeling the edge from vertex $k-1$ to k. Then we label sub-simplices of a product $(a_*) \times (b_*)$ by: if $r_k = r_{k-1} + 1$ then label the edge from $k-1$ to k with a_{r_k}, otherwise label it with b_{s_k}. This identifies the sub-simplices as corresponding to i, j–shuffles: orderings of the union $(a_*) \cup (b_*)$ which restrict to the given orderings of a_* and b_*. Thus we can write

$$\Delta^i \times \Delta^j = \cup_s s(\Delta^{i+j})$$

where the union is over i, j shuffles s. For future reference we mention that the orientations don't all agree: the orientation on $s(\Delta^{i+j})$ is $(-1)^s$ times the orientation on the product, where $(-1)^s$ indicates the parity of s as a permutation.

Now the product on B_G is defined by: if s is a shuffle the sub-simplex $s(\Delta^{i+j}) \times (a_*) \times (b_*)$ goes to $\Delta^{i+j} \times s(a_*, b_*)$. It is a standard fact that this is well-defined on intersections of sub-simplices.

As before the n–simplices of $B(B_G$ are indexed by points in the n–fold product $\times^n B_G$. The realization is again

$$B^2_G = \big(\cup_n \Delta^n \times (\times^n B_G)\big) / \simeq .$$

The equivalence relation identifies points in the boundary of Δ^n with points in lower-dimensional pieces. Specifically we identify $\partial_k\Delta^n \times (\times^n B_G)$ with its image in $\Delta^{n-1}(\times^{n-1}B_G)$, via the map which is the "identity" on the simplices, and on the B_G part multiplies the $k-1$ and k entries if $0<k<n$, omits the first if $0=k$, and omits the last if $k=n$.

This definition gives a cell complex model for B_G^2. Unraveling, we find the cells are of the form

$$\Delta^n \times \left(\Delta^{i_1} \times \cdots \times \Delta^{i_n}\right) \times \left((a_*^1) \times \cdots \times (a_*^n)\right),$$

where (a_*^k) is a sequence of length i_k.

The cell structure on the space gives standard models for the chain and cochain complexes. The first comment about the chain complex is that the cells that involve the 0–cell of B_G form a contractible subcomplex. The union is not a topological subcomplex because these cells have faces that are not of this type. However if a face does not involve a 0–cell then there is an adjacent face with the same image but opposite sign, so they algebraically cancel in the chain complex. Dividing out this subcomplex leaves "non-trivial" cells, corresponding to non-empty sequences (a_*).

We use this to describe the cohomology group H^4. Eventually the coefficients will be units(R), but to keep the notation standard we start with a group J with group operation written as addition. Nontrivial 4–cells are in two families:

$$\begin{aligned} \Delta^1 \times (\Delta^3) &\text{ indexed by } (a,b,c), \text{ and} \\ \Delta^2 \times (\Delta^1 \times \Delta^1) &\text{ indexed by } ((a),(b)) \end{aligned}$$

Denote the cochain $C_4 \to J$ by $\alpha(a,b,c)$ on the first family, and $\sigma(a,b)$ on the second.

The cocycle condition on (α,σ) comes from boundaries of 5–cells. Nontrivial 5–cells are in families:

$$\begin{aligned} \Delta^1 \times (\Delta^4) &\text{ indexed by } (a,b,c,d) \\ \Delta^2 \times (\Delta^1 \times \Delta^2) &\text{ indexed by } ((a),(b,c)), \text{ and} \\ \Delta^2 \times (\Delta^2 \times \Delta^1) &\text{ indexed by } ((a,b),(c)) \end{aligned}$$

In the first family the boundary of the Δ^1 factor is trivial, so the boundary is the boundary of (a,b,c,d) as a 4–cell of B_G. As before this gives the pentagon axiom for α. Now consider $((a),(b,c))$ in the second family. Boundaries of products are given by $\partial(x \times y) = \partial(x) \times y + (-1)^{\dim(y)} x \times \partial y$. In $\Delta^2 \times (\Delta^1 \times \Delta^2)$ the boundary on the middle piece vanishes so the total boundary is $\partial \times \mathrm{id} \times \mathrm{id} - \mathrm{id} \times \mathrm{id} \times \partial$. In the first factor the boundary is $\partial_0 - \partial_1 + \partial_2$. The first and last use projection of $B_G \times B_G$ to one factor, so map to cells of dimension less than 3 and are trivial algebraically. ∂_1 uses multiplication in B_G so is given by $(1,2)$ shuffles. This contribution to the boundary is thus $-\big((a,b,c)-(b,a,c)+(b,c,a)\big)$. The boundary in the last coordinate applies the B_G boundary to (b,c). This contribution is $-\big(((a),(c))-((a),(bc))+((a),(b))\big)$. Applying the cochain and setting it to zero gives

$$\alpha(a,b,c)-\alpha(b,a,c)+\alpha(b,c,a)+\sigma(a,c)-\sigma(a,bc)+\sigma(a,b)=0.$$

This is exactly the hexagon axiom for α and σ^{-1}, written additively. Boundaries of cells in the third family give the hexagon axiom for α and σ.

The conclusion is that 4–dimensional cellular cochains in B_G^2 correspond exactly to associativity and commutativity isomorphisms (α,σ) satisfying the pentagon and hexagon axioms, for the standard product on the category $\mathcal{R}[G]$.

The final step in the proof of Lemma 2.4.2 is seeing that coboundaries correspond to endomorphisms, or more precisely natural transformations of the standard product to itself.

The only nontrivial 3–cells in B_G^2 are of the form $\Delta^1\times\Delta^2\times(a,b)$. 3–cochains therefore correspond to functions $\mu(a,b)$. Boundaries of 4–cells are given by: in the $\Delta^1\times\Delta^3$ case, the negative of the B_G boundary (the negative comes from the preceeding Δ^1 factor). In the $\Delta^2\times(\Delta^1\times\Delta^1)$ case all terms vanish except the $-\partial_1$ term in the first factor, which gives shuffles $-\big((a,b)-(b,a)\big)$. Therefore changing a 4–cocycle (α,σ) by the coboundary of μ changes $\alpha(a,b,c)$ just as in the monoidal category case, and changes σ by conjugation by μ.

Finally we come to the symmetric monoidal case, using B_G^3. This is a further bar construction obtained as

$$B_G^3=\big(\cup_n(\Delta^n\times(\times^n B_G^2))\big)/\simeq$$

where the identifications in $\simeq$ involve a product structure on B_G^2. We indicate the source of the new information (symmetry of σ) without going into details.

We are concerned with H^5, so functions on the 5–cells. Again we can divide out the "trivial" ones involving 0–cells of B_G at the lowest level. The only nontrivial cells are products of Δ^1 and 4–cells of B_G^2, so these use the same data (α,σ) as 4–cochains on B_G^2. Boundaries of 6–cells of the form Δ^1 times a 5–cell of B_G^2 involve only the second factor, so give the same relations as in B^2 (namely, the pentagon and hexagon axioms). The only other source of relations are nontrivial 6–cells of the form $\Delta^2\times((2\text{–cell})\times(2\text{–cell}))$, where each of these 2–cells (in B^2) is of the form $\Delta^1\times\Delta^1_{(a)}$. The only nonzero term in the boundary of such a 6–cell comes from ∂_1 in Δ^2, which goes to Δ^1 times the product of the two 2–cells in B^2. We won't describe this in detail, but multiplying two cells of the form $\Delta^1\times\Delta^1$ involves multiplying the first two

Δ^1 factors to get a square, then subdividing this into two Δ^2. These two subsimplices have opposite orientation, so the product is a difference of cells of B^2 of the form $\Delta^2 \times (\Delta^1 \times \Delta^1)$. Vanishing of the cocycle on this therefore is a relation of the form (written additively) $\sigma(*) - \sigma(**) = 0$. So much follows from generalities. We don't do it here, but explicit description of the product structure shows the indices on the two final Δ^1 factors is interchanged, so we get exactly the symmetry relation

$$\sigma(a,b) = \sigma(b,a). \qquad \square$$

2.5 Numerical presentations

Here we get explicit "numerical presentations" of group–categories in the sense of [3]. This amounts to direct computation of group cohomology, and we interpret some of the formulae in terms of cohomology operations. We consider the symmetric and braided–commutative cases in detail, and only remark on the general monoidal case.

2.5.1 Proposition *Suppose G is an abelian group with generators g_i of order n_i, and R is a commutative ring.*

(1) *Braided–commutative group–categories over R with underlying group G correspond to*

 i) *σ_* with $\sigma_i^{2n_i} = 1$, and $\sigma_i^{n_i} = 1$ if n_i is odd; and*

 ii) *$\sigma_{i,j}$ for $i > j$, with $\sigma_{i,j}^{n_i} = \sigma_{i,j}^{n_j} = 1$.*

(2) *These categories are all tortile, and any tortile structure is obtained by scaling a standard one by a homomorphism from G to the units of R.*

(3) *The symmetric monoidal categories correspond to $\sigma_i^2 = \sigma_{i,j} = 1$.*

Given a group–category we extract the invariants as follows: Choose a simple object $\hat{g}_i$ in the equivalence class g_i. The commuting isomorphism $\sigma_{\hat{g}_i,\hat{g}_i}$ is an endomorphism of the object $\hat{g}_i \diamond \hat{g}_i$, so is multiplication by an element of R. Define this to be σ_i. If $i > j$ the double commuting isomorphism $\hat{g}_i \diamond \hat{g}_j \to \hat{g}_j \diamond \hat{g}_i \to \hat{g}_i \diamond \hat{g}_j$ is also an endomorphism, so is multiplication by an element of R. Define this to be $\sigma_{i,j}$.

Conversely given invariants we define a group–category by defining associating and commuting isomorphisms for the standard product on the standard group–category $\mathcal{R}[G]$. The content of 2.5.1 is then that there is a braided–monoidal equivalence from a general group–category to the standard one with the same invariants.

2.5.2 The inverse construction

Suppose data as in 2.5.1 are given. If a is an element of G we let a_i denote the exponent of g_i in a, so we have $a = \Pi_i g_i^{a_i}$. Then define:

$$\alpha\colon (ab)c \to a(bc) \text{ is multiplication by } \Pi_i \begin{cases} 1 & \text{if } b_i + c_i < n_i \\ \sigma_i^{n_i a_i} & \text{if } b_i + c_i \geq n_i \end{cases}$$

$$\sigma\colon ab \to ba \text{ is multiplication by } \Pi_{i \leq j} \sigma_{i,j}^{a_i b_j}$$

In the second expression $\sigma_{i,j}$ means σ_i if $i = j$. Recall the exponent of σ_i is at most $2n_i$, so the terms $\sigma_i^{n_i a_i}$ are 1 if n_i is odd, and depend at most on the parity of a_i in general.

