Invitation to higher local fields

Conference in Münster, August–September 1999

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Introduction

This volume is a result of the conference on higher local fields in Münster, August 29–September 5, 1999, which was supported by SFB 478 "Geometrische Strukturen in der Mathematik". The conference was organized by I. Fesenko and F. Lorenz. We gratefully acknowledge great hospitality and tremendous efforts of Falko Lorenz which made the conference vibrant.

Class field theory as developed in the first half of this century is a fruitful generalization and extension of Gauss reciprocity law; it describes abelian extensions of number fields in terms of objects associated to these fields. Since its construction, one of the important themes of number theory was its generalizations to other classes of fields or to non-abelian extensions.

In modern number theory one encounters very naturally schemes of finite type over $\mathbb{Z}$. A very interesting direction of generalization of class field theory is to develop a theory for higher dimensional fields — finitely generated fields over their prime subfields (or schemes of finite type over $\mathbb{Z}$ in the geometric language). Work in this subject, higher (dimensional) class field theory, was initiated by A.N. Parshin and K. Kato independently about twenty five years ago. For an introduction into several global aspects of the theory see W. Raskind's review on abelian class field theory of arithmetic schemes.

One of the first ideas in higher class field theory is to work with the Milnor K-groups instead of the multiplicative group in the classical theory. It is one of the principles of class field theory for number fields to construct the reciprocity map by some blending of class field theories for local fields. Somewhat similarly, higher dimensional class field theory is obtained as a blending of higher dimensional *local* class field theories, which treat abelian extensions of *higher local fields*. In this way, the higher local fields were introduced in mathematics.

A precise definition of higher local fields will be given in section 1 of Part I; here we give an example. A complete discrete valuation field K whose residue field is isomorphic to a usual local field with finite residue field is called a two-dimensional local field. For example, fields $\mathbb{F}_p((T))((S))$, $\mathbb{Q}_p((S))$ and

$$\mathbb{Q}_p\{\{T\}\} = \left\{ \sum_{-\infty}^{+\infty} a_i T^i : a_i \in \mathbb{Q}_p, \inf v_p(a_i) > -\infty, \lim_{i\to-\infty} v_p(a_i) = +\infty \right\}$$

(v_p is the p-adic valuation map) are two-dimensional local fields. Whereas the first two fields above can be viewed as generalizations of functional local fields, the latter field comes in sight as an arithmetical generalization of $\mathbb{Q}_p$.

In the classical local case, where K is a complete discrete valuation field with finite residue field, the Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ of the maximal abelian extension of K is approximated by the multiplicative group K^*; and the reciprocity map

$$K^* \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

is close to an isomorphism (it induces an isomorphism between the group $K^*/N_{L/K}L^*$ and $\mathrm{Gal}(L/K)$ for a finite abelian extension L/K, and it is injective with everywhere dense image). For two-dimensional local fields K as above, instead of the multiplicative group K^*, the Milnor K-group $K_2(K)$ (cf. Some Conventions and section 2 of Part I) plays an important role. For these fields there is a reciprocity map

$$K_2(K) \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

which is approximately an isomorphism (it induces an isomorphism between the group $K_2(K)/N_{L/K}K_2(L)$ and $\mathrm{Gal}(L/K)$ for a finite abelian extension L/K, and it has everywhere dense image; but it is not injective: the quotient of $K_2(K)$ by the kernel of the reciprocity map can be described in terms of topological generators, see section 6 Part I).

Similar statements hold in the general case of an n-dimensional local field where one works with the Milnor K_n-groups and their quotients (sections 5,10,11 of Part I); and even class field theory of more general classes of complete discrete valuation fields can be reasonably developed (sections 13,16 of Part I).

Since $K_1(K) = K^*$, higher local class field theory contains the classical local class field theory as its one-dimensional version.

The aim of this book is to provide an introduction to higher local fields and render the main ideas of this theory. The book grew as an extended version of talks given at the conference in Münster. Its expository style aims to introduce the reader into the subject and explain main ideas, methods and constructions (sometimes omitting details). The contributors applied essential efforts to explain the most important features of their subjects.

Hilbert's words in Zahlbericht that precious treasures are hidden in the theory of abelian extensions are still up-to-date. We hope that this volume, as the first collection of main strands of higher local field theory, will be useful as an introduction and guide on the subject.

The first part presents the theory of higher local fields, very often in the more general setting of complete discrete valuation fields.

Section 1, written by I. Zhukov, introduces higher local fields and topologies on their additive and multiplicative groups. Subsection 1.1 contains all basic definitions and is referred to in many other sections of the volume. The topologies are defined in such a way that the topology of the residue field is taken into account; the price one pays is that multiplication is not continuous in general, however it is sequentially continuous which allows one to expand elements into convergent power series or products.

Section 2, written by O. Izhboldin, is a short review of the Milnor K-groups and Galois cohomology groups. It discusses p-torsion and cotorsion of the groups $K_n(F)$ and $K_n^t(F) = K_n(F)/\cap_{l\geqslant 1} lK_n(F)$, an analogue of Satz 90 for the groups $K_n(F)$ and $K_n^t(F)$, and computation of $H_m^{n+1}(F)$ where F is either the rational function field in one variable $F = k(t)$ or the formal power series $F = k((t))$.

Appendix to Section 2, written by M. Kurihara and I. Fesenko, contains some basic definitions and properties of differential forms and Kato's cohomology groups in characteristic p and a sketch of the proof of Bloch–Kato–Gabber's theorem which describes the differential symbol from the Milnor K-group $K_n(F)/p$ of a field F of positive characteristic p to the differential module Ω_F^n.

Section 4, written by J. Nakamura, presents main steps of the proof of Bloch–Kato's theorem which states that the norm residue homomorphism

$$K_q(K)/m \to H^q(K, \mathbb{Z}/m\,(q))$$

is an isomorphism for a henselian discrete valuation field K of characteristic 0 with residue field of positive characteristic. This theorem and its proof allows one to simplify Kato's original approach to higher local class field theory.

Section 5, written by M. Kurihara, is a presentation of main ingredients of Kato's higher local class field theory.

Section 6, written by I. Fesenko, is concerned with certain topologies on the Milnor K-groups of higher local fields K which are related to the topology on the multiplicative group; their properties are discussed and the structure of the quotient of the Milnor K-groups modulo the intersection of all neighbourhoods of zero is described. The latter quotient is called a topological Milnor K-group; it was first introduced by Parshin.

Section 7, written by I. Fesenko, describes Parshin's higher local class field theory in characteristic p, which is relatively easy in comparison with the cohomological approach.

Section 8, written by S. Vostokov, is a review of known approaches to explicit formulas for the (wild) Hilbert symbol not only in the one-dimensional case but in the higher dimensional case as well. One of them, Vostokov's explicit formula, is of importance for the study of topological Milnor K-groups in section 6 and the existence theorem in section 10.

Section 9, written by M. Kurihara, introduces his exponential homomorphism for a complete discrete valuation field of characteristic zero, which relates differential forms and the Milnor K-groups of the field, thus helping one to get an additional information on the structure of the latter. An application to explicit formulas is discussed in subsection 9.2.

Section 10, written by I. Fesenko, presents his explicit method to construct higher local class field theory by using topological K-groups and a generalization of Neukirch–

Hazewinkel's axiomatic approaches to class field theory. Subsection 10.2 presents another simple approach to class field theory in the characteristic p case. The case of characteristic 0 is sketched using a concept of Artin–Schreir trees of extensions (as those extensions in characteristic 0 which are twinkles of the characteristic p world). The existence theorem is discussed in subsection 10.5, being built upon the results of sections 6 and 8.

Section 11, written by M. Spieß, provides a glimpse of Koya's and his approach to the higher local reciprocity map as a generalization of the classical class formations approach to the level of complexes of Galois modules.

Section 12, written by M. Kurihara, sketches his classification of complete discrete valuation fields K of characteristic 0 with residue field of characteristic p into two classes depending on the behaviour of the torsion part of a differential module. For each of these classes, subsection 12.1 characterizes the quotient filtration of the Milnor K-groups of K, for all sufficiently large members of the filtration, as a quotient of differential modules. For a higher local field the previous result and higher local class field theory imply certain restrictions on types of cyclic extensions of the field of sufficiently large degree. This is described in 12.2.

Section 13, written by M. Kurihara, describes his theory of cyclic p-extensions of an absolutely unramified complete discrete valuation field K with *arbitrary* residue field of characteristic p. In this theory a homomorphism is constructed from the p-part of the group of characters of K to Witt vectors over its residue field. This homomorphism satisfies some important properties listed in the section.

Section 14, written by I. Zhukov, presents some explicit methods of constructing abelian extensions of complete discrete valuation fields. His approach to explicit equations of a cyclic extension of degree p^n which contains a given cyclic extension of degree p is explained.

Section 15, written by J. Nakamura, contains a list of all known results on the quotient filtration on the Milnor K-groups (in terms of differential forms of the residue field) of a complete discrete valuation field. It discusses his recent study of the case of a tamely ramified field of characteristic 0 with residue field of characteristic p by using the exponential map of section 9 and a syntomic complex.

Section 16, written by I. Fesenko, is devoted to his generalization of one-dimensional class field theory to a description of abelian totally ramified p-extensions of a complete discrete valuation field with arbitrary non separably-p-closed residue field. In particular, subsection 16.3 shows that two such extensions coincide if and only if their norm groups coincide. An illustration to the theory of section 13 is given in subsection 16.4.

Section 17, written by I. Zhukov, is a review of his recent approach to ramification theory of a complete discrete valuation field with residue field whose p-basis consists of at most one element. One of important ingredients of the theory is Epp's theorem on elimination of wild ramification (subsection 17.1). New lower and upper filtrations are defined (so that cyclic extensions of degree p may have non-integer ramification breaks, see examples in subsection 17.2). One of the advantages of this theory is its compatibility with the reciprocity map. A refinement of the filtration for two-dimensional local fields which is compatible with the reciprocity map is discussed.

Section 18, written by L. Spriano, presents ramification theory of monogenic extensions of complete discrete valuation fields; his recent study demonstrates that in this case there is a satisfactory theory if one systematically uses a generalization of the function i and not s (see subsection 18.0 for definitions). Relations to Kato's conductor are discussed in 18.2 and 18.3.

These sections 17 and 18 can be viewed as the rudiments of higher ramification theory; there are several other approaches. Still, there is no satisfactory general ramification theory for complete discrete valuation fields in the imperfect residue field case; to construct such a theory is a challenging problem.

Without attempting to list all links between the sections we just mention several paths (2 means Section 2 and Appendix to Section 2)

$1 \to 6 \to 7$	(leading to Parshin's approach in positive characteristic),
$2 \to 4 \to 5 \to 11$	(leading to Kato's cohomological description
	of the reciprocity map and generalized class formations),
$8.3 \to 6 \to 10$	(explicit construction of the reciprocity map),
$5 \to 12 \to 13 \to 15$,	(structure of the Milnor K-groups of the fields
$1 \to 10 \to 14, 16$	and more explicit study of abelian extensions),
$8, 9$	(explicit formulas for the Hilbert norm symbol
	and its generalizations),
$1 \to 10 \to 17, 18$	(aspects of higher ramification theory).

A special place in this volume (between Part I and Part II) is occupied by the work of K. Kato on the existence theorem in higher local class field theory which was produced in 1980 as an IHES preprint and has never been published. We are grateful to K. Kato for his permission to include this work in the volume. In it, viewing higher local fields as ring objects in the category of iterated pro-ind-objects, a definition of open subgroups in the Milnor K-groups of the fields is given. The self-duality of the additive group of a higher local field is proved. By studying norm groups of cohomological objects and using cohomological approach to higher local class field theory the existence theorem is proved. An alternative approach to the description of norm subgroups of Galois extensions of higher local fields and the existence theorem is contained in sections 6 and 10.

The second part is concerned with various applications and connections of higher local fields with several other areas.

Section 1, written by A.N. Parshin, describes some first steps in extending Tate–Iwasawa's analytic method to define an L-function in higher dimensions; historically the latter problem was one of the stimuli of the work on higher class field theory. For generalizing this method the author advocates the usefulness of the classical Riemann–Hecke approach (subsection 1.1), his adelic complexes (subsection 1.2.2) together with his generalization of Krichever's correspondence (subsection 1.2.1). He analyzes dimension 1 types of functions in subsection 1.3 and discusses properties of the lattice of commensurable classes of subspaces in the adelic space associated to a divisor on an algebraic surface in subsection 1.4.

Section 2, written by D. Osipov, is a review of his recent work on adelic constructions of direct images of differentials and symbols in the two-dimensional case in the relative situation. In particular, reciprocity laws for relative residues of differentials and symbols are introduced and applied to a construction of the Gysin map for Chow groups.

Section 3, written by A.N. Parshin, presents his theory of Bruhat–Tits buildings over higher dimensional local fields. The theory is illustrated with the buildings for $PGL(2)$ and $PGL(3)$ for one- and two-dimensional local fields.

Section 4, written by E.-U. Gekeler, provides a survey of relations between Drinfeld modules and higher dimensional fields of positive characteristic. The main new result stated is the expression of vanishing orders of certain modular forms through partial zeta values.

Section 5, written by M. Kapranov, sketches his recent approach to elements of harmonic analysis on algebraic groups over functional two-dimensional local fields. For a two-dimensional local field subsection 5.4 introduces a Hecke algebra which is formed by operators which integrate pro-locally-constant complex functions over a non-compact domain.

Section 6, written by L. Herr, is a survey of his recent study of applications of Fontaine's theory of p-adic representations of local fields ($\Phi - \Gamma$-modules) to Galois cohomology of local fields and explicit formulas for the Hilbert symbol (subsections 6.4–6.6). The two Greek letters lead to two-dimensional local objects (like $\mathcal{O}_{\mathcal{E}(K)}$ introduced in subsection 6.3).

Section 7, written by I. Efrat, introduces recent advances in the zero-dimensional anabelian geometry, that is a characterization of fields by means of their absolute Galois group (for finitely generated fields and for higher local fields). His method of construction of henselian valuations on fields which satisfy some K-theoretical properties is presented in subsection 10.3, and applications to an algebraic proof of the local correspondence part of Pop's theorem and to higher local fields are given.

Section 8, written by A. Zheglov, presents his study of two dimensional local skew fields which was initiated by A.N. Parshin. If the skew field has one-dimensional residue field which is in its centre, then one is naturally led to the study of automorphisms of the residue field which are associated to a local parameter of the skew field. Results on such automorphisms are described in subsections 8.2 and 8.3.

Section 9, written by I. Fesenko, is an exposition of his recent work on noncommutative local reciprocity maps for totally ramified Galois extensions with arithmetically profinite group (for instance p-adic Lie extensions). These maps in general are not homomorphisms but Galois cycles; a description of their image and kernel is included.

Section 10, written by B. Erez, is a concise survey of Galois module theory links with class field theory; it lists several open problems.

The theory of higher local fields has several interesting aspects and applications which are not contained in this volume. One of them is the work of Kato on applications of an explicit formula for the reciprocity map in higher local fields to calculations of special values of the L-function of a modular form. There is some interest in two-dimensional local fields (especially of the functional type) in certain parts of mathematical physics, infinite group theory and topology where formal power series objects play a central role.

Prerequisites for most sections in the first part of the book are small: local fields and local class field theory, for instance, as presented in Serre's "Local Fields", Iwasawa's "Local Class Field Theory" or Fesenko–Vostokov's "Local Fields and Their Extensions" (the first source contains a cohomological approach whereas the last two are cohomology free) and some basic knowledge of Milnor K-theory of discrete valuation fields (for instance Chapter IX of the latter book). See also Some Conventions and Appendix to Section 2 of Part I where we explain several notions useful for reading Part I.

We thank P. Schneider for his support of the conference and work on this volume. The volume is typed using a modified version of osudeG style (written by Walter Neumann and Larry Siebenmann and available from the public domain of Department of Mathematics of Ohio State University, pub/osutex); thanks are due to Larry for his advice on aspects of this style and to both Walter and Larry for permission to use it.

Ivan Fesenko Masato Kurihara September 2000

Contents

Some Conventions

The notation $X \subset Y$ means that X is a subset of Y.

For an abelian group A written additively denote by A/m the quotient group A/mA where $mA = \{ma : a \in A\}$ and by ${}_mA$ the subgroup of elements of order dividing m. The subgroup of torsion elements of A is denoted by $\operatorname{Tors} A$.

For an algebraic closure $F^{\rm alg}$ of F denote the separable closure of the field F by $F^{\rm sep}$; let $G_F = \operatorname{Gal}(F^{\rm sep}/F)$ be the absolute Galois group of F. Often for a G_F-module M we write $H^i(F, M)$ instead of $H^i(G_F, M)$.

For a positive integer l which is prime to characteristic of F (if the latter is non-zero) denote by $\mu_l = \langle \zeta_l \rangle$ the group of l th roots of unity in $F^{\rm sep}$.

If l is prime to $\operatorname{char}(F)$, for $m \geqslant 0$ denote by $\mathbb{Z}/l(m)$ the G_F-module $\mu_l^{\otimes m}$ and put $\mathbb{Z}_l(m) = \varprojlim_r \mathbb{Z}/l^r(m)$; for $m < 0$ put $\mathbb{Z}_l(m) = \operatorname{Hom}(\mathbb{Z}_l, \mathbb{Z}_l(-m))$.

Let A be a commutative ring. The group of invertible elements of A is denoted by A^*. Let B be an A-algebra. $\Omega^1_{B/A}$ denotes as usual the B-module of regular differential forms of B over A; $\Omega^n_{B/A} = \wedge^n \Omega^1_{B/A}$. In particular, $\Omega^n_A = \Omega^n_{A/\mathbb{Z}1_A}$ where 1_A is the identity element of A with respect to multiplication. For more on differential modules see subsection A1 of the appendix to the section 2 in the first part.

Let $K_n(k) = K_n^M(k)$ be the Milnor K-group of a field k (for the definition see subsection 2.0 in the first part).

For a complete discrete valuation field K denote by $\mathcal{O} = \mathcal{O}_K$ its *ring of integers*, by $\mathcal{M} = \mathcal{M}_K$ the *maximal ideal* of $\mathcal{O}$ and by $k = k_K$ its *residue field*. If k is of characteristic p, denote by $\mathcal{R}$ the set of *Teichmüller representatives* (or *multiplicative representatives*) in $\mathcal{O}$. For θ in the maximal perfect subfield of k denote by $[\theta]$ its Teichmüller representative.

For a field k denote by $W(k)$ the ring of Witt vectors (more precisely, Witt p-vectors where p is a prime number) over k. Denote by $W_r(k)$ the ring of Witt vectors of length r over k. If $\operatorname{char}(k) = p$ denote by $\mathbf{F}\colon W(k) \to W(k)$, $\mathbf{F}\colon W_r(k) \to W_r(k)$ the map $(a_0, \dots) \mapsto (a_0^p, \dots)$.

Denote by v_K the surjective discrete valuation $K^* \to \mathbb{Z}$ (it is sometimes called the *normalized discrete valuation* of K). Usually $\pi = \pi_K$ denotes a *prime element* of K: $v_K(\pi_K) = 1$.

Denote by $K_{\rm ur}$ the *maximal unramified extension* of K. If k_K is finite, denote by Frob_K the *Frobenius automorphism* of $K_{\rm ur}/K$.

For a finite extension L of a complete discrete valuation field K $\mathcal{D}_{L/K}$ denotes its different.

If $\operatorname{char}(K) = 0$, $\operatorname{char}(k_K) = p$, then K is called a field of *mixed characteristic*. If $\operatorname{char}(K) = 0 = \operatorname{char}(k_K)$, then K is called a field of *equal characteristic*.

If k_K is perfect, K is called a *local field*.

Part I

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1. Higher dimensional local fields

Igor Zhukov

We give here basic definitions related to n-dimensional local fields. For detailed exposition, see [P] in the equal characteristic case, [K1, §8] for the two-dimensional case and [MZ1], [MZ2] for the general case. Several properties of the topology on the multiplicative group are discussed in [F].

1.1. Main definitions

Suppose that we are given a surface S over a finite field of characteristic p, a curve $C \subset S$, and a point $x \in C$ such that both S and C are regular at x. Then one can attach to these data the quotient field of the completion $\widehat{(\widehat{\mathcal{O}}_{S,x})_C}$ of the localization at C of the completion $\widehat{\mathcal{O}}_{S,x}$ of the local ring $\mathcal{O}_{S,x}$ of S at x. This is a two-dimensional local field over a finite field, i.e., a complete discrete valuation field with local residue field. More generally, an n-dimensional local field F is a complete discrete valuation field with $(n-1)$-dimensional residue field. (Finite fields are considered as 0-dimensional local fields.)

Definition. A complete discrete valuation field K is said to have the structure of an n-dimensional local field if there is a chain of fields $K = K_n, K_{n-1}, \dots, K_1, K_0$ where K_{i+1} is a complete discrete valuation field with residue field K_i and K_0 is a finite field. The field $k_K = K_{n-1}$ (resp. K_0) is said to be the *first* (resp. the *last*) residue field of K.

Remark. Most of the properties of n-dimensional local fields do not change if one requires that the last residue K_0 is *perfect* rather than finite. To specify the exact meaning of the word, K can be referred to as an n-dimensional local field *over* a finite (resp. perfect) field. One can consider an n-dimensional local field over an arbitrary field K_0 as well. However, in this volume mostly the higher local fields over finite fields are considered.

Examples. 1. $\mathbb{F}_q((X_1))\dots((X_n))$. 2. $k((X_1))\dots((X_{n-1}))$, k a finite extension of $\mathbb{Q}_p$.

3. For a complete discrete valuation field F let

$$K = F\{\{T\}\} = \left\{\sum_{-\infty}^{+\infty} a_i T^i : a_i \in F,\ \inf v_F(a_i) > -\infty,\ \lim_{i\to-\infty} v_F(a_i) = +\infty\right\}.$$

Define $v_K(\sum a_i T^i) = \min v_F(a_i)$. Then K is a complete discrete valuation field with residue field $k_F((t))$.

Hence for a local field k the fields

$$k\{\{T_1\}\}\dots\{\{T_m\}\}((T_{m+2}))\dots((T_n)), \quad 0 \leqslant m \leqslant n-1$$

are n-dimensional local fields (they are called *standard fields*).

Remark. $K((X))\{\{Y\}\}$ is isomorphic to $K((Y))((X))$.

Definition. An n-tuple of elements $t_1,\dots,t_n \in K$ is called a *system of local parameters* of K, if t_n is a prime element of K_n, t_{n-1} is a unit in $\mathcal{O}_K$ but its residue in K_{n-1} is a prime element of K_{n-1}, and so on.

For example, for $K = k\{\{T_1\}\}\dots\{\{T_m\}\}((T_{m+2}))\dots((T_n))$, a convenient system of local parameter is $T_1,\dots,T_m,\pi,T_{m+2},\dots,T_n$, where π is a prime element of k.

Consider the maximal m such that $\operatorname{char}(K_m) = p$; we have $0 \leqslant m \leqslant n$. Thus, there are $n+1$ types of n-dimensional local fields: fields of characteristic p and fields with $\operatorname{char}(K_{m+1}) = 0$, $\operatorname{char}(K_m) = p$, $0 \leqslant m \leqslant n-1$. Thus, the mixed characteristic case is the case $m = n-1$.

Suppose that $\operatorname{char}(k_K) = p$, i.e., the above m equals either $n-1$ or n. Then the set of Teichmüller representatives $\mathcal{R}$ in $\mathcal{O}_K$ is a field isomorphic to K_0.

Classification Theorem. *Let K be an n-dimensional local field. Then*

(1) *K is isomorphic to $\mathbb{F}_q((X_1))\dots((X_n))$ if* $\operatorname{char}(K) = p$;

(2) *K is isomorphic to $k((X_1))\dots((X_{n-1}))$, k is a local field, if* $\operatorname{char}(K_1) = 0$;

(3) *K is a finite extension of a standard field $k\{\{T_1\}\}\dots\{\{T_m\}\}((T_{m+2}))\dots((T_n))$ and there is a finite extension of K which is a standard field if* $\operatorname{char}(K_{m+1}) = 0$, $\operatorname{char}(K_m) = p$.

Proof. In the equal characteristic case the statements follow from the well known classification theorem for complete discrete valuation fields of equal characteristic. In the mixed characteristic case let k_0 be the fraction field of $W(\mathbb{F}_q)$ and let $T_1,\dots,T_{n-1},\pi$ be a system of local parameters of K. Put

$$K' = k_0\{\{T_1\}\}\dots\{\{T_{n-1}\}\}.$$

Then K' is an absolutely unramified complete discrete valuation field, and the (first) residue fields of K' and K coincide. Therefore, K can be viewed as a finite extension of K' by [FV, II.5.6].

Alternatively, let $t_1, \dots, t_{n-1}$ be any liftings of a system of local parameters of k_K. Using the canonical lifting $h_{t_1,\dots,t_{n-1}}$ defined below, one can construct an embedding $K' \hookrightarrow K$ which identifies T_i with t_i.

To prove the last assertion of the theorem, one can use *Epp's theorem on elimination of wild ramification* (see 17.1) which asserts that there is a finite extension l/k_0 such that $e\left(lK/lK'\right) = 1$. Then lK' is standard and lK is standard, so K is a subfield of lK. See [Z] or [KZ] for details and a stronger statement. □

Definition. The *lexicographic order* of $\mathbb{Z}^n$: $\mathbf{i} = (i_1, \dots, i_n) \leqslant \mathbf{j} = (j_1, \dots, j_n)$ if and only if

$$i_l \leqslant j_l,\ i_{l+1} = j_{l+1}, \dots, i_n = j_n \text{ for some } l \leqslant n .$$

Introduce $\mathbf{v} = (v_1, \dots, v_n)\colon K^* \to \mathbb{Z}^n$ as $v_n = v_{K_n}$, $v_{n-1}(\alpha) = v_{K_{n-1}}(\alpha_{n-1})$ where α_{n-1} is the residue of $\alpha t_n^{-v_n(\alpha)}$ in K_{n-1}, and so on. The map $\mathbf{v}$ is a valuation; this is a so called discrete valuation of rank n. Observe that for $n > 1$ the valuation $\mathbf{v}$ does depend on the choice of $t_2, \dots, t_n$. However, all the valuations obtained this way are in the same class of equivalent valuations.

Now we define several objects which do not depend on the choice of a system of local parameters.

Definition.
$O_K = \{\alpha \in K : \mathbf{v}(\alpha) \geqslant 0\}$, $M_K = \{\alpha \in K : \mathbf{v}(\alpha) > 0\}$, so $O_K/M_K \simeq K_0$. The group of *principal units* of K with respect to the valuation $\mathbf{v}$ is $V_K = 1 + M_K$.

Definition.

$$P(i_l, \dots, i_n) = P_K(i_l, \dots, i_n) = \{\alpha \in K : (v_l(\alpha), \dots, v_n(\alpha)) \geqslant (i_l, \dots, i_n)\}.$$

In particular, $O_K = P\big(\underbrace{0, \dots, 0}_{n}\big)$, $M_K = P\big(1, \underbrace{0, \dots, 0}_{n-1}\big)$, whereas $\mathcal{O}_K = P(0)$, $\mathcal{M}_K = P(1)$. Note that if $n > 1$, then

$$\cap_i M_K^i = P\big(1, \underbrace{0, \dots, 0}_{n-2}\big),$$

since $t_2 = t_1^{i-1}(t_2/t_1^{i-1})$.

Lemma. *The set of all non-zero ideals of O_K consists of all*

$$\{P(i_l, \dots, i_n) : (i_l, \dots, i_n) \geqslant (0, \dots, 0), \quad 1 \leqslant l \leqslant n\}.$$

The ring O_K is not Noetherian for $n > 1$.

