ISSN 1464-8997 (on line) 1464-8989 (printed)

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part I, section 12, pages 109–112

12. Two types of complete discrete valuation fields

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In this section we discuss results of a paper [Ku1] which is an attempt to understand the structure of the Milnor K-groups of complete discrete valuation fields of mixed characteristics in the case of an arbitrary residue field.

12.0. Definitions

Let K be a complete discrete valuation field of mixed characteristics $(0,p)$ with the ring of integers $\mathcal{O}_K$. We consider the p-adic completion $\widehat{\Omega}^1_{\mathcal{O}_K}$ of $\Omega^1_{\mathcal{O}_K/\mathbb{Z}}$ as in section 9.

Note that

(a) If K is a finite extension of $\mathbb{Q}_p$, then

$$\widehat{\Omega}^1_{\mathcal{O}_K} = (\mathcal{O}_K/\mathcal{D}_{K/\mathbb{Q}_p})d\pi$$

where $\mathcal{D}_{K/\mathbb{Q}_p}$ is the different of $K/\mathbb{Q}_p$, and π is a prime element of K.

(b) If $K = k\{\{t_1\}\}\dots\{\{t_{n-1}\}\}$ with $|k : \mathbb{Q}_p| < \infty$ (for the definition see subsection 1.1), then

$$\widehat{\Omega}^1_{\mathcal{O}_K} = (\mathcal{O}_k/\mathcal{D}_{k/\mathbb{Q}_p})d\pi \oplus \mathcal{O}_K dt_1 \oplus \dots \oplus \mathcal{O}_K dt_{n-1}$$

where π is a prime element of $\mathcal{O}_k$.

But in general, the structure of $\widehat{\Omega}^1_{\mathcal{O}_K}$ is a little more complicated. Let F be the residue field of K, and consider a natural map

$$\varphi\colon \widehat{\Omega}^1_{\mathcal{O}_K} \longrightarrow \Omega^1_F.$$

Definition. Let $\operatorname{Tors}\widehat{\Omega}^1_{\mathcal{O}_K}$ be the torsion part of $\widehat{\Omega}^1_{\mathcal{O}_K}$. If $\varphi(\operatorname{Tors}\widehat{\Omega}^1_{\mathcal{O}_K}) = 0$, K is said to be of *type I*, and said to be of *type II* otherwise.

So if K is a field in (a) or (b) as above, K is of type I.

Let π be a prime element and $\{t_i\}$ be a lifting of a p-base of F. Then, there is a relation

$$a d\pi + \sum b_i dt_i = 0$$

with $a,\ b_i \in \mathcal{O}_K$. The field K is of type I if and only if $v_K(a) < \min_i v_K(b_i)$, where v_K is the normalized discrete valuation of K.

Examples.

(1) If $v_K(p)$ is prime to p, or if F is perfect, then K is of type I.

(2) The field $K = \mathbb{Q}_p\{\{t\}\}(\pi)$ with $\pi^p = pt$ is of type II. In this case we have

$$\widehat{\Omega}^1_{\mathcal{O}_K} \simeq \mathcal{O}_K/p \oplus \mathcal{O}_K.$$

The torsion part is generated by $dt - \pi^{p-1}d\pi$ (we have $pdt - p\pi^{p-1}d\pi = 0$), so $\varphi(dt - \pi^{p-1}d\pi) = dt \neq 0$.

12.1. The Milnor K-groups

Let π be a prime element, and put $e = v_K(p)$. Section 4 contains the definition of the homomorphism

$$\rho_m \colon \Omega_F^{q-1} \oplus \Omega_F^{q-2} \longrightarrow \mathrm{gr}_m K_q(K).$$

Theorem. *Put* $\ell = \mathrm{length}_{\mathcal{O}_K}(\mathrm{Tors}\,\widehat{\Omega}^1_{\mathcal{O}_K})$.

(a) *If K is of type I, then for $m \geqslant \ell + 1 + 2e/(p-1)$*

$$\rho_m|_{\Omega_F^{q-1}} \colon \Omega_F^{q-1} \longrightarrow \mathrm{gr}_m K_q(K)$$

is surjective.

(b) *If K is of type II, then for $m \geqslant \ell + 2e/(p-1)$ and for $q \geqslant 2$*

$$\rho_m|_{\Omega_F^{q-2}} \colon \Omega_F^{q-2} \longrightarrow \mathrm{gr}_m K_q(K)$$

is surjective.