2.5.3 Example

Suppose $G = Z/2Z$ and $i \in R$ is a primitive 4^{th} root of unity. Then $\sigma = \pm 1$ and $\sigma = \pm i$ give four group–categories that are not braided commutative equivalent. The ± 1 cases are symmetric, and monoidally equivalent (ignoring commutativity). The $\pm i$ cases are genuinely braided. In these the associativity $(gg)g \to g(gg)$ is multiplication by -1, so they are monoidally equivalent to each other but distinct from the standard category.

2.5.4 A relation to cohomology

Since group–categories correspond to cohomology classes, Proposition 2.5.1 amounts to an explicit calculation of cohomology. We discuss only a piece of this: the associativity structure is the image of the braided structure under the suspension

$$\Sigma\colon H^4(B_G^2; \text{units}(R)) \to H^3(B_G; \text{units}(R)).$$

Elements of $H^3(B_G; \text{units}(R))$ can be obtained as follows:

(1) take homomorphisms $G \to J \to Z/2 \to \text{units}(R)$, with J cyclic;

(2) the identity homomorphisms defines a class $\iota \in H^1(B_J; J)$;

(3) the Bockstein is an operation $\beta\colon H^1(B_J; J) \to H^2(B_J; J)$;

(4) applying the Bockstein to ι and then cup product with ι gives $\iota \cup \beta(\iota) \in H^3(B_J; J)$;

(5) applying $B_G \to B_J$ in the space argument, and $J \to \text{units}(R)$ in the coefficients gives an element in $H^3(B_G; \text{units}(R))$.

Working out the Bockstein and cup product on the chain level gives exactly the formulas in the description of α above when $\sigma_i^{n_i} \neq 1$.

2.5.5 Representatives and products

To begin the construction we need:

(1) a standard representative for each isomorphism class of simple object; and

(2) an algorithm for finding a parameterization of an arbitrary iterated product by the standard representative.

Here we will use the solution to the word problem in the abelian group G. The analysis of other categories uses the same approach, as far as it can be taken. Descriptions of representations of $sl(2)$, cf [4], and other small algebras [13] depend on the description of specific representatives for simples using projections on iterated products of "fundamental" representations. When special information of this type is not available numerical presentations can be obtained by numerically describing representatives and then parameterizing iterated products by direct computation [2, 3].

Choose representatives as follows: choose simple objects $\hat{g}_i$ in the equivalence class of the generator $g_i \in G$, for each i. A general element $a \in G$ has a unique representation of the form $a = g_1^{r_1} g_2^{r_2} \cdots g_k^{r_k}$, where $0 \le r_i < n_i$. We want to get an object in the category by subsituting the simple object $\hat{g}_i$ for the group element g_i, but for this to be well-defined we must specify a way to associate the product. Associate as follows: each $g_i^{r_i}$ is nested left (ie, $g^4 = ((gg)g)g)$, and then the product of these pieces is also nested left. Now subsituting standard representatives for generators gives a standard simple object in each equivalence class.

Next fix for each i an isomorphism $\Lambda_i \colon 1 \to g_i^{n_i}$. Suppose W is a word with associations, in the generators g_i. W specifies an iterated product, and we want an algorithm describing a morphism from the standard representative for this simple object into the product of the word W. Proceed as follows:

(1) If there is a pair $g_j g_1$ with $j > 1$ in the word (ignoring associations), then associate to pair them, and apply σ_{g_1,g_j}^{-1} to interchange them. The result is a simpler word W' with a morphism (of products) $W' \to W$ formed by composing associations and σ_{g_i,g_j};

(2) when (1) is no longer possible, then all g_1 occur first. Repeat to move all g_2 just after the g_1, etc. Then associate to the left to obtain $g_1^{r_1} g_2^{r_2} \cdots g_n^{r_n}$.

(3) after (2) is done, if any r_i is too large, compose with $\Lambda_i \diamond \mathrm{id} \colon g_i^{r_i - n_i} \to g_i^{r_i}$.

When this process terminates the result is a morphism from a standard representative to the product of the word W.

Lemma *The morphisms resulting from this algorithm are well defined.*

The point is that there are choices, but the final result is independent of these choices. Suppose that we have two sequences of operations as described in the algorithm. The coherence theorem for associations shows the outcome does not depend on the order of associations, so problems can come only from the Λ in (3) and the commuting isomorphisms in (1) and (2). There is no choice about which operations are needed, but some choice in the order. If there is a choice then the operations do not overlap, in the sense that each is of the form $\mathrm{id}\diamond\sigma\diamond\mathrm{id}$, and the nontrival part of one operation takes place in an identity factor of the other. Thus the operations commute, and the result is well-defined.

2.5.6 The functor

We use the choices of 2.5.5 to define a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{R}[G]$, and a natural transformation between the two products.

Suppose a is an object of $\mathcal{C}$. $\mathcal{F}(a)$ is supposed to be a function from G to $R-$modules. Define $\mathcal{F}(a)(g) = \hom_{\mathcal{C}}(\hat{g}, a)$, where $\hat{g}$ denotes the standard simple object in the equivalence class g. The natural transformation from the product in $\mathcal{R}[G]$ to the one in $\mathcal{C}$ is given by natural homomorphisms

$$\oplus_{\{r,s|rs=g\}}\hom(\hat{r}, a) \otimes \hom(\hat{s}, b) \to \hom(\hat{g}, a \diamond b).$$

These are defined by $h_1 \otimes h_2 \mapsto (h_1 \diamond h_2)m$, where $m\colon \widehat{rs} \to \hat{r} \diamond \hat{s}$ is the standard parameterization, ie, the morphism from the standard representative of the product to the product of representatives.

The proposition is proved by showing this functor and transformation commute with commutativity and associativity isomorphisms when the twisted structure 2.5.2 is used in $\mathcal{R}[G]$. we begin with very special cases. Consider the commutativity $\sigma_{\hat{g}_i,\hat{g}_j} : \hat{g}_i \diamond \hat{g}_j \to g_j \diamond \hat{g}_i$. If $i < j$ then the right term is already canonical and the algorithm gives $\sigma^{-1}_{\hat{g}_j,\hat{g}_i}$ as parameterization of the left. The diagram

$$\begin{array}{ccc}
\hat{g}_i \diamond \hat{g}_j & \xrightarrow{\sigma_{\hat{g}_i,\hat{g}_j}} & g_j \diamond \hat{g}_i \\
\uparrow \mathrm{id} & & \uparrow \sigma^{-1}_{\hat{g}_j,\hat{g}_i} \\
\widehat{g_i g_j} & \longrightarrow & \widehat{g_i g_j}
\end{array}$$

commutes if we put $\sigma_{i,j} = \sigma_{\hat{g}_j,\hat{g}_i}\sigma_{\hat{g}_i,\hat{g}_j}$ across the bottom.

If $i > j$ in the same situation then the left term is canonical and we get the diagram

$$\begin{array}{ccc} \hat{g}_i \diamond \hat{g}_j & \xrightarrow{\sigma_{\hat{g}_i,\hat{g}_j}} & g_j \diamond \hat{g}_i \\ \uparrow{\scriptstyle \sigma^{-1}_{\hat{g}_i,\hat{g}_j}} & & \uparrow{\scriptstyle \mathrm{id}} \\ \widehat{g_i g_j} & \longrightarrow & \widehat{g_i g_j} \end{array}$$

which commutes with the identity across the bottom. The commutativity required in the model is therefore multiplication by

$$\begin{cases} \sigma_{i,j} & \text{if } i < j \\ \sigma_i & \text{if } i = j \\ 1 & \text{if } i > j \end{cases}$$

which is the factor specified in 2.5.2.

Associativity terms come from different ways of reducing excessively large powers. Fix a particular generator g_i, drop i from the notation, and consider the association $(\hat{g}^r \hat{g}^s)\hat{g}^t \to \hat{g}^r(\hat{g}^s \hat{g}^t)$. If $s + t < n$ then the parameterization algorithm gives the same thing on the two sides, and the associativity is the identity. If $s + t \geq n$ the reductions using Λ are different:

$$\begin{array}{ccc} (\hat{g}^n) \diamond \hat{g}^{r+s+t-n} & \xrightarrow{\alpha} & \hat{g}^r \diamond (\hat{g}^n) \diamond \hat{g}^{s+t-n} \\ \uparrow{\scriptstyle \Lambda\diamond\mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}\diamond\Lambda\diamond\mathrm{id}} \\ (1) \diamond \hat{g}^{r+s+t-n} & \longrightarrow & \hat{g}^r \diamond (1) \diamond \hat{g}^{s+t-n} \end{array}$$

Putting multiplication by $\sigma_i^{n_i r}$ on the bottom makes the diagram commute. This corresponds to commuting g^r past g^n one g factor at a time. The point is that this is different from commuting the full products, which wouldn't contribute anything since $g^n = 1$.

We now claim the hexagon axiom and these special cases imply the general case, ie, the associativity and commutativity isomorphisms in $\mathcal{C}$ commute with the natural transformation between products and the twisted associativity and commutativity morphisms 2.5.2. The new feature in the general case is that different associations change the way a product is reduced to standard form. Specifically, $g_3(g_2g_1)$ follows the standard algorithm in first commuting the g_1 all the way to the left, while in $(g_3g_2)g_1$ the g_3g_2 are commuted first. However the fact that both orders give the same final morphism is exactly the standard crossing identity for braided–commutative categories. Independence of association in arbitrary products follows from this by induction on the number of out-of-order commutes. Once one can choose associations arbitrarily it is straightforward to check the general associativity and commutativity formulae by choosing special association patterns.

The general associative (ie, non-braided) case of classification is not considered in this section, but at this point we can indicate what is involved when the underlying group is abelian. As above choose simple objects representing generators, and reduction isomorphisms $\Lambda_i\colon 1 \to \hat{g}_i^{n_i}$. Since the underlying group is abelian there are isomorphisms $s_{i,j}\colon \hat{g}_i \diamond \hat{g}_j \to \hat{g}_j \diamond \hat{g}_i$. Use these in place of the commuting isomorphisms in defining morphisms to products via the standard algorithm. The same proof shows the morphisms produced by the algorithm are well-defined, since special properties of σ were not used. The difference comes in associations. As above, when products are reduced in blocks specified by associations rather than all at once, the "commuting" isomorphisms $s_{i,j}$ occur out of the standard order. Now, however, the crossing identity is no longer valid so each of these out-of-order interchange contributes a correction factor. These are the new ingredients of the general case.

2.5.7 Order conditions

The arguments of 2.5.6 give uniqueness, ie, that there is a braided–monoidal equivalence from a group–category $\mathcal{C}$ to the standard one with the same invariants. However this implicitly uses the existence assertions, that the invariants of $\mathcal{C}$ satisfy the order conditions, and conversely if a set of invariants satisfy the order conditions then the twisted structure on $\mathcal{R}[G]$ does in fact give a braided–monoidal category. We will discuss the cyclic case, ie, the σ_i which commute a generator with itself, since this has the extra factor of 2 and the connection to associativity. The conditions on $\sigma_{i,j}$ which commutes distinct generators are more routine and are omitted.