Proof. Let J be a non-zero ideal of O_K. Put $i_n = \min\{v_n(\alpha) : \alpha \in J\}$. If $J = P(i_n)$, then we are done. Otherwise, it is clear that

$$i_{n-1} := \inf\{v_{n-1}(\alpha) : \alpha \in J, v_n(\alpha) = i_n\} > -\infty.$$

If $i_n = 0$, then obviously $i_{n-1} \geqslant 0$. Continuing this way, we construct $(i_l, \dots, i_n) \geqslant (0, \dots, 0)$, where either $l = 1$ or

$$i_{l-1} = \inf\{v_{l-1}(\alpha) : \alpha \in J, v_n(\alpha) = i_n, \dots, v_l(\alpha) = i_l\} = -\infty.$$

In both cases it is clear that $J = P(i_l, \dots, i_n)$.

The second statement is immediate from $P(0,1) \subset P(-1,1) \subset P(-2,1) \dots$. □

For more on ideals in O_K see subsection 3.0 of Part II.

1.2. Extensions

Let L/K be a finite extension. If K is an n-dimensional local field, then so is L.

Definition. Let $t_1, \dots, t_n$ be a system of local parameters of K and let $t_1^{'}, \dots, t_n^{'}$ be a system of local parameters of L. Let $\mathbf{v}, \mathbf{v}'$ be the corresponding valuations. Put

$$E(L|K) := \left(v_j^{'}(t_i)\right)_{i,j} = \begin{pmatrix} e_1 & 0 & \dots & 0 \\ \dots & e_2 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & e_n \end{pmatrix},$$

where $e_i = e_i(L|K) = e(L_i|K_i)$, $i = 1, \dots, n$. Then e_i do not depend on the choice of parameters, and $|L : K| = f(L|K) \prod_{i=1}^n e_i(L|K)$, where $f(L|K) = |L_0 : K_0|$.

The expression "unramified extension" can be used for extensions L/K with $e_n(L|K) = 1$ and L_{n-1}/K_{n-1} separable. It can be also used in a narrower sense, namely, for extensions L/K with $\prod_{i=1}^n e_i(L|K) = 1$. To avoid ambiguity, sometimes one speaks of a "semiramified extension" in the former case and a "purely unramified extension" in the latter case.

1.3. Topology on K

Consider an example of n-dimensional local field

$$K = k\,\{\{T_1\}\}\,\dots\{\{T_m\}\}\,((T_{m+2}))\dots((T_n)).$$

Expanding elements of k into power series in π with coefficients in $\mathcal{R}_k$, one can write elements of K as formal power series in n parameters. To make them convergent power series we should introduce a topology in K which takes into account topologies of the residue fields. We do not make K a topological field this way, since multiplication is only sequentially continuous in this topology. However, for class field theory sequential continuity seems to be more important than continuity.

1.3.1.

Definition.

(a) If F has a topology, consider the following topology on $K = F((X))$. For a sequence of neighbourhoods of zero $(U_i)_{i\in\mathbb{Z}}$ in F, $U_i = F$ for $i \gg 0$, denote $U_{\{U_i\}} = \left\{\sum a_i X^i : a_i \in U_i\right\}$. Then all $U_{\{U_i\}}$ constitute a base of open neighbourhoods of 0 in $F((X))$. In particular, a sequence $u^{(n)} = \sum a_i^{(n)} X^i$ tends to 0 if and only if there is an integer m such that $u^{(n)} \in X^m F[[X]]$ for all n and the sequences $a_i^{(n)}$ tend to 0 for every i.

Starting with the discrete topology on the last residue field, this construction is used to obtain a well-defined topology on an n-dimensional local field of characteristic p.

(b) Let K_n be of mixed characteristic. Choose a system of local parameters $t_1, \dots, t_n = \pi$ of K. The choice of $t_1, \dots, t_{n-1}$ determines a canonical lifting

$$h = h_{t_1,\dots,t_{n-1}} : K_{n-1} \to \mathcal{O}_K$$

(see below). Let $(U_i)_{i\in\mathbb{Z}}$ be a system of neighbourhoods of zero in K_{n-1}, $U_i = K_{n-1}$ for $i \gg 0$. Take the system of all $U_{\{U_i\}} = \left\{\sum h(a_i)\pi^i,\ a_i \in U_i\right\}$ as a base of open neighbourhoods of 0 in K. This topology is well defined.

(c) In the case $\operatorname{char}(K) = \operatorname{char}(K_{n-1}) = 0$ we apply constructions (a) and (b) to obtain a topology on K which depends on the choice of the coefficient subfield of K_{n-1} in $\mathcal{O}_K$.

The definition of the canonical lifting $h_{t_1,\dots,t_{n-1}}$ is rather complicated. In fact, it is worthwhile to define it for any $(n-1)$-tuple $(t_1, \dots, t_{n-1})$ such that $v_i(t_i) > 0$ and $v_j(t_i) = 0$ for $i < j \leqslant n$. We shall give an outline of this construction, and the details can be found in [MZ1, §1].

Let $F = K_0((\overline{t_1}))\dots((\overline{t_{n-1}})) \subset K_{n-1}$. By a lifting we mean a map $h\colon F \to \mathcal{O}_K$ such that the residue of $h(a)$ coincides with a for any $a \in F$.

Step 1. An auxiliary lifting $H_{t_1,\dots,t_{n-1}}$ is uniquely determined by the condition

$$H_{t_1,\dots,t_{n-1}}\left(\sum_{i_1=0}^{p-1}\cdots\sum_{i_{n-1}=0}^{p-1} \overline{t_1}^{\,i_1}\dots\overline{t_{n-1}}^{\,i_{n-1}} a^p_{i_1,\dots,i_{n-1}}\right)$$
$$= \sum_{i_1=0}^{p-1}\cdots\sum_{i_{n-1}=0}^{p-1} t_1^{i_1}\dots t_{n-1}^{i_{n-1}}(H_{t_1,\dots,t_{n-1}}(a_{i_1,\dots,i_{n-1}}))^p.$$

Step 2. Let k_0 be the fraction field of $W(K_0)$. Then $K' = k_0\{\{T_1\}\}\dots\{\{T_{n-1}\}\}$ is an n-dimensional local field with the residue field F. Comparing the lifting $H = H_{T_1,\dots,T_{n-1}}$ with the lifting h defined by

$$h\Big(\sum_{\mathbf{r}\in\mathbb{Z}^{n-1}} \theta_{\mathbf{r}}\overline{T_1}^{r_1}\dots\overline{T_{n-1}}^{r_{n-1}}\Big) = \sum_{\mathbf{r}\in\mathbb{Z}^{n-1}} [\theta_{\mathbf{r}}]T_1^{r_1}\dots T_{n-1}^{r_{n-1}},$$

we introduce the maps $\lambda_i\colon F \to F$ by the formula

$$h(a) = H(a) + pH(\lambda_1(a)) + p^2H(\lambda_2(a)) + \dots$$

Step 3. Introduce $h_{t_1,\dots,t_{n-1}}\colon F \to \mathcal{O}_K$ by the formula

$$h_{t_1,\dots,t_{n-1}}(a) = H_{t_1,\dots,t_{n-1}}(a) + pH_{t_1,\dots,t_{n-1}}(\lambda_1(a)) + p^2H_{t_1,\dots,t_{n-1}}(\lambda_2(a)) + \dots.$$

Remarks. 1. Observe that for a standard field $K = k\,\{\{T_1\}\}\dots\{\{T_{n-1}\}\}$, we have

$$h_{T_1,\dots,T_{n-1}}\colon \sum \theta_{\mathbf{i}}\overline{T_1}^{i_1}\dots\overline{T_{n-1}}^{i_{n-1}} \mapsto \sum [\theta_{\mathbf{i}}]T_1^{i_1}\dots T_{n-1}^{i_{n-1}},$$

where $\overline{T_j}$ is the residue of T_j in k_K, $j = 1,\dots,n-1$.

2. The idea of the above construction is to find a field $k_0\{\{t_1\}\}\dots\{\{t_{n-1}\}\}$ isomorphic to K' inside K without a priori given topologies on K and K'. More precisely, let $t_1,\dots,t_{n-1}$ be as above. For $a = \sum_{-\infty}^{\infty} p^i h(a_i) \in K'$, let

$$f_{t_1,\dots,t_{n-1}}(a) = \sum_{-\infty}^{\infty} p^i h_{t_1,\dots,t_{n-1}}(a_i)$$

Then $f_{t_1,\dots,t_{n-1}}\colon K' \to K$ is an embedding of n-dimensional complete fields such that

$$f_{t_1,\dots,t_{n-1}}(T_j) = t_j, \qquad j = 1,\dots,n-1$$

(see [MZ1, Prop. 1.1]).

3. In the case of a standard mixed characteristic field the following alternative construction of the same topology is very useful.

Let $K = E\{\{X\}\}$, where E is an $(n-1)$-dimensional local field; assume that the topology of E is already defined. Let $\{V_i\}_{i\in\mathbb{Z}}$ be a sequence of neighbourhoods of zero in E such that
(i) there is $c \in \mathbb{Z}$ such that $P_E(c) \subset V_i$ for all $i \in \mathbb{Z}$;
(ii) for every $l \in \mathbb{Z}$ we have $P_E(l) \subset V_i$ for all sufficiently large i.

Put

$$\mathcal{V}_{\{V_i\}} = \Big\{\sum b_i X^i : b_i \in V_i\Big\}.$$

Then all the sets $\mathcal{V}_{\{V_i\}}$ form a base of neighbourhoods of 0 in K. (This is an easy but useful exercise in the 2-dimensional case; in general, see Lemma 1.6 in [MZ1]).

4. The formal construction of $h_{t_1,\dots,t_{n-1}}$ works also in case $\operatorname{char}(K) = p$, and one need not consider this case separately. However, if one is interested in equal

characteristic case only, all the treatment can be considerably simplified. (In fact, in this case $h_{t_1,\dots,t_{n-1}}$ is just the obvious embedding of $F \subset k_K$ into $\mathcal{O}_K = k_K[[t_n]]$.)

1.3.2. Properties.

(1) K is a topological group which is complete and separated.

(2) If $n > 1$, then every base of neighbourhoods of 0 is uncountable. In particular, there are maps which are sequentially continuous but not continuous.

(3) If $n > 1$, multiplication in K is not continuous. In fact, $UU = K$ for every open subgroup U, since $U \supset P(c)$ for some c and $U \not\subset P(s)$ for any s. However, multiplication is sequentially continuous:

$$\alpha_i \to \alpha, \quad 0 \neq \beta_i \to \beta \neq 0 \Longrightarrow \alpha_i\beta_i^{-1} \to \alpha\beta^{-1}.$$

(4) The map $K \to K$, $\quad \alpha \mapsto c\alpha$ for $c \neq 0$ is a homeomorphism.

(5) For a finite extension L/K the topology of L = the topology of finite dimensional vector spaces over K (i.e., the product topology on $K^{|L:K|}$). Using this property one can redefine the topology first for "standard" fields

$$k\,\{\{T_1\}\}\,\dots\{\{T_m\}\}\,((T_{m+2}))\,\dots((T_n))$$

using the canonical lifting h, and then for arbitrary fields as the topology of finite dimensional vector spaces.

(6) For a finite extension L/K the topology of K = the topology induced from L. Therefore, one can use the Classification Theorem and define the topology on K as induced by that on L, where L is taken to be a standard n-dimensional local field.

Remark. In practical work with higher local fields, both (5) and (6) enables one to use the original definition of topology only in the simple case of a standard field.

1.3.3. About proofs. The outline of the proof of assertions in 1.3.1–1.3.2 is as follows. (Here we concentrate on the most complicated case $\mathrm{char}(K) = 0$, $\mathrm{char}(K_{n-1}) = p$; the case of $\mathrm{char}(K) = p$ is similar and easier, for details see [P]).

Step 1 (see [MZ1, §1]). Fix first $n-1$ local parameters (or, more generally, any elements $t_1, \dots, t_{n-1} \in K$ such that $v_i(t_i) > 0$ and $v_j(t_i) = 0$ for $j > i$).

Temporarily fix $\pi_i \in K$ $(i \in \mathbb{Z})$, $v_n(\pi_i) = i$, and $e_j \in P_K(0)$, $j = 1, \dots, d$, so that $\{\overline{e_j}\}_{j=1}^d$ is a basis of the F-linear space K_{n-1}. (Here F is as in 1.3.1, and $\bar{\alpha}$ denotes the residue of α in K_{n-1}.) Let $\{U_i\}_{i\in\mathbb{Z}}$ be a sequence of neighbourhoods of zero in F, $U_i = F$ for all sufficiently large i. Put

$$\mathcal{U}_{\{U_i\}} = \Big\{\sum_{i\geqslant i_0} \pi_i \cdot \sum_{j=1}^{d} e_j h_{t_1,\dots,t_{n-1}}(a_{ij}) : a_{ij} \in U_i, i_0 \in \mathbb{Z}\Big\}.$$

The collection of all such sets $\mathcal{U}_{\{U_i\}}$ is denoted by B_U.

Step 2 ([MZ1, Th. 1.1]). In parallel one proves that

– the set B_U has a cofinal subset which consists of subgroups of K; thus, B_U is a base of neighbourhoods of zero of a certain topological group $K_{t_1,\dots,t_{n-1}}$ with the underlying (additive) group K;

– $K_{t_1,\dots,t_{n-1}}$ does not depend on the choice of $\{\pi_i\}$ and $\{e_j\}$;

– property (4) in 1.3.2 is valid for $K_{t_1,\dots,t_{n-1}}$.

Step 3 ([MZ1, §2]). Some properties of $K_{t_1,\dots,t_{n-1}}$ are established, in particular, (1) in 1.3.2, the sequential continuity of multiplication.

Step 4 ([MZ1, §3]). The independence from the choice of $t_1, \dots, t_{n-1}$ is proved.

We give here a short proof of some statements in Step 3.

Observe that the topology of $K_{t_1,\dots,t_{n-1}}$ is essentially defined as a topology of a finite-dimensional vector space over a standard field $k_0\{\{t_1\}\}\dots\{\{t_{n-1}\}\}$. (It will be precisely so, if we take $\{\pi_i e_j : 0 \leqslant i \leqslant e-1, 1 \leqslant j \leqslant d\}$ as a basis of this vector space, where e is the absolute ramification index of K, and $\pi_{i+e} = p\pi_i$ for any i.) This enables one to reduce the statements to the case of a standard field K.

If K is standard, then either $K = E((X))$ or $K = E\{\{X\}\}$, where E is of smaller dimension. Looking at expansions in X, it is easy to construct a limit of any Cauchy sequence in K and to prove the uniqueness of it. (In the case $K = E\{\{X\}\}$ one should use the alternative construction of topology in Remark 3 in 1.3.1.) This proves (1) in 1.3.2.

To prove the sequential continuity of multiplication in the mixed characteristic case, let $\alpha_i \to 0$ and $\beta_i \to 0$, we shall show that $\alpha_i\beta_i \to 0$.

Since $\alpha_i \to 0, \beta_i \to 0$, one can easily see that there is $c \in \mathbb{Z}$ such that $v_n(\alpha_i) \geqslant c$, $v_n(\beta_i) \geqslant c$ for $i \geqslant 1$.

By the above remark, we may assume that K is standard, i.e., $K = E\{\{t\}\}$. Fix an open subgroup U in K; we have $P(d) \subset U$ for some integer d. One can assume that $U = \mathcal{V}_{\{V_i\}}$, V_i are open subgroups in E. Then there is m_0 such that $P_E(d-c) \subset V_m$ for $m > m_0$. Let

$$\alpha_i = \sum_{-\infty}^{\infty} a_i^{(r)} t^r, \quad \beta_i = \sum_{-\infty}^{\infty} b_i^{(l)} t^l, \qquad a_i^{(r)}, b_i^{(l)} \in E.$$

Notice that one can find an r_0 such that $a_i^{(r)} \in P_E(d-c)$ for $r < r_0$ and all i. Indeed, if this were not so, one could choose a sequence $r_1 > r_2 > \dots$ such that $a_{i_j}^{(r_j)} \notin P_E(d-c)$ for some i_j. It is easy to construct a neighbourhood of zero V'_{r_j} in E such that $P_E(d-c) \subset V'_{r_j}$, $a_{i_j}^{(r_j)} \notin V_{r_j}$. Now put $V'_r = E$ when r is distinct from any of r_j, and $U' = \mathcal{V}_{\{V'_r\}}$. Then $a_{i_j} \notin U'$, $j = 1, 2, \dots$ The set $\{i_j\}$ is obviously infinite, which contradicts the condition $\alpha_i \to 0$.

Similarly, $b_i^{(l)} \in P_E(d-c)$ for $l < l_0$ and all i. Therefore,

$$\alpha_i\beta_i \equiv \sum_{r=r_0}^{m_0} a_i^{(r)} t^r \cdot \sum_{l=l_0}^{m_0} b_i^{(l)} t^l \mod U,$$

and the condition $a_i^{(r)} b_i^{(l)} \to 0$ for all r and l immediately implies $\alpha_i \beta_i \to 0$.

1.3.4. Expansion into power series. Let $n = 2$. Then in characteristic p we have $\mathbb{F}_q((X))((Y)) = \{\sum \theta_{ij} X^j Y^i\}$, where θ_{ij} are elements of $\mathbb{F}_q$ such that for some i_0 we have $\theta_{ij} = 0$ for $i \leqslant i_0$ and for every i there is $j(i)$ such that $\theta_{ij} = 0$ for $j \leqslant j(i)$.

On the other hand, the definition of the topology implies that for every neighbourhood of zero U there exists i_0 and for every $i < i_0$ there exists $j(i)$ such that $\theta X^j Y^i \in U$ whenever either $i \geqslant i_0$ or $i < i_0, j \geqslant j(i)$.

So every formal power series has only finitely many terms $\theta X^j Y^i$ outside U. Therefore, it is in fact a *convergent* power series in the just defined topology.

Definition. $\Omega \subset \mathbb{Z}^n$ is called admissible if for every $1 \leqslant l \leqslant n$ and every $j_{l+1}, \dots, j_n$ there is $i = i(j_{l+1}, \dots, j_n) \in \mathbb{Z}$ such that

$$(i_1, \dots, i_n) \in \Omega,\ i_{l+1} = j_{l+1}, \dots, i_n = j_n \Rightarrow i_l \geqslant i.$$

Theorem. *Let $t_1, \dots, t_n$ be a system of local parameters of K. Let s be a section of the residue map $O_K \to O_K/M_K$ such that $s(0) = 0$. Let Ω be an admissible subset of $\mathbb{Z}^n$. Then the series*

$$\sum_{(i_1,\dots,i_n)\in\Omega} b_{i_1,\dots,i_n} t_1^{i_1} \dots t_n^{i_n} \text{ converges } \quad (b_{i_1,\dots,i_n} \in s(O_K/M_K))$$

and every element of K can be uniquely written this way.

Remark. In this statement it is essential that the last residue field is finite. In a more general setting, one should take a "good enough" section. For example, for $K = k\{\{T_1\}\} \dots \{\{T_m\}\}((T_{m+2}))\dots((T_n))$, where k is a finite extension of the fraction field of $W(K_0)$ and K_0 is perfect of prime characteristic, one may take the Teichmüller section $K_0 \to K_{m+1} = k\{\{T_1\}\} \dots \{\{T_m\}\}$ composed with the obvious embedding $K_{m+1} \hookrightarrow K$.

Proof. We have

$$\sum_{(i_1,\dots,i_n)\in\Omega} b_{i_1,\dots,i_n} t_1^{i_1} \dots t_n^{i_n} = \sum_{b\in s(O_K/M_K)} \Big(b \cdot \sum_{(i_1,\dots,i_n)\in\Omega_b} t_1^{i_1} \dots t_n^{i_n}\Big),$$

where $\Omega_b = \{(i_1, \dots, i_n) \in \Omega : b_{i_1,\dots,i_n} = b\}$. In view of the property (4), it is sufficient to show that the inner sums converge. Equivalently, one has to show that given a neighbourhood of zero U in K, for almost all $(i_1, \dots, i_n) \in \Omega$ we have $t_1^{i_1} \dots t_n^{i_n} \in U$. This follows easily by induction on n if we observe that $t_1^{i_1} \dots t_{n-1}^{i_{n-1}} = h_{t_1,\dots,t_{n-1}}(\overline{t_1}^{i_1} \dots \overline{t_{n-1}}^{i_{n-1}})$.

To prove the second statement, apply induction on n once again. Let $r = v_n(\alpha)$, where α is a given element of K. Then by the induction hypothesis

$$\overline{t_n^{-r}\alpha} = \sum_{(i_1,\dots,i_{n-1})\in\Omega_r} \overline{b_{i_1,\dots,i_n}}\,(\overline{t_1})^{i_1}\dots(\overline{t_{n-1}})^{i_{n-1}},$$

where $\Omega_r \subset \mathbb{Z}^{n-1}$ is a certain admissible set. Hence

$$\alpha = \sum_{(i_1,\dots,i_{n-1})\in\Omega_r} b_{i_1,\dots,i_n} t_1^{i_1}\dots t_{n-1}^{i_{n-1}} t_n^r + \alpha',$$

where $v_n(\alpha') > r$. Continuing this way, we obtain the desired expansion into a sum over the admissible set $\Omega = (\Omega_r \times \{r\}) \cup (\Omega_{r+1} \times \{r+1\}) \cup \dots$

The uniqueness follows from the continuity of the residue map $\mathcal{O}_K \to K_{n-1}$. $\square$

1.4. Topology on K^*

1.4.1. 2-dimensional case, $\operatorname{char}(k_K) = p$**.**

Let A be the last residue field K_0 if $\operatorname{char}(K) = p$, and let $A = W(K_0)$ if $\operatorname{char}(K) = 0$. Then A is canonically embedded into $\mathcal{O}_K$, and it is in fact the subring generated by the set $\mathcal{R}$.

For a 2-dimensional local field K with a system of local parameters t_2, t_1 define a base of neighbourhoods of 1 as the set of all $1 + t_2^i\mathcal{O}_K + t_1^jA[[t_1,t_2]]$, $i \geqslant 1$, $j \geqslant 1$. Then every element $\alpha \in K^*$ can be expanded as a convergent (with respect to the just defined topology) product

$$\alpha = t_2^{a_2}t_1^{a_1}\theta\prod(1+\theta_{ij}t_2^it_1^j)$$

with $\theta \in \mathcal{R}^*, \theta_{ij} \in \mathcal{R}, a_1, a_2 \in \mathbb{Z}$. The set $S = \{(j,i) : \theta_{ij} \neq 0\}$ is admissible.

1.4.2. In the general case, following Parshin's approach in characteristic p [P], we define the topology τ on K^* as follows.

Definition. If $\operatorname{char}(K_{n-1}) = p$, then define the topology τ on

$$K^* \simeq V_K \times \langle t_1\rangle \times \dots \times \langle t_n\rangle \times \mathcal{R}^*$$

as the product of the induced from K topology on the group of principal units V_K and the discrete topology on $\langle t_1\rangle \times \dots \times \langle t_n\rangle \times \mathcal{R}^*$.

If $\operatorname{char}(K) = \operatorname{char}(K_{m+1}) = 0$, $\operatorname{char}(K_m) = p$, where $m \leqslant n-2$, then we have a canonical exact sequence

$$1 \to 1 + P_K(1, \underbrace{0,\dots,0}_{n-m-2}) \to O_K^* \to O_{K_{m+1}}^* \to 1.$$

Define the topology τ on $K^* \simeq O_K^* \times \langle t_1 \rangle \times \cdots \times \langle t_n \rangle$ as the product of the discrete topology on $\langle t_1 \rangle \times \cdots \times \langle t_n \rangle$ and the inverse image of the topology τ on $O^*_{K_{m+1}}$. Then the intersection of all neighbourhoods of 1 is equal to $1 + P_K\big(1, \underbrace{0, \dots, 0}_{n-m-2}\big)$ which is a uniquely divisible group.

Remarks. 1. Observe that K_{m+1} is a mixed characteristic field and therefore its topology is well defined. Thus, the topology τ is well defined in all cases.

2. A base of neighbourhoods of 1 in V_K is formed by the sets

$$h(U_0) + h(U_1)t_n + \dots + h(U_{c-1})t_n^{c-1} + P_K(c),$$

where $c \geqslant 1$, U_0 is a neighbourhood of 1 in V_{k_K}, $U_1, \dots, U_{c-1}$ are neighbourhoods of zero in k_K, h is the canonical lifting associated with some local parameters, t_n is the last local parameter of K. In particular, in the two-dimensional case τ coincides with the topology of 1.4.1.

Properties.

(1) Each Cauchy sequence with respect to the topology τ converges in K^*.

(2) Multiplication in K^* is sequentially continuous.

(3) If $n \leqslant 2$, then the multiplicative group K^* is a topological group and it has a countable base of open subgroups. K^* is not a topological group with respect to τ if $m \geqslant 3$.

Proof. (1) and (2) follow immediately from the corresponding properties of the topology defined in subsection 1.3. In the 2-dimensional case (3) is obvious from the description given in 1.4.1. Next, let $m \geqslant 3$, and let U be an arbitrary neighbourhood of 1. We may assume that $n = m$ and $U \subset V_K$. From the definition of the topology on V_K we see that $U \supset 1 + h(U_1)t_n + h(U_2)t_n^2$, where U_1, U_2 are neighbourhoods of 0 in k_K, t_n a prime element in K, and h the canonical lifting corresponding to some choice of local parameters. Therefore,

$$\begin{aligned} UU + P(4) &\supset (1 + h(U_1)t_n)(1 + h(U_2)t_n^2) + P(4) \\ &= \{1 + h(a)t_n + h(b)t_n^2 + h(ab)t_n^3 : a \in U_1, b \in U_2\} + P(4). \end{aligned}$$

(Indeed, $h(a)h(b) - h(ab) \in P(1)$.) Since $U_1 U_2 = k_K$ (see property (3) in 1.3.2), it is clear that UU cannot lie in a neighbourhood of 1 in V_K of the form $1 + h(k_K)t_n + h(k_K)t_n^2 + h(U')t_n^3 + P(4)$, where $U' \neq k_K$ is a neighbourhood of 0 in k_K. Thus, K^* is not a topological group. $\square$

Remarks. 1. From the point of view of class field theory and the existence theorem one needs a stronger topology on K^* than the topology τ (in order to have more open subgroups). For example, for $n \geqslant 3$ each open subgroup A in K^* with respect to the topology τ possesses the property: $1 + t_n^2 \mathcal{O}_K \subset (1 + t_n^3 \mathcal{O}_K)A$.

A topology λ_* which is the sequential saturation of τ is introduced in subsection 6.2; it has the same set of convergence sequences as τ but more open subgroups. For example [F1], the subgroup in $1 + t_n \mathcal{O}_K$ topologically generated by $1 + \theta t_n^{i_n} \dots t_1^{i_1}$ with $(i_1, \dots, i_n) \neq (0, 0, \dots, 1, 2)$, $i_n \geqslant 1$ (i.e., the sequential closure of the subgroup generated by these elements) is open in λ_* and does not satisfy the above-mentioned property.

One can even introduce a topology on K^* which has the same set of convergence sequences as τ and with respect to which K^* is a topological group, see [F2].

2. For another approach to define open subgroups of K^* see the paper of K. Kato in this volume.

1.4.3. Expansion into convergent products. To simplify the following statements we assume here char $k_K = p$. Let B be a fixed set of representatives of non-zero elements of the last residue field in K.

Lemma. *Let* $\{\alpha_i : i \in I\}$ *be a subset of* V_K *such that*

$$(*) \qquad \alpha_i = 1 + \sum_{\mathbf{r} \in \Omega_i} b_{\mathbf{r}}^{(i)} t_1^{r_1} \dots t_n^{r_n},$$

where $b \in B$, *and* $\Omega_i \subset \mathbb{Z}_+^n$ *are admissible sets satisfying the following two conditions:*

(i) $\Omega = \bigcup_{i \in I} \Omega_i$ *is an admissible set;*

(ii) $\bigcap_{j \in J} \Omega_j = \emptyset$, *where* J *is any infinite subset of* I.