For the proof we used the exponential homomorphism for the Milnor K-groups defined in section 9.

Corollary. *Define the subgroup $U_i K_q(K)$ of $K_q(K)$ as in section 4, and define the subgroup $V_i K_q(K)$ as generated by $\{1 + \mathcal{M}_K^i, \mathcal{O}_K^*, \dots, \mathcal{O}_K^*\}$ where $\mathcal{M}_K$ is the maximal ideal of $\mathcal{O}_K$.*

(a) *If K is of type I, then for sufficiently large m we have $U_m K_q(K) = V_m K_q(K)$.*

(b) *If K is of type II, then for sufficiently large m, we have $V_m K_q(K) = U_{m+1} K_q(K)$. Especially, $\mathrm{gr}_m K_q(K) = 0$ for sufficiently large m prime to p.*

Example. Let $K = \mathbb{Q}_p\{\{t\}\}(\pi)$ where $\pi^p = pt$ as in Example (2) of subsection 12.0, and assume $p > 2$. Then, we can determine the structures of $\mathrm{gr}_m K_q(K)$ as follows ([Ku2]).

For $m \leqslant p+1$, $\mathrm{gr}_m K_q(K)$ is determined by Bloch and Kato ([BK]). We have an isomorphism $\mathrm{gr}_0 K_2(K) = K_2(K)/U_1K_2(K) \simeq K_2(F) \oplus F^*$, and $\mathrm{gr}_p K_q(K)$ is a certain quotient of $\Omega^1_F/dF \oplus F$ (cf. [BK]). The homomorphism ρ_m induces an isomorphism from

$$\begin{cases} \Omega^1_F & \text{if } 1 \leqslant m \leqslant p-1 \text{ or } m = p+1 \\ 0 & \text{if } i \geqslant p+2 \text{ and } i \text{ is prime to } p \\ F/F^p & \text{if } m = 2p \\ & (x \mapsto \{1 + p\pi^p x, \pi\} \text{ induces this isomorphism}) \\ F^{p^{n-2}} & \text{if } m = np \text{ with } n \geqslant 3 \\ & (x \mapsto \{1 + p^n x, \pi\} \text{ induces this isomorphism}) \end{cases}$$

onto $\mathrm{gr}_m K_2(K)$.

12.2. Cyclic extensions

For cyclic extensions of K, by the argument using higher local class field theory and the theorem of 12.1 we have (cf. [Ku1])

Theorem. *Let ℓ be as in the theorem of 12.1.*

(a) *If K is of type I and $i \geqslant 1+\ell+2e/(p-1)$, then K does not have ferociously ramified cyclic extensions of degree p^i. Here, we call an extension L/K ferociously ramified if $|L:K| = |k_L : k_K|_{\mathrm{ins}}$ where k_L (resp. k_K) is the residue field of L (resp. K).*

(b) *If K is of type II and $i \geqslant \ell + 2e/(p-1)$, then K does not have totally ramified cyclic extensions of degree p^i.*

The bounds in the theorem are not so sharp. By some consideration, we can make them more precise. For example, using this method we can give a new proof of the following result of Miki.

Theorem (Miki, [M]). *If $e < p-1$ and L/K is a cyclic extension, the extension of the residue fields is separable.*

For $K = \mathbb{Q}_p\{\{t\}\}(\sqrt[p]{pt})$ with $p > 2$, we can show that it has no cyclic extensions of degree p^3.

Miki also showed that for any K, there is a constant c depending only on K such that K has no ferociously ramified cyclic extensions of degree p^i with $i > c$.

For totally ramified extensions, we guess the following. Let F^{p^∞} be the maximal perfect subfield of F, namely $F^{p^\infty} = \bigcap F^{p^n}$. We regard the ring of Witt vectors $W(F^{p^\infty})$ as a subring of $\mathcal{O}_K$, and write k_0 for the quotient field of $W(F^{p^\infty})$, and write k for the algebraic closure of k_0 in K. Then, k is a finite extension of k_0, and is a complete discrete valuation field of mixed characteristics $(0,p)$ with residue field F^{p^∞}.

Conjecture. *Suppose that $e(K|k) > 1$, i.e. a prime element of $\mathcal{O}_k$ is not a prime element of $\mathcal{O}_K$. Then there is a constant c depending only on K such that K has no totally ramified cyclic extension of degree p^i with $i > c$.*

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