Fix a generator of G, and drop the index i from the notation g_i. Thus the generator is g, its order is n, $\hat{g}$ is the chosen simple object in the equivalence class, $\Lambda\colon 1 \to \hat{g}^n$ is the chosen isomorphism implementing the order, and the commutativity isomorphism $\hat{g} \diamond \hat{g} \to \hat{g} \diamond \hat{g}$ is multiplication by σ. Finally define $\alpha \in R$ so that the diagram

$$\begin{tikzcd} \hat{g} \arrow[r, "\alpha"] \arrow[d, "\Lambda\diamond\mathrm{id}"] & \hat{g} \arrow[d, "\mathrm{id}\diamond\Lambda"] \\ (\hat{g} \diamond \hat{g}^{n-1}) \diamond \hat{g} \arrow[r] & \hat{g} \diamond (\hat{g}^{n-1} \diamond \hat{g}) \end{tikzcd}$$

commutes, where the top morphism is multiplication by α and the bottom is the associativity isomorphism in the category. The conditions in 2.5.1 for a single generator are equivalent to:

Lemma $\sigma^n = \alpha = \alpha^{-1}$

The hexagon axiom for commutativity isomorphisms asserts that the diagram commutes (where unmarked arrows are associativities):

$$\begin{array}{ccc}
(g \diamond g^{k-1}) \diamond g & \longrightarrow & g \diamond (g^{k-1} \diamond g) \\
\big\downarrow{\scriptstyle \sigma_{g,g^{k-1}}} & & \big\downarrow{\scriptstyle \sigma_{g,g^k}} \\
(g^{k-1} \diamond g) \diamond g & & (g^{k-1} \diamond g) \diamond g \\
\big\downarrow & & \big\downarrow \\
g^{k-1} \diamond (g \diamond g) & \xrightarrow{\sigma_{g,g}} & g^{k-1} \diamond (g \diamond g)
\end{array}$$

The reduction algorithm of 2.5.5 give canonical maps from a standard g^{k+1} into these objects, and we think of these as bases for $\hom(g^{k+1}, *)$. If $k < n$ then the associativities are all "identities" (preserve these canonical bases). The commuting maps multiply by elements of R, so for these elements the diagram gives a relation $\sigma_{g,g^k} = \sigma_{g,g^{k-1}} \sigma_{g,g}$. $\sigma_{g,g}$ is multiplication by σ, so this subsitution and induction gives $\sigma_{g^j,g^k} = \sigma^{jk}$, if $j, k < n$.

Now consider the diagram with $k = n$. The previous argument still applies to the left side and bottom, and shows the diagonal composition is σ^n. α is defined so the top associativity takes $\Lambda \diamond \mathrm{id}$ to $\alpha(\mathrm{id} \diamond \Lambda)$. We can evaluate the upper right σ term using the unit condition. This condition requires that the diagram commutes:

$$\begin{array}{ccc}
g \diamond 1 & \xrightarrow{\sigma_{g,1}} & 1 \diamond g \\
\big\downarrow{\scriptstyle \nu_g} & & \big\downarrow{\scriptstyle {}_g\nu} \\
g & \xrightarrow{=} & g
\end{array}$$

Composing the inverse of this with $\Lambda \colon 1 \to g^n$ and using naturality gives

$$\begin{array}{ccc}
g & \xrightarrow{=} & g \\
\big\downarrow{\scriptstyle \mathrm{id} \diamond \Lambda} & & \big\downarrow{\scriptstyle \Lambda \diamond \mathrm{id}} \\
g \diamond g^n & \xrightarrow{\sigma_{g,g^n}} & g^n \diamond g
\end{array}$$

This shows the upper right side in the main diagram takes $\mathrm{id} \diamond \Lambda$ to $\Lambda \diamond \mathrm{id}$. Therefore going across the top and down the right side takes the standard generator to α times the standard generator. Comparing with the other composition gives $\sigma^n = \alpha$.

There is a second hexagon axiom in which $\sigma_{a,b}$ is replaced by $\sigma_{b,a}^{-1}$. The same argument applies to this diagram to give $(\sigma^{-1})^n = \alpha$. This completes the proof of the identity. In fact this proof shows that the identities are exactly equivalent to commutativity of the diagrams above, so the identity implies the hexagon axioms. To complete the argument it must be verified that the formula for association in 2.5.2 satisfies the pentagon axiom if $\alpha^2 = 1$. This is straightforward so is omitted.

2.5.8 Balance

The final task is to show that braided group–categories are balanced, ie, there is a functor $\tau(a)$ so that (writing the operations multiplicatively)

$$\sigma(a,b)^{-1} = (\tau(a) \diamond \tau(b))\sigma(b,a)\tau(ab)^{-1}.$$

In fact $\tau(a) = \sigma(a,a)$ works.

Lemma *The commuting isomorphism σ in a braided group–category satisfies*

$$\sigma(ab,ab) = \sigma(a,a)\sigma(b,b)\sigma(a,b)\sigma(b,a)$$

Note this relation would follow if σ were bilinear, but this is usually not the case.

Proof In the following we use freely the fact that R is a commutative ring, so even though the identities are written multiplicatively they can be reordered at will. First, the hexagon axiom for (ab,a,b) gives

$$\sigma_{ab,ab} = \sigma_{ab,a}\sigma_{ab,b}\alpha^{-1}_{ab,a,b}\alpha^{-1}_{a,b,ab} \tag{1}$$

Next the pentagon axiom for (a,b,a,b) gives

$$\alpha_{a,ab,b}\alpha^{-1}_{ab,a,b}\alpha^{-1}_{a,b,ab} = \alpha^{-1}_{a,b,a}\alpha^{-1}_{b,a,b}.$$

Subsituting this into (1) gives

$$\sigma_{ab,ab} = (\sigma_{ab,a}\alpha^{-1}_{a,b,a})(\sigma_{ab,b}\alpha^{-1}_{b,a,b}) \tag{2}$$

In the inverse hexagon for (a,b,a) two α terms cancel to give

$$\sigma_{a,ba}\alpha^{-1}_{a,b,a} = \sigma_{a,b}\sigma_{a,a}$$

Subsituting this, and the similar formula obtained by interchanging a and b, into (2) gives the identity of the lemma. □

3 Homological field theories

The "theory" based on n^{th} homology is described in 3.1. It is defined for general topological spaces, but is not a field theory in this generality. Criteria for this are given in 3.1.3. In particular the H_n theory is modular on $(n+1)$–complexes, but is a nonmodular field theory on $(n+2)$–manifolds. In 3.2 the H_1 theory on 2–complexes is shown to agree with the categorical construction using a group–category. More general theories are obtained in Section 4 by twisting the dual cohomology-based theories.

3.1 The H_n field theory

The objective is to use homology groups to define a topological field theory. The definition is given in 3.1.1, and hypotheses implying the field theory axioms are given in 3.1.3. Examples are given in 3.1.4, and in particular the H_n theory is a non-modular field theory on M^{n+2} manifolds. In 3.1.5 the H_1 theory on 2–complexes is shown to be the category-based theory defined using the canonical group–category. In the following "space" will mean finite CW complex, "subspace" means subcomplex. These assumptions imply that homology groups are finitely generated, and pairs satisfy excision, long exact sequences, etc.

3.1.1 Definition

Fix a commutative ring R, a finite abelian group G and a dimension n. For a pair (Y,W) define the "state space" by

$$Z(Y,W) = R[H_n(Y,W;G)].$$

Next suppose $X \supset Y_0 \cup Y_1$ and $Y_0 \cap Y_1 = W$. Then the induced homomorphism $Z_X\colon Z(Y_0,W) \to Z(Y_1,W)$ is defined by: for $y \in H_n(Y_0,W;G)$,

$$Z_X(y) = \Sigma_{\{x \mid \partial_0 x = -y\}} \partial_1 x.$$

The x in the sum are elements of $H_{n+1}(X, Y_0 \cup Y_1; G)$, and the ∂_i are boundary homomorphisms $\partial_i\colon H_{n+1}(X, Y_0 \cup Y_n; G) \to H_n(Y_i, W; G)$.

Z_X can be described a bit more explicitly using the exact sequence

$$H_{n+1}(X) \to H_{n+1}(X, Y_0 \cup Y_1) \xrightarrow{\partial} H_n(Y_0) \oplus H_n(Y_1) \xrightarrow{i} H_n(X).$$

Let k be the order of the image of $H_{n+1}(X)$ in $H_{n+1}(X, Y_0 \cup Y_1)$. Then

$$Z_X(y) = k\Sigma\{y_1 \in H_n(Y_1) \mid i(y_1) = i(y)\}.$$

We want to find conditions under which this defines a topological field theory, and when the theory is modular.

3.1.2 Axioms

Domain categories are defined in [17] as the appropriate setting for topological field theories, but full details are not needed here. We take the objects (space-times) of the category to be a subcategory $\mathcal{T}$ of topological pairs (X, Y). The boundary objects are the possible second elements Y. The definition above satisfies the tensor property (disjoint unions give tensor products of state spaces, morphisms) on any $\mathcal{T}$ because disjoint unions give direct sums in homology. The composition property requires that if $X_1 \colon Y_0 \to Y_1$ and $X_2 \colon Y_1 \to Y_2$ are bordisms then $Z_{X_2} Z_{X_1} = Z_{X_1 \cup X_2}$. This is not satisfied for completely general $\mathcal{T}$.

In a *modular* domain category three levels of objects are specified. Boundary objects have corner objects as their boundaries and certain identifications are allowed. A field theory on a modular domain category has relative state spaces $Z(Y, W)$ defined for a (boundary, corner) pair, and induced homomorphisms defined for boundaries with corners. Here we assume the extended boundary objects (Y, W) are certain specified topological pairs, glueing is the standard topological operation, etc, and then definition 3.1.1 is given in the modular formulation. If Z is a field theory on a modular domain category then for each corner object W the state space $Z(W \times I, W \times \partial I)$ has a natural ring structure, and if Y is a boundary object with boundary $W_1 \cup W_2$ then the state space $Z(Y, W_1 \cup W_2)$ has natural module structures over the corner algebras $Z(W_i \times I, W_i \times \partial I)$. A field theory is modular if the state space of a glued object is obtained by "algebraically" glueing the state space of the original object. More specifically suppose $(Y, \partial Y)$ is a boundary object with a decomposition of its boundary in the corner category, $\partial Y = W_1 \cup W_2 \cup V$, and $W_1 \simeq \bar{W}_2$. Then there is a glueing in the category, $(\cup_W Y, V)$, and a natural homomorphism of state spaces

$$Z(Y, W_1 \cup W_2 \cup V) \to Z(\cup_W Y, V).$$

The two copies of W give two module structures on $Z(Y, W_1 \cup W_2 \cup V)$ over the ring $Z(W \times I, W \times \partial I)$, and the difference between the two vanishes in $Z(\cup_W Y, V)$. This gives a factorization of the natural homomorphism through

$$(*) \qquad \otimes_{Z(W \times I, W \times \partial I)} Z(Y, W_1 \cup W_2 \cup V) \to Z(\cup_W Y, V).$$

The field theory is said to be modular if this homomorphism is an isomorphism.