Then $\prod_{i \in I} \alpha_i$ *converges.*

Proof. Fix a neighbourhood of 1 in V_K; by definition it is of the form $(1 + U) \cap V_K$, where U is a neighbourhood of 0 in K. Consider various finite products of $b_{\mathbf{r}}^{(i)} t_1^{r_1} \dots t_n^{r_n}$ which occur in $(*)$. It is sufficient to show that almost all such products belong to U.

Any product under consideration has the form

$$(**) \qquad \gamma = b_1^{k_1} \dots b_s^{k_s} t_1^{l_1} \dots t_n^{l_n}$$

with $l_n > 0$, where $B = \{b_1, \dots, b_s\}$. We prove by induction on j the following claim: for $0 \leqslant j \leqslant n$ and fixed $l_{j+1}, \dots, l_n$ the element γ almost always lies in U (in case $j = n$ we obtain the original claim). Let

$$\hat{\Omega} = \{\mathbf{r}_1 + \dots + \mathbf{r}_t : t \geqslant 1, \mathbf{r}_1, \dots, \mathbf{r}_t \in \Omega\}.$$

It is easy to see that $\hat{\Omega}$ is an admissible set and any element of $\hat{\Omega}$ can be written as a sum of elements of Ω in finitely many ways only. This fact and condition (ii) imply that any particular n-tuple $(l_1, \dots, l_n)$ can occur at the right hand side of $(**)$ only finitely many times. This proves the base of induction ($j = 0$).

For $j > 0$, we see that l_j is bounded from below since $(l_1, \dots, l_n) \in \hat{\Omega}$ and $l_{j+1}, \dots, l_n$ are fixed. On the other hand, $\gamma \in U$ for sufficiently large l_j and arbitrary $k_1, \dots, k_s, l_1, \dots, l_{j-1}$ in view of [MZ1, Prop. 1.4] applied to the neighbourhood of zero $t_{j+1}^{-l_{j+1}} \dots t_n^{-l_n} U$ in K. Therefore, we have to consider only a finite range of values $c \leqslant l_j \leqslant c'$. For any l_j in this range the induction hypothesis is applicable. $\square$

Theorem. *For any* $\mathbf{r} \in \mathbb{Z}_+^n$ *and any* $b \in B$ *fix an element*

$$a_{\mathbf{r},b} = \sum_{\mathbf{s}\in\Omega_{\mathbf{r},b}} b_{\mathbf{s}}^{\mathbf{r},b} t_1^{s_1} \dots t_n^{s_n},$$

such that $b_{\mathbf{r}}^{\mathbf{r},b} = b$, *and* $b_{\mathbf{s}}^{\mathbf{r},b} = 0$ *for* $\mathbf{s} < \mathbf{r}$. *Suppose that the admissible sets*

$$\{\Omega_{\mathbf{r},b} : \mathbf{r} \in \Omega_*, b \in B\}$$

satisfy conditions (i) *and* (ii) *of the Lemma for any given admissible set* Ω_*.

1. Every element $a \in K$ *can be uniquely expanded into a convergent series*

$$a = \sum_{\mathbf{r}\in\Omega_a} a_{\mathbf{r},b_{\mathbf{r}}},$$

where $b_{\mathbf{r}} \in B$, $\Omega_a \subset \mathbb{Z}_n$ *is an admissible set.*

2. Every element $\alpha \in K^*$ *can be uniquely expanded into a convergent product:*

$$\alpha = t_n^{a_n} \dots t_1^{a_1} b_0 \prod_{\mathbf{r}\in\Omega_\alpha} \left(1 + a_{\mathbf{r},b_{\mathbf{r}}}\right),$$

where $b_0 \in B$, $b_{\mathbf{r}} \in B$, $\Omega_\alpha \subset \mathbb{Z}_n^+$ *is an admissible set.*

Proof. The additive part of the theorem is [MZ2, Theorem 1]. The proof of it is parallel to that of Theorem 1.3.4.

To prove the multiplicative part, we apply induction on n. This reduces the statement to the case $\alpha \in 1 + P(1)$. Here one can construct an expansion and prove its uniqueness applying the additive part of the theorem to the residue of $t_n^{-v_n(\alpha-1)}(\alpha - 1)$ in k_K. The convergence of all series which appear in this process follows from the above Lemma. For details, see [MZ2, Theorem 2]. $\square$

Remarks. 1. Conditions (i) and (ii) in the Lemma are essential. Indeed, the infinite products $\prod_{i=1}^{\infty}(1 + t_1^i + t_1^{-i} t_2)$ and $\prod_{i=1}^{\infty}(1 + t_1^i + t_2)$ do not converge. This means that the statements of Theorems 2.1 and 2.2 in [MZ1] have to be corrected and conditions (i) and (ii) for elements $\varepsilon_{\mathbf{r},\theta}$ $(\mathbf{r} \in \Omega_*)$ should be added.

2. If the last residue field is not finite, the statements are still true if the system of representatives B is not too pathological. For example, the system of Teichmüller representatives is always suitable. The above proof works with the only ammendment: instead of Prop. 1.4 of [MZ1] we apply the definition of topology directly.

Corollary. *If* $\mathrm{char}(K_{n-1}) = p$*, then every element* $\alpha \in K^*$ *can be expanded into a convergent product:*

$$(***) \qquad \alpha = t_n^{a_n} \dots t_1^{a_1} \theta \prod (1 + \theta_{i_1,\dots,i_n} t_1^{i_1} \dots t_n^{i_n}), \quad \theta \in \mathcal{R}^*, \quad \theta_{i_1,\dots,i_n} \in \mathcal{R},$$

with $\{(i_1, \dots, i_n) : \theta_{i_1,\dots,i_n} \neq 0\}$ *being an admissible set. Any series* $(***)$ *converges.*

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2. p-primary part of the Milnor K-groups and Galois cohomologies of fields of characteristic p

Oleg Izhboldin

2.0. Introduction

Let F be a field and F^{sep} be the separable closure of F. Let F^{ab} be the maximal abelian extension of F. Clearly the Galois group $G^{\mathrm{ab}} = \mathrm{Gal}(F^{\mathrm{ab}}/F)$ is canonically isomorphic to the quotient of the absolute Galois group $G = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ modulo the closure of its commutant. By Pontryagin duality, a description of G^{ab} is equivalent to a description of

$$\mathrm{Hom}_{\mathrm{cont}}\big(G^{\mathrm{ab}}, \mathbb{Z}/m\big) = \mathrm{Hom}_{\mathrm{cont}}\big(G, \mathbb{Z}/m\big) = H^1(F, \mathbb{Z}/m).$$

where m runs over all positive integers. Clearly, it suffices to consider the case where m is a power of a prime, say $m = p^i$. The main cohomological tool to compute the group $H^1(F, \mathbb{Z}/m)$ is a pairing

$$(\ ,\ )_m \colon H^1(F, \mathbb{Z}/m) \otimes K_n(F)/m \to H_m^{n+1}(F)$$

where the right hand side is a certain cohomological group discussed below.

Here $K_n(F)$ for a field F is the n th Milnor K-group $K_n(F) = K_n^M(F)$ defined as

$$(F^*)^{\otimes n}/J$$

where J is the subgroup generated by the elements of the form $a_1 \otimes \ldots \otimes a_n$ such that $a_i + a_j = 1$ for some $i \neq j$. We denote by $\{a_1, \ldots, a_n\}$ the class of $a_1 \otimes \ldots \otimes a_n$. Namely, $K_n(F)$ is the abelian group defined by the following
generators: symbols $\{a_1, \ldots, a_n\}$ with $a_1, \ldots, a_n \in F^*$
and relations:

$$\{a_1, \ldots, a_i a_i', \ldots a_n\} = \{a_1, \ldots, a_i, \ldots a_n\} + \{a_1, \ldots, a_i', \ldots a_n\}$$
$$\{a_1, \ldots, a_n\} = 0 \quad \text{if } a_i + a_j = 1 \text{ for some } i \text{ and } j \text{ with } i \neq j.$$

We write the group law additively.

Consider the following example (definitions of the groups will be given later).

Example. Let F be a field and let p be a prime integer. Assume that there is an integer n with the following properties:
(i) the group $H_p^{n+1}(F)$ is isomorphic to $\mathbb{Z}/p$,
(ii) the pairing

$$(\; , \;)_p: H^1(F,\mathbb{Z}/p) \otimes K_n(F)/p \to H_p^{n+1}(F) \simeq \mathbb{Z}/p$$

is non-degenerate in a certain sense.

Then the $\mathbb{Z}/p$-linear space $H^1(F,\mathbb{Z}/p)$ is obviously dual to the $\mathbb{Z}/p$-linear space $K_n(F)/p$. On the other hand, $H^1(F,\mathbb{Z}/p)$ is dual to the $\mathbb{Z}/p$-space $G^{\text{ab}}/(G^{\text{ab}})^p$. Therefore there is an isomorphism

$$\Psi_{F,p}: K_n(F)/p \simeq G^{\text{ab}}/(G^{\text{ab}})^p.$$

It turns out that this example can be applied to computations of the group $G^{\text{ab}}/(G^{\text{ab}})^p$ for multidimensional local fields. Moreover, it is possible to show that the homomorphism $\Psi_{F,p}$ can be naturally extended to a homomorphism $\Psi_F: K_n(F) \to G^{\text{ab}}$ (the so called reciprocity map). Since G^{ab} is a profinite group, it follows that the homomorphism $\Psi_F: K_n(F) \to G^{\text{ab}}$ factors through the homomorphism $K_n(F)/DK_n(F) \to G^{\text{ab}}$ where the group $DK_n(F)$ consists of all divisible elements:

$$DK_n(F) := \cap_{m \geqslant 1} m K_n(F).$$

This observation makes natural the following notation:

Definition (cf. section 6 of Part I)**.** For a field F and integer $n \geqslant 0$ set

$$K_n^t(F) := K_n(F)/DK_n(F),$$

where $DK_n(F) := \bigcap_{m \geqslant 1} m K_n(F)$.

The group $K_n^t(F)$ for a higher local field F endowed with a certain topology (cf. section 6 of this part of the volume) is called a topological Milnor K-group $K^{\text{top}}(F)$ of F.

The example shows that computing the group G^{ab} is closely related to computing the groups $K_n(F)$, $K_n^t(F)$, and $H_m^{n+1}(F)$. The main purpose of this section is to explain some basic properties of these groups and discuss several classical conjectures. Among the problems, we point out the following:

- discuss p-torsion and cotorsion of the groups $K_n(F)$ and $K_n^t(F)$,
- study an analogue of Satz 90 for the groups $K_n(F)$ and $K_n^t(F)$,
- compute the group $H_m^{n+1}(F)$ in two "classical" cases where F is either the rational function field in one variable $F = k(t)$ or the formal power series $F = k((t))$.

We shall consider in detail the case (so called "non-classical case") of a field F of characteristic p and $m = p$.

2.1. Definition of $H_m^{n+1}(F)$ and pairing $(\ ,\ )_m$

To define the group $H_m^{n+1}(F)$ we consider three cases depending on the characteristic of the field F.

Case 1 (Classical)**.** Either $\operatorname{char}(F) = 0$ or $\operatorname{char}(F) = p$ is prime to m.

In this case we set

$$H_m^{n+1}(F) := H^{n+1}(F, \mu_m^{\otimes n}).$$

The Kummer theory gives rise to the well known natural isomorphism $F^*/F^{*m} \to H^1(F, \mu_m)$. Denote the image of an element $a \in F^*$ under this isomorphism by (a). The cup product gives the homomorphism

$$\underbrace{F^* \otimes \cdots \otimes F^*}_{n} \to H^n(F, \mu_m^{\otimes n}), \qquad a_1 \otimes \cdots \otimes a_n \to (a_1, \dots, a_n)$$

where $(a_1, \dots, a_n) := (a_1) \cup \cdots \cup (a_n)$. It is well known that the element $(a_1, \dots, a_n)$ is zero if $a_i + a_j = 1$ for some $i \neq j$. From the definition of the Milnor K-group we get the homomorphism

$$\eta_m \colon K_n^M(F)/m \to H^n(F, \mu_m^{\otimes n}), \qquad \{a_1, \dots, a_n\} \to (a_1, \dots, a_n).$$

Now, we define the pairing $(\ ,\ )_m$ as the following composite

$$H^1(F, \mathbb{Z}/m) \otimes K_n(F)/m \xrightarrow{\mathrm{id} \otimes \eta_m} H^1(F, \mathbb{Z}/m) \otimes H^n(F, \mu_m^{\otimes n}) \xrightarrow{\cup} H_m^{n+1}(F, \mu_m^{\otimes n}).$$

Case 2. $\operatorname{char}(F) = p \neq 0$ and m is a power of p.

To simplify the exposition we start with the case $m = p$. Set

$$H_p^{n+1}(F) = \operatorname{coker}\bigl(\Omega_F^n \xrightarrow{\wp} \Omega_F^n / d\Omega_F^{n-1}\bigr)$$

where

$$d\,(a db_2 \wedge \cdots \wedge db_n) = da \wedge db_2 \wedge \cdots \wedge db_n,$$
$$\wp\Bigl(a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}\Bigr) = (a^p - a)\,\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} + d\Omega_F^{n-1}$$

($\wp = \mathrm{C}^{-1} - 1$ where C^{-1} is the inverse Cartier operator defined in subsection 4.2). The pairing $(\ ,\ )_p$ is defined as follows:

$$(\ ,\ )_p \colon F/\wp(F) \times K_n(F)/p \to H_p^{n+1}(F),$$
$$(a, \{b_1, \dots, b_n\}) \mapsto a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}$$

where $F/\wp(F)$ is identified with $H^1(F, \mathbb{Z}/p)$ via Artin–Schreier theory.

To define the group $H^{n+1}_{p^i}(F)$ for an arbitrary $i \geqslant 1$ we note that the group $H^{n+1}_p(F)$ is the quotient group of Ω^n_F. In particular, generators of the group $H^{n+1}_p(F)$ can be written in the form $adb_1 \wedge \dots \wedge db_n$. Clearly, the natural homomorphism

$$F \otimes \underbrace{F^* \otimes \dots \otimes F^*}_{n} \to H^{n+1}_p(F), \qquad a \otimes b_1 \otimes \dots \otimes b_n \mapsto a\frac{db_1}{b_1} \wedge \dots \wedge \frac{db_n}{b_n}$$

is surjective. Therefore the group $H^{n+1}_p(F)$ is naturally identified with the quotient group $F \otimes F^* \otimes \dots \otimes F^*/J$. It is not difficult to show that the subgroup J is generated by the following elements:

$(a^p - a) \otimes b_1 \otimes \dots \otimes b_n$,
$a \otimes a \otimes b_2 \otimes \dots \otimes b_n$,
$a \otimes b_1 \otimes \dots \otimes b_n$, where $b_i = b_j$ for some $i \neq j$.

This description of the group $H^{n+1}_p(F)$ can be easily generalized to define $H^{n+1}_{p^i}(F)$ for an arbitrary $i \geqslant 1$. Namely, we define the group $H^{n+1}_{p^i}(F)$ as the quotient group

$$W_i(F) \otimes \underbrace{F^* \otimes \dots \otimes F^*}_{n}/J$$

where $W_i(F)$ is the group of Witt vectors of length i and J is the subgroup of $W_i(F) \otimes F^* \otimes \dots \otimes F^*$ generated by the following elements:

$(\mathbf{F}(w) - w) \otimes b_1 \otimes \dots \otimes b_n$,
$(a, 0, \dots, 0) \otimes a \otimes b_2 \otimes \dots \otimes b_n$,
$w \otimes b_1 \otimes \dots \otimes b_n$, where $b_i = b_j$ for some $i \neq j$.

The pairing $(\ ,\ )_{p^i}$ is defined as follows:

$$(\ ,\ )_p \colon W_i(F)/\wp(W_i(F)) \times K_n(F)/p^i \to H^{n+1}_{p^i}(F),$$
$$(w, \{b_1, \dots, b_n\}) \mapsto w \otimes b_1 \otimes \dots \otimes b_n$$

where $\wp = \mathbf{F} - \mathrm{id} \colon W_i(F) \to W_i(F)$ and the group $W_i(F)/\wp(W_i(F))$ is identified with $H^1(F, \mathbb{Z}/p^i)$ via Witt theory. This completes definitions in Case 2.

Case 3. $\operatorname{char}(F) = p \neq 0$ and $m = m'p^i$ where $m' > 1$ is an integer prime to p and $i \geqslant 1$.

The groups $H^{n+1}_{m'}(F)$ and $H^{n+1}_{p^i}(F)$ are already defined (see Cases 1 and 2). We define the group $H^{n+1}_m(F)$ by the following formula:

$$H^{n+1}_m(F) := H^{n+1}_{m'}(F) \oplus H^{n+1}_{p^i}(F)$$

Since $H^1(F, \mathbb{Z}/m) \simeq H^1(F, \mathbb{Z}/m') \oplus H^1(F, \mathbb{Z}/p^i)$ and $K_n(F)/m \simeq K_n(F)/m' \oplus K_n(F)/p^i$, we can define the pairing $(\ ,\ )_m$ as the direct sum of the pairings $(\ ,\ )_{m'}$ and $(\ ,\ )_{p^i}$. This completes the definition of the group $H^{n+1}_m(F)$ and of the pairing $(\ ,\ )_m$.

Remark 1. In the case $n = 1$ or $n = 2$ the group $H^n_m(F)$ can be determined as follows:

$$H^1_m(F) \simeq H^1(F, \mathbb{Z}/m) \qquad \text{and} \qquad H^2_m(F) \simeq {}_m\,\mathrm{Br}(F).$$

Remark 2. The group $H^{n+1}_m(F)$ is often denoted by $H^{n+1}(F, \mathbb{Z}/m\,(n))$.

2.2. The group $H^{n+1}(F)$

In the previous subsection we defined the group $H^{n+1}_m(F)$ and the pairing $(\ ,\ )_m$ for an arbitrary m. Now, let m and m' be positive integers such that m' is divisible by m. In this case there exists a canonical homomorphism

$$i_{m,m'}\colon H^{n+1}_m(F) \to H^{n+1}_{m'}(F).$$

To define the homomorphism $i_{m,m'}$ it suffices to consider the following two cases:

Case 1. Either $\mathrm{char}\,(F) = 0$ or $\mathrm{char}\,(F) = p$ is prime to m and m'.

This case corresponds to Case 1 in the definition of the group $H^{n+1}_m(F)$ (see subsection 2.1). We identify the homomorphism $i_{m,m'}$ with the homomorphism

$$H^{n+1}(F, \mu_m^{\otimes n}) \to H^{n+1}(F, \mu_{m'}^{\otimes n})$$

induced by the natural embedding $\mu_m \subset \mu_{m'}$.

Case 2. m and m' are powers of $p = \mathrm{char}\,(F)$.

We can assume that $m = p^i$ and $m' = p^{i'}$ with $i \leqslant i'$. This case corresponds to Case 2 in the definition of the group $H^{n+1}_m(F)$. We define $i_{m,m'}$ as the homomorphism induced by

$$W_i(F) \otimes F^* \otimes \dots F^* \to W_{i'}(F) \otimes F^* \otimes \dots F^*,$$
$$(a_1, \dots, a_i) \otimes b_1 \otimes \dots \otimes b_n \mapsto (0, \dots, 0, a_1, \dots, a_i) \otimes b_1 \otimes \dots \otimes b_n.$$

The maps $i_{m,m'}$ (where m and m' run over all integers such that m' is divisible by m) determine the inductive system of the groups.

Definition. For a field F and an integer n set

$$H^{n+1}(F) = \varinjlim{}_m\, H^{n+1}_m(F).$$

Conjecture 1. *The natural homomorphism $H^{n+1}_m(F) \to H^{n+1}(F)$ is injective and the image of this homomorphism coincides with the m-torsion part of the group $H^{n+1}(F)$.*

This conjecture follows easily from the Milnor–Bloch–Kato conjecture (see subsection 4.1) in degree n. In particular, it is proved for $n \leqslant 2$. For fields of characteristic p we have the following theorem.

Theorem 1. *Conjecture 1 is true if* $\operatorname{char}(F) = p$ *and* $m = p^i$.

2.3. Computing the group $H_m^{n+1}(F)$ for some fields

We start with the following well known result.

Theorem 2 (classical)**.** *Let F be a perfect field. Suppose that* $\operatorname{char}(F) = 0$ *or* $\operatorname{char}(F)$ *is prime to m. Then*

$$H_m^{n+1}\big(F((t))\big) \simeq H_m^{n+1}(F) \oplus H_m^n(F)$$

$$H_m^{n+1}\big(F(t)\big) \simeq H_m^{n+1}(F) \oplus \coprod_{\text{monic irred } f(t)} H_m^n\big(F[t]/f(t)\big).$$

It is known that we cannot omit the conditions on F and m in the statement of Theorem 2. To generalize the theorem to the arbitrary case we need the following notation. For a complete discrete valuation field K and its maximal unramified extension K_{ur} define the groups $H_{m,\mathrm{ur}}^n(K)$ and $\widetilde{H}_m^n(K)$ as follows:

$$H_{m,\mathrm{ur}}^n(K) = \ker\big(H_m^n(K) \to H_m^n(K_{\mathrm{ur}})\big) \qquad \text{and} \qquad \widetilde{H}_m^n(K) = H_m^n(K)/H_{m,\mathrm{ur}}^n(K).$$

Note that for a field $K = F((t))$ we obviously have $K_{\mathrm{ur}} = F^{\mathrm{sep}}((t))$. We also note that under the hypotheses of Theorem 2 we have $H^n(K) = H_{m,\mathrm{ur}}^n(K)$ and $H^n(K) = 0$. The following theorem is due to Kato.

Theorem 3 (Kato, [K1, Th. 3 §0])**.** *Let K be a complete discrete valuation field with residue field k. Then*

$$H_{m,\mathrm{ur}}^{n+1}(K) \simeq H_m^{n+1}(k) \oplus H_m^n(k).$$

In particular, $H_{m,\mathrm{ur}}^{n+1}\big(F((t))\big) \simeq H_m^{n+1}(F) \oplus H_m^n(F)$.

This theorem plays a key role in Kato's approach to class field theory of multidimensional local fields (see section 5 of this part).

To generalize the second isomorphism of Theorem 2 we need the following notation. Set

$$H_{m,\mathrm{sep}}^{n+1}(F(t)) = \ker\big(H_m^{n+1}(F(t)) \to H_m^{n+1}(F^{\mathrm{sep}}(t))\big) \text{ and}$$

$$\widetilde{H}_m^{n+1}(F(t)) = H_m^{n+1}(F(t))/H_{m,\mathrm{sep}}^{n+1}(F(t)).$$

If the field F satisfies the hypotheses of Theorem 2, we have $H_{m,\mathrm{sep}}^{n+1}(F(t)) = H_m^{n+1}(F(t))$ and $\widetilde{H}_m^{n+1}(F(t)) = 0$.

In the general case we have the following statement.

Theorem 4 (Izhboldin, [I2, Introduction])**.**

$$H^{n+1}_{m,\mathrm{sep}}(F(t)) \simeq H^{n+1}_m(F) \oplus \coprod_{\text{monic irred } f(t)} H^n_m\left(F[t]/f(t)\right),$$

$$\widetilde{H}^{n+1}_m(F(t)) \simeq \coprod_v \widetilde{H}^{n+1}_m(F(t)_v)$$

where v runs over all normalized discrete valuations of the field $F(t)$ and $F(t)_v$ denotes the v-completion of $F(t)$.

2.4. On the group $K_n(F)$

In this subsection we discuss the structure of the torsion and cotorsion in Milnor K-theory. For simplicity, we consider the case of prime $m = p$. We start with the following fundamental theorem concerning the quotient group $K_n(F)/p$ for fields of characteristic p.

Theorem 5 (Bloch–Kato–Gabber, [BK, Th. 2.1])**.** *Let F be a field of characteristic p. Then the differential symbol*

$$d_F\colon K_n(F)/p \to \Omega^n_F, \qquad \{a_1, \dots, a_n\} \mapsto \frac{da_1}{a_1} \wedge \dots \wedge \frac{da_n}{a_n}$$

is injective and its image coincides with the kernel $\nu_n(F)$ of the homomorphism $\wp$ (for the definition see Case 2 of 2.1). In other words, the sequence

$$0 \longrightarrow K_n(F)/p \xrightarrow{d_F} \Omega^n_F \xrightarrow{\wp} \Omega^n_F/d\Omega^{n-1}_F$$

is exact.

This theorem relates the Milnor K-group modulo p of a field of characteristic p with a submodule of the differential module whose structure is easier to understand. The theorem is important for Kato's approach to higher local class field theory. For a sketch of its proof see subsection A2 in the appendix to this section.

There exists a natural generalization of the above theorem for the quotient groups $K_n(F)/p^i$ by using De Rham–Witt complex ([BK, Cor. 2.8]).

Now, we recall well known Tate's conjecture concerning the torsion subgroup of the Milnor K-groups.

Conjecture 2 (Tate)**.** *Let F be a field and p be a prime integer.*
(i) *If* $\mathrm{char}(F) \neq p$ *and* $\zeta_p \in F$, *then* ${}_pK_n(F) = \{\zeta_p\} \cdot K_{n-1}(F)$.
(ii) *If* $\mathrm{char}(F) = p$ *then* ${}_pK_n(F) = 0$.

This conjecture is trivial in the case where $n \leqslant 1$. In the other cases we have the following theorem.

Theorem 6. *Let F be a field and n be a positive integer.*
(1) *Tate's Conjecture holds if $n \leqslant 2$ (Suslin,* [S]),
(2) *Part* (ii) *of Tate's Conjecture holds for all n (Izhboldin,* [I1]).

The proof of this theorem is closely related to the proof of Satz 90 for K-groups. Let us recall two basic conjectures on this subject.

Conjecture 3 (Satz 90 for K_n)**.** *If L/F is a cyclic extension of degree p with the Galois group $G = \langle\sigma\rangle$ then the sequence*

$$K_n(L) \xrightarrow{1-\sigma} K_n(L) \xrightarrow{N_{L/F}} K_n(F)$$

is exact.

There is an analogue of the above conjecture for the quotient group $K_n(F)/p$. Fix the following notation till the end of this section:

Definition. For a field F set

$$k_n(F) = K_n(F)/p.$$

Conjecture 4 (Small Satz 90 for k_n)**.** *If L/F is a cyclic extension of degree p with the Galois group $G = \langle\sigma\rangle$, then the sequence*

$$k_n(F) \oplus k_n(L) \xrightarrow{i_{F/L} \oplus (1-\sigma)} k_n(L) \xrightarrow{N_{L/F}} k_n(F)$$

is exact.

The conjectures 2,3 and 4 are not independent:

Lemma (Suslin)**.** *Fix a prime integer p and integer n. Then in the category of all fields* (*of a given characteristic*) *we have*

$$(\textit{Small Satz 90 for } k_n) + (\textit{Tate conjecture for } {}_pK_n) \iff (\textit{Satz 90 for } K_n).$$

Moreover, for a given field F we have

$$(\textit{Small Satz 90 for } k_n) + (\textit{Tate conjecture for } {}_pK_n) \Rightarrow (\textit{Satz 90 for } K_n)$$

and

$$(\textit{Satz 90 for } K_n) \Rightarrow (\textit{small Satz 90 for } k_n).$$

Satz 90 conjectures are proved for $n \leqslant 2$ (Merkurev-Suslin, [MS1]). If $p = 2$, $n = 3$, and $\operatorname{char}(F) \neq 2$, the conjectures were proved by Merkurev and Suslin [MS] and Rost. For $p = 2$ the conjectures follow from recent results of Voevodsky. For fields of characteristic p the conjectures are proved for all n:

Theorem 7 (Izhboldin, [I1]). *Let F be a field of characteristic p and L/F be a cyclic extension of degree p. Then the following sequence is exact:*

$$0 \to K_n(F) \to K_n(L) \xrightarrow{1-\sigma} K_n(L) \xrightarrow{N_{L/F}} K_n(F) \to H_p^{n+1}(F) \to H_p^{n+1}(L)$$

2.5. On the group $K_n^t(F)$

In this subsection we discuss the same issues, as in the previous subsection, for the group $K_n^t(F)$.

Definition. Let F be a field and p be a prime integer. We set

$$DK_n(F) = \bigcap_{m\geqslant 1} mK_n(F) \quad \text{and} \quad D_pK_n(F) = \bigcap_{i\geqslant 0} p^iK_n(F).$$

We define the group $K_n^t(F)$ as the quotient group:

$$K_n^t(F) = K_n(F)/DK_n(F) = K_n(F)/\bigcap_{m\geqslant 1} mK_n(F).$$

The group $K_n^t(F)$ is of special interest for higher class field theory (see sections 6, 7 and 10). We have the following evident isomorphism (see also 2.0):

$$K_n^t(F) \simeq \operatorname{im}\left(K_n(F) \to \varprojlim_m K_n(F)/m\right).$$

The quotient group $K_n^t(F)/m$ is obviously isomorphic to the group $K_n(F)/m$. As for the torsion subgroup of $K_n^t(F)$, it is quite natural to state the same questions as for the group $K_n(F)$.

Question 1. Are the K^t-analogue of Tate's conjecture and Satz 90 Conjecture true for the group $K_n^t(F)$?

If we know the (positive) answer to the corresponding question for the group $K_n(F)$, then the previous question is equivalent to the following:

Question 2. Is the group $DK_n(F)$ divisible?

At first sight this question looks trivial because the group $DK_n(F)$ consists of all divisible elements of $K_n(F)$. However, the following theorem shows that the group $DK_n(F)$ is not necessarily a divisible group!

Theorem 8 (Izhboldin, [I3]). *For every $n \geqslant 2$ and prime p there is a field F such that* $\operatorname{char}(F) \neq p$, $\zeta_p \in F$ *and*

(1) *The group $DK_n(F)$ is not divisible, and the group $D_pK_2(F)$ is not p-divisible,*

(2) *The* K^t*-analogue of Tate's conjecture is false for* K_n^t*:*

$$_pK_n^t(F) \neq \{\zeta_p\} \cdot K_{n-1}^t(F).$$

(3) *The* K^t*-analogue of Hilbert 90 conjecture is false for group* $K_n^t(F)$*.*

Remark 1. The field F satisfying the conditions of Theorem 8 can be constructed as the function field of some infinite dimensional variety over any field of characteristic zero whose group of roots of unity is finite.

Quite a different construction for irregular prime numbers p and $F = \mathbb{Q}(\mu_p)$ follows from works of G. Banaszak [B].

Remark 2. If F is a field of characteristic p then the groups $D_pK_n(F)$ and $DK_n(F)$ are p-divisible. This easily implies that $_pK_n^t(F) = 0$. Moreover, Satz 90 theorem holds for K_n^t in the case of cyclic p-extensions.

Remark 3. If F is a multidimensional local fields then the group $K_n^t(F)$ is studied in section 6 of this volume. In particular, Fesenko (see subsections 6.3–6.8 of section 6) gives positive answers to Questions 1 and 2 for multidimensional local fields.

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A. Appendix to Section 2

Masato Kurihara and Ivan Fesenko

This appendix aims to provide more details on several notions introduced in section 2, as well as to discuss some basic facts on differentials and to provide a sketch of the proof of Bloch–Kato–Gabber's theorem. The work on it was completed after sudden death of Oleg Izhboldin, the author of section 2.

A1. Definitions and properties of several basic notions (by M. Kurihara)

Before we proceed to our main topics, we collect here the definitions and properties of several basic notions.

A1.1. Differential modules.

Let A and B be commutative rings such that B is an A-algebra. We define $\Omega^1_{B/A}$ to be the B-module of regular differentials over A. By definition, this B-module $\Omega^1_{B/A}$ is a unique B-module which has the following property. For a B-module M we denote by $\mathrm{Der}_A(B,M)$ the set of all A-derivations (an A-homomorphism $\varphi\colon B \to M$ is called an A-derivation if $\varphi(xy) = x\varphi(y) + y\varphi(x)$ and $\varphi(x) = 0$ for any $x \in A$). Then, φ induces $\overline{\varphi}\colon \Omega^1_{B/A} \to M$ ($\varphi = \overline{\varphi} \circ d$ where d is the canonical derivation $d\colon B \to \Omega^1_{B/A}$), and $\varphi \mapsto \overline{\varphi}$ yields an isomorphism

$$\mathrm{Der}_A(B,M) \xrightarrow{\sim} \mathrm{Hom}_B(\Omega^1_{B/A}, M).$$

In other words, $\Omega^1_{B/A}$ is the B-module defined by the following
generators: dx for any $x \in B$
and relations:

$$d(xy) = xdy + ydx$$

$$dx = 0 \quad \text{for any } x \in A\,.$$

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If $A = \mathbb{Z}$, we simply denote $\Omega^1_{B/\mathbb{Z}}$ by Ω^1_B.

When we consider Ω^1_A for a local ring A, the following lemma is very useful.

Lemma. *If A is a local ring, we have a surjective homomorphism*

$$A \otimes_{\mathbb{Z}} A^* \longrightarrow \Omega^1_A$$

$$a \otimes b \mapsto a d \log b = a \frac{db}{b}.$$

The kernel of this map is generated by elements of the form

$$\sum_{i=1}^{k} (a_i \otimes a_i) - \sum_{i=1}^{l} (b_i \otimes b_i)$$

for $a_i,\ b_i \in A^*$ such that $\Sigma_{i=1}^{k} a_i = \Sigma_{i=1}^{l} b_i$.

Proof. First, we show the surjectivity. It is enough to show that xdy is in the image of the above map for $x,\ y \in A$. If y is in A^*, xdy is the image of $xy \otimes y$. If y is not in A^*, y is in the maximal ideal of A, and $1 + y$ is in A^*. Since $xdy = xd(1 + y)$, xdy is the image of $x(1 + y) \otimes (1 + y)$.

Let J be the subgroup of $A \otimes A^*$ generated by the elements

$$\sum_{i=1}^{k} (a_i \otimes a_i) - \sum_{i=1}^{l} (b_i \otimes b_i)$$

for $a_i,\ b_i \in A^*$ such that $\Sigma_{i=1}^{k} a_i = \Sigma_{i=1}^{l} b_i$. Put $M = (A \otimes_{\mathbb{Z}} A^*)/J$. Since it is clear that J is in the kernel of the map in the lemma, $a \otimes b \mapsto a d \log b$ induces a surjective homomorphism $M \to \Omega^1_A$, whose injectivity we have to show.

We regard $A \otimes A^*$ as an A-module via $a(x \otimes y) = ax \otimes y$. We will show that J is a sub A-module of $A \otimes A^*$. To see this, it is enough to show

$$\sum_{i=1}^{k} (xa_i \otimes a_i) - \sum_{i=1}^{l} (xb_i \otimes b_i) \in J$$

for any $x \in A$. If $x \notin A^*$, x can be written as $x = y + z$ for some $y,\ z \in A^*$, so we may assume that $x \in A^*$. Then,

$$\begin{aligned}
&\sum_{i=1}^{k} (xa_i \otimes a_i) - \sum_{i=1}^{l} (xb_i \otimes b_i) \\
&= \sum_{i=1}^{k} (xa_i \otimes xa_i - xa_i \otimes x) - \sum_{i=1}^{l} (xb_i \otimes xb_i - xb_i \otimes x) \\
&= \sum_{i=1}^{k} (xa_i \otimes xa_i) - \sum_{i=1}^{l} (xb_i \otimes xb_i) \in J.
\end{aligned}$$

Thus, J is an A-module, and $M = (A \otimes A^*)/J$ is also an A-module.

In order to show the bijectivity of $M \to \Omega^1_A$, we construct the inverse map $\Omega^1_A \to M$. By definition of the differential module (see the property after the definition), it is enough to check that the map

$$\varphi\colon A \longrightarrow M \qquad \begin{array}{l} x \mapsto x \otimes x \quad (\text{if } x \in A^*) \\ x \mapsto (1+x) \otimes (1+x) \quad (\text{if } x \notin A^*) \end{array}$$

is a $\mathbb{Z}$-derivation. So, it is enough to check $\varphi(xy) = x\varphi(y) + y\varphi(x)$. We will show this in the case where both x and y are in the maximal ideal of A. The remaining cases are easier, and are left to the reader. By definition, $x\varphi(y) + y\varphi(x)$ is the class of

$$\begin{aligned} & x(1+y) \otimes (1+y) + y(1+x) \otimes (1+x) \\ &= (1+x)(1+y) \otimes (1+y) - (1+y) \otimes (1+y) \\ &\quad + (1+y)(1+x) \otimes (1+x) - (1+x) \otimes (1+x) \\ &= (1+x)(1+y) \otimes (1+x)(1+y) - (1+x) \otimes (1+x) \\ &\quad - (1+y) \otimes (1+y). \end{aligned}$$

But the class of this element in M is the same as the class of $(1+xy) \otimes (1+xy)$. Thus, φ is a derivation. This completes the proof of the lemma. □

By this lemma, we can regard Ω^1_A as a group defined by the following generators: symbols $[a, b\}$ for $a \in A$ and $b \in A^*$
and relations:

$$\begin{array}{l} [a_1 + a_2, b\} = [a_1, b\} + [a_2, b\} \\ [a, b_1 b_2\} = [a, b_1\} + [a_2, b_2\} \\ \displaystyle\sum_{i=1}^{k} [a_i, a_i\} = \sum_{i=1}^{l} [b_i, b_i\} \quad \text{where } a_i\text{'s and } b_i\text{'s satisfy} \quad \sum_{i=1}^{k} a_i = \sum_{i=1}^{l} b_i. \end{array}$$

A1.2. n-th differential forms.

Let A and B be commutative rings such that B is an A-algebra. For a positive integer $n > 0$, we define $\Omega^n_{B/A}$ by

$$\Omega^n_{B/A} = \bigwedge_B \Omega^1_{B/A}.$$

Then, d naturally defines an A-homomorphism $d\colon \Omega^n_{B/A} \to \Omega^{n+1}_{B/A}$, and we have a complex

$$\dots \longrightarrow \Omega^{n-1}_{B/A} \longrightarrow \Omega^n_{B/A} \longrightarrow \Omega^{n+1}_{B/A} \longrightarrow \dots$$

which we call the *de Rham complex*.

For a commutative ring A, which we regard as a $\mathbb{Z}$-module, we simply write Ω^n_A for $\Omega^n_{A/\mathbb{Z}}$. For a local ring A, by Lemma A1.1, we have $\Omega^n_A = \bigwedge^n_A((A \otimes A^*)/J)$, where J is the group as in the proof of Lemma A1.1. Therefore we obtain

Lemma. *If A is a local ring, we have a surjective homomorphism*

$$A \otimes (A^*)^{\otimes n} \longrightarrow \Omega^n_{A/\mathbb{Z}}$$

$$a \otimes b_1 \otimes \dots \otimes b_n \mapsto a\frac{db_1}{b_1} \wedge \dots \wedge \frac{db_n}{b_n}.$$

The kernel of this map is generated by elements of the form

$$\sum_{i=1}^{k}(a_i \otimes a_i \otimes b_1 \otimes \dots \otimes b_{n-1}) - \sum_{i=1}^{l}(b_i \otimes b_i \otimes b_1 \otimes \dots \otimes b_{n-1})$$

(where $\Sigma_{i=1}^k a_i = \Sigma_{i=1}^l b_i$)
and

$$a \otimes b_1 \otimes \dots \otimes b_n \quad \text{with } b_i = b_j \text{ for some } i \neq j\,.$$

A1.3. Galois cohomology of $\mathbb{Z}/p^n(r)$ for a field of characteristic $p > 0$.

Let F be a field of characteristic $p > 0$. We denote by F^{sep} the separable closure of F in an algebraic closure of F.

We consider Galois cohomology groups $H^q(F, -) := H^q(\mathrm{Gal}(F^{\mathrm{sep}}/F), -)$. For an integer $r \geqslant 0$, we define

$$H^q(F, \mathbb{Z}/p(r)) = H^{q-r}(\mathrm{Gal}(F^{\mathrm{sep}}/F), \Omega^r_{F^{\mathrm{sep}},\log})$$

where $\Omega^r_{F^{\mathrm{sep}},\log}$ is the logarithmic part of $\Omega^r_{F^{\mathrm{sep}}}$, namely the subgroup generated by $d\log a_1 \wedge \dots \wedge d\log a_r$ for all $a_i \in (F^{\mathrm{sep}})^*$.

We have an exact sequence (cf. [I, p.579])

$$0 \longrightarrow \Omega^r_{F^{\mathrm{sep}},\log} \longrightarrow \Omega^r_{F^{\mathrm{sep}}} \xrightarrow{\mathbf{F}-1} \Omega^r_{F^{\mathrm{sep}}}/d\Omega^{r-1}_{F^{\mathrm{sep}}} \longrightarrow 0$$

where $\mathbf{F}$ is the map

$$\mathbf{F}(a\frac{db_1}{b_1} \wedge \dots \wedge \frac{db_r}{b_r}) = a^p\frac{db_1}{b_1} \wedge \dots \wedge \frac{db_r}{b_r}.$$

Since $\Omega^r_{F^{\mathrm{sep}}}$ is an F-vector space, we have

$$H^n(F, \Omega^r_{F^{\mathrm{sep}}}) = 0$$

for any $n > 0$ and $r \geqslant 0$. Hence, we also have

$$H^n(F, \Omega^r_{F^{\mathrm{sep}}}/d\Omega^{r-1}_{F^{\mathrm{sep}}}) = 0$$

for $n > 0$. Taking the cohomology of the above exact sequence, we obtain

$$H^n(F, \Omega^r_{F^{\mathrm{sep}},\log}) = 0$$

for any $n \geqslant 2$. Further, we have an isomorphism

$$H^1(F, \Omega^r_{F^{\mathrm{sep}},\log}) = \operatorname{coker}(\Omega^r_F \xrightarrow{\mathbf{F}-1} \Omega^r_F/d\Omega^{r-1}_F)$$

and

$$H^0(F, \Omega^r_{F^{\mathrm{sep}},\log}) = \ker(\Omega^r_F \xrightarrow{\mathbf{F}-1} \Omega^r_F/d\Omega^{r-1}_F).$$

Lemma. *For a field F of characteristic $p > 0$ and $n > 0$, we have*

$$H^{n+1}(F, \mathbb{Z}/p(n)) = \operatorname{coker}(\Omega^n_F \xrightarrow{\mathbf{F}-1} \Omega^n_F/d\Omega^{n-1}_F)$$

and

$$H^n(F, \mathbb{Z}/p(n)) = \ker(\Omega^n_F \xrightarrow{\mathbf{F}-1} \Omega^n_F/d\Omega^{n-1}_F).$$

Furthermore, $H^n(F, \mathbb{Z}/p(n-1))$ is isomorphic to the group which has the following generators: symbols $[a, b_1, ..., b_{n-1}\}$ where $a \in F$, and $b_1, ..., b_{n-1} \in F^*$ and relations:

$$[a_1 + a_2, b_1, ..., b_{n-1}\} = [a_1, b_1, ..., b_{n-1}\} + [a_2, b_1, ..., b_{n-1}\}$$
$$[a, b_1, ..., b_i b_i', ...b_{n-1}\} = [a, b_1, ..., b_i, ...b_{n-1}\} + [a, b_1, ..., b_i', ...b_{n-1}\}$$
$$[a, a, b_2, ..., b_{n-1}\} = 0$$
$$[a^p - a, b_1, b_2, ..., b_{n-1}\} = 0$$
$$[a, b_1, ..., b_{n-1}\} = 0 \quad \textit{where } b_i = b_j \textit{ for some } i \neq j.$$

Proof. The first half of the lemma follows from the computation of $H^n(F, \Omega^r_{F^{\mathrm{sep}},\log})$ above and the definition of $H^q(F, \mathbb{Z}/p(r))$. Using

$$H^n(F, \mathbb{Z}/p(n-1)) = \operatorname{coker}(\Omega^{n-1}_F \xrightarrow{\mathbf{F}-1} \Omega^{n-1}_F/d\Omega^{n-2}_F)$$

and Lemma A1.2 we obtain the explicit description of $H^n(F, \mathbb{Z}/p(n-1))$. □

We sometimes use the notation $H^n_p(F)$ which is defined by

$$H^n_p(F) = H^n(F, \mathbb{Z}/p(n-1)).$$

Moreover, for any $i > 1$, we can define $\mathbb{Z}/p^i(r)$ by using the de Rham–Witt complexes instead of the de Rham complex. For a positive integer $i > 0$, following Illusie [I], define $H^q(F, \mathbb{Z}/p^i(r))$ by

$$H^q(F, \mathbb{Z}/p^i(r)) = H^{q-r}(F, W_i\Omega^r_{F^{\mathrm{sep}},\log})$$

where $W_i\Omega^r_{F^{\mathrm{sep}},\log}$ is the logarithmic part of $W_i\Omega^r_{F^{\mathrm{sep}}}$.

Though we do not give here the proof, we have the following explicit description of $H^n(F, \mathbb{Z}/p^i(n-1))$ using the same method as in the case of $i = 1$.

Lemma. *For a field F of characteristic $p > 0$ let $W_i(F)$ denote the ring of Witt vectors of length i, and let $\mathbf{F}: W_i(F) \to W_i(F)$ denote the Frobenius endomorphism. For any $n > 0$ and $i > 0$, $H^n(F, \mathbb{Z}/p^i(n-1))$ is isomorphic to the group which has the following*
generators: symbols $[a, b_1, ..., b_{n-1}\}$ where $a \in W_i(F)$, and $b_1, ..., b_{n-1} \in F^*$
and relations:

$$[a_1 + a_2, b_1, ..., b_{n-1}\} = [a_1, b_1, ..., b_{n-1}\} + [a_2, b_1, ..., b_{n-1}\}$$
$$[a, b_1, ..., b_j b_j', ...b_{n-1}\} = [a, b_1, ..., b_j, ...b_{n-1}\} + [a, b_1, ..., b_j', ...b_{n-1}\}$$
$$[(0, ..., 0, a, 0, ..., 0), a, b_2, ..., b_{n-1}\} = 0$$
$$[\mathbf{F}(a) - a, b_1, b_2, ..., b_{n-1}\} = 0$$
$$[a, b_1, ..., b_{n-1}\} = 0 \quad \text{where } b_j = b_k \text{ for some } j \neq k.$$

We sometimes use the notation
$$H^n_{p^i}(F) = H^n(F, \mathbb{Z}/p^i(n-1)).$$

A2. Bloch–Kato–Gabber's theorem (by I. Fesenko)

For a field k of characteristic p denote
$$\nu_n = \nu_n(k) = H^n(k, \mathbb{Z}/p(n)) = \ker(\wp: \Omega^n_k \to \Omega^n_k / d\Omega^{n-1}_k),$$
$$\wp = \mathbf{F} - 1: \left(a \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}\right) \mapsto (a^p - a) \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} + d\Omega^{n-1}_k.$$

Clearly, the image of the differential symbol
$$d_k: K_n(k)/p \to \Omega^n_k, \qquad \{a_1, \ldots, a_n\} \mapsto \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}$$
is inside $\nu_n(k)$. We shall sketch the proof of Bloch–Kato–Gabber's theorem which states that d_k is an isomorphism between $K_n(k)/p$ and $\nu_n(k)$.

A2.1. Surjectivity of the differential symbol $d_k: K_n(k)/p \to \nu_n(k)$.

It seems impossible to suggest a shorter proof than original Kato's proof in [K, §1]. We can argue by induction on n; the case of $n = 1$ is obvious, so assume $n > 1$.

Definitions–Properties.

(1) Let $\{b_i\}_{i \in I}$ be a p-base of k (I is an ordered set). Let S be the set of all strictly increasing maps
$$s: \{1, \ldots, n\} \to I.$$
For two maps $s, t: \{1, \ldots, n\} \to I$ write $s < t$ if $s(i) \leqslant t(i)$ for all i and $s(i) \neq t(i)$ for some i.

(2) Denote $d\log a := a^{-1}da$. Put

$$\omega_s = d\log b_{s(1)} \wedge \cdots \wedge d\log b_{s(n)}.$$

Then $\{\omega_s : s \in S\}$ is a basis of Ω^n_k over k.

(3) For a map $\theta\colon I \to \{0, 1, \ldots, p-1\}$ such that $\theta(i) = 0$ for almost all i set

$$b_\theta = \prod b_i^{\theta(i)}.$$

Then $\{b_\theta \omega_s\}$ is a basis of Ω^n_k over k^p.

(4) Denote by $\Omega^n_k(\theta)$ the k^p-vector space generated by $b_\theta\omega_s, s \in S$. Then $\Omega^n_k(0) \cap d\Omega^{n-1}_k = 0$. For an extension l of k, such that $k \supset l^p$, denote by $\Omega^n_{l/k}$ the module of relative differentials. Let $\{b_i\}_{i\in I}$ be a p-base of l over k. Define $\Omega^n_{l/k}(\theta)$ for a map $\theta\colon I \to \{0, 1, \ldots, p-1\}$ similarly to the previous definition. The cohomology group of the complex

$$\Omega^{n-1}_{l/k}(\theta) \to \Omega^n_{l/k}(\theta) \to \Omega^{n+1}_{l/k}(\theta)$$

is zero if $\theta \neq 0$ and is $\Omega^n_{l/k}(0)$ if $\theta = 0$.

We shall use *Cartier's theorem* (which can be more or less easily proved by induction on $|l : k|$): the sequence

$$0 \to l^*/k^* \to \Omega^1_{l/k} \to \Omega^1_{l/k}/dl$$

is exact, where the second map is defined as $b \mod k^* \to d\log b$ and the third map is the map $a\,d\log b \mapsto (a^p - a)d\log b + dl$.

Proposition. *Let $\Omega^n_k(<s)$ be the k-subspace of Ω^n_k generated by all ω_t for $s > t \in S$. Let $k^{p-1} = k$ and let a be a non-zero element of k. Let I be finite. Suppose that*

$$(a^p - a)\omega_s \in \Omega^n_k(<s) + d\Omega^{n-1}_k.$$

Then there are $v \in \Omega^n_k(<s)$ and

$$x_i \in k^p(\{b_j : j \leqslant s(i)\}) \quad \textit{for} \quad 1 \leqslant i \leqslant n$$

such that

$$a\omega_s = v + d\log x_1 \wedge \cdots \wedge d\log x_n.$$

Proof of the surjectivity of the differential symbol. First, suppose that $k^{p-1} = k$ and I is finite. Let $S = \{s_1, \ldots, s_m\}$ with $s_1 > \cdots > s_m$. Let $s_0\colon \{1, \ldots, n\} \to I$ be a map such that $s_0 > s_1$. Denote by A the subgroup of Ω^n_k generated by $d\log x_1 \wedge \cdots \wedge d\log x_n$. Then $A \subset \nu_n$. By induction on $0 \leqslant j \leqslant m$ using the proposition it is straightforward to show that $\nu_n \subset A + \Omega^n_k(<s_j)$, and hence $\nu_n = A$.

To treat the general case put $c(k) = \operatorname{coker}(k_n(k) \to \nu_n(k))$. Since every field is the direct limit of finitely generated fields and the functor c commutes with direct limits, it is sufficient to show that $c(k) = 0$ for a finitely generated field k. In particular, we may assume that k has a finite p-base. For a finite extension k' of k there is a commutative diagram

$$\begin{tikzcd} k_n(k') \arrow[r] \arrow[d, "N_{k'/k}"'] & \nu_n(k') \arrow[d, "\mathrm{Tr}_{k'/k}"'] \\ k_n(k) \arrow[r] & \nu_n(k). \end{tikzcd}$$

Hence the composite $c(k) \to c(k') \xrightarrow{\mathrm{Tr}_{k'/k}} c(k)$ is multiplication by $|k' : k|$. Therefore, if $|k' : k|$ is prime to p then $c(k) \to c(k')$ is injective.

Now pass from k to a field l which is the compositum of all l_i where $l_{i+1} = l_i(\sqrt[p-1]{l_{i-1}})$, $l_0 = k$. Then $l = l^{p-1}$. Since l/k is separable, l has a finite p-base and by the first paragraph of this proof $c(l) = 0$. The degree of every finite subextension in l/k is prime to p, and by the second paragraph of this proof we conclude $c(k) = 0$, as required. $\square$

Proof of Proposition. First we prove the following lemma which will help us later for fields satisfying $k^{p-1} = k$ to choose a specific p-base of k.

Lemma. *Let l be a purely inseparable extension of k of degree p and let $k^{p-1} = k$. Let $f: l \to k$ be a k-linear map. Then there is a non-zero $c \in l$ such that $f(c^i) = 0$ for all $1 \leqslant i \leqslant p-1$.*

Proof of Lemma. The l-space of k-linear maps from l to k is one-dimensional, hence $f = ag$ for some $a \in l$, where $g: l = k(b) \to \Omega^1_{l/k}/dl \xrightarrow{\sim} k$, $x \mapsto x d\log b \bmod dl$ for every $x \in l$. Let $\alpha = g d\log b$ generate the one-dimensional space $\Omega^1_{l/k}/dl$ over k. Then there is $h \in k$ such that $g^p d\log b - h\alpha \in dl$. Let $z \in k$ be such that $z^{p-1} = h$. Then $((g/z)^p - g/z) d\log b \in dl$ and by Cartier's theorem we deduce that there is $w \in l$ such that $(g/z) d\log b = d\log w$. Hence $\alpha = z d\log w$ and $\Omega^1_{l/k} = dl \cup k d\log l$.

If $f(1) = a d\log b \neq 0$, then $f(1) = g d\log c$ with $g \in k, c \in l^*$ and hence $f(c^i) = 0$ for all $1 \leqslant i \leqslant p-1$. $\square$

Now for $s: \{1, \dots, n\} \to I$ as in the statement of the Proposition denote

$$k_0 = k^p(\{b_i : i < s(1)\}), \quad k_1 = k^p(\{b_i : i \leqslant s(1)\}), \quad k_2 = k^p(\{b_i : i \leqslant s(n)\}).$$

Let $|k_2 : k_1| = p^r$.

Let $a = \sum_\theta x_\theta^p b_\theta$. Assume that $a \notin k_2$. Then let θ, j be such that $j > s(n)$ is the maximal index for which $\theta(j) \neq 0$ and $x_\theta \neq 0$.

$\Omega^n_k(\theta)$-projection of $(a^p - a)\omega_s$ is equal to $-x_\theta^p b_\theta \omega_s \in \Omega^n_k(<s)(\theta) + d\Omega^{n-1}_k(\theta)$. Log differentiating, we get

$$-x_\theta^p \Big(\sum_i \theta(i) d\log b_i\Big) b_\theta \wedge \omega_s \in d\Omega^n_k(<s)(\theta)$$

which contradicts $-x_\theta^p\theta(j)b_\theta d\log b_j\wedge\omega_s\notin d\Omega^n_k(<s)(\theta)$. Thus, $a\in k_2$.

Let $m(1)<\dots<m(r-n)$ be integers such that the union of m's and s's is equal to $[s(1),s(n)]\cap\mathbb{Z}$. Apply the Lemma to the linear map

$$f\colon k_1\to\Omega^r_{k_2/k_0}/d\Omega^{r-1}_{k_2/k_0}\xrightarrow{\sim}k_0,\quad b\mapsto ba\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}.$$

Then there is a non-zero $c\in k_1$ such that

$$c^ia\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}\in d\Omega^{r-1}_{k_2/k_0}\quad\text{for } 1\leqslant i\leqslant p-1\,.$$

Hence $\Omega^r_{k_2/k_0}(0)$-projection of $c^ia\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}$ for $1\leqslant i\leqslant p-1$ is zero.

If $c\in k_0$ then $\Omega^r_{k_2/k_0}(0)$-projection of $a\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}$ is zero. Due to the definition of k_0 we get

$$\beta=(a^p-a)\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}\in d\Omega^{r-1}_{k_2/k_0}.$$

Then $\Omega^r_{k_2/k_0}(0)$-projection of β is zero, and so is $\Omega^r_{k_2/k_0}(0)$-projection of

$$a^p\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)},$$

a contradiction. Thus, $c\notin k_0$.