In the following $\mathcal{T}$ is a domain category whose objects are (certain specified) topological spaces. Examples are given in 3.1.4.

3.1.3 Lemma *Z satisfies the composition property (so defines a field theory) on $\mathcal{T}$ provided: if (X,Y) is a $\mathcal{T}$ pair and $Y=Y_1\cup_W Y_2$ is a $\mathcal{T}$ decomposition then*

$$H_{n+2}(X,Y_1\cup Y_2;G)\xrightarrow{\partial}H_{n+1}(Y_1,W;G)$$

is onto. If $\mathcal{T}$ is a modular topological domain category then Z is modular provided in addition: if $\cup_W Y$ is a glueing in the boundary category, with boundary V, then the homomorphism

$$H_{n+1}(\cup_W Y,V;G)\xrightarrow{\partial}H_n(W;G)$$

is onto.

3.1.4 Examples

(1) Z is a modular field theory on the modular domain category of $(n+1)$–complexes, ie, with (objects, boundaries, corners) $=$ ($(n+1)$–complexes, (n)–complexes, $(n-1)$–complexes). Slightly more generally, it is sufficient to have the homotopy type of complexes of the indicated dimensions. The composition and modularity conditions are satisfied because the groups involved are all trivial.

(2) Z is a field theory on the domain category of oriented $(n+2)$–manifolds, ie, with (objects, boundaries) $=$ ($(n+2)$–manifolds, $(n+1)$–manifolds). In this case $H_{n+2}(X,Y_1\cup Y_2;G)$ and $H_{n+1}(Y_1,W;G)$ are both isomorphic to G generated by the respective fundamental classes, and the boundary homomorphism is an isomorphism. However the theory is not modular on the modular domain category with corners n–manifolds. The criterion given in the lemma fails because $H_n(W;G)\simeq G$, and when Y is obtained by identifying two copies of W the boundary homomorphism $\partial\colon H_{n+1}(Y,\partial Y)\to H_n(W)$ is trivial. More directly, the theory is not modular because the modularity construction does not account for the image of the fundamental class of W in $H_n(Y,\partial Y)$.

Proof of 3.1.3 The composition property for $(X_1,Y_0\cup_W Y_1)$ and $(X_2,Y_1\cup_W Y_2)$ is that the functions $Z_{X_1\cup_{Y_1}X_2}$ and $Z_{X_2}Z_{X_1}$ agree. Both are defined as sums of ∂_2 of homology classes, so we need to show there is an appropriate bijection between the index sets.

There is a commutative diagram with excision isomorphisms on the top and bottom,

$$\begin{array}{ccc}
H_k(X_1\cup_{Y_1}X_2,Y_0\cup Y_1\cup Y_2) & \xrightarrow{\simeq} & H_k(X_1,Y_0\cup Y_1)\oplus H_k(X_2,Y_1\cup Y_2)\\
\big\downarrow{\scriptstyle\partial} & & \big\downarrow{\scriptstyle\partial_1^{X_1}-\partial_1^{X_2}}\\
H_{k-1}(Y_0\cup Y_1\cup Y_2,Y_0\cup Y_2) & \xrightarrow{\simeq} & H_{k-1}(Y_1,W)
\end{array}$$

Using this to replace terms in the long exact sequence of the triple $X_1 \cup_{Y_1} X_2 \supset Y_0 \cup Y_1 \cup Y_2 \supset Y_0 \cup Y_2$ gives

$$\begin{aligned} &H_{n+2}(X_1, Y_0 \cup Y_1) \oplus H_{n+2}(X_2, Y_1 \cup Y_2) \xrightarrow{\partial_1 - \partial_1} H_{n+1}(Y_1, W) \xrightarrow{i} \\ &\quad H_{n+1}(X_1 \cup_{Y_1} X_2, Y_0 \cup Y_2) \xrightarrow{j} H_{n+1}(X_1, Y_0 \cup Y_1) \oplus H_{n+1}(X_2, Y_1 \cup Y_2) \\ &\qquad \xrightarrow{\partial_1 - \partial_1} H_n(Y_1, W) \end{aligned}$$

The index set for the sum in $Z_{X_1 \cup X_2}$ is the middle term, while the index set for the composition is the kernel of the lower boundary homomorphism. The function j between these is onto by exactness. For it to also be one-to-one we need $i = 0$, or equivalently the upper boundary homomorphism is onto. But this is the sum of two morphisms, both of which are onto by the hypothesis of the lemma, so it is onto.

Now consider the modular case. The ring structures are obtained by applying Z to $(W \times I) \times I$, regarded as a bordism rel ends from $W \times I \cup W \times I$ to $W \times I$. Similarly $Y \cup_W W \times I \simeq Y$, so $Y \times I$ can be regarded as a bordism $Y \cup W \times I \to Y$. Applying Z to this gives the module structure. In the case at hand $Z(W \times I, W \times \partial I) = R[H_n(W \times I, W \times \partial I)]$, and the ring structure is pointwise multiplication in the free module. (This means if v, w are basis elements then $vw = 0$ if $v \neq w$, and $vw = w$ if $v = w$.) There are isomorphisms $H_n(W \times I, W \times \partial I) \xrightarrow{\partial_i} H_{n-1}(W)$ for $i = 0, 1$, and $\partial_0 = -\partial_1$. The module structure on $R[H_n(Y, V \cup W)]$ using the 1 end of $W \times I$ is: if $y \in H_n(Y, V \cup W)$, $v \in H_{n-1}(W)$ then $vy = 0$ if $\partial_W y \neq v$, and $vy = y$ if $\partial_W y = v$. Using the other end of $W \times I$ gives 0 or y depending on whether or not $\partial_W y = -v$.

This description of the ring and module structures identifies the algebraic glueing on the left in the modularity criterion $(*)$ as the free module generated by $y \in H_n(Y, W_1 \cup W_2 \cup V; G)$ satisfying $\partial_{W_1} y = -\partial_{W_2} y$.

Now consider the long exact sequence of the triple $\cup_W Y \supset V \cup W \supset V$:

$$\begin{aligned} H_{n+1}(\cup_W Y, V \cup W) \xrightarrow{\partial} H_n(W) \to H_n(\cup_W Y, V) \to & H_n(\cup_W Y, V \cup W) \\ & \xrightarrow{\partial} H_{n-1}(W) \end{aligned}$$

The state space of the geometric glueing is generated by the third term, while we have identified the algebraic glueing as generated by the kernel of ∂ in the fourth term. The homomorphism of $(*)$ is induced by the set-level inverse of the third homomorphism, so we need to show the third homomorphism is an isomorphism onto the kernel of ∂. Exactness implies it is onto. For injectivity we need the second homomorphism to be 0, or equivalently the first ∂ to be onto. But this is exactly the hypothesis of the lemma. $\square$

3.2 Connections to categories

This gives the first direct connection between the homological theories and categorical constructions. The general case is in Section 4.

Proposition *The canonical untwisted group–categories are the only ones that define modular field theories on 2–complexes, and the corresponding field theories are the H_1 theories of 3.1.1.*

Proof The categorical input for fields on 2–complexes is a symmetric monoidal category satisfying a symmetry condition. Symmetric monoidal group–categories are classified in 2.3(3), or 2.5.1(3). The first part of 3.1.4 corresponds to the fact that of these only the canonical examples satisfy the symmetry conditions.

The symmetry condition concerns nondegenerate pairings. A nondegenerate pairing on a is another object $\bar{a}$ and morphisms

$$\Lambda_a\colon 1 \to \bar{a} \diamond a$$
$$\lambda_a\colon a \diamond \bar{a} \to 1$$

satisfying

$$a \simeq a \diamond 1 \xrightarrow{\mathrm{id}\diamond\Lambda_a} a \diamond (\bar{a} \diamond a) \xrightarrow{\text{associate}} (a \diamond \bar{a}) \diamond a \xrightarrow{\lambda_a\diamond\mathrm{id}} 1 \diamond a \simeq a$$
$$\bar{a} \simeq 1 \diamond a \xrightarrow{\Lambda_a\diamond\mathrm{id}} (\bar{a} \diamond a) \diamond \bar{a} \xrightarrow{\text{associate}} \bar{a} \diamond (a \diamond \bar{a}) \xrightarrow{\mathrm{id}\diamond\lambda_a} \bar{a} \diamond 1 \simeq \bar{a}$$

are both identity maps. The construction requires a fixed choice of pairings on the simple objects. This is equivalent to an additive assignment of pairings to all objects, and this in turn is equivalent to a "duality" functor making the category "autonomous", [21] or a nondegenerate trace function.

The construction requires the symmetry condition $\lambda_{\bar{a}} = \lambda_a \sigma_{\bar{a},a}$. If $a \neq \bar{a}$ then we can arrange this to hold by taking it as the definition of $\lambda_{\bar{a}}$. If $a = \bar{a}$ the condition is equivalent to $\sigma_{a,a}$ being the identity. But this is the only possibly nontrivial invariant in 2.5.1(3), so the category is standard.

Now we show that the H_1 theory corresponds to the standard group–category. One way to do this is to go through the construction [17, 2] and see homology emerge. This is illuminating but too long to reproduce here. Instead we use the reverse construction, extracting a category from a field theory. This goes as follows: let "pt" denote the connected corner object in the domain category. The state space of $Z(\mathrm{pt} \times I)$ has a natural ring structure, and additively the category is the category of modules over this ring. The state space of the cone on three points $Z(c(3))$ has three module structures over the ring. The product on the category is defined by tensoring with this trimodule.

$Z(\mathrm{pt}\times I) = R[H_1(I,\partial I;G)] = R[G]$. The ring structure is obtained by considering the boundary of I^2 as the union of three intervals, with two incoming and one outgoing. If $(g,h)\in H_1(I,\partial I)\oplus H_1(I,\partial I)$ then the image in $R[H_1(I,\partial I)]$ is obtained by summing over elements of $H_2(I\times I,\partial I\times I;G)$ whose restrictions to the incoming boundary intervals is g and h. $H_2(I\times I,\partial I\times I;G)=G$, and the restrictions are identities. Thus (g,h) goes to 0 if $g\neq h$, and to g if they are the same. The ring is therefore $R[G]$ with componentwise multiplication. There is an antiinvolution on this ring induced by interchanging ends of the interval. This is the involution on $R[G]$ induced by inverse in G. The category of modules over this ring is exactly the G–graded (left) R–modules. Denote the category by $\mathcal{C}$. Simple objects are $R[g]$ as in 2.2.1: a copy of R on which multiplication by $h\in G$ is zero if $h\neq g$ and is the identity if $h=g$.