From $dk_0\subset\sum_{i<s(1)}k^pdb_i$ we deduce $dk_0\wedge\Omega^{n-1}_k\subset\Omega^n_k(<s)$. Since $k_0(c)=k_0(b_{s(1)})$, there are $a_i\in k_0$ such that $b_{s(1)}=\sum_{i=0}^{p-1}a_ic^i$. Then

$$ad\log b_{s(1)}\wedge\dots\wedge d\log b_{s(n)}\equiv a'd\log b_{s(2)}\dots\wedge d\log b_{s(n)}\wedge d\log c\mod\Omega^n_k(<s).$$

Define $s'\colon\{1,\dots,n-1\}\to I$ by $s'(j)=s(j+1)$. Then

$$a\omega_s=v_1+a'\omega_{s'}\wedge d\log c\quad\text{with } v_1\in\Omega^n_k(<s)$$

and $c^ia'\omega_{s'}\wedge d\log c\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}\in d\Omega^{r-1}_{k_2/k_0}$. The set

$$I'=\{c\}\cup\{b_i: s(1)<i\leqslant s(n)\}$$

is a p-base of k_2/k_0. Since c^ia' for $1\leqslant i\leqslant p-1$ have zero $k_2(0)$-projection with respect to I', there are $a'_0\in k_0$, $a'_1\in\oplus_{\theta\neq0}k_1b'_\theta$ with $b'_\theta=\prod_{s(1)<i\leqslant s(n)}b_i^{\theta(i)}$ such that $a'=a'_0+a'_1$.

The image of $a\omega_s\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}$ with respect to the Artin–Schreier map belongs to $\Omega^r_{k_2/k_0}$ and so is

$$(a'^p-a')d\log c\wedge\omega_{s'}\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}$$

which is the image of

$$a'd\log c\wedge\omega_{s'}\wedge d\log b_{m(1)}\wedge\dots\wedge d\log b_{m(r-n)}.$$

Then $a'^p-a'_0$, as $k_0(0)$-projection of a'^p-a', is zero. So $a'-a'^p=a'_1$.

Note that $d(a'_1\omega_{s'})\wedge d\log c\in d\Omega^n_{k/k_0}(<s)=d\Omega^{n-1}_{k/k_0}(<s)\wedge d\log c$.

Hence $d(a_1'\omega_{s'}) \in d\Omega^{n-1}_{k/k_0}(<s) + d\log c \wedge d\Omega^{n-2}_{k/k_0}$. Therefore $d(a_1'\omega_{s'}) \in d\Omega^{n-1}_{k/k_1}(<s)$ and $a_1'\omega_{s'} = \alpha + \beta$ with $\alpha \in \Omega^{n-1}_{k/k_1}(<s)$, $\beta \in \ker(d\colon \Omega^{n-1}_{k/k_1} \to \Omega^n_{k/k_1})$.

Since $k(0)$-projection of a_1' is zero, $\Omega^{n-1}_{k/k_1}(0)$-projection of $a_1'\omega_{s'}$ is zero. Then we deduce that $\beta(0) = \sum_{x_t\in k_1, t<s'} x_t^p\omega_t$, so $a_1'\omega_{s'} = \alpha + \beta(0) + (\beta - \beta(0))$. Then $\beta - \beta(0) \in \ker(d\colon \Omega^{n-1}_{k/k_1} \to \Omega^n_{k/k_1})$, so $\beta - \beta(0) \in d\Omega^{n-2}_{k/k_1}$. Hence $(a' - a'^p)\omega_{s'} = a_1'\omega_{s'}$ belongs to $\Omega^{n-1}_{k/k_1}(<s') + d\Omega^{n-2}_k$. By induction on n, there are $v' \in \Omega^{n-1}_k(<s')$, $x_i \in k^p\{b_j : j \leqslant s(i)\}$ such that $a'\omega_{s'} = v' + d\log x_2 \wedge \dots \wedge d\log x_n$. Thus, $a\omega_s = v_1 \pm d\log c \wedge v' \pm d\log c \wedge d\log x_2 \wedge \dots \wedge d\log x_n$. □

A2.2. Injectivity of the differential symbol.

We can assume that k is a finitely generated field over $\mathbb{F}_p$. Then there is a finitely generated algebra over $\mathbb{F}_p$ with a local ring being a discrete valuation ring $\mathcal{O}$ such that $\mathcal{O}/\mathcal{M}$ is isomorphic to k and the field of fractions E of $\mathcal{O}$ is purely transcendental over $\mathbb{F}_p$.

Using standard results on $K_n(l(t))$ and $\Omega^n_{l(t)}$ one can show that the injectivity of d_l implies the injectivity of $d_{l(t)}$. Since $d_{\mathbb{F}_p}$ is injective, so is d_E.

Define $k_n(\mathcal{O}) = \ker(k_n(E) \to k_n(k))$. Then $k_n(\mathcal{O})$ is generated by symbols and there is a homomorphism

$$k_n(\mathcal{O}) \to k_n(k), \quad \{a_1, \dots, a_n\} \to \{\overline{a_1}, \dots, \overline{a_n}\},$$

where $\bar{a}$ is the residue of a. Let $k_n(\mathcal{O}, \mathcal{M})$ be its kernel.

Define $\nu_n(\mathcal{O}) = \ker(\Omega^n_{\mathcal{O}} \to \Omega^n_{\mathcal{O}}/d\Omega^{n-1}_{\mathcal{O}})$, $\nu_n(\mathcal{O}, \mathcal{M}) = \ker(\nu_n(\mathcal{O}) \to \nu_n(k))$. There is a homomorphism $k_n(\mathcal{O}) \to \nu_n(\mathcal{O})$ such that

$$\{a_1, \dots, a_n\} \mapsto d\log a_1 \wedge \dots \wedge d\log a_n.$$

So there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & k_n(\mathcal{O},\mathcal{M}) \arrow[r] \arrow[d, "\varphi"'] & k_n(\mathcal{O}) \arrow[r] \arrow[d] & k_n(k) \arrow[r] \arrow[d, "d_k"'] & 0 \\
0 \arrow[r] & \nu_n(\mathcal{O},\mathcal{M}) \arrow[r] & \nu_n(\mathcal{O}) \arrow[r] & \nu_n(k) & .
\end{tikzcd}$$

Similarly to A2.1 one can show that φ is surjective [BK, Prop. 2.4]. Thus, d_k is injective. □

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4. Cohomological symbol for henselian discrete valuation fields of mixed characteristic

Jinya Nakamura

4.1. Cohomological symbol map

Let K be a field. If m is prime to the characteristic of K, there exists an isomorphism

$$h_{1,K}: K^*/m \to H^1(K, \mu_m)$$

supplied by Kummer theory. Taking the cup product we get

$$(K^*/m)^q \to H^q(K, \mathbb{Z}/m(q))$$

and this factors through (by [T])

$$h_{q,K}: K_q(K)/m \to H^q(K, \mathbb{Z}/m(q)).$$

This is called the cohomological symbol or norm residue homomorphism.

Milnor–Bloch–Kato Conjecture. *For every field K and every positive integer m which is prime to the characteristic of K the homomorphism $h_{q,K}$ is an isomorphism.*

This conjecture is shown to be true in the following cases:

(i) K is an algebraic number field or a function field of one variable over a finite field and $q = 2$, by Tate [T].

(ii) Arbitrary K and $q = 2$, by Merkur'ev and Suslin [MS1].

(iii) $q = 3$ and m is a power of 2, by Rost [R], independently by Merkur'ev and Suslin [MS2].

(iv) K is a henselian discrete valuation field of mixed characteristic $(0, p)$ and m is a power of p, by Bloch and Kato [BK].

(v) (K, q) arbitrary and m is a power of 2, by Voevodsky [V].

For higher dimensional local fields theory Bloch–Kato's theorem is very important and the aim of this text is to review its proof.

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Theorem (Bloch–Kato). *Let K be a henselian discrete valuation fields of mixed characteristic $(0,p)$ (i.e., the characteristic of K is zero and that of the residue field of K is $p > 0$), then*

$$h_{q,K}\colon K_q(K)/p^n \longrightarrow H^q(K, \mathbb{Z}/p^n(q))$$

is an isomorphism for all n.

Till the end of this section let K be as above, $k = k_K$ the residue field of K.

4.2. Filtration on $K_q(K)$

Fix a prime element π of K.

Definition.

$$U_m K_q(K) = \begin{cases} K_q(K), & m = 0 \\ \langle \{1 + \mathcal{M}_K^m\} \cdot K_{q-1}(K) \rangle, & m > 0. \end{cases}$$

Put $\mathrm{gr}_m K_q(K) = U_m K_q(K)/U_{m+1} K_q(K)$.

Then we get an isomorphism by [FV, Ch. IX sect. 2]

$$K_q(k) \oplus K_{q-1}(k) \xrightarrow{\rho_0} \mathrm{gr}_0 K_q(K)$$
$$\rho_0\left(\{x_1, \dots, x_q\}, \{y_1, \dots, y_{q-1}\}\right) = \{\widetilde{x_1}, \dots, \widetilde{x_q}\} + \{\widetilde{y_1}, \dots, \widetilde{y_{q-1}}, \pi\}$$

where $\widetilde{x}$ is a lifting of x. This map ρ_0 depends on the choice of a prime element π of K.

For $m \geqslant 1$ there is a surjection

$$\Omega_k^{q-1} \oplus \Omega_k^{q-2} \xrightarrow{\rho_m} \mathrm{gr}_m K_q(K)$$

defined by

$$\left(x\frac{dy_1}{y_1} \wedge \dots \wedge \frac{dy_{q-1}}{y_{q-1}}, 0\right) \longmapsto \{1 + \pi^m \widetilde{x}, \widetilde{y_1}, \dots, \widetilde{y_{q-1}}\},$$
$$\left(0, x\frac{dy_1}{y_1} \wedge \dots \wedge \frac{dy_{q-2}}{y_{q-2}}\right) \longmapsto \{1 + \pi^m \widetilde{x}, \widetilde{y_1}, \dots, \widetilde{y_{q-2}}, \pi\}.$$

Definition.

$$k_q(K) = K_q(K)/p, h_q(K) = H^q(K, \mathbb{Z}/p(q)),$$
$$U_m k_q(K) = \mathrm{im}(U_m K_q(K)) \text{ in } k_q(K), \qquad U_m h^q(K) = h_{q,K}(U_m k_q(K)),$$
$$\mathrm{gr}_m h^q(K) = U_m h^q(K)/U_{m+1} h^q(K).$$

□

Proposition. *Denote* $\nu_q(k) = \ker(\Omega_k^q \xrightarrow{1-\mathrm{C}^{-1}} \Omega_k^q/d\Omega_k^{q-1})$ *where* C^{-1} *is the inverse Cartier operator:*

$$x\frac{dy_1}{y_1}\wedge\dots\wedge\frac{dy_q}{y_q} \longmapsto x^p\frac{dy_1}{y_1}\wedge\dots\wedge\frac{dy_q}{y_q}.$$

Put $e' = pe/(p-1)$*, where* $e = v_K(p)$*.*

(i) *There exist isomorphisms* $\nu_q(k) \to k_q(k)$ *for any* q*; and the composite map denoted by* $\widetilde{\rho}_0$

$$\widetilde{\rho}_0\colon \nu_q(k)\oplus\nu_{q-1}(k) \xrightarrow{\sim} k_q(k)\oplus k_{q-1}(k) \xrightarrow{\sim} \mathrm{gr}_0 k_q(K)$$

is also an isomorphism.

(ii) *If* $1 \leqslant m < e'$ *and* $p \nmid m$*, then* ρ_m *induces a surjection*

$$\widetilde{\rho}_m\colon \Omega_k^{q-1} \to \mathrm{gr}_m k_q(K).$$

(iii) *If* $1 \leqslant m < e'$ *and* $p \mid m$*, then* ρ_m *factors through*

$$\widetilde{\rho}_m\colon \Omega_k^{q-1}/Z_1^{q-1} \oplus \Omega_k^{q-2}/Z_1^{q-2} \to \mathrm{gr}_m k_q(K)$$

and $\widetilde{\rho}_m$ *is a surjection. Here we denote* $Z_1^q = Z_1\Omega_k^q = \ker(d\colon \Omega_k^q \to \Omega_k^{q+1})$*.*

(iv) *If* $m = e' \in \mathbb{Z}$*, then* $\rho_{e'}$ *factors through*

$$\widetilde{\rho}_{e'}\colon \Omega_k^{q-1}/(1+a\,\mathrm{C})Z_1^{q-1} \oplus \Omega_k^{q-2}/(1+a\,\mathrm{C})Z_1^{q-2} \to \mathrm{gr}_{e'} k_q(K)$$

and $\widetilde{\rho}_{e'}$ *is a surjection.*

Here a *is the residue class of* $p\pi^{-e}$*, and* C *is the Cartier operator*

$$x^p\frac{dy_1}{y_1}\wedge\dots\wedge\frac{dy_q}{y_q} \mapsto x\frac{dy_1}{y_1}\wedge\dots\wedge\frac{dy_q}{y_q}, \quad d\Omega_k^{q-1} \to 0.$$

(v) *If* $m > e'$*, then* $\mathrm{gr}_m k_q(K) = 0$*.*

Proof. (i) follows from Bloch–Gabber–Kato's theorem (subsection 2.4). The other claims follow from calculations of symbols. □

Definition. Denote the left hand side in the definition of $\widetilde{\rho}_m$ by G_m^q. We denote the composite map $G_m^q \xrightarrow{\widetilde{\rho}_m} \mathrm{gr}_m k_q(K) \xrightarrow{h_{q,K}} \mathrm{gr}_m h^q(K)$ by $\overline{\rho}_m$; the latter is also surjective.

4.3

In this and next section we outline the proof of Bloch–Kato's theorem.

4.3.1. Norm argument.

We may assume $\zeta_p \in K$ to prove Bloch–Kato's theorem.
Indeed, $|K(\zeta_p) : K|$ is a divisor of $p-1$ and therefore is prime to p. There exists a norm homomorphism $N_{L/K}: K_q(L) \to K_q(K)$ (see [BT, Sect. 5]) such that the following diagram is commutative:

$$\begin{array}{ccccc} K_q(K)/p^n & \longrightarrow & K_q(L)/p^n & \xrightarrow{N_{L/K}} & K_q(K)/p^n \\ \Big\downarrow h_{q,K} & & \Big\downarrow h_{q,L} & & \Big\downarrow h_{q,K} \\ H^q(K,\mathbb{Z}/p^n(q)) & \xrightarrow{\text{res}} & H^q(L,\mathbb{Z}/p^n(q)) & \xrightarrow{\text{cor}} & H^q(K,\mathbb{Z}/p^n(q)) \end{array}$$

where the left horizontal arrow of the top row is the natural map, and res (resp. cor) is the restriction (resp. the corestriction). The top row and the bottom row are both multiplication by $|L:K|$, thus they are isomorphisms. Hence the bijectivity of $h_{q,K}$ follows from the bijectivity of $h_{q,L}$ and we may assume $\zeta_p \in K$.

4.3.2. Tate's argument.

To prove Bloch–Kato's theorem we may assume that $n = 1$.
Indeed, consider the cohomological long exact sequence

$$\dots \to H^{q-1}(K,\mathbb{Z}/p(q)) \xrightarrow{\delta} H^q(K,\mathbb{Z}/p^{n-1}(q)) \xrightarrow{p} H^q(K,\mathbb{Z}/p^n(q)) \to \dots$$

which comes from the Bockstein sequence

$$0 \longrightarrow \mathbb{Z}/p^{n-1} \xrightarrow{p} \mathbb{Z}/p^n \xrightarrow{\text{mod } p} \mathbb{Z}/p \longrightarrow 0.$$

We may assume $\zeta_p \in K$, so $H^{q-1}(K,\mathbb{Z}/p(q)) \simeq h_{q-1}(K)$ and the following diagram is commutative (cf. [T, §2]):

$$\begin{array}{ccccccc} k_{q-1}(K) & \xrightarrow{\{*,\zeta_p\}} & K_q(K)/p^{n-1} & \xrightarrow{p} & K_q(K)/p^n & \xrightarrow{\text{mod } p} & k_q(K) \\ \Big\downarrow h_{q-1,K} & & h_{q,K}\Big\downarrow & & h_{q,K}\Big\downarrow & & h_{q,K}\Big\downarrow \\ h^{q-1}(K) & \xrightarrow{\cup\zeta_p} & H^q(K,\mathbb{Z}/p^{n-1}(q)) & \xrightarrow{p} & H^q(K,\mathbb{Z}/p^n(q)) & \xrightarrow{\text{mod } p} & h^q(K). \end{array}$$

The top row is exact except at $K_q(K)/p^{n-1}$ and the bottom row is exact. By induction on n, we have only to show the bijectivity of $h_{q,K}: k_q(K) \to h^q(K)$ for all q in order to prove Bloch–Kato's theorem.

4.4. Bloch–Kato's Theorem

We review the proof of Bloch–Kato's theorem in the following four steps.

I $\quad \overline{\rho}_m\colon \mathrm{gr}_m k_q(K) \to \mathrm{gr}_m h^q(K)$ is injective for $1 \leqslant m < e'$.

II $\quad \overline{\rho}_0\colon \mathrm{gr}_0 k_q(K) \to \mathrm{gr}_0 h^q(K)$ is injective.

III $\quad h^q(K) = U_0 h^q(K)$ if k is separably closed.

IV $\quad h^q(K) = U_0 h^q(K)$ for general k.

4.4.1. Step I.

Injectivity of $\overline{\rho}_m$ is preserved by taking inductive limit of k. Thus we may assume k is finitely generated over $\mathbb{F}_p$ of transcendence degree $r < \infty$. We also assume $\zeta_p \in K$. Then we get

$$\mathrm{gr}_{e'} h^{r+2}(K) = U_{e'} h^{r+2}(K) \neq 0.$$

For instance, if $r = 0$, then K is a local field and $U_{e'} h^2(K) = {}_p\mathrm{Br}(K) = \mathbb{Z}/p$. If $r \geqslant 1$, one can use a cohomological residue to reduce to the case of $r = 0$. For more details see [K1, Sect. 1.4] and [K2, Sect. 3].

For $1 \leqslant m < e'$, consider the following diagram:

$$\begin{array}{ccc}
G_m^q \times G_{e'-m}^{r+2-q} & \xrightarrow{\overline{\rho}_m \times \overline{\rho}_{e'-m}} & \mathrm{gr}_m h^q(K) \oplus \mathrm{gr}_{e'-m} h^{r+2-q}(K) \\
{\scriptstyle \varphi_m}\big\downarrow & & \big\downarrow{\scriptstyle \text{cup product}} \\
\Omega_k^r / d\Omega_k^{r-1} \to G_{e'}^{r+2} & \xrightarrow{\overline{\rho}_{e'}} & \mathrm{gr}_{e'} h^{r+2}(K)
\end{array}$$

where φ_m is, if $p \nmid m$, induced by the wedge product $\Omega_k^{q-1} \times \Omega_k^{r+1-q} \to \Omega_k^r / d\Omega_k^{r-1}$, and if $p \mid m$,

$$\frac{\Omega_k^{q-1}}{Z_1^{q-1}} \oplus \frac{\Omega_k^{q-2}}{Z_1^{q-2}} \times \frac{\Omega_k^{r+1-q}}{Z_1^{r+1-q}} \oplus \frac{\Omega_k^{r-q}}{Z_1^{r-q}} \xrightarrow{\varphi_m} \Omega_k^q / d\Omega_k^{q-1}$$

$$(x_1, x_2, y_1, y_2) \longmapsto x_1 \wedge dy_2 + x_2 \wedge dy_1,$$

and the first horizontal arrow of the bottom row is the projection

$$\Omega_k^q / d\Omega_k^{q-1} \longrightarrow \Omega_k^r / (1 + a\,\mathrm{C}) Z_1^r = G_{e'}^{r+2}$$

since $\Omega_k^{r+1} = 0$ and $d\Omega_k^{q-1} \subset (1 + a\,\mathrm{C}) Z_1^r$. The diagram is commutative, $\Omega_k^r / d\Omega_k^{r-1}$ is a one-dimensional k^p-vector space and φ_m is a perfect pairing, the arrows in the bottom row are both surjective and $\mathrm{gr}_{e'} h^{r+2}(K) \neq 0$, thus we get the injectivity of $\overline{\rho}_m$.

4.4.2. Step II.

Let K' be a henselian discrete valuation field such that $K \subset K'$, $e(K'|K) = 1$ and $k_{K'} = k(t)$ where t is an indeterminate. Consider

$$\mathrm{gr}_0 h_q(K) \xrightarrow{\cup 1+\pi t} \mathrm{gr}_1 h^{q+1}(K').$$

The right hand side is equal to $\Omega^q_{k(t)}$ by (I). Let ψ be the composite

$$\nu_q(k) \oplus \nu_{q-1}(k) \xrightarrow{\overline{\rho}_0} \mathrm{gr}_0 h^q(K) \xrightarrow{\cup 1+\pi t} \mathrm{gr}_1 h^{q+1}(K') \simeq \Omega^q_{k(t)}.$$

Then

$$\psi\left(\frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_q}{x_q}, 0\right) = t\frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_q}{x_q},$$
$$\psi\left(0, \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_{q-1}}{x_{q-1}}\right) = \pm dt \wedge \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_{q-1}}{x_{q-1}}.$$

Since t is transcendental over k, ψ is an injection and hence $\overline{\rho}_0$ is also an injection.

4.4.3. Step III.

Denote $sh^q(K) = U_0 h^q(K)$ (the letter s means the symbolic part) and put

$$C(K) = h^q(K)/sh^q(K).$$

Assume $q \geqslant 2$. The purpose of this step is to show $C(K) = 0$. Let $\widetilde{K}$ be a henselian discrete valuation field with algebraically closed residue field $k_{\widetilde{K}}$ such that $K \subset \widetilde{K}$, $k \subset k_{\widetilde{K}}$ and the valuation of K is the induced valuation from $\widetilde{K}$. By Lang [L], $\widetilde{K}$ is a C_1-field in the terminology of [S]. This means that the cohomological dimension of $\widetilde{K}$ is one, hence $C(\widetilde{K}) = 0$. If the restriction $C(K) \to C(\widetilde{K})$ is injective then we get $C(K) = 0$. To prove this, we only have to show the injectivity of the restriction $C(K) \to C(L)$ for any $L = K(b^{1/p})$ such that $b \in \mathcal{O}^*_K$ and $\overline{b} \notin k^p_K$.

We need the following lemmas.

Lemma 1. *Let K and L be as above. Let $G = \mathrm{Gal}(L/K)$ and let $sh^q(L)^G$ (resp. $sh^q(L)_G$) be G-invariants (resp. G-coinvariants). Then*

(i) $sh^q(K) \xrightarrow{\mathrm{res}} sh^q(L)^G \xrightarrow{\mathrm{cor}} sh^q(K)$ *is exact.*

(ii) $sh^q(K) \xrightarrow{\mathrm{res}} sh^q(L)_G \xrightarrow{\mathrm{cor}} sh^q(K)$ *is exact.*

Proof. A nontrivial calculation with symbols, for more details see ([BK, Prop. 5.4]. □

Lemma 2. *Let K and L be as above. The following conditions are equivalent:*

(i) $h^{q-1}(K) \xrightarrow{\mathrm{res}} h^{q-1}(L)_G \xrightarrow{\mathrm{cor}} h^{q-1}(K)$ *is exact.*

(ii) $h^{q-1}(K) \xrightarrow{\cup b} h^q(K) \xrightarrow{\mathrm{res}} h^q(L)$ *is exact.*

Proof. This is a property of the cup product of Galois cohomologies for L/K. For more details see [BK, Lemma 3.2]. $\square$

By induction on q we assume $sh^{q-1}(K) = h^{q-1}(K)$. Consider the following diagram with exact rows:

$$\begin{tikzcd}
 & & h^{q-1}(K) \arrow[d, "\cup b"'] & & \\
0 \arrow[r] & sh^q(K) \arrow[r] \arrow[d, "\mathrm{res}"'] & h^q(K) \arrow[r] \arrow[d, "\mathrm{res}"'] & C(K) \arrow[r] \arrow[d, "\mathrm{res}"'] & 0 \\
0 \arrow[r] & sh^q(L)^G \arrow[r] \arrow[d, "\mathrm{cor}"'] & h^q(L)^G \arrow[r] \arrow[d, "\mathrm{cor}"'] & C(L)^G & \\
0 \arrow[r] & sh^q(K) \arrow[r] & h^q(K). & &
\end{tikzcd}$$

By Lemma 1 (i) the left column is exact. Furthermore, due to the exactness of the sequence of Lemma 1 (ii) and the inductional assumption we have an exact sequence

$$h^{q-1}(K) \xrightarrow{\mathrm{res}} h^{q-1}(L)_G \longrightarrow h^{q-1}(K).$$

So by Lemma 2

$$h^{q-1}(K) \xrightarrow{\cup b} h^q(K) \xrightarrow{\mathrm{res}} h^q(L)$$

is exact. Thus, the upper half of the middle column is exact. Note that the lower half of the middle column is at least a complex because the composite map $\mathrm{cor} \circ \mathrm{res}$ is equal to multiplication by $|L : K| = p$. Chasing the diagram, one can deduce that all elements of the kernel of $C(K) \to C(L)^G$ come from $h^{q-1}(K)$ of the top group of the middle column. Now $h^{q-1}(K) = sh^{q-1}(K)$, and the image of

$$sh^{q-1}(K) \xrightarrow{\cup b} h^q(K)$$

is also included in the symbolic part $sh^q(K)$ in $h^q(K)$. Hence $C(K) \to C(L)^G$ is an injection. The claim is proved.

4.4.4. Step IV.

We use the Hochschild–Serre spectral sequence

$$H^r(G_k, h^q(K_{\mathrm{ur}})) \Longrightarrow h^{q+r}(K).$$

For any q,

$$\Omega^q_{k^{\mathrm{sep}}} \simeq \Omega^q_k \otimes_k k^{\mathrm{sep}}, \qquad Z_1\Omega^q_{k^{\mathrm{sep}}} \simeq Z_1\Omega^q_k \otimes_{k^p} (k^{\mathrm{sep}})^p.$$

Thus, $\mathrm{gr}_m h^q(K_{\mathrm{ur}}) \simeq \mathrm{gr}_m h^q(K) \otimes_{k^p} (k^{\mathrm{sep}})^p$ for $1 \leqslant m < e'$. This is a direct sum of copies of k^{sep}, hence we have

$$\begin{aligned} &H^0(G_k, U_1 h^q(K_{\mathrm{ur}})) \simeq U_1 h^q(K)/U_{e'} h^q(K),\\ &H^r(G_k, U_1 h^q(K_{\mathrm{ur}})) = 0 \end{aligned}$$

for $r \geqslant 1$ because $H^r(G_k, k^{\mathrm{sep}}) = 0$ for $r \geqslant 1$. Furthermore, taking cohomologies of the following two exact sequences

$$0 \longrightarrow U_1 h^q(K_{\mathrm{ur}}) \longrightarrow h^q(K_{\mathrm{ur}}) \longrightarrow \nu^q_{k^{\mathrm{sep}}} \oplus \nu^{q-1}_{k^{\mathrm{sep}}} \longrightarrow 0,$$

$$0 \longrightarrow \nu^q_{k^{\mathrm{sep}}} \xrightarrow{\mathrm{C}} Z_1\Omega^q_{k^{\mathrm{sep}}} \xrightarrow{1-\mathrm{C}^{-1}} \Omega^q_{k^{\mathrm{sep}}} \longrightarrow 0,$$

we have

$$\begin{aligned} H^0(G_k, h^q(K_{\mathrm{ur}})) &\simeq s h^q(K)/U_{e'} h^q(K) \simeq k^q(K)/U_{e'} k^q(K),\\ H^1(G_k, h^q(K_{\mathrm{ur}})) &\simeq H^1(G_k, \nu^q_{k^{\mathrm{sep}}} \oplus \nu^{q-1}_{k^{\mathrm{sep}}})\\ &\simeq (\Omega^q_k/(1-\mathrm{C})Z_1\Omega^q_k) \oplus (\Omega^{q-1}_k/(1-\mathrm{C})Z_1\Omega^{q-1}_k),\\ H^r(G_k, h^q(K_{\mathrm{ur}})) &= 0 \end{aligned}$$

for $r \geqslant 2$, since the cohomological p-dimension of G_k is less than or equal to one (cf. [S, II-2.2]). By the above spectral sequence, we have the following exact sequence

$$\begin{aligned} 0 &\longrightarrow (\Omega^{q-1}_k/(1-\mathrm{C})Z^{q-1}_1) \oplus (\Omega^{q-2}_k/(1-\mathrm{C})Z^{q-2}_1) \longrightarrow h^q(K)\\ &\longrightarrow k_q(K)/U_{e'} k_q(K) \longrightarrow 0. \end{aligned}$$

Multiplication by the residue class of $(1-\zeta_p)^p/\pi^{e'}$ gives an isomorphism

$$\begin{aligned} &(\Omega^{q-1}_k/(1-\mathrm{C})Z^{q-1}_1) \oplus (\Omega^{q-2}_k/(1-\mathrm{C})Z^{q-2}_1)\\ &\longrightarrow (\Omega^{q-1}_k/(1+a\,\mathrm{C})Z^{q-1}_1) \oplus (\Omega^{q-2}_k/(1+a\,\mathrm{C})Z^{q-2}_1) = \mathrm{gr}_{e'} k_q(K), \end{aligned}$$

hence we get $h^q(K) \simeq k_q(K)$.

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5. Kato's higher local class field theory

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5.0. Introduction

We first recall the classical local class field theory. Let K be a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_q((X))$. The main theorem of local class field theory consists of the isomorphism theorem and existence theorem. In this section we consider the isomorphism theorem.

An outline of one of the proofs is as follows. First, for the Brauer group $\mathrm{Br}(K)$, an isomorphism

$$\mathrm{inv}\colon \mathrm{Br}(K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

is established; it mainly follows from an isomorphism

$$H^1(F, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

where F is the residue field of K.

Secondly, we denote by $X_K = \mathrm{Hom}_{\mathrm{cont}}(G_K, \mathbb{Q}/\mathbb{Z})$ the group of continuous homomorphisms from $G_K = \mathrm{Gal}(\overline{K}/K)$ to $\mathbb{Q}/\mathbb{Z}$. We consider a pairing

$$K^* \times X_K \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$(a, \chi) \mapsto \mathrm{inv}(\chi, a)$$

where (χ, a) is the cyclic algebra associated with χ and a. This pairing induces a homomorphism

$$\Psi_K\colon K^* \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K) = \mathrm{Hom}(X_K, \mathbb{Q}/\mathbb{Z})$$

which is called the reciprocity map.

Thirdly, for a finite abelian extension L/K, we have a diagram

$$\begin{array}{ccc} L^* & \xrightarrow{\Psi_L} & \mathrm{Gal}(L^{\mathrm{ab}}/L) \\ {\scriptstyle N}\downarrow & & \downarrow \\ K^* & \xrightarrow{\Psi_K} & \mathrm{Gal}(K^{\mathrm{ab}}/K) \end{array}$$

which is commutative by the definition of the reciprocity maps. Here, N is the norm map and the right vertical map is the canonical map. This induces a homomorphism

$$\Psi_{L/K}\colon K^*/NL^* \longrightarrow \mathrm{Gal}(L/K).$$

The isomorphism theorem tells us that the above map is bijective.

To show the bijectivity of $\Psi_{L/K}$, we can reduce to the case where $|L:K|$ is a prime ℓ. In this case, the bijectivity follows immediately from a famous exact sequence

$$L^* \xrightarrow{N} K^* \xrightarrow{\cup\chi} \mathrm{Br}(K) \xrightarrow{\mathrm{res}} \mathrm{Br}(L)$$

for a cyclic extension L/K (where $\cup\chi$ is the cup product with χ, and res is the restriction map).

In this section we sketch a proof of the isomorphism theorem for a higher dimensional local field as an analogue of the above argument. For the existence theorem see the paper by Kato in this volume and subsection 10.5.

5.1. Definition of $H^q(k)$

In this subsection, for any field k and $q > 0$, we recall the definition of the cohomology group $H^q(k)$ ([K2], see also subsections 2.1 and 2.2 and A1 in the appendix to section 2). If $\mathrm{char}\,(k) = 0$, we define $H^q(k)$ as a Galois cohomology group

$$H^q(k) = H^q(k, \mathbb{Q}/\mathbb{Z}(q-1))$$

where $(q-1)$ is the $(q-1)$ st Tate twist.

If $\mathrm{char}\,(k) = p > 0$, then following Illusie [I] we define

$$H^q(k, \mathbb{Z}/p^n(q-1)) = H^1(k, W_n\Omega^{q-1}_{k^{\mathrm{sep}},\log}).$$

We can explicitly describe $H^q(k, \mathbb{Z}/p^n(q-1))$ as the group isomorphic to

$$W_n(k) \otimes (k^*)^{\otimes(q-1)}/J$$

where $W_n(k)$ is the ring of Witt vectors of length n, and J is the subgroup generated by elements of the form

$w \otimes b_1 \otimes \cdots \otimes b_{q-1}$ such that $b_i = b_j$ for some $i \neq j$, and
$(0, \dots, 0, a, 0, \dots, 0) \otimes a \otimes b_1 \otimes \cdots \otimes b_{q-2}$, and
$(\mathbf{F} - 1)(w) \otimes b_1 \otimes \cdots \otimes b_{q-1}$ ($\mathbf{F}$ is the Frobenius map on Witt vectors).

We define $H^q(k, \mathbb{Q}_p/\mathbb{Z}_p(q-1)) = \varinjlim H^q(k, \mathbb{Z}/p^n(q-1))$, and define

$$H^q(k) = \bigoplus_{\ell} H^q(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(q-1))$$

where ℓ ranges over all prime numbers. (For $\ell \neq p$, the right hand side is the usual Galois cohomology of the $(q-1)$ st Tate twist of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$.)

Then for any k we have

$$H^1(k) = X_k \qquad (X_k \text{ is as in 5.0, the group of characters}),$$
$$H^2(k) = \mathrm{Br}(k) \qquad (\text{Brauer group}).$$

We explain the second equality in the case of $\mathrm{char}(k) = p > 0$. The relation between the Galois cohomology group and the Brauer group is well known, so we consider only the p-part. By our definition,

$$H^2(k, \mathbb{Z}/p^n(1)) = H^1(k, W_n\Omega^1_{k^{\mathrm{sep}},\log}).$$

From the bijectivity of the differential symbol (Bloch–Gabber–Kato's theorem in subsection A2 in the appendix to section 2), we have

$$H^2(k, \mathbb{Z}/p^n(1)) = H^1(k, (k^{\mathrm{sep}})^*/((k^{\mathrm{sep}})^*)^{p^n}).$$

From the exact sequence

$$0 \longrightarrow (k^{\mathrm{sep}})^* \xrightarrow{p^n} (k^{\mathrm{sep}})^* \longrightarrow (k^{\mathrm{sep}})^*/((k^{\mathrm{sep}})^*)^{p^n} \longrightarrow 0$$

and an isomorphism $\mathrm{Br}(k) = H^2(k, (k^{\mathrm{sep}})^*)$, $H^2(k, \mathbb{Z}/p^n(1))$ is isomorphic to the p^n-torsion points of $\mathrm{Br}(k)$. Thus, we get $H^2(k) = \mathrm{Br}(k)$.

If K is a henselian discrete valuation field with residue field F, we have a canonical map

$$i_F^K \colon H^q(F) \longrightarrow H^q(K).$$

If $\mathrm{char}(K) = \mathrm{char}(F)$, this map is defined naturally from the definition of H^q (for the Galois cohomology part, we use a natural map $\mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow \mathrm{Gal}(K_{\mathrm{ur}}/K) = \mathrm{Gal}(F^{\mathrm{sep}}/F)$) . If K is of mixed characteristics $(0,p)$, the prime-to-p-part is defined naturally and the p-part is defined as follows. For the class $[w \otimes b_1 \otimes \cdots \otimes b_{q-1}]$ in $H^q(F, \mathbb{Z}/p^n(q-1))$ we define $i_F^K([w \otimes b_1 \otimes \cdots \otimes b_{q-1}])$ as the class of

$$i(w) \otimes \widetilde{b_1} \otimes \cdots \otimes \widetilde{b_{q-1}}$$

in $H^1(K, \mathbb{Z}/p^n(q-1))$, where $i\colon W_n(F) \to H^1(F, \mathbb{Z}/p^n) \to H^1(K, \mathbb{Z}/p^n)$ is the composite of the map given by Artin–Schreier–Witt theory and the canonical map, and $\widetilde{b_i}$ is a lifting of b_i to K.

Theorem (Kato [K2, Th. 3])**.** *Let K be a henselian discrete valuation field, π be a prime element, and F be the residue field. We consider a homomorphism*

$$i = (i_F^K, i_F^K \cup \pi)\colon H^q(F) \oplus H^{q-1}(F) \longrightarrow H^q(K)$$
$$(a, b) \mapsto i_F^K(a) + i_F^K(b) \cup \pi$$

where $i_F^K(b) \cup \pi$ is the element obtained from the pairing

$$H^{q-1}(K) \times K^* \longrightarrow H^q(K)$$

which is defined by Kummer theory and the cup product, and the explicit description of $H^q(K)$ *in the case of* $\operatorname{char}(K) > 0$. *Suppose* $\operatorname{char}(F) = p$. *Then* i *is bijective in the prime-to-* p *component. In the* p*-component,* i *is injective and the image coincides with the* p*-component of the kernel of* $H^q(K) \to H^q(K_{\mathrm{ur}})$ *where* K_{ur} *is the maximal unramified extension of* K.

From this theorem and Bloch–Kato's theorem in section 4, we obtain

Corollary. *Assume that* $\operatorname{char}(F) = p > 0$, $|F : F^p| = p^{d-1}$, *and that there is an isomorphism* $H^d(F) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$.

Then, i *induces an isomorphism*

$$H^{d+1}(K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

A typical example which satisfies the assumptions of the above corollary is a d-dimensional local field (if the last residue field is quasi-finite (not necessarily finite), the assumptions are satisfied).

5.2. Higher dimensional local fields

We assume that K is a d-dimensional local field, and F is the residue field of K, which is a $(d-1)$-dimensional local field. Then, by the corollary in the previous subsection and induction on d, there is a canonical isomorphism

$$\operatorname{inv}: H^{d+1}(K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

This corresponds to the first step of the proof of the classical isomorphism theorem which we described in the introduction.

The cup product defines a pairing

$$K_d(K) \times H^1(K) \longrightarrow H^{d+1}(K) \simeq \mathbb{Q}/\mathbb{Z}.$$

This pairing induces a homomorphism

$$\Psi_K : K_d(K) \longrightarrow \operatorname{Gal}(K^{\mathrm{ab}}/K) \simeq \operatorname{Hom}(H^1(K), \mathbb{Q}/\mathbb{Z})$$

which we call the *reciprocity map*. Since the isomorphism $\operatorname{inv}: H^d(K) \longrightarrow \mathbb{Q}/\mathbb{Z}$ is naturally constructed, for a finite abelian extension L/K we have a commutative diagram

$$\begin{array}{ccc} H^{d+1}(L) & \xrightarrow{\operatorname{inv}_L} & \mathbb{Q}/\mathbb{Z} \\ {\scriptstyle\mathrm{cor}}\downarrow & & \downarrow \\ H^{d+1}(K) & \xrightarrow{\operatorname{inv}_K} & \mathbb{Q}/\mathbb{Z}. \end{array}$$

So the diagram

$$\begin{array}{ccc} K_d(L) & \xrightarrow{\Psi_L} & \mathrm{Gal}(L^{\mathrm{ab}}/L) \\ {\scriptstyle N}\big\downarrow & & \big\downarrow \\ K_d(K) & \xrightarrow{\Psi_K} & \mathrm{Gal}(K^{\mathrm{ab}}/K) \end{array}$$

is commutative where N is the norm map and the right vertical map is the canonical map. So, as in the classical case, we have a homomorphism

$$\Psi_{L/K}\colon K_d(K)/NK_d(L) \longrightarrow \mathrm{Gal}(L/K).$$

Isomorphism Theorem. $\Psi_{L/K}$ *is an isomorphism.*

We outline a proof. We may assume that L/K is cyclic of degree ℓ. As in the classical case in the introduction, we may study a sequence

$$K_d(L) \xrightarrow{N} K_d(K) \xrightarrow{\cup\chi} H^{d+1}(K) \xrightarrow{\mathrm{res}} H^{d+1}(L),$$

but here we describe a more elementary proof.

First of all, using the argument in [S, Ch.5] by calculation of symbols one can obtain

$$|K_d(K) : NK_d(L)| \leqslant \ell.$$

We outline a proof of this inequality.

It is easy to see that it is sufficient to consider the case of prime ℓ. (For another calculation of the index of the norm group see subsection 6.7).

Recall that $K_d(K)$ has a filtration $U_mK_d(K)$ as in subsection 4.2. We consider $\mathrm{gr}_mK_d(K) = U_mK_d(K)/U_{m+1}K_d(K)$.

If L/K is unramified, the norm map $N\colon K_d(L) \to K_d(K)$ induces surjective homomorphisms $\mathrm{gr}_mK_d(L) \to \mathrm{gr}_mK_d(K)$ for all $m > 0$. So $U_1K_d(K)$ is in $NK_d(L)$. If we denote by F_L and F the residue fields of L and K respectively, the norm map induces a surjective homomorphism $K_d(F_L)/\ell \to K_d(F)/\ell$ because $K_d(F)/\ell$ is isomorphic to $H^d(F, \mathbb{Z}/\ell(d))$ (cf. sections 2 and 3) and the cohomological dimension of F [K2, p.220] is d. Since $\mathrm{gr}_0K_d(K) = K_d(F) \oplus K_{d-1}(F)$ (see subsection 4.2), the above implies that $K_d(K)/NK_d(L)$ is isomorphic to $K_{d-1}(F)/NK_{d-1}(F_L)$, which is isomorphic to $\mathrm{Gal}(F_L/F)$ by class field theory of F (we use induction on d). Therefore $|K_d(K) : NK_d(L)| = \ell$.

If L/K is totally ramified and ℓ is prime to $\mathrm{char}(F)$, by the same argument (cf. the argument in [S, Ch.5]) as above, we have $U_1K_d(K) \subset NK_d(L)$. Let π_L be a prime element of L, and $\pi_K = N_{L/K}(\pi_L)$. Then the element $\{\alpha_1, ..., \alpha_{d-1}, \pi_K\}$ for $\alpha_i \in K^*$ is in $NK_d(L)$, so $K_d(K)/NK_d(L)$ is isomorphic to $K_d(F)/\ell$, which is isomorphic to $H^d(F, \mathbb{Z}/\ell(d))$, so the order is ℓ. Thus, in this case we also have $|K_d(K) : NK_d(L)| = \ell$.

Hence, we may assume L/K is not unramified and is of degree $\ell = p = \mathrm{char}\,(F)$. Note that $K_d(F)$ is p-divisible because of $\Omega_F^d = 0$ and the bijectivity of the differential symbol.

Assume that L/K is totally ramified. Let π_L be a prime element of L, and σ a generator of $\mathrm{Gal}(L/K)$, and put $a = \sigma(\pi_L)\pi_L^{-1} - 1$, $b = N_{L/K}(a)$, and $v_K(b-1) = i$. We study the induced maps $\mathrm{gr}_{\psi(m)} K_d(L) \to \mathrm{gr}_m K_d(K)$ from the norm map N on the subquotients by the argument in [S, Ch.5]. We have $U_{i+1}K_d(K) \subset NK_d(L)$, and can show that there is a surjective homomorphism (cf. [K1, p.669])

$$\Omega_F^{d-1} \longrightarrow K_d(K)/NK_d(L)$$

such that

$$x d\log y_1 \wedge \ldots \wedge d\log y_{d-1} \mapsto \{1 + \widetilde{x}b, \widetilde{y_1}, \ldots, \widetilde{y_{d-1}}\}$$

($\widetilde{x}, \widetilde{y_i}$ are liftings of x and y_i). Furthermore, from

$$N_{L/K}(1 + xa) \equiv 1 + (x^p - x)b \pmod{U_{i+1}K^*},$$

the above map induces a surjective homomorphism

$$\Omega_F^{d-1}/\big((\mathbf{F} - 1)\Omega_F^{d-1} + d\Omega_F^{d-2}\big) \longrightarrow K_d(K)/NK_d(L).$$

The source group is isomorphic to $H^d(F, \mathbb{Z}/p(d-1))$ which is of order p. So we obtain $|K_d(K) : NK_d(L)| \leqslant p$.

Now assume that L/K is ferociously ramified, i.e. F_L/F is purely inseparable of degree p. We can use an argument similar to the previous one. Let h be an element of $\mathcal{O}_L$ such that $F_L = F(\overline{h})$ ($\overline{h} = h \mod \mathcal{M}_L$). Let σ be a generator of $\mathrm{Gal}(L/K)$, and put $a = \sigma(h)h^{-1} - 1$, and $b = N_{L/K}(a)$. Then we have a surjective homomorphism (cf. [K1, p.669])

$$\Omega_F^{d-1}/\big((\mathbf{F} - 1)\Omega_F^{d-1} + d\Omega_F^{d-2}\big) \longrightarrow K_d(K)/NK_d(L)$$

such that

$$x d\log y_1 \wedge \ldots \wedge d\log y_{d-2} \wedge d\log N_{F_L/F}(\overline{h}) \mapsto \{1 + \widetilde{x}b, \widetilde{y_1}, \ldots, \widetilde{y_{d-2}}, \pi\}$$

(π is a prime element of K). So we get $|K_d(K) : NK_d(L)| \leqslant p$.

So in order to obtain the bijectivity of $\Psi_{L/K}$, we have only to check the surjectivity. We consider the most interesting case $\mathrm{char}\,(K) = 0$, $\mathrm{char}\,(F) = p > 0$, and $\ell = p$. To show the surjectivity of $\Psi_{L/K}$, we have to show that there is an element $x \in K_d(K)$ such that $\chi \cup x \neq 0$ in $H^{d+1}(K)$ where χ is a character corresponding to L/K. We may assume a primitive p-th root of unity is in K. Suppose that L is given by an equation $X^p = a$ for some $a \in K \setminus K^p$. By Bloch–Kato's theorem (bijectivity of the cohomological symbols in section 4), we identify the kernel of multiplication by p on $H^{d+1}(K)$ with $H^{d+1}(K, \mathbb{Z}/p(d))$, and with $K_{d+1}(K)/p$. Then our aim is to show that there is an element $x \in K_d(K)$ such that $\{x, a\} \neq 0$ in $k_{d+1}(K) = K_{d+1}(K)/p$.

(Remark. The pairing $K_1(K)/p \times K_d(K)/p \to K_{d+1}(K)/p$ coincides up to a sign with Vostokov's symbol defined in subsection 8.3 and the latter is non-degenerate which provides an alternative proof).

We use the notation of section 4. By the Proposition in subsection 4.2, we have

$$K_{d+1}(K)/p = k_{d+1}(K) = U_{e'}k_{d+1}(K)$$

where $e' = v_K(p)p/(p-1)$. Furthermore, by the same proposition there is an isomorphism

$$H^d(F, \mathbb{Z}/p(d-1)) = \Omega_F^{d-1}/\big((\mathbf{F}-1)\Omega_F^{d-1} + d\Omega_F^{d-2}\big) \longrightarrow k_{d+1}(K)$$

such that

$$x d\log y_1 \wedge \ldots \wedge d\log y_{d-1} \mapsto \{1 + \widetilde{x}b, \widetilde{y_1}, \ldots, \widetilde{y_{d-1}}, \pi\}$$

where π is a uniformizer, and b is a certain element of K such that $v_K(b) = e'$. Note that $H^d(F, \mathbb{Z}/p(d-1))$ is of order p.

This shows that for any uniformizer π of K, and for any lifting $t_1, \ldots, t_{d-1}$ of a p-base of F, there is an element $x \in \mathcal{O}_K$ such that

$$\{1 + \pi^{e'}x, t_1, \ldots, t_{d-1}, \pi\} \neq 0$$

in $k_{d+1}(K)$.

If the class of a is not in $U_1k_1(K)$, we may assume a is a uniformizer or a is a part of a lifting of a p-base of F. So it is easy to see by the above property that there exists an x such that $\{a, x\} \neq 0$. If the class of a is in $U_{e'}k_1(K)$, it is also easily seen from the description of $U_{e'}k_{d+1}(K)$ that there exists an x such that $\{a, x\} \neq 0$.

Suppose $a \in U_ik_1(K) \setminus U_{i+1}k_1(K)$ such that $0 < i < e'$. We write $a = 1 + \pi^i a'$ for a prime element π and $a' \in \mathcal{O}_K^*$. First, we assume that p does not divide i. We use a formula (which holds in $K_2(K)$)

$$\{1-\alpha, 1-\beta\} = \{1-\alpha\beta, -\alpha\} + \{1-\alpha\beta, 1-\beta\} - \{1-\alpha\beta, 1-\alpha\}$$

for $\alpha \neq 0, 1$, and $\beta \neq 1, \alpha^{-1}$. From this formula we have in $k_2(K)$

$$\{1 + \pi^i a', 1 + \pi^{e'-i}b\} = \{1 + \pi^{e'}a'b, \pi^i a'\}$$

for $b \in \mathcal{O}_K$. So for a lifting $t_1, \ldots, t_{d-1}$ of a p-base of F we have

$$\begin{aligned}\{1 + \pi^i a', 1 + \pi^{e'-i}b, t_1, \ldots, t_{d-1}\} &= \{1 + \pi^{e'}a'b, \pi^i, t_1, \ldots, t_{d-1}\} \\ &= i\{1 + \pi^{e'}a'b, \pi, t_1, \ldots, t_{d-1}\}\end{aligned}$$

in $k_{d+1}(K)$ (here we used $\{1 + \pi^{e'}x, u_1, \ldots, u_d\} = 0$ for any units u_i in $k_{d+1}(K)$ which follows from $\Omega_F^d = 0$ and the calculation of the subquotients $\mathrm{gr}_m k_{d+1}(K)$ in subsection 4.2). So we can take $b \in \mathcal{O}_K$ such that the above symbol is non-zero in $k_{d+1}(K)$. This completes the proof in the case where i is prime to p.

Next, we assume p divides i. We also use the above formula, and calculate

$$\begin{aligned}\{1+\pi^i a', 1+(1+b\pi)\pi^{e'-i-1}, \pi\} &= \{1+\pi^{e'-1}a'(1+b\pi), 1+b\pi, \pi\}\\ &= \{1+\pi^{e'}a'b(1+b\pi), a'(1+b\pi), \pi\}.\end{aligned}$$

Since we may think of a' as a part of a lifting of a p-base of F, we can take some $x = \{1+(1+b\pi)\pi^{e'-i-1}, \pi, t_1, ..., t_{d-2}\}$ such that $\{a, x\} \neq 0$ in $k_{d+1}(K)$.

If ℓ is prime to $\mathrm{char}(F)$, for the extension L/K obtained by an equation $X^\ell = a$, we can find x such that $\{a, x\} \neq 0$ in $K_{d+1}(K)/\ell$ in the same way as above, using $K_{d+1}(K)/\ell = \mathrm{gr}_0 K_{d+1}(K)/\ell = K_d(F)/\ell$. In the case where $\mathrm{char}(K) = p > 0$ we can use Artin–Schreier theory instead of Kummer theory, and therefore we can argue in a similar way to the previous method. This completes the proof of the isomorphism theorem.

Thus, the isomorphism theorem can be proved by computing symbols, once we know Bloch–Kato's theorem. See also a proof in [K1].

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6. Topological Milnor K-groups of higher local fields

Ivan Fesenko

Let $F = K_n, \dots, K_0 = \mathbb{F}_q$ be an n-dimensional local field. We use the notation of section 1.

In this section we describe properties of certain quotients $K^{\mathrm{top}}(F)$ of the Milnor K-groups of F by using in particular topological considerations. This is an updated and simplified summary of relevant results in [F1–F5]. Subsection 6.1 recalls well-known results on K-groups of classical local fields. Subsection 6.2 discusses so called sequential topologies which are important for the description of subquotients of $K^{\mathrm{top}}(F)$ in terms of a simpler objects endowed with sequential topology (Theorem 1 in 6.6 and Theorem 1 in 7.2 of section 7). Subsection 6.3 introduces $K^{\mathrm{top}}(F)$, 6.4 presents very useful pairings (including Vostokov's symbol which is discussed in more detail in section 8), subsection 6.5–6.6 describe the structure of $K^{\mathrm{top}}(F)$ and 6.7 deals with the quotients $K(F)/l$; finally, 6.8 presents various properties of the norm map on K-groups. Note that subsections 6.6–6.8 are not required for understanding Parshin's class field theory in section 7.

6.0. Introduction

Let A be a commutative ring and let X be an A-module endowed with some topology. A set $\{x_i\}_{i\in I}$ of elements of X is called a set of *topological generators* of X if the sequential closure of the submodule of X generated by this set coincides with X. A set of topological generators is called a *topological basis* if for every $j \in I$ and every non-zero $a \in A$ ax_j doesn't belong to the sequential closure of the submodule generated by $\{x_i\}_{i\neq j}$.

Let I be a countable set. If $\{x_i\}$ is set of topological generators of X then every element $x \in X$ can be expressed as a convergent sum $\sum a_i x_i$ with some $a_i \in A$ (note that it is not necessarily the case that for all $a_i \in A$ the sum $\sum a_i x_i$ converges). This expression is unique if $\{x_i\}$ is a topological basis of X; then provided addition in X is sequentially continuous, we get $\sum a_i x_i + \sum b_i x_i = \sum (a_i + b_i) x_i$.

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Recall that in the one-dimensional case the group of principal units $U_{1,F}$ is a multiplicative $\mathbb{Z}_p$-module with finitely many topological generators if $\operatorname{char}(F) = 0$ and infinitely many topological generators if $\operatorname{char}(F) = p$ (see for instance [FV, Ch. I §6]). This representation of $U_{1,F}$ and a certain specific choice of its generators is quite important if one wants to deduce the Shafarevich and Vostokov explicit formulas for the Hilbert symbol (see section 8).

Similarly, the group V_F of principal units of an n-dimensional local field F is topologically generated by $1 + \theta t_n^{i_n} \dots t_1^{i_1}$, $\theta \in \mu_{q-1}$ (see subsection 1.4.2). This leads to a natural suggestion to endow the Milnor K-groups of F with an appropriate topology and use the sequential convergence to simplify calculations in K-groups.

On the other hand, the reciprocity map

$$\Psi_F \colon K_n(F) \to \operatorname{Gal}(F^{\mathrm{ab}}/F)$$

is not injective in general, in particular $\ker(\Psi_F) \supset \bigcap_{l \geqslant 1} lK_n(F) \neq 0$. So the Milnor K-groups are too large from the point of view of class field theory, and one can pass to the quotient $K_n(F)/\bigcap_{l \geqslant 1} lK_n(F)$ without loosing any arithmetical information on F. The latter quotient coincides with $K_n^{\mathrm{top}}(F)$ (see subsection 6.6) which is defined in subsection 6.3 as the quotient of $K_n(F)$ by the intersection $\Lambda_n(F)$ of all neighbourhoods of 0 in $K_n(F)$ with respect to a certain topology. The existence theorem in class field theory uses the topology to characterize norm subgroups $N_{L/F}K_n(L)$ of finite abelian extensions L of F as open subgroups of finite index of $K_n(F)$ (see subsection 10.5). As a corollary of the existence theorem in 10.5 one obtains that in fact

$$\bigcap_{l \geqslant 1} lK_n(F) = \Lambda_n(F) = \ker(\Psi_F).$$

However, the class of open subgroups of finite index of $K_n(F)$ can be defined without introducing the topology on $K_n(F)$, see the paper of Kato in this volume which presents a different approach.