Now let $c(3)$ denote the cone on three points. The standard cell structure is three invervals joined at a point. Using cellular chains gives an explicit description of $H_1(c(3),3;G)$ as $\{(a,b,c)\in G^3 \mid abc=1\}$ (the three generators correspond to the three 1–cells, the relation comes from the boundary homomorphism to the chains on the vertex). The three (left) module structures over $R[H_1(I,\partial I;G)]$ are defined by glueing intervals on the three endpoints. Thus in the first structure g in the ring takes (a,b,c) to 0 if $g\neq a$, and (a,b,c) if $g=a$. The product on the category

$$\mathcal{C}\times\mathcal{C}\to\mathcal{C}$$

is defined by: begin with M and N left modules over the ring. Convert these to right modules using the antiinvolution in the ring, and tensor with the first two module structures on $Z(c(3))$. Then $M\diamond N$ is the result, with respect to the third module structure. Now we can work out the product of two simple objects $R[g]\diamond R[h]$. The involution converts these to right modules on which g^{-1}, h^{-1} respectively act nontrivially. Tensoring with the first two coordinates in $R[\{(a,b,c)\in G^3 \mid abc=1\}]$ kills everything with $a\neq g^{-1}$, $b\neq h^{-1}$, so leaves exactly $R[gh]$. Therefore the product is the standard product in $\mathcal{R}[G]$.

This does not yet identify the category as standard: according to 2.3.2 any group–category is equivalent to the standard one with the standard product. The differences are in the associativity and commutativity structures. Here commutativity comes from the involution on the cone on three points that interchanges the two "incoming" ends. This interchanges two of the 1–cells in the cell structure, so interchanges the corresponding generators in the cellular 1–chains. Thus in homology it interchanges the first two coordinates in $\{(a,b,c)\in G^3 \mid abc=1\}$. Following through the tensor product gives the standard "trivial" commuting isomorphism for $R[g]\diamond R[h]=R[g]\otimes R[h]$. This finishes the argument because the commutativity determines the associativity.

Standardness of associativity is also easy to see directly: associating isomorphisms come from two ways to glue together two cones on three points to get (up to homotopy) the cone on four points. Following this through gives the standard trivial associations. □

4 Cohomological field theories

Homology will probably be the most natural setting for field theories, but so far only the fields for standard group–categories can be described this way. In this section we restrict to manifolds and show how to twist the dual theory theory based on cohomology. More specifically fix a space E with two nonvanishing homotopy groups $\pi_d E = G$ and $\pi_{n+d}E = \mathrm{units}(R)$, and suppose E is simple if $d = 1$. We construct state spaces and induced homomorphisms from homotopy classes of maps to this space. A simple case is described in 4.1.1 to show this gives a twisted version of the $H_n(*; G)$ theory. The full definition occupies the rest of 4.1. The field axioms and modularity are verified in 4.2. The $n = 1$ cases are shown to be Reshetikhin–Turaev constructions from group–categories in 4.3.

4.1 The definition

The general construction is a bit complicated so we begin with a special case in 4.1.1. The domain category for the theory is defined in 4.1.2; the special case of 4.1.1 is supposed to explain why this is the right choice. Once the objects are known the full definition can be presented.

4.1.1 A special case

The Postnikov decomposition for the fixed space E is

$$B^{n+d}_{\mathrm{units}(R)} \to E \to B^d_G \xrightarrow{k} B^{n+d+1}_{\mathrm{units}(R)}.$$

The first space $B^{n+d}_{\mathrm{units}(R)}$ has the structure of a topological abelian group, and the last space is the classifying space for principal bundles with this group. In particular E is a principal bundle with an action of $B^{n+d}_{\mathrm{units}(R)}$.

Now suppose Y is a connected oriented manifold of dimension $n + d$. The group $[Y/\partial Y, B^{n+d}_{\mathrm{units}(R)}] = H^{n+d}(Y, \partial Y; \mathrm{units}(R))$ is dual to $H_0(Y; \mathrm{units}(R)) = \mathrm{units}(R)$. This acts on the set of homotopy classes $[Y/\partial Y, E]$ and the quotient of this action is (when G is abelian)

$$[Y/\partial Y, B^d_G] = H^d(Y, \partial y; G) \simeq H_n(Y; G).$$

Define the state space $Z(Y)$ to be the set of functions $[Y/\partial Y, E] \to R$ that commute with the action of units(R).

If E is the product $B^{n+d}_{\mathrm{units}(R)} \times B^d_G$ then the homotopy classes are also a product $[Y/\partial Y, E] = \mathrm{units}(R) \times H_n(Y;G)$ and the set of units(R)–maps is $R[H_n(Y;G)]$, exactly the definition of Section 3. Thus the k–invariant of E gives a way to twist the R–module generated by $H_n(Y;G)$. In the present case (Y connected) this can also be described as: $[Y/\partial Y, E]$ is a principal units(R) bundle over $H_n(Y;G)$. The state space is the space of sections of the associated R–bundle.

Note to get the key canonical identification of $[Y/\partial Y, B^{n+d}_{\mathrm{units}(R)}]$ with units(R) we needed the boundary objects to be oriented manifolds of dimension $n+d$.

4.1.2 The domain category

The field theory will be defined on $(n+1+d)$–dimensional thickenings of $(n+1)$–complexes. The definitions of state spaces and induced homomorphisms use only the manifold structure. Restrictions on the homotopy dimension are needed for the field axioms to be satisfied.

(1) Corner objects are compact oriented $(n+d-1)$–manifolds with the homotopy type of an $(n-1)$–complex, together with a set of maps $w_i\colon W/\partial W \to E$, one in each homotopy class;

(2) relative boundary objects are compact oriented $(n+d)$–manifolds with the homotopy type of an n–complex, with boundary given as a union $\partial Y = \hat{\partial} Y \cup W$ of submanifolds, and W has the structure of a corner object (ie, homotopy dimension $n-1$ and a choice of maps w_i); and

(3) "spacetime" objects are compact oriented $(n+d+1)$–manifolds with homotopy type of $(n+1)$–complexes, boundary given as a union $\partial X = \hat{\partial} X \cup Y$ of submanifolds, and Y having the homotopy type of an n–complex.

The "internal" boundary (in the domain category) of an object $(X, \hat{\partial} X \cup Y)$ is Y with $\hat{\partial} Y = \partial Y$ and $W = \emptyset$. The internal boundary of Y with $\partial Y = \hat{\partial} Y \cup W$ is W. Morphisms are orientation-preserving homeomorphisms, required to commute with the fixed reference maps on corners. The choices of maps in (1) are typical of the rigidity seen in corner objects, see [17]. The involution $X \mapsto \overline{X}$ is defined by reversing the orientation.

4.1.3 The definition

Suppose Y with $\partial Y = \hat{\partial} Y \cup W$ and $w_i\colon W/\partial W \to E$ is a relative boundary object. Define $[Y/\hat{\partial}Y, E]_0$ to be maps that agree with one of the standard choices on W, modulo homotopy rel ∂Y. The group $[Y/\partial Y, B^{n+d}_{\text{units}(R)}] = H^{n+d}(Y, \partial Y; \text{units}(R))$ acts on this set, as in 4.1.1. *Caution*: the group operation in units(R) is written *multiplicatively*. The operations in cohomology groups and their action on homotopy classes into E are therefore also written multiplicatively. Define

$$\epsilon\colon H^{n+d}(Y, \partial Y; \text{units}(R)) \to \text{units}(R)$$

by evaluation on the fundamental class of Y. When Y is connected (as in 4.1.1) this is an isomorphism, but we do not assume that here. Define the state space for the theory by

$$Z(Y, W) = \hom_\epsilon([Y/\hat{\partial}Y, E]_0, R)$$

where $\hom_\epsilon$ indicates functions $\alpha\colon [Y/\hat{\partial}Y, E]_0 \to R$ so that if $f \in [Y/\hat{\partial}Y, E]_0$ and $a \in H^{n+d}(Y, \partial Y; \text{units}(R))$ then $\alpha(af) = \epsilon(a)\alpha(f)$.

Now we define induced homomorphisms. The general modular setting is an object with boundary divided into "incoming" and "outgoing" pieces, and the incoming boundary further subdivided. Specifically suppose Y_1 is a relative boundary object with corner a disjoint union $W_1 \sqcup W_1' \sqcup W_2$, an isomorphism $W_1' \simeq \overline{W}_1$ is given, and $\cup_{W_1} Y_1$ is the object obtained by identifying W_1' and W_1. Suppose Y_2 is a boundary object with corner W_2, and finally X is an object with internal boundary $(\cup_{W_1}\overline{Y}_1) \cup_{W_2} Y_2$. Then we define

$$Z_X\colon Z(Y_1, W_1 \sqcup W_1' \sqcup W_2) \to Z(Y_2, W_2)$$

as follows. An element in the domain is a function $\alpha\colon [Y_1/\hat{\partial}Y_1, E]_0 \to R$. The output is a function $[Y_2/\hat{\partial}Y_2, E]_0 \to R$, so we can define it by specifying its value on a map $f\colon Y_2/\hat{\partial}Y_2 \to E$. We first suppose each component of X intersects either Y_1 or Y_2. Then

$$Z_X(\alpha)(f) = \Sigma_{[g]}\epsilon_2(a)\alpha(\hat{g}|Y_1).$$

The sum is over homotopy classes of $g\colon X/\hat{\partial}X \to B^d_G$ whose restriction to Y_2 is homotopic to the projection of f. When G is abelian (eg if $d > 1$) this is dual to index set used in the homological version. $\hat{g}\colon X/\hat{\partial}X \to E$ is a lift of g which is standard on W_1 and W_2, and $a \in H^{n+d}(Y_2, \partial Y_2; \text{units}(R))$ so that $a \cdot \hat{g}|Y_2 \sim f$. When each component of X intersects either Y_1 or Y_2 such a lift exists, and since $\hat{g}|Y_2$ and f project to homotopic maps in B^d_G they differ by the action of some such element a.

If Y_1 and Y_2 are empty then we define an element of R by

$$\hat{Z}_X = \Sigma_{[g]} k(g)([X]).$$

Here the sum is again over $X/\partial X \to B^d_G$, $k\colon B^d_G \to B^{n+d+1}_{\mathrm{units}(R)}$ is the k–invariant of E, (see 2.3.2 and 4.1.1) and $k(g)([X])$ is the evaluation of the resulting cohomology class on the fundamental class of X.

Now define Z_X for general X. Write X as $X_1 \sqcup X_2$, where X_1 are the components intersecting $Y_1 \cup Y_2$ and X_2 are the others. If X_1 is nonempty define Z_X as Z_{X_1} multiplied by $\hat{Z}_{X_2}$. If X_1 is empty then $Z(Y_i)$ are canonically identified with R and Z_X is multiplication by $\hat{Z}_X$.