6.1. K-groups of one-dimensional local fields

The structure of the Milnor K-groups of a one-dimensional local field F is completely known.

Recall that using the Hilbert symbol and multiplicative $\mathbb{Z}_p$-basis of the group of principal units of F one obtains that

$$K_2(F) \simeq \operatorname{Tors} K_2(F) \oplus mK_2(F), \qquad \text{where } m = |\operatorname{Tors} F^*|, \ \operatorname{Tors} K_2(F) \simeq \mathbb{Z}/m$$

and $mK_2(F)$ is an uncountable uniquely divisible group (Bass, Tate, Moore, Merkur'ev; see for instance [FV, Ch. IX §4]). The groups $K_m(F)$ for $m \geqslant 3$ are uniquely divisible uncountable groups (Kahn [Kn], Sivitsky [FV, Ch. IX §4]).

6.2. Sequential topology

We need slightly different topologies from the topology of F and F^* introduced in section 1.

Definition. Let X be a topological space with topology τ. Define its *sequential saturation* λ:
a subset U of X is open with respect to λ if for every $\alpha \in U$ and a convergent (with respect to τ) sequence $X \ni \alpha_i$ to α almost all α_i belong to U. Then $\alpha_i \underset{\tau}{\to} \alpha \Leftrightarrow \alpha_i \underset{\lambda}{\to} \alpha$.

Hence the sequential saturation is the strongest topology which has the same convergent sequences and their limits as the original one. For a very elementary introduction to sequential topologies see [S].

Definition. For an n-dimensional local field F denote by λ the sequential saturation of the topology on F defined in section 1.

The topology λ is different from the old topology on F defined in section 1 for $n \geqslant 2$: for example, if $F = \mathbb{F}_p((t_1))((t_2))$ then $Y = F \setminus \{t_2^i t_1^{-j} + t_2^{-i} t_1^j : i, j \geqslant 1\}$ is open with respect to λ and is not open with respect to the topology of F defined in section 1.

Let λ_* on F^* be the sequential saturation of the topology τ on F^* defined in section 1. It is a shift invariant topology.

If $n = 1$, the restriction of λ_* on V_F coincides with the induced from λ.

The following properties of λ (λ_*) are similar to those in section 1 and/or can be proved by induction on dimension.

Properties.

(1) $\alpha_i, \beta_i \underset{\lambda}{\to} 0 \Rightarrow \alpha_i - \beta_i \underset{\lambda}{\to} 0$;
(2) $\alpha_i, \beta_i \underset{\lambda_*}{\longrightarrow} 1 \Rightarrow \alpha_i \beta_i^{-1} \underset{\lambda_*}{\longrightarrow} 1$;
(3) for every $\alpha_i \in U_F$, $\alpha_i^{p^i} \underset{\lambda_*}{\longrightarrow} 1$;
(4) multiplication is not continuous in general with respect to λ_*;
(5) every fundamental sequence with respect to λ (resp. λ_*) converges;
(6) V_F and F^{*m} are closed subgroups of F^* for every $m \geqslant 1$;
(7) The intersection of all open subgroups of finite index containing a closed subgroup H coincides with H.

Definition. For topological spaces $X_1, \dots, X_j$ define the $*$-product topology on $X_1 \times \dots \times X_j$ as the sequential saturation of the product topology.

6.3. K^{top}-groups

Definition. Let λ_m be the strongest topology on $K_m(F)$ such that subtraction in $K_m(F)$ and the natural map

$$\varphi\colon (F^*)^m \to K_m(F), \quad \varphi(\alpha_1, \dots, \alpha_m) = \{(\alpha_1, \dots, \alpha_m\}$$

are sequentially continuous. Then the topology λ_m coincides with its sequential saturation. Put

$$\Lambda_m(F) = \bigcap \text{ open neighbourhoods of } 0.$$

It is straightforward to see that $\Lambda_m(F)$ is a subgroup of $K_m(F)$.

Properties.

(1) $\Lambda_m(F)$ is closed: indeed $\Lambda_m(F) \ni x_i \to x$ implies that $x = x_i + y_i$ with $y_i \to 0$, so $x_i, y_i \to 0$, hence $x = x_i + y_i \to 0$, so $x \in \Lambda_m(F)$.

(2) Put $VK_m(F) = \langle \{V_F\} \cdot K_{m-1}(F) \rangle$ (V_F is defined in subsection 1.1). Since the topology with $VK_m(F)$ and its shifts as a system of fundamental neighbourhoods satisfies two conditions of the previous definition, one obtains that $\Lambda_m(F) \subset VK_m(F)$.

(3) $\lambda_1 = \lambda_*$.

Following the original approach of Parshin [P1] introduce now the following:

Definition. Set

$$K_m^{\mathrm{top}}(F) = K_m(F)/\Lambda_m(F)$$

and endow it with the quotient topology of λ_m which we denote by the same notation.

This new group $K_m^{\mathrm{top}}(F)$ is sometimes called the *topological Milnor K-group* of F.

If $\mathrm{char}(K_{n-1}) = p$ then $K_1^{\mathrm{top}} = K_1$.

If $\mathrm{char}(K_{n-1}) = 0$ then $K_1^{\mathrm{top}}(K) \neq K_1(K)$, since $1 + \mathcal{M}_{K_n}$ (which is uniquely divisible) is a subgroup of $\Lambda_1(K)$.

6.4. Explicit pairings

Explicit pairings of the Milnor K-groups of F are quite useful if one wants to study the structure of K^{top}-groups.

The general method is as follows. Assume that there is a pairing

$$\langle\ ,\ \rangle\colon A \times B \to \mathbb{Z}/m$$

of two $\mathbb{Z}/m$-modules A and B. Assume that A is endowed with a topology with respect to which it has topological generators α_i where i runs over elements of a totally ordered countable set I. Assume that for every $j \in I$ there is an element $\beta_j \in B$ such that

$$\langle \alpha_j, \beta_j \rangle = 1 \mod m, \qquad \langle \alpha_i, \beta_j \rangle = 0 \mod m \quad \text{for all } i > j\,.$$

Then if a convergent sum $\sum c_i \alpha_i$ is equal to 0, assume that there is a minimal j with non-zero c_j and deduce that

$$0 = \sum c_i \langle \alpha_i, \beta_j \rangle = c_j,$$

a contradiction. Thus, $\{\alpha_i\}$ form a topological basis of A.

If, in addition, for every $\beta \in B \setminus \{0\}$ there is an $\alpha \in A$ such that $\langle \alpha, \beta \rangle \neq 0$, then the pairing $\langle\ ,\ \rangle$ is obviously non-degenerate.

Pairings listed below satisfy the assumptions above and therefore can be applied to study the structure of quotients of the Milnor K-groups of F.

6.4.1. "Valuation map".

Let $\partial: K_r(K_s) \to K_{r-1}(K_{s-1})$ be the border homomorphism (see for example [FV, Ch. IX §2]). Put

$$\mathfrak{v} = \mathfrak{v}_F : K_n(F) \xrightarrow{\partial} K_{n-1}(K_{n-1}) \xrightarrow{\partial} \dots \xrightarrow{\partial} K_0(K_0) = \mathbb{Z}, \quad \mathfrak{v}(\{t_1, \dots, t_n\}) = 1$$

for a system of local parameters $t_1, \dots, t_n$ of F. The valuation map $\mathfrak{v}$ doesn't depend on the choice of a system of local parameters.

6.4.2. Tame symbol.

Define

$$t: K_n(F)/(q-1) \times F^*/F^{*\,q-1} \to K_{n+1}(F)/(q-1) \to \mathbb{F}_q^* \to \mu_{q-1}, \quad q = |K_0|$$

by

$$K_{n+1}(F)/(q-1) \xrightarrow{\partial} K_n(K_{n-1})/(q-1) \xrightarrow{\partial} \dots \xrightarrow{\partial} K_1(K_0)/(q-1) = \mathbb{F}_q^* \to \mu_{q-1}.$$

Here the map $\mathbb{F}_q^* \to \mu_{q-1}$ is given by taking multiplicative representatives.

An explicit formula for this symbol (originally asked for in [P2] and suggested in [F1]) is simple: let $t_1, \dots, t_n$ be a system of local parameters of F and let $\mathbf{v} = (v_1, \dots, v_n)$ be the associated valuation of rank n (see section 1 of this volume). For elements $\alpha_1, \dots, \alpha_{n+1}$ of F^* the value $t(\alpha_1, \alpha_2, \dots, \alpha_{n+1})$ is equal to the $(q-1)$th root of unity whose residue is equal to the residue of

$$\alpha_1^{b_1} \dots \alpha_{n+1}^{b_{n+1}} (-1)^b$$

in the last residue field $\mathbb{F}_q$, where $b = \sum_{s, i<j} v_s(b_i) v_s(b_j) b^s_{i,j}$, b_j is the determinant of the matrix obtained by cutting off the jth column of the matrix $A = (v_i(\alpha_j))$ with the sign $(-1)^{j-1}$, and $b^s_{i,j}$ is the determinant of the matrix obtained by cutting off the ith and jth columns and sth row of A.

6.4.3. Artin–Schreier–Witt pairing in characteristic p.

Define, following [P2], the pairing

$$(\ , \ ]_r\colon K_n(F)/p^r \times W_r(F)/(\mathbf{F}-1)W_r(F) \to W_r(\mathbb{F}_p) \simeq \mathbb{Z}/p^r$$

by ($\mathbf{F}$ is the map defined in the section, Some Conventions)

$$(\alpha_0, \dots, \alpha_n, (\beta_0, \dots, \beta_r)]_r = \operatorname{Tr}_{K_0/\mathbb{F}_p} \quad (\gamma_0, \dots, \gamma_r)$$

where the i th ghost component $\gamma^{(i)}$ is given by res_{K_0} $(\beta^{(i)}\alpha_1^{-1}d\alpha_1 \wedge \dots \wedge \alpha_n^{-1}d\alpha_n)$.

For its properties see [P2, sect. 3]. In particular,

(1) for $x \in K_n(F)$

$$(x, \mathbf{V}(\beta_0, \dots, \beta_{r-1})]_r = \mathbf{V}(x, (\beta_0, \dots, \beta_{r-1})]_{r-1}$$

where as usual for a field K

$$\mathbf{V}\colon W_{r-1}(K) \to W_r(K), \quad \mathbf{V}(\beta_0, \dots, \beta_{r-1}) = (0, \beta_0, \dots, \beta_{r-1});$$

(2) for $x \in K_n(F)$

$$(x, \mathbf{A}(\beta_0, \dots, \beta_r)]_{r-1} = \mathbf{A}(x, (\beta_0, \dots, \beta_r)]_r$$

where for a field K

$$\mathbf{A}\colon W_r(K) \to W_{r-1}(K), \quad \mathbf{A}(\beta_0, \dots, \beta_{r-1}, \beta_r) = (\beta_0, \dots, \beta_{r-1}).$$

(3) If $\operatorname{Tr}\theta_0 = 1$ then $\big(\{t_1, \dots, t_n\}, \theta_0\big]_1 = 1$. If i_l is prime to p then

$$\big(\{1+\theta t_n^{i_n}\dots t_1^{i_1}, t_1, \dots, \widehat{t_l}, \dots, t_n\}, \theta_0\theta^{-1}i_l^{-1}t_1^{-i_1}\dots t_n^{-i_n}]_1 = 1.$$

6.4.4. Vostokov's symbol in characteristic 0.

Suppose that $\mu_{p^r} \leqslant F^*$ and $p > 2$. Vostokov's symbol

$$V(\ , \ )_r\colon K_m(F)/p^r \times K_{n+1-m}(F)/p^r \to K_{n+1}(F)/p^r \to \mu_{p^r}$$

is defined in section 8.3. For its properties see 8.3.

Each pairing defined above is sequentially continuous, so it induces the pairing of $K_m^{\text{top}}(F)$.

6.5. Structure of $K^{\text{top}}(F)$. I

Denote $VK_m^{\text{top}}(F) = \big\langle \{V_F\} \cdot K_{m-1}^{\text{top}}(F)\big\rangle$. Using the tame symbol and valuation $\mathfrak{v}$ as described in the beginning of 6.4 it is easy to deduce that

$$K_m(F) \simeq VK_m(F) \oplus \mathbb{Z}^{a(m)} \oplus (\mathbb{Z}/(q-1))^{b(m)}$$

with appropriate integer $a(m), b(m)$ (see [FV, Ch. IX, §2]); similar calculations are applicable to $K_m^{\mathrm{top}}(F)$. For example, $\mathbb{Z}^{a(m)}$ corresponds to $\oplus\langle\{t_{j_1}, \dots, t_{j_m}\}\rangle$ with $1 \leqslant j_1 < \dots < j_m \leqslant n$.

To study $VK_m(F)$ and $VK_m^{\mathrm{top}}(F)$ the following elementary equality is quite useful

$$\{1-\alpha, 1-\beta\} = \left\{\alpha, 1+\frac{\alpha\beta}{1-\alpha}\right\} + \left\{1-\beta, 1+\frac{\alpha\beta}{1-\alpha}\right\}.$$

Note that $\mathbf{v}(\alpha\beta/(1-\alpha)) = \mathbf{v}(\alpha) + \mathbf{v}(\beta)$ if $\mathbf{v}(\alpha), \mathbf{v}(\beta) > (0, \dots, 0)$.

For $\varepsilon, \eta \in V_F$ one can apply the previous formula to $\{\varepsilon, \eta\} \in K_2^{\mathrm{top}}(F)$ and using the topological convergence deduce that

$$\{\varepsilon, \eta\} = \sum \{\rho_i, t_i\}$$

with units $\rho_i = \rho_i(\varepsilon, \eta)$ sequentially continuously depending on ε, η.

Therefore $VK_m^{\mathrm{top}}(F)$ is *topologically generated* by symbols

$$\left\{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_{j_1} \dots, t_{j_{m-1}}\right\}, \quad \theta \in \mu_{q-1}.$$

In particular, $K_{n+2}^{\mathrm{top}}(F) = 0$.

Lemma. $\bigcap_{l \geqslant 1} lK_m(F) \subset \Lambda_m(F)$.

Proof. First, $\bigcap lK_m(F) \subset VK_m(F)$. Let $x \in VK_m(F)$. Write

$$x = \sum \left\{\alpha_J, t_{j_1}, \dots, t_{j_{m-1}}\right\} \mod \Lambda_m(F), \quad \alpha_J \in V_F.$$

Then

$$p^r x = \sum \left\{\alpha_J^{p^r}\right\} \cdot \left\{t_{j_1}, \dots, t_{j_{m-1}}\right\} + \lambda_r, \quad \lambda_r \in \Lambda_m(F).$$

It remains to apply property (3) in 6.2. □

6.6. Structure of $K^{\mathrm{top}}(F)$. II

This subsection 6.6 and the rest of this section are not required for understanding Parshin's class field theory theory of higher local fields of characteristic p which is discussed in section 7.

The next theorem relates the structure of $VK_m^{\mathrm{top}}(F)$ with the structure of a simpler object.

Theorem 1 ([F5, Th. 4.6]). *Let* $\mathrm{char}\,(K_{n-1}) = p$. *The homomorphism*

$$g\colon \prod_J V_F \to VK_m(F), \quad (\beta_J) \mapsto \sum_{J=\{j_1,\dots,j_{m-1}\}} \left\{\beta_J, t_{j_1}, \dots, t_{j_{m-1}}\right\}$$

induces a homeomorphism between $\prod V_F/g^{-1}(\Lambda_m(F))$ *endowed with the quotient of the* $*$*-topology and* $VK_m^{\mathrm{top}}(F)$*;* $g^{-1}(\Lambda_m(F))$ *is a closed subgroup.*

Since every closed subgroup of V_F is the intersection of some open subgroups of finite index in V_F (property (7) of 6.2), we obtain the following:

Corollary. $\Lambda_m(F) = \bigcap$ *open subgroups of finite index in* $K_m(F)$.

Remarks. 1. If F is of characteristic p, there is a complete description of the structure of $K_m^{\mathrm{top}}(F)$ in the language of topological generators and relations due to Parshin (see subsection 7.2).

2. If $\mathrm{char}(K_{n-1}) = 0$, then the border homomorphism in Milnor K-theory (see for instance [FV, Ch. IX §2]) induces the homomorphism

$$VK_m(F) \to VK_m(K_{n-1}) \oplus VK_{m-1}(K_{n-1}).$$

Its kernel is equal to the subgroup of $VK_m(F)$ generated by symbols $\{u, \dots\}$ with u in the group $1 + \mathcal{M}_F$ which is uniquely divisible. So

$$VK_m^{\mathrm{top}}(F) \simeq VK_m^{\mathrm{top}}(K_{n-1}) \oplus VK_{m-1}^{\mathrm{top}}(K_{n-1})$$

and one can apply Theorem 1 to describe $VK_m^{\mathrm{top}}(F)$.

Proof of Theorem 1. Recall that every symbol $\{\alpha_1, \dots, \alpha_m\}$ in $K_m^{\mathrm{top}}(F)$ can be written as a convergent sum of symbols $\{\beta_J, t_{j_1}, \dots, t_{j_{m-1}}\}$ with β_J sequentially continuously depending on α_i (subsection 6.5). Hence there is a sequentially continuous map $f: V_F \times F^{*\oplus m-1} \to \prod_J V_F$ such that its composition with g coincides with the restriction of the map $\varphi: (F^*)^m \to K_m^{\mathrm{top}}(F)$ on $V_F \oplus F^{*\oplus m-1}$.

So the quotient of the $*$-topology of $\prod_J V_F$ is $\leqslant \lambda_m$, as follows from the definition of λ_m. Indeed, the sum of two convergent sequences x_i, y_i in $\prod_J V_F/g^{-1}(\Lambda_m(F))$ converges to the sum of their limits.

Let U be an open subset in $VK_m(F)$. Then $g^{-1}(U)$ is open in the $*$-product of the topology $\prod_J V_F$. Indeed, otherwise for some J there were a sequence $\alpha_J^{(i)} \notin g^{-1}(U)$ which converges to $\alpha_J \in g^{-1}(U)$. Then the properties of the map φ of 6.3 imply that the sequence $\varphi(\alpha_J^{(i)}) \notin U$ converges to $\varphi(\alpha_J) \in U$ which contradicts the openness of U. □

Theorem 2 ([F5, Th. 4.5])**.** *If* $\mathrm{char}(F) = p$ *then* $\Lambda_m(F)$ *is equal to* $\bigcap_{l \geqslant 1} lK_m(F)$ *and is a divisible group.*

Proof. Bloch–Kato–Gabber's theorem (see subsection A2 in the appendix to section 2) shows that the differential symbol

$$d: K_m(F)/p \longrightarrow \Omega_F^m, \qquad \{\alpha_1, \dots, \alpha_m\} \longmapsto \frac{d\alpha_1}{\alpha_1} \wedge \dots \wedge \frac{d\alpha_m}{\alpha_m}$$

is injective. The topology of Ω_F^m induced from F (as a finite dimensional vector space) is Hausdorff, and d is continuous, so $\Lambda_m(F) \subset pK_m(F)$.

Since $VK_m(F)/\Lambda_m(F) \simeq \prod \mathcal{E}_J$ doesn't have p-torsion by Theorem 1 in subsection 7.2, $\Lambda_m(F) = p\Lambda_m(F)$. $\square$

Theorem 3 ([F5, Th. 4.7]). *If* $\mathrm{char}(F) = 0$ *then* $\Lambda_m(F)$ *is equal to* $\bigcap_{l\geqslant 1} lK_m(F)$ *and is a divisible group. If a primitive* l*th root* ζ_l *belongs to* F, *then* ${}_lK_m^{\mathrm{top}}(F) = \{\zeta_l\} \cdot K_{m-1}^{\mathrm{top}}(F)$.

Proof. To show that $p^rVK_m(F) \supset \Lambda_m(F)$ it suffices to check that $p^rVK_m(F)$ is the intersection of open neighbourhoods of $p^rVK_m(F)$.

We can assume that μ_p is contained in F applying the standard argument by using $(p, |F(\mu_p) : F|) = 1$ and l-divisibility of $VK_m(F)$ for l prime to p.

If $r = 1$ then one can use Bloch–Kato's description of

$$U_iK_m(F) + pK_m(F)/U_{i+1}K_m(F) + pK_m(F)$$

in terms of products of quotients of $\Omega^j_{K_{n-1}}$ (section 4). $\Omega^j_{K_{n-1}}$ and its quotients are finite-dimensional vector spaces over K_{n-1}/K_{n-1}^p, so the intersection of all neighborhoods of zero there with respect to the induced from K_{n-1} topology is trivial. Therefore the injectivity of d implies $\Lambda_m(F) \subset pK_m(F)$.

Thus, the intersection of open subgroups in $VK_m(F)$ containing $pVK_m(F)$ is equal to $pVK_m(F)$.

Induction Step.

For a field F consider the pairing

$$(\ ,\ )_r \colon K_m(F)/p^r \times H^{n+1-m}(F, \mu_{p^r}^{\otimes n-m}) \to H^{n+1}(F, \mu_{p^r}^{\otimes n})$$

given by the cup product and the map $F^* \to H^1(F, \mu_{p^r})$. If $\mu_{p^r} \subset F$, then Bloch–Kato's theorem shows that $(\ ,\ )_r$ can be identified (up to sign) with Vostokov's pairing $V_r(\ ,\ )$.

For $\chi \in H^{n+1-m}(F, \mu_{p^r}^{\otimes n-m})$ put

$$A_\chi = \{\alpha \in K_m(F) : (\alpha, \chi)_r = 0\}.$$

One can show [F5, Lemma 4.7] that A_χ is an open subgroup of $K_m(F)$.

Let α belong to the intersection of all open subgroups of $VK_m(F)$ which contain $p^rVK_m(F)$. Then $\alpha \in A_\chi$ for every $\chi \in H^{n+1-m}(F, \mu_{p^r}^{\otimes n-m})$.

Set $L = F(\mu_{p^r})$ and $p^s = |L : F|$. From the induction hypothesis we deduce that $\alpha \in p^sVK_m(F)$ and hence $\alpha = N_{L/F}\beta$ for some $\beta \in VK_m(L)$. Then

$$0 = (\alpha, \chi)_{r,F} = (N_{L/F}\beta, \chi)_{r,F} = (\beta, i_{F/L}\chi)_{r,L}$$

where $i_{F/L}$ is the natural map. Keeping in mind the identification between Vostokov's pairing V_r and $(\ ,\ )_r$ for the field L we see that β is annihilated by $i_{F/L}K_{n+1-m}(F)$

with respect to Vostokov's pairing. Using explicit calculations with Vostokov's pairing one can directly deduce that

$$\beta \in (\sigma - 1)K_m(L) + p^{r-s} i_{F/L} K_m(F) + p^r K_m(L),$$

and therefore $\alpha \in p^r K_m(F)$, as required.

Thus $p^r K_m(F) = \bigcap$ open neighbourhoods of $p^r V K_m(F)$.

To prove the second statement we can assume that l is a prime. If $l \neq p$, then since $K_m^{\text{top}}(F)$ is the direct sum of several cyclic groups and $VK_m^{\text{top}}(F)$ and since l-torsion of $K_m^{\text{top}}(F)$ is p-divisible and $\cap_r p^r V K_m^{\text{top}}(F) = \{0\}$, we deduce the result.

Consider the most difficult case of $l = p$. Use the exact sequence

$$0 \to \mu_{p^s}^{\otimes n} \to \mu_{p^{s+1}}^{\otimes n} \to \mu_p^{\otimes n} \to 0$$

and the following commutative diagram (see also subsection 4.3.2)

$$\begin{array}{ccccc}
\mu_p \otimes K_{m-1}(F)/p & \longrightarrow & K_m(F)/p^s & \xrightarrow{p} & K_m(F)/p^{s+1} \\
\downarrow & & \downarrow & & \downarrow \\
H^{m-1}(F, \mu_p^{\otimes m}) & \longrightarrow & H^m(F, \mu_{p^s}^{\otimes m}) & \longrightarrow & H^m(F, \mu_{p^{s+1}}^{\otimes m}).
\end{array}$$

We deduce that $px \in \Lambda_m(F)$ implies $px \in \bigcap p^r K_m(F)$, so $x = \{\zeta_p\} \cdot a_{r-1} + p^{r-1} b_{r-1}$ for some $a_i \in K_{m-1}^{\text{top}}(F)$ and $b_i \in K_m^{\text{top}}(F)$.

Define $\psi \colon K_{m-1}^{\text{top}}(F) \to K_m^{\text{top}}(F)$ as $\psi(\alpha) = \{\zeta_p\} \cdot \alpha$; it is a continuous map. Put $D_r = \psi^{-1}(p^r K_m^{\text{top}}(F))$. The group $D = \cap D_r$ is the kernel of ψ. One can show [F5, proof of Th. 4.7] that $\{a_r\}$ is a Cauchy sequence in the space $K_{m-1}^{\text{top}}(F)/D$ which is complete. Hence there is $y \in \bigcap \left(a_{r-1} + D_{r-1}\right)$. Thus, $x = \{\zeta_p\} \cdot y$ in $K_m^{\text{top}}(F)$.

Divisibility follows. □

Remarks. 1. Compare with Theorem 8 in 2.5.

2. For more properties of $K_m^{\text{top}}(F)$ see [F5].

3. Zhukov [Z, §7–10] gave a description of $K_n^{\text{top}}(F)$ in terms of topological generators and relations for some fields F of characteristic zero with small $v_F(p)$.

6.7. The group $K_m(F)/l$

6.7.1. If a prime number l is distinct from p, then, since V_F is l-divisible, we deduce from 6.5 that

$$K_m(F)/l \simeq K_m^{\text{top}}(F)/l \simeq (\mathbb{Z}/l)^{a(m)} \oplus (\mathbb{Z}/d)^{b(m)}$$

where $d = \gcd(q-1, l)$.

6.7.2. The case of $l = p$ is more interesting and difficult. We use the method described at the beginning of 6.4.

If $\operatorname{char}(F) = p$ then the Artin–Schreier pairing of 6.4.3 for $r = 1$ helps one to show that $K_n^{\mathrm{top}}(F)/p$ has the following topological $\mathbb{Z}/p$-basis:

$$\{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_n, \dots, \widehat{t_l}, \dots, t_1\}$$

where $p \nmid \gcd(i_1, \dots, i_n)$, $0 < (i_1, \dots, i_n)$, $l = \min\{k : p \nmid i_k\}$ and θ runs over all elements of a fixed basis of K_0 over $\mathbb{F}_p$.

If $\operatorname{char}(F) = 0$, $\zeta_p \in F^*$, then using Vostokov's symbol (6.4.4 and 8.3) one obtains that $K_n^{\mathrm{top}}(F)/p$ has the following topological $\mathbb{Z}_p$-basis consisting of elements of two types:

$$\omega_*(j) = \{1 + \theta_* t_n^{pe_n/(p-1)} \dots t_1^{pe_1/(p-1)}, t_n, \dots, \widehat{t_j}, \dots, t_1\}$$

where $1 \leqslant j \leqslant n$, $(e_1, \dots, e_n) = \mathbf{v}_F(p)$ and $\theta_* \in \mu_{q-1}$ is such that $1 + \theta_* t_n^{pe_n/(p-1)} \dots t_1^{pe_1/(p-1)}$ doesn't belong to F^{*p}
and

$$\{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_n, \dots, \widehat{t_l}, \dots, t_1\}$$

where $p \nmid \gcd(i_1, \dots, i_n)$, $0 < (i_1, \dots, i_n) < p(e_1, \dots, e_n)/(p-1)$,
$l = \min\{k : p \nmid i_k\}$, where θ runs over all elements of a fixed basis of K_0 over $\mathbb{F}_p$.