4.1.4 Lemma *Z_X is well-defined, and takes values in $Z(Y_2, W_2)$.*

Proof The things to be checked are that $\epsilon_2(a)\alpha(\hat{g}|Y_1)$ does not depend on the choice of lift $\hat{g}$ and a, and that the resulting function $[Y_2/\hat{\partial}Y_2, E]_0 \to R$ commutes appropriately with the action of units(R).

Suppose g' is another lift of a map g. There is $b \in H^{n+d}(X, \hat{\partial}X; \mathrm{units}(R))$ with $g' = b \cdot \hat{g}$. Denote the restrictions of b to Y_1 and Y_2 by b_1 and b_2 respectively, then we have $f \sim a\cdot\hat{g}|Y_2 \sim a(b_2)^{-1}(b_2)\cdot(\hat{g}|Y_2) \sim a(b_2)^{-1}\cdot(b\cdot\hat{g})|Y_2 \sim a(b_2)^{-1}\cdot(g')|Y_2$. Therefore the element of $H^{n+d}(Y_2, \partial Y_2; \mathrm{units}(R))$ associated to g' is ab_2^{-1}, and the corresponding contribution to Z_X is $\epsilon_2(ab_2^{-1})\alpha(g'|Y_1)$. Since ϵ_2 is a homomorphism and α is an ϵ_1–homomorphism,

$$\epsilon_2(ab_2^{-1})\alpha((b\cdot\hat{g})|Y_1) = \epsilon_2(a)\epsilon_2(b_2^{-1})\epsilon_1(b_1)\alpha(\hat{g}|Y_1).$$

Thus we have to show $\epsilon_2(b_2)^{-1}\epsilon_1(b_1) = 1$. ϵ is defined by evaluation on fundamental classes. The orientation of Y_2 is the opposite of the induced orientation of ∂X, and the complement of $Y_1 \cup Y_2$ in ∂X is taken to the basepoint. Thus $\epsilon_2(b|Y_2)^{-1}\epsilon_1(b|Y_1)$ is obtained by evaluating $b|\partial X$ on the fundamental class of ∂X. But $b|\partial X$ extends to a cohomology class (b) on X, and the image of $[\partial X]$ in the homology of X is trivial (it is the boundary of the fundamental class of X). Thus the evaluation is trivial; 1 since we are writing the structure multiplicatively.

To complete the lemma we show $Z_X(\alpha)$ is an ϵ_2–morphism. Suppose f, α as above, and $c \in H^{n+d}(Y_2, \partial Y_2; \mathrm{units}(R))$. Then

$$\begin{aligned}
\epsilon_2(c)Z_X(\alpha)(f) &= \epsilon_2(c)\Sigma_{[g]}\epsilon_2(a)\alpha(\hat{g}|Y_1)\\
&= \Sigma_{[g]}\epsilon_2(ac)\alpha(\hat{g}|Y_1)\\
&= Z_X(\alpha)(c\cdot f)
\end{aligned}$$

The third line is justified by the fact that $a\cdot\hat{g}|y \simeq f$ if and only if $ac\cdot\hat{g}|y \simeq c\cdot f$ $\square$

4.2 The field axioms

We will not be as precise as in 3.1.3 about the exact conditions for field axioms, but concentrate on the case of interest. We continue the standard assumption that E has two nontrivial homotopy groups, G in dimension d and units(R) in dimension $n+d$.

4.2.1 Proposition *Z defined in 4.2 is a modular field theory on $(n+d+1)$–dimensional thickenings of $(n+1)$–complexes. If E is a product (and G abelian if $d=1$) then Z is equal to the homological theory of 3.1.*

Proof Consider the composition of $X_1\colon Y_1 \to Y_2$ and $X_2\colon Y_2 \to Y_3$. Suppose first that each component of $X_1 \cup_{Y_2} X_2$ intersects either Y_1 or Y_3. In this case $Z_{X_1 \cup X_2}$ and $Z_{X_2} Z_{X_1}$ are given by sums over $[(X_1 \cup X_2)/\hat\partial, B^d_G]$ and $[X_1/\hat\partial, B^d_G] \times [X_2/\hat\partial, B^d_G]$ respectively. Since these are dual to the index sets used in the homological theory, that proof shows that under the given dimension restrictions the natural function between the two is a bijection.Thus we need only show that the corresponding terms in the sum are equal. Suppose $\alpha \in Z(Y_1)$ and $f \in [Y_3/\hat\partial Y_3, E]_0$, and consider the image of α evaluated on f. Choose an element $g\colon (X_1 \cup X_2)/\hat\partial \to B^d_G$ in the index set, and let $\hat g$ be a lift, with $a \in H^{n+d}(Y_3, \partial Y_3; \mathrm{units}(R))$ so that $a \cdot (\hat g | Y_3) \sim f$. The term in $Z_{X_1 \cup X_2}$ is $\epsilon_3(a)\alpha(\hat g|Y_1)$. Use restrictions of $\hat g$ as lifts of the restrictions of g to X_1 and X_2. Since these agree on Y_2 there is no ϵ_2 correction factor, and the corresponding term in $Z_{X_2} Z_{X_1}$ is exactly the same.

Now consider a component of $X_1 \cup X_2$ disjoint from Y_1 and Y_3, so we want to show that $Z_{X_2} Z_{X_1}$ is multiplication by the ring element $\hat Z_{X_1 \cup X_2}$. If the union is disjoint from Y_2 as well then it lies entirely in one piece and this is trivially true. Thus suppose $X_1\colon \emptyset \to Y_2$, $X_2\colon Y_2 \to \emptyset$, and Y_2 intersects each component of the union. Again the index sets match up so we show corresponding terms are equal. Choose a map $g\colon (X_1 \cup X_2)/\partial(X_1 \cup X_2) \to B^d_G$, and choose lifts $\hat g_1$ and $\hat g_2$ of the restrictions to the two pieces. Note g itself may not lift, so the two lifts may not agree on Y_2. Let a be a class with $a \cdot (\hat g_1 | Y_2) = \hat g_2 | Y_2$. Then we want to show a evaluated on the fundamental class of Y_2 is the same as kg evaluated on the fundamental class of $X_1 \cup X_2$.

For convenience insert a collar on Y_2, so the union is $X_1 \cup Y_2 \times I \cup X_2$. Now consider the lifts on the pieces as a lift of g on the disjoint union. This lift gives a factorization of kg through $Y_2 \times I/\partial$:

$$\begin{array}{ccc}
(X_1 \sqcup X_2)/\hat{\partial} & \xrightarrow{\hat{g}_1 \sqcup \hat{g}_2} & E \\
\downarrow & & \downarrow \\
(X_1 \cup Y_2 \times I \cup X_2)/\partial & \xrightarrow{g} & B^d_G \\
\downarrow & & \downarrow{\scriptstyle k} \\
Y_2 \times I/\partial & \longrightarrow & B^{n+d+1}_{\mathrm{units}(R)}
\end{array}$$

The lower map gives kg as the image of an element in $H^{n+d+1}(Y_2 \times I, \partial(Y_2 \times I; \mathrm{units}(R))$. The suspension isomorphism

$$H^{n+d}(Y_2, \partial Y_2; \mathrm{units}(R)) \to H^{n+d+1}(Y_2 \times I, \partial(Y_2 \times I; \mathrm{units}(R))$$

takes a to this element. To see this, interpret the first class as the classifying map for a principal bundle over $Y_2 \times I$ filling in between the restrictions of $\hat{g}_i$ to Y_2. The homotopy extension property for principal bundles shows this is the mapping cylinder of a bundle isomorphism, which must be the one classified by a. Since evaluation of a on $[Y_2]$ is equal to the evaluation of the suspension of a on $[Y_2 \times I]$, it follows that $\epsilon_2(a) = kg([X])$.

This completes the proof of the composition property for induced homomorphisms. The proof of modularity is similar to the homology case, and in fact the algebra associated to a corner object is exactly the same.

Suppose W is a corner object, so an oriented $(n+d-1)$–manifold with the homotopy type of a $(d-1)$–complex and chosen representatives w_i for homotopy classes $[W/\partial W, E]$. The first claim is that there is a canonical isomorphism

$$Z(W \times I) \xrightarrow{\sim} \mathrm{functions}([W/\partial W, B^d_G], R),$$

and this takes the corner algebra structure to the product induced by multiplication in R. The definition of $Z(W \times I)$ is $\hom_\epsilon([(W/\partial W) \times I, B^d_G]_0$, where the subscript 0 indicates that the restrictions to $W \times \{0,1\}$ are images of standard representatives w_i. The first point is that the homotopy $(d-1)$–dimensionality of W implies that $[W/\partial W, E] \to [W/\partial W, B^d_G]$ is a bijection. Since the restrictions to the ends of a map $(W/\partial W) \times I \to B^d_G$ are homotopic, this means the maps on the ends are actually equal. Next, again using dimensionality, a map $(W/\partial W) \times I \to B^d_G$ which is equal to pw_i on each end is itself homotopic rel ends to the map which is constant in the I coordinate. This map has a canonical lift to $(W/\partial W) \times I \to E$ which is standard on the ends, namely w_i applied to projection to the first coordinate. Applying $H^{n+d}(W \times I, \partial(W \times I), \mathrm{units}(R)$ to this gives a surjection

$[W/\partial W, B^d_G] \times H^{n+d}(W \times I, \partial(W \times I), \text{units}(R) \to [(W/\partial w) \times I, E]_0$. Applying $\hom_\epsilon(*, R)$ to this gives the required bijection.

The algebra structure in the algebra, or more generally the action on a state space, is described as follows: Suppose Y is a relative boundary object with $\partial Y = \hat\partial Y \cup W_1 \cup W_2$. A function $\tau\colon [W_1/\partial W_1, B^d_G] \to R$ acts on $\alpha\colon [Y/\hat\partial Y, E]_0 \to R$ to give another function like α. The new function can be specified by its action on $f \in [Y/\hat\partial Y, E]_0$, by

$$(\tau \cdot \alpha)(f) = \tau(f|W_1)\alpha(f).$$

Finally we prove modularity. Suppose Y has corners $W \sqcup \overline{W} \sqcup W_2$, and let $\cup_W Y$ be the boundary object obtained by identifying the copies of W. The homomorphism of state spaces induced by this glueing is

$$\hom_\epsilon([Y/\hat\partial Y, E]_0, R) \to \hom_\epsilon([\cup_W Y/\hat\partial Y, E]_0, R).$$

This is induced by a "splitting" function

$$[\cup_W Y/\hat\partial Y, E]_0 \to [Y/\hat\partial Y, E]_0$$

defined as follows. Suppose $f\colon \cup_W Y/\hat\partial Y \to E$ is standard on W_2. The restriction to W is homotopic to a standard map. Use this to make f standard on W, then split along W to obtain $f'\colon Y/\hat\partial Y \to E$ standard on $W \sqcup \overline{W} \sqcup W_2$. The dimensionality hypotheses can be used as above to show f' is well-defined up to homotopy rel boundary, and the splitting function is a bijection onto the subset of $g\colon Y/\hat\partial Y \to E$ satisfying $g|W = g|(\overline{W})$. Therefore to show the algebraic glueing map

$$\otimes_{Z(W\times I)} Z(Y) \to Z(\cup_W Y)$$

is an isomorphism we need to show that dividing by the difference between the two $Z(W \times I)$–module structures divides out exactly the functions supported on the complement of the image of the splitting function. These functions are sums of "delta" functions: suppose g has $g|W \neq g|(\overline{W})$. Define α_g to take g to 1, extend to an ϵ–morphism on $H^(Y/\partial Y; \text{units}(R)) \cdot g$, and define it to be 0 elsewhere. It is sufficient to show these functions get divided out. Dividing by the difference between the module structures divides all elements of the form $f \mapsto (\tau(f|W) - \tau(f|\overline{W}))\alpha(f)$. For the particular g under consideration there is a function τ with $\tau(g|W) = 1$ and $\tau(g|\overline{W}) = 0$. Using this τ and the delta function α_g gives

$$f \mapsto \begin{cases} \alpha_g(f) & \text{if } f \text{ is a multiple of } g, \text{ since } \tau(f|W) - \tau(f|\overline{W}) = 1 \\ 0 & \text{otherwise , since } \alpha_g(f) = 0 \end{cases}$$

But this is exactly α_g, so α_g is divided out. □

4.3 Relations to group–categories

Suppose E is a space as above with $n=1$, so $\pi_d(E)=G$ and $\pi_{d+1}(E)=$ units(R). In Section 2.3 these spaces are shown to correspond to group–categories with various degrees of commutativity. The cohomological construction of Proposition 4.2.1 gives a modular field theory on the domain category whose objects, boundaries, corners are manifolds of dimension $(d+2,d+1,d)$ and homotopy type of complexes of dimension $(2,1,0)$ respectively. Specifically we have:

d	category structure	fields on
1	associative	$(3,2,1)$–thickenings
2	braided–commutative	$(4,3,2)$–thickenings
≥ 3	symmetric	$(d+2,d+1,d)$–thickenings

On the category side the independence of d when $d\geq 3$ comes from stability of group cohomology under suspension. For fields, cartesian product with I gives a "suspension" functor of domain categories, from d–thickenings to $(d+1)$–thickenings. Composition with this gives a suspension function on field theories, from ones on $(d+1)$–thickenings to ones on d–thickenings. When $d\geq 3$, suspension is an equivalence of domain categories (ie, all thickenings are isomorphic to products in an appropriately canonical way), so it induces a bijection of field theories.

There is also a Reshetikhin–Turaev type construction that uses a category to define a field theory on the same objects. Here we show that the two field theories agree.

Proposition *Suppose $\mathcal{G}$ is a group–category corresponding to a space E. The cohomological field theory defined in 4.2 using E is the same as the Reshetikhin–Turaev theory defined using $\mathcal{G}$.*

Proof We will not prove this directly, but rather use the fact (as in 3.2) that the category can be recovered (up to equivalence) from the field theory. Specifically the category $\mathcal{G}$ is additively equivalent to the category of representations of the corner algebra of a thickening of a point, with product structure induced by the state space of thickenings of the cone on three points.

Fix a connected corner object: a copy of D^d with specific choices of representatives $\hat{g}\colon D^d/\partial D^d\to E$ for each homotopy class $g\in[D^d/\partial D^d,E]=\pi_d(E)=G$. The algebra structure on $Z(D^d\times I,D^d\times\{0,1\})$ is identified in the proof of 4.2.1 as the set of functions $G\to R$, with product given by product in R, or alternatively $R[G]$ with componentwise multiplication. Representations of this are exactly $\mathcal{R}[G]$, so this gives an additive equivalence $\mathcal{G} \to \mathcal{R}[G]$.

To determine the product structure we choose data as in 2.3.3–4: for each pair g, h choose a homotopy $m_{g,h}\colon \hat{g}\hat{h} \simeq \widehat{gh}$. The left side of this expression is the product of homotopy classes in $\pi_d(E)$, while the right side is the given representative of the product in G. Let Y denote the thickening of the cone on three points, so $Y \simeq D^{d+1}$ with internal boundary $3D^d \subset \partial D^{d+1}$ and $\hat{\partial} Y$ the complement. The next object is to describe $Z(Y)$ with its three module structures over the corner algebra. Reverse the orientation on two of the boundary components (to switch the module structure from left to right). A map $Y/\hat{\partial}Y \to E$ that restricts to $\hat{g} \sqcup \hat{h}$ on the incoming boundaries of Y gives a homotopy to the restriction to the third component. This identifies the third restriction as $\widehat{(gh)^{-1}}$. The inverse comes from the fact that all the components of ∂Y have the induced orientation, while in $D^d \times I$ one of the ends has the reverse orientation. We have specified one such map, namely $m_{g,h}$, and all others with this restriction are obtained (up to homotopy) by the action of $H^{d+1}(Y, \partial Y; \text{units}(R)) = \text{units}(R)$. Thus the choices m_{gh} give a bijection

$$[Y/\hat{\partial}Y, E]_0 \sim \cup_{g,h} \text{units}(R).$$

The state space $Z(Y)$ is the set $\hom_\epsilon([Y/\hat{\partial}Y, E]_0, R)$, so the bijection gives an identification $Z(Y) = R[G \times G]$. The three (left) module structures are: on a summand $R[(g,h)]$, $f \in G$ acts by the delta function $\delta_{f,g}$, $\delta_{f,h}$, and $\delta_{f,(gh)^{-1}}$. Switch the first two to right structures by reversing the orientation, and replace g, h by g^{-1}, h^{-1}. This gives an identification in which the right structures on $R[(g,h)]$ are $\delta_{f,g}$ and $\delta_{f,h}$ respectively, and the left structure is $\delta_{f,gh}$. Now suppose a_g and a_h are simple modules in $\mathcal{R}[G]$. Their Z–product is $Z(Y) \otimes_{Z(W \times I)^2} (a_g \otimes a_h)$. The description of $Z(Y)$ shows this is a free based module of rank 1, canonically isomorphic to a_{gh}. This gives a natural isomorphism between the Z product in $\mathcal{G}$ and the standard product in $\mathcal{R}[G]$.

The category structure shows up in the reassociating and (when $d > 1$) commuting isomorphisms. Specifically the isomorphism $\alpha\colon (a_f \diamond a_g) \diamond a_h \simeq a_f \diamond (a_g \diamond a_h)$ comes from the thickening of the cone on four points decomposed in two ways as union of cones on three points. The two decompositions give two basepoints in $[Y/\hat{\partial}Y, E]_0$, namely the homotopies $m_{fg,h} m_{f,g}$ and $m_{f,gh} m_{g,h}$. These differ by a unit in R, which gives the difference between the identifications of the iterated products with a_{fgh}. But according to 2.3.2 this unit is exactly the associativity isomorphism in the category associated with E. Thus the natural isomorphism between the products in $\mathcal{G}$ and $\mathcal{R}[G]$ takes the associativity isomorphisms in $\mathcal{G}$ to the E–twisted ones in $\mathcal{R}[G]$.

A similar argument shows that the commutativity isomorphisms agree too, when $d > 1$. □

5 Modular field theories on 3–manifolds

Modular theories on 3–manifolds with a little extra data can be obtained as follows: start with a theory on 4–dimensional thickenings of 2–complexes, corresponding to some braided–symmetric category. Restrict to a subcategory of objects that are almost determined by their boundaries. Then normalize using an Euler-characteristic theory to remove most of the remaining dependence on interiors. Here we carry this through for group–categories. The untwisted theories (which are H_1 theories in the sense of Section 3) can be normalized if the order of the underlying group is invertible. For cyclic groups we determine exactly which group–categories give normalizable theories: in most cases it requires a certain divisor of the group order to have a square root. However there are cases, including the category with group $Z/2Z$ and $\sigma = -1$, that cannot be normalized.

5.1 Extended, or weighted, 3–manifolds

There is a domain category (in the sense of [17]) with

(1) corners are closed 1–manifolds, with a parametrization of each component by S^1;

(2) boundaries are oriented surfaces with boundary, the boundary is a corner object (ie, has parameterized components, with correct orientation), and a lagrangian subspace of $H_1(\overline{Y}; Z)$; and

(3) spacetimes are 3–manifolds whose boundaries are boundary objects (ie, have lagrangian subspaces), together with an integer (the "index").

In (2) $\overline{Y}$ denotes the closed surface obtained by glueing copies of D^2 to Y via the given parameterizations of the boundary components. A "lagrangian subspace" is a Z–summand of half the rank on which the intersection pairing vanishes. These objects are the "extended" or "e–manifolds" of Walker [23], and special cases of the "weighted" manifolds of Turaev [22]. Turaev allows lagrangian subspaces of the real rather than integer cohomology.

A domain category comes with cylinder functors and glueing operations. Most of these are pretty clear. For instance when glueing spacetimes along closed (no corners) boundaries, the weights add. Glueing when corners are involved requires Wall's formula for modified additivity of the index, using the Shale–Weyl cocycle [23, 22].

The geometric basis for the construction is:

5.1.1 Theorem

(1) *If U is an oriented 3–dimensional thickening of a 1–complex, then the kernel of the inclusion $H_1(\partial U, Z) \to H_1(U; Z)$ is a lagrangian subspace. Every lagrangian subspace arises this way, and the manifold U is unique up to diffeomorphism rel boundary; and*

(2) *(Kirby [12]) A connected oriented 3–manifold is the boundary of a smooth 4–manifold with the homotopy type of a 1–point union of copies of S^2. If X_1 and X_2 are two such manifolds with the same boundary, then for some m_1, n_1, m_2, n_2 there is a diffeomorphism*

$$X_1 \# m_1 CP^2 \# n_1 \overline{CP}^2 \simeq X_2 \# m_2 CP^2 \# n_2 \overline{CP}^2$$

which is the identity on the boundary.

Some of the modifications in (2) can be tracked with the index of the 4–manifold: adding CP^2 increases it by 1, while $\overline{CP}^2$ decreases it by 1. Doing both leaves the index unchanged. This gives a refinement of (2):

5.1.2 Corollary *Suppose X_1 and X_2 are 4–manifolds as in 5.1.1(2) and the indexes are the same. Then for some p_1, p_2 there is a diffeomorphism*

$$X_1 \# p_1 (CP^2 \# \overline{CP}^2) \simeq X_2 \# p_2 (CP^2 \# \overline{CP}^2).$$

5.2 Construction of field theories

Now suppose Z is a field theory on 4–dimensional thickenings of 2–complexes. Suppose Y is a extended boundary object, so a surface with parameterized boundary and homology lagrangian. According to 5.1.1(1) this data is the same as a 3-d thickening U of a 1–complex with $\partial U = \overline{Y}$, together with parameterized 2–disks in the boundary. This is a boundary object of the category of thickenings, so we can define $\hat{Z}(Y) = Z(U)$.

If induced homomorphisms Z_V are unchanged by connected sum with CP^2 and $\overline{CP}^2$ then we can define $\hat{Z}_X$ to be Z_V for one of the 4–manifolds of 5.1.1(2) with $\partial V = X$. Usually these operations do change Z_V; specifically there are elements $\tau, \overline{\tau} \in R$ so that

$$Z_{V \# CP^2} = \tau Z_V \quad \text{and} \quad Z_{V \# \overline{CP}^2} = \overline{\tau} Z_V. \qquad (5.2.1)$$

These changes were called "anomalies" by physicists. Usually the changes are too strong to fix, but sometimes we can fix the changes caused by adding both

CP^2 and $\overline{CP}^2$ together. Specifically, suppose there is an inverse square root for $\tau\overline{\tau}$: an element r such that

$$r^2\tau\overline{\tau} = 1. \tag{5.2.2}$$

Connected sum with $CP^2\#\overline{CP}^2$ changes Z_V by $\tau\overline{\tau}$ and increases the Euler characteristic of V by 2. Thus if we multiply by r to the power $\mathcal{X}(V)$ the changes cancel. More specifically if $(X, n)\colon Y_1 \to Y_2$ is a bordism in the extended 3–manifold category, and $V\colon U_1 \to U_2$ is a corresponding 4-d morphism of thickenings with index n define

$$\hat{Z}_X = r^{\mathcal{X}(V,U_1)} Z_V. \tag{5.2.3}$$

Proposition *If an element r satisfying 5.2.2 exists, then $\hat{Z}$ is a modular field theory on extended 3–manifolds.*

Note that adding 1 to the index of an extended 3–manifold X corresponds to changing the bounding 4–manifold by $\#CP^2$. This adds 1 to the Euler characteristic so changes $\hat{Z}_X$ by $r\tau$. This is the "anomaly" of the normalized theory. In particular it is nontrivial if $\tau \neq \overline{\tau}$.

Proof Multiplication by r raised to the relative Euler chacteristic gives a modular field theory with all state spaces R, defined on all finite complexes [17]. The product in 5.2.3 is the tensor product of this Euler theory with Z, so defines a theory on 4-d thickenings. Restricting to the simply-connected thickenings obtained from extended 3–manifolds therefore is a modular field theory. By construction it is insensitive to the difference between different V with fixed boundary and index, so it is a well-defined theory on extended 3–manifolds. □

5.3 Normalization of group–category field theories

Here we describe the "anomalies" of the field theory associated to a group–category in terms of the category structure. In the cyclic case this is explicit enough to completely determine when the field theory can be normalized to give one on extended 3–manifolds.

5.3.1 Proposition *Suppose $\mathcal{G}$ is a braided–commutative group–category over R, with finite underlying group G. Then the associated field theory Z has $Z_{CP^2} = \Sigma_{g\in G}\sigma_g$ and $Z_{\overline{CP}^2} = \Sigma_{g\in G}\sigma_g^{-1}$. If G is cyclic of order n, σ ($= \sigma_g$ for some generator g) has order exactly ℓ, and R has no zero divisors then*

$$Z_{CP^2\#\overline{CP}^2} = \begin{cases} n^2/\ell & \text{if } \ell \text{ is odd} \\ 2n^2/\ell & \text{if } 4|\ell \\ 0 & \text{otherwise } (\ell \text{ is even and } \ell/2 \text{ is odd}). \end{cases}$$

We recall $\sigma_g \in \mathrm{units}(R)$ is the number so that the commuting endomorphism $\sigma_{g,g}\colon g \diamond g \to g \diamond g$ is multiplication by σ_g. The order of σ_g divides the order of g if this order is odd, and twice this order if it is even.

5.3.2 Example

If $G = Z/2Z$ then $Z_{CP^2} = 1+\sigma$ and $Z_{\overline{CP}^2} = 1+\sigma^{-1}$. Since $\sigma^4 = 1$ there are three cases: $\sigma = 1$, $\sigma = -1$, and $\sigma = i$ (a primitive 4^{th} root of unity).

(1) When $\sigma = 1$ (the standard untwisted category) both Z are 2, so the inverse square root of the product is $1/2$. Thus the theory can be normalized over $R[1/2]$ and gives an anomaly-free theory ($\hat{Z}_X$ doesn't depend on the index of X).

(2) When $\sigma = -1$ (the nontrivial symmetric category) both Z are 0, and no extended 3–manifold theory can be obtained.

(3) When $\sigma = i$ (a non-symmetric braided category) the Z are $1+i$ and $1-i$ respectively. The product is 2, so the theory can be normalized over $R[1/\sqrt{2}]$.

Note that the $Z/2Z$ category with $\sigma = -1$ is a (possibly twisted) tensor factor of the quantum categories coming from $sl(2)$ at roots of unity. This should mean that on 4-d thickenings the field theory is a (possibly twisted) tensor product. The non-normalizability of the $Z/2Z$ factor would explain why it has been so hard to normalize the full $sl(2)$ theory.

Proof of 5.3.1 In general we want $Z_{CP^2-D^4}$, where $CP^2 - D^4$ is regarded as a bordism $D^3 \to D^3$ (relative to the corner $S^2 = \partial D^3$). In the group–category case this is the same as the closed case (CP^2 as a bordism from the empty set to itself). This can either be seen directly, or more generally induced homomorphisms can be seen to be multiplicative with respect to connected sums. Thus we consider the closed case.

Let $k \in H^4(B^2_G; \mathrm{units}(R))$ be the class corresponding to the group–category $\mathcal{G}$. Z_{CP^2} is multiplication by the sum over $[CP^2, B^2_G] = H^2(CP^2; G) = G$ of k evaluated on the image of the fundamental class of CP^2. We claim this evaluation for a single $g \in G$ is σ_g, so the sum is as indicated in 5.3.1. The element for $\overline{CP}^2$ is obtained by evaluating on the negative of the fundamental class of CP^2, so gives σ_g^{-1}.

This claim is verified using a geometric argument and the description of 2.3.4. Suppose data $\tilde{g}\colon D^2/S^1 \to E$ and $\tilde{m}_{g,h}$ has been chosen. Then $\sigma_{g,g}$ is obtained by glueing together $\tilde{m}_{g,g}$, its inverse, and the standard commuting homotopy in π_2 to get $D^2 \times S^1 \to E$. Consider this as a neighborhood of a standard circle in D^3 and extend to $D^3/S^2 \to E$ by taking the complement to the basepoint. $\sigma_{g,g}$ is the resulting element in $\pi_3(E) = \text{units}(R)$. We manipulate this a little. $\tilde{m}_{g,g}$ and its inverse cancel to leave just the standard commuting homotopy. This gives the following description: take an embedding $\mu\colon D^2 \times S^1 \to D^2 \times S^1$ that goes twice around the S^1, and locally preserves products. The element of π_3 is obtained by

$$D^3/S^2 \to D^2 \times S^1/(\partial D^2 \times S^1) \xrightarrow{\mu^{-1}} D^2 \times S^1/(\partial D^2 \times S^1) \xrightarrow{p} D^2/S^1 \xrightarrow{\tilde{g}} E$$

where the first map divides out the complement of the standard $D^2 \times S^1 \subset D^3$, and p projects to the D^2 factor of the product. According to 2.3.4 the image of this in $\pi_3(E) = \text{units}(R)$ is σ_g. Denote the composition $D^3/S^2 \to D^2/S^1$ by h. This is homotopic to the Hopf map. This can be checked using Hopf's original description: the inverses of two points in the interior of D^2 give two unknotted circles in S^3 with linking number 1. But $S^2 \cup_h D^4 \simeq CP^2$. The vanishing higher homotopy of B^2_G implies there is an extension (unique up to homotopy) of g over the 4–cell to give $CP^2 \to B^2_G$. General principles imply that the k–invariant evaluated on the homology image of the 4–cell (the orientation class of CP^2) is equal to the homotopy class of the attaching map in $\pi_3 E$, so the evaluation does give σ_g.

The numerical presentation material of 2.5 can be used to make these conclusions more concrete. We carry this out for cyclic groups. Suppose G is cyclic of order n with generator g, and $\sigma = \sigma_g$. Suppose σ has order ℓ. From 2.5 we know ℓ divides n if n is odd, and $2n$ if n is even. Further (see 2.5.2), $\sigma_{g^r} = (\sigma_g)^{r^2}$. Therefore

$$Z_{CP^2} = \Sigma_{r=0}^{n-1}\sigma^{r^2} \quad \text{and} \quad Z_{\overline{CP}^2} = \Sigma_{r=0}^{n-1}\sigma^{-r^2}.$$

The product of these is

$$\Sigma_{r,s=0}^{n-1}\sigma^{r^2-s^2} = \sigma_{r,s=0}^{n-1}\sigma^{(r+s)(r-s)}$$

Reindex this by setting $r - s = t$, and use the fact that r and s can be changed by multiples of n to get

$$\Sigma_{s,t=0}^{n-1}\sigma^{t(t+2s)} = \Sigma_{t=0}^{n-1}\sigma^{t^2}\Sigma_{s=0}^{n-1}(\sigma^{2t})^s. \tag{5.3.3}$$

We have assumed R has no zero divisors. This means if $(\rho^n - 1) = (\rho - 1)(\Sigma_{s=0}^{n-1}\rho^s) = 0$ then one of the factors is 0. This implies

$$\Sigma_{s=0}^{n-1}\rho^s) = \begin{cases} n & \text{if } \rho = 1 \\ 0 & \text{if } \rho \neq 1. \end{cases}$$

Applying this with $\rho = \sigma^{2t}$ to (5.3.3) gives the sum over t with $\sigma^{2t} = 1$ of $n\sigma^{t^2}$.

If ℓ (the order of σ) is odd or divisible by 4, then $\sigma^{2t}=1$ implies $\sigma^{t^2}=1$. In this case the sum is just n times the number of such t between 0 and $n-1$. This number is n/ℓ if n is odd, $2n/\ell$ if n is even. This gives the conclusion of the proposition in these cases. If ℓ is even but $\ell/2$ is odd then $\sigma^{2t}=1$ implies $\sigma^{t^2}=1$ if t is even, and $\sigma^{t^2}=-1$ if t is odd. The sum is thus n times the difference between the number of even and odd t with $\sigma^{2t}=1$. These are $t=(\ell/2)j$ for $0\leq j<2n/\ell$, so they exactly cancel if $2n/\ell$ is even, or equivalently if ℓ divides n. We are in the case with ℓ even so n is even and 4 divides $2n$. But $\ell/2$ odd, so if ℓ divides $2n$ it must also divide n. This completes the proof of the proposition. □

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