If $\zeta_p \notin F^*$, then pass to the field $F(\zeta_p)$ and then go back, using the fact that the degree of $F(\zeta_p)/F$ is relatively prime to p. One deduces that $K_n^{\mathrm{top}}(F)/p$ has the following topological $\mathbb{Z}_p$-basis:

$$\{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_n, \dots, \widehat{t_l}, \dots, t_1\}$$

where $p \nmid \gcd(i_1, \dots, i_n)$, $0 < (i_1, \dots, i_n) < p(e_1, \dots, e_n)/(p-1)$,
$l = \min\{k : p \nmid i_k\}$, where θ runs over all elements of a fixed basis of K_0 over $\mathbb{F}_p$.

6.8. The norm map on K^{top}-groups

Definition. Define the norm map on $K_n^{\mathrm{top}}(F)$ as induced by $N_{L/F}\colon K_n(L) \to K_n(F)$.

Alternatively in characteristic p one can define the norm map as in 7.4.

6.8.1. Put $u_{i_1,\dots,i_n} = U_{i_1,\dots,i_n}/U_{i_1+1,\dots,i_n}$.

Proposition ([F2, Prop. 4.1] and [F3, Prop. 3.1])**.** *Let L/F be a cyclic extension of prime degree l such that the extension of the last finite residue fields is trivial. Then there is s and a local parameter $t_{s,L}$ of L such that $L = F(t_{s,L})$. Let $t_1, \dots, t_n$ be a system of local parameters of F, then $t_1, \dots, t_{s,L}, \dots, t_n$ is a system of local parameters of L.*

Let $l = p$. For a generator σ of $\mathrm{Gal}(L/F)$ *let*

$$\frac{\sigma t_{s,L}}{t_{s,L}} = 1 + \theta_0 t_n^{r_n} \cdots t_{s,L}^{r_s} \cdots t_1^{r_1} + \cdots$$

Then

(1) *if* $(i_1, \dots, i_n) < (r_1, \dots, r_n)$ *then*

$$N_{L/F}\colon u_{i_1,\dots,i_n,L} \to u_{pi_1,\dots,i_s,\dots,pi_n,F}$$

sends $\theta \in K_0$ *to* θ^p*;*

(2) *if* $(i_1, \dots, i_n) = (r_1, \dots, r_n)$ *then*

$$N_{L/F}\colon u_{i_1,\dots,i_n,L} \to u_{pi_1,\dots,i_s,\dots,pi_n,F}$$

sends $\theta \in K_0$ *to* $\theta^p - \theta\theta_0^{p-1}$*;*

(3) *if* $(j_1, \dots, j_n) > 0$ *then*

$$N_{L/F}\colon u_{j_1+r_1,\dots,pj_s+r_s,\dots,j_n+r_n,L} \to u_{j_1+pr_1,\dots,j_s+r_s,\dots,j_n+pr_n,F}$$

sends $\theta \in K_0$ *to* $-\theta\theta_0^{p-1}$.

Proof. Similar to the one-dimensional case [FV, Ch. III §1]. □

6.8.2. If L/F is cyclic of prime degree l then

$$K_n^{\mathrm{top}}(L) = \left\langle \{L^*\} \cdot i_{F/L} K_{n-1}^{\mathrm{top}}(F) \right\rangle$$

where $i_{F/L}$ is induced by the embedding $F^* \to L^*$. For instance (we use the notations of section 1), if $f(L|F) = l$ then L is generated over F by a root of unity of order prime to p; if $e_i(L|F) = l$, then use the previous proposition.

Corollary 1. *Let L/F be a cyclic extension of prime degree l. Then*

$$|K_n^{\mathrm{top}}(F) : N_{L/F} K_n^{\mathrm{top}}(L)| = l.$$

If L/F is as in the preceding proposition, then the element

$$\left\{1 + \theta_* t_n^{pr_n} \cdots t_{s,F}^{r_s} \cdots t_1^{pr_1}, t_1, \dots, \widehat{t_s}, \dots, t_n\right\},$$

where the residue of θ_* in K_0 doesn't belong to the image of the map

$$\mathcal{O}_F \xrightarrow{\theta \mapsto \theta^p - \theta\theta_0^{p-1}} \mathcal{O}_F \to K_0,$$

is a generator of $K_n^{\mathrm{top}}(F)/N_{L/F} K_n^{\mathrm{top}}(L)$.

If $f(L|F) = 1$ and $l \neq p$, then

$$\left\{\theta_*, t_1, \dots, \widehat{t_s}, \dots, t_n\right\}$$

where $\theta_* \in \mu_{q-1} \setminus \mu_{q-1}^l$ is a generator of $K_n^{\mathrm{top}}(F)/N_{L/F} K_n^{\mathrm{top}}(L)$.

If $f(L|F) = l$, then

$$\{t_1, \dots, t_n\}$$

is a generator of $K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L)$.

Corollary 2. *$N_{L/F}$ (closed subgroup) is closed and $N_{L/F}^{-1}$ (open subgroup) is open.*

Proof. Sufficient to show for an extension of prime degree; then use the previous proposition and Theorem 1 of 6.6. □

6.8.3.

Theorem 4 ([F2, §4], [F3, §3]). *Let L/F be a cyclic extension of prime degree l with a generator σ then the sequence*

$$K_n^{\mathrm{top}}(F)/l \oplus K_n^{\mathrm{top}}(L)/l \xrightarrow{i_{F/L}\oplus(1-\sigma)} K_n^{\mathrm{top}}(L)/l \xrightarrow{N_{L/F}} K_n^{\mathrm{top}}(F)/l$$

is exact.

Proof. Use the explicit description of K_n^{top}/l in 6.7. □

This theorem together with the description of the torsion of $K_n^{\mathrm{top}}(F)$ in 6.6 imply:

Corollary. *Let L/F be cyclic with a generator σ then the sequence*

$$K_n^{\mathrm{top}}(L) \xrightarrow{1-\sigma} K_n^{\mathrm{top}}(L) \xrightarrow{N_{L/F}} K_n^{\mathrm{top}}(F)$$

is exact.

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7. Parshin's higher local class field theory in characteristic p

Ivan Fesenko

Parshin's theory in characteristic p is a remarkably simple and effective approach to all the main theorems of class field theory by using relatively few ingredients.

Let $F = K_n, \dots, K_0$ be an n-dimensional local field of characteristic p.

In this section we use the results and definitions of 6.1–6.5; we don't need the results of 6.6 – 6.8.

7.1

Recall that the group V_F is topologically generated by

$$1 + \theta t_n^{i_n} \dots t_1^{i_1}, \quad \theta \in \mathcal{R}^*, p \nmid (i_n, \dots, i_1)$$

(see 1.4.2). Note that

$$\begin{aligned} &i_1 \dots i_n \{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_1, \dots, t_n\} = \{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_1^{i_1}, \dots, t_n^{i_n}\} \\ &= \{1 + \theta t_n^{i_n} \dots t_1^{i_1}, t_1^{i_1} \cdots t_n^{i_n}, \dots, t_n^{i_n}\} = \{1 + \theta t_n^{i_n} \dots t_1^{i_1}, -\theta, \dots, t_n^{i_n}\} = 0, \end{aligned}$$

since $\theta^{q-1} = 1$ and V_F is $(q-1)$-divisible. We deduce that

$$K_{n+1}^{\mathrm{top}}(F) \simeq \mathbb{F}_q^*, \quad \{\theta, t_1, \dots, t_n\} \mapsto \theta, \quad \theta \in \mathcal{R}^*.$$

Recall that (cf. 6.5)

$$K_n^{\mathrm{top}}(F) \simeq \mathbb{Z} \oplus \left(\mathbb{Z}/(q-1)\right)^n \oplus V K_n^{\mathrm{top}}(F),$$

where the first group on the RHS is generated by $\{t_n, \dots, t_1\}$, and the second by $\{\theta, \dots, \widehat{t_l}, \dots\}$ (apply the tame symbol and valuation map of subsection 6.4).

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7.2. The structure of $VK_n^{\mathrm{top}}(F)$

Using the Artin–Schreier–Witt pairing (its explicit form in 6.4.3)

$$(\ ,\ ]_r\colon K_n^{\mathrm{top}}(F)/p^r \times W_r(F)/(\mathbf{F}-1)W_r(F) \to \mathbb{Z}/p^r,\ r \geqslant 1$$

and the method presented in subsection 6.4 we deduce that every element of $VK_n^{\mathrm{top}}(F)$ is uniquely representable as a convergent series

$$\sum a_{\theta,i_n,\dots,i_1}\{1+\theta t_n^{i_n}\dots t_1^{i_1}, t_1, \dots, \widehat{t_l}, \dots, t_n\}, \quad a_{\theta,i_n,\dots,i_1} \in \mathbb{Z}_p,$$

where θ runs over a basis of the $\mathbb{F}_p$-space K_0, $p \nmid \gcd(i_n, \dots, i_1)$ and $l = \min\{k : p \nmid i_k\}$. We also deduce that the pairing $(\ ,\ ]_r$ is non-degenerate.

Theorem 1 (Parshin, [P2]). *Let $J = \{j_1, \dots, j_{m-1}\}$ run over all $(m-1)$-elements subsets of $\{1, \dots, n\}$, $m \leqslant n+1$. Let $\mathcal{E}_J$ be the subgroups of V_F generated by $1+\theta t_n^{i_n}\dots t_1^{i_1}$, $\theta \in \mu_{q-1}$ such that $p \nmid \gcd(i_1, \dots, i_n)$ and $\min\{l : p \nmid i_l\} \notin J$. Then the homomorphism*

$$h\colon \prod_J^{*-\text{topology}} \mathcal{E}_J \to VK_m^{\mathrm{top}}(F), \quad (\varepsilon_J) \mapsto \sum_{J=\{j_1,\dots,j_{m-1}\}} \{\varepsilon_J, t_{j_1}, \dots, t_{j_{m-1}}\}$$

is a homeomorphism.

Proof. There is a sequentially continuous map $f\colon V_F \times F^{*\oplus m-1} \to \prod_J \mathcal{E}_J$ such that its composition with h coincides with the restriction of the map $\varphi\colon (F^*)^m \to K_m^{\mathrm{top}}(F)$ of 6.3 on $V_F \oplus F^{*\oplus m-1}$.

So the topology of $\prod_J^{*-\text{topology}} \mathcal{E}_J$ is $\leqslant \lambda_m$, as follows from the definition of λ_m.

Let U be an open subset in $VK_m(F)$. Then $h^{-1}(U)$ is open in the $*$-product of the topology $\prod_J \mathcal{E}_J$. Indeed, otherwise for some J there were a sequence $\alpha_J^{(i)} \notin h^{-1}(U)$ which converges to $\alpha_J \in h^{-1}(U)$. Then the sequence $\varphi(\alpha_J^{(i)}) \notin U$ converges to $\varphi(\alpha_J) \in U$ which contradicts the openness of U. □

Corollary. $K_m^{\mathrm{top}}(F)$ *has no nontrivial p-torsion*; $\cap p^r VK_m^{\mathrm{top}}(F) = \{0\}$.

7.3

Put $\widetilde{W}(F) = \varinjlim W_r(F)/(\mathbf{F}-1)W_r(F)$ with respect to the homomorphism $\mathbf{V}\colon (a_0, \dots, a_{r-1}) \to \big(0, a_0, \dots, a_{r-1}\big)$. From the pairings (see 6.4.3)

$$K_n^{\mathrm{top}}(F)/p^r \times W_r(F)/(\mathbf{F}-1)W_r(F) \xrightarrow{(\ ,\ ]_r} \mathbb{Z}/p^r \to \frac{1}{p^r}\mathbb{Z}/\mathbb{Z}$$

one obtains a non-degenerate pairing

$$(\ ,\ ]\colon \widetilde{K}_n(F) \times \widetilde{W}(F) \to \mathbb{Q}_p/\mathbb{Z}_p$$

where $\widetilde{K}_n(F) = K_n^{\mathrm{top}}(F)/\bigcap_{r\geqslant 1} p^r K_n^{\mathrm{top}}(F)$. From 7.1 and Corollary of 7.2 we deduce

$$\bigcap_{r\geqslant 1} p^r K_n^{\mathrm{top}}(F) = \mathrm{Tors}_{p'} K_n^{\mathrm{top}}(F) = \mathrm{Tors}\, K_n^{\mathrm{top}}(F),$$

where $\mathrm{Tors}_{p'}$ is prime-to-p-torsion.

Hence

$$\widetilde{K}_n(F) = K_n^{\mathrm{top}}(F)/\,\mathrm{Tors}\, K_n^{\mathrm{top}}(F).$$

7.4. The norm map on K^{top}-groups in characteristic p

Following Parshin we present an alternative description (to that one in subsection 6.8) of the norm map on K^{top}-groups in characteristic p.

If L/F is cyclic of prime degree l, then it is more or less easy to see that

$$K_n^{\mathrm{top}}(L) = \big\langle \{L^*\} \cdot i_{F/L} K_{n-1}^{\mathrm{top}}(F) \big\rangle$$

where $i_{F/L}$ is induced by the embedding $F^* \to L^*$. For instance, if $f(L|F) = l$ then L is generated over F by a root of unity of order prime to p; if $e_i(L|F) = l$, then there is a system of local parameters $t_1, \dots, t_i', \dots, t_n$ of L such that $t_1, \dots, t_i, \dots, t_n$ is a system of local parameters of F.

For such an extension L/F define [P2]

$$N_{L/F}\colon K_n^{\mathrm{top}}(L) \to K_n^{\mathrm{top}}(F)$$

as induced by $N_{L/F}\colon L^* \to F^*$. For a separable extension L/F find a tower of subextensions

$$F = F_0 - F_1 - \dots - F_{r-1} - F_r = L$$

such that F_i/F_{i-1} is a cyclic extension of prime degree and define

$$N_{L/F} = N_{F_1/F_0} \circ \dots \circ N_{F_r/F_{r-1}}.$$

To prove correctness use the non-degenerate pairings of subsection 6.4 and the properties

$$(N_{L/F}\alpha, \beta]_{F,r} = (\alpha, i_{F/L}\beta]_{L,r}$$

for p-extensions;

$$t\left(N_{L/F}\alpha, \beta\right)_F = t(\alpha, i_{F/L}\beta)_L$$

for prime-to-p-extensions (t is the tame symbol of 6.4.2).

7.5. Parshin's reciprocity map

Parshin's theory [P2], [P3] deals with three partial reciprocity maps which then can be glued together.

Proposition ([P3])**.** *Let L/F be a cyclic p-extension. Then the sequence*

$$0 \to \widetilde{K}_n(F) \xrightarrow{i_{F/L}} \widetilde{K}_n(L) \xrightarrow{1-\sigma} \widetilde{K}_n(L) \xrightarrow{N_{L/F}} \widetilde{K}_n(F)$$

is exact and the cokernel of $N_{L/F}$ is a cyclic group of order $|L : F|$.

Proof. The sequence is dual (with respect to the pairing of 7.3) to

$$\widetilde{\widetilde{W}}(F) \to \widetilde{\widetilde{W}}(L) \xrightarrow{1-\sigma} \widetilde{\widetilde{W}}(L) \xrightarrow{\mathrm{Tr}_{L/F}} \widetilde{\widetilde{W}}(F) \to 0.$$

The norm group index is calculated by induction on degree. □

Hence the class of p-extensions of F and $\widetilde{K}_n(F)$ satisfy the classical class formation axioms. Thus, one gets a homomorphism $\widetilde{K}_n(F) \to \mathrm{Gal}(F^{\mathrm{abp}}/F)$ and

$$\Psi_F^{(p)}\colon K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F^{\mathrm{abp}}/F)$$

where F^{abp} is the maximal abelian p-extension of F. In the one-dimensional case this is Kawada–Satake's theory [KS].

The valuation map $\mathfrak{v}$ of 6.4.1 induces a homomorphism

$\Psi_F^{(\mathrm{ur})}\colon K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F_{\mathrm{ur}}/F)$,

$\{t_1, \dots, t_n\} \to$ the lifting of the Frobenius automorphism of K_0^{sep}/K_0;

and the tame symbol t of 6.4.2 together with Kummer theory induces a homomorphism

$$\Psi_F^{(p')}\colon K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F(\sqrt[q-1]{t_1}, \dots, \sqrt[q-1]{t_n})/F).$$

The three homomorphisms $\Psi_F^{(p)}$, $\Psi_F^{(\mathrm{ur})}$, $\Psi_F^{(p')}$ agree [P2], so we get the reciprocity map

$$\Psi_F\colon K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

with all the usual properties.

Remark. For another rather elementary approach [F1] to class field theory of higher local fields of positive characteristic see subsection 10.2. For Kato's approach to higher class field theory see section 5 above.

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8. Explicit formulas for the Hilbert symbol

Sergei V. Vostokov

Recall that the Hilbert symbol for a local field K with finite residue field which contains a primitive p^n th root of unity ζ_{p^n} is a pairing

$$(\ ,\ )_{p^n}\colon K^*/K^{*\,p^n} \times K^*/K^{*\,p^n} \to \langle \zeta_{p^n} \rangle, \quad (\alpha,\beta)_{p^n} = \gamma^{\Psi_K(\alpha)-1}, \quad \gamma^{p^n} = \beta,$$

where $\Psi_K\colon K^* \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is the reciprocity map.

8.1. History of explicit formulas for the Hilbert symbol

There are two different branches of explicit reciprocity formulas (for the Hilbert symbol).

8.1.1. The first branch (Kummer's type formulas).

Theorem (E. Kummer 1858)**.** *Let $K = \mathbb{Q}_p(\zeta_p)$, $p \neq 2$. Then for principal units ε, η*

$$\boxed{(\varepsilon,\eta)_p = \zeta_p^{\operatorname{res}(\log\, \eta(X)\, d\log\, \varepsilon(X) X^{-p})}}$$

where $\varepsilon(X)|_{X=\zeta_p-1} = \varepsilon$, $\eta(X)|_{X=\zeta_p-1} = \eta$, $\varepsilon(X), \eta(X) \in \mathbb{Z}_p[[X]]^*$.

The important point is that one associates to the elements ε, η the series $\varepsilon(X), \eta(X)$ in order to calculate the value of the Hilbert symbol.

Theorem (I. Shafarevich 1950)**.** *Complete explicit formula for the Hilbert norm residue symbol $(\alpha,\beta)_{p^n}$, $\alpha,\beta \in K^*$, $K \supset \mathbb{Q}_p(\zeta_{p^n})$, $p \neq 2$, using a special basis of the group of principal units.*

This formula is not very easy to use because of the special basis of the group of units and certain difficulties with its verification for $n > 1$. One of applications of this formula was in the work of Yakovlev on the description of the absolute Galois group of a local field in terms of generators and relations.

Complete formulas, which are simpler that Shafarevich's formula, were discovered in the seventies:

Theorem (S. Vostokov 1978), (H. Brückner 1979). *Let a local field K with finite residue field contain $\mathbb{Q}_p(\zeta_{p^n})$ and let $p \neq 2$. Denote $\mathcal{O}_0 = W(k_K)$, $\mathrm{Tr} = \mathrm{Tr}_{\mathcal{O}_0/\mathbb{Z}_p}$. Then for $\alpha, \beta \in K^*$*

$$\boxed{(\alpha,\beta)_{p^n} = \zeta_{p^n}^{\mathrm{Tr\ res}\ \Phi(\alpha,\beta)/\underline{s}}, \quad \Phi(\alpha,\beta) = l(\underline{\beta})\underline{\alpha}^{-1}d\underline{\alpha} - l(\underline{\alpha})\frac{1}{p}\underline{\beta}^{-\triangle}d\underline{\beta}^{\triangle}}$$

where $\underline{\alpha} = \theta X^m(1+\psi(X))$, $\theta \in \mathcal{R}$, $\psi \in X\mathcal{O}_0[[X]]$, *is such that* $\underline{\alpha}(\pi) = \alpha$, $\underline{s} = \underline{\zeta_{p^n}}^{p^n} - 1$,

$$l(\underline{\alpha}) = \frac{1}{p}\log\left(\underline{\alpha}^p/\underline{\alpha}^{\triangle}\right),$$

$$\left(\sum a_i X^i\right)^{\triangle} = \sum \mathrm{Frob}_K(a_i)X^{pi}, \quad a_i \in \mathcal{O}_0.$$

Note that for the term X^{-p} in Kummer's theorem can be written as $X^{-p} = 1/(\underline{\zeta_p}^p - 1) \mod p$, since $\zeta_p = 1+\pi$ and so $\underline{s} = \underline{\zeta_p}^p - 1 = (1+X)^p - 1 = X^p \mod p$.

The works [V1] and [V2] contain two different proofs of this formula. One of them is to construct the explicit pairing

$$(\alpha,\beta) \to \zeta_{p^n}^{\mathrm{Tr\ res}\ \Phi(\alpha,\beta)/\underline{s}}$$

and check the correctness of the definition and all the properties of this pairing completely independently of class field theory (somewhat similarly to how one works with the tame symbol), and only at the last step to show that the pairing coincides with the Hilbert symbol. The second method, also followed by Brükner, is different: it uses Kneser's (1951) calculation of symbols and reduces the problem to a simpler one: to find a formula for $(\varepsilon,\pi)_{p^n}$ where π is a prime element of K and ε is a principal unit of K. Whereas the first method is very universal and can be extended to formal groups and higher local fields, the second method works well in the classical situation only.

For $p = 2$ explicit formulas were obtained by G. Henniart (1981) who followed to a certain extent Brückner's method, and S. Vostokov and I. Fesenko (1982, 1985).

8.1.2. The second branch (Artin–Hasse's type formulas).

Theorem (E. Artin and H. Hasse 1928). *Let $K = \mathbb{Q}_p(\zeta_{p^n})$, $p \neq 2$. Then for a principal unit ε and prime element $\pi = \zeta_{p^n} - 1$ of K*

$$\boxed{(\varepsilon,\zeta_{p^n})_{p^n} = \zeta_{p^n}^{\mathrm{Tr}(-\log\varepsilon)/p^n}, \quad (\varepsilon,\pi)_{p^n} = \zeta_{p^n}^{\mathrm{Tr}(\pi^{-1}\zeta_{p^n}\log\varepsilon)/p^n}}$$

where $\mathrm{Tr} = \mathrm{Tr}_{K/\mathbb{Q}_p}$.

Theorem (K. Iwasawa 1968). *Formula for $(\varepsilon,\eta)_{p^n}$ where $K = \mathbb{Q}_p(\zeta_{p^n})$, $p \neq 2$, ε,η are principal units of K and $v_K(\eta - 1) > 2v_K(p)/(p-1)$.*

To some extent the following formula can be viewed as a formula of Artin–Hasse's type. Sen deduced it using his theory of continuous Galois representations which itself is a generalization of a part of Tate's theory of p-divisible groups. The Hilbert symbol is interpreted as the cup product of H^1.

Theorem (Sh. Sen 1980). *Let $|K:\mathbb{Q}_p|<\infty$, $\zeta_{p^n}\in K$, and let π be a prime element of $\mathcal{O}_K$. Let $g(T),h(T)\in W(k_K)[T]$ be such that $g(\pi)=\beta\neq 0$, $h(\pi)=\zeta_{p^m}$. Let $\alpha\in\mathcal{O}_K$, $v_K(\alpha)\geqslant 2v_K(p)/(p-1)$. Then*

$$\boxed{(\alpha,\beta)_{p^m}=\zeta_{p^m}^c,\quad c=\frac{1}{p^m}\operatorname{Tr}_{K/\mathbb{Q}_p}\left(\frac{\zeta_{p^m}}{h'(\pi)}\frac{g'(\pi)}{\beta}\log\alpha\right).}$$

R. Coleman (1981) gave a new form of explicit formulas which he proved for $K=\mathbb{Q}_p(\zeta_{p^n})$. He uses formal power series associated to norm compatible sequences of elements in the tower of finite subextensions of the p-cyclotomic extension of the ground field and his formula can be viewed as a generalization of Iwasawa's formula.

8.2. History: Further developments

8.2.1. Explicit formulas for the (generalized) Hilbert symbol in the case where it is defined by an appropriate class field theory.

Definition. Let K be an n-dimensional local field of characteristic 0 which contains a primitive p^m th root of unity. The p^m th Hilbert symbol is defined as

$$K_n^{\mathrm{top}}(K)/p^m\times K^*/K^{*p^m}\to\langle\zeta_{p^m}\rangle,\quad(\alpha,\beta)_{p^m}=\gamma^{\Psi_K(\alpha)-1},\quad\gamma^{p^m}=\beta,$$

where $\Psi_K\colon K_n^{\mathrm{top}}(K)\to\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is the reciprocity map.

For higher local fields and $p>2$ complete formulas of Kummer's type were constructed by S. Vostokov (1985). They are discussed in subsections 8.3 and their applications to K-theory of higher local fields and p-part of the existence theorem in characteristic 0 are discussed in subsections 6.6, 6.7 and 10.5. For higher local fields, $p>2$ and Lubin–Tate formal group complete formulas of Kummer's type were deduced by I. Fesenko (1987).

Relations of the formulas with syntomic cohomologies were studied by K. Kato (1991) in a very important work where it is suggested to use Fontaine–Messing's syntomic cohomologies and an interpretation of the Hilbert symbol as the cup product explicitly computable in terms of the cup product of syntomic cohomologies; this approach implies Vostokov's formula. On the other hand, Vostokov's formula appropriately generalized defines a homomorphism from the Milnor K-groups to cohomology groups of a syntomic complex (see subsection 15.1.1). M. Kurihara (1990) applied syntomic cohomologies to deduce Iwasawa's and Coleman's formulas in the multiplicative case.

For higher local fields complete formulas of Artin–Hasse's type were constructed by M. Kurihara (1998), see section 9.

8.2.2. Explicit formulas for p-divisible groups.

Definition. Let F be a formal p-divisible group over the ring $\mathcal{O}_{K_0}$ where K_0 is a subfield of a local field K. Let K contain p^n-division points of F. Define the Hilbert symbol by

$$K^* \times F(\mathcal{M}_K) \to \ker[p^n], \quad (\alpha,\beta)_{p^n} = \Psi_K(\alpha)(\gamma) -_F \gamma, \quad [p^n](\gamma) = \beta,$$

where $\Psi_K\colon K^* \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is the reciprocity map.

For formal Lubin–Tate groups, complete formulas of Kummer's type were obtained by S. Vostokov (1979) for odd p and S. Vostokov and I. Fesenko (1983) for even p. For relative formal Lubin–Tate groups complete formulas of Kummer's type were obtained by S. Vostokov and A. Demchenko (1995).

For local fields with finite residue field and formal Lubin–Tate groups formulas of Artin–Hasse's type were deduced by A. Wiles (1978) for K equal to the $[\pi^n]$-division field of the isogeny $[\pi]$ of a formal Lubin–Tate group; by V. Kolyvagin (1979) for K containing the $[\pi^n]$-division field of the isogeny $[\pi]$; by R. Coleman (1981) in the multiplicative case and some partial cases of Lubin–Tate groups; his conjectural formula in the general case of Lubin–Tate groups was proved by E. de Shalit (1986) for K containing the $[\pi^n]$-division field of the isogeny $[\pi]$. This formula was generalized by Y. Sueyoshi (1990) for relative formal Lubin–Tate groups. F. Destrempes (1995) extended Sen's formulas to Lubin–Tate formal groups.

J.–M. Fontaine (1991) used his crystalline ring and his and J.–P. Wintenberger's theory of field of norms for the p-cyclotomic extension to relate Kummer theory with Artin–Schreier–Witt theory and deduce in particular some formulas of Iwasawa's type using Coleman's power series. D. Benois (1998) further extended this approach by using Fontaine–Herr's complex and deduced Coleman's formula. V. Abrashkin (1997) used another arithmetically profinite extension ($L = \cup F_i$ of F, $F_i = F_{i-1}(\pi_i)$, $\pi_i^p = \pi_{i-1}$, π_0 being a prime element of F) to deduce the formula of Brückner–Vostokov.

For formal groups which are defined over an absolutely unramified local field K_0 ($e(K_0|\mathbb{Q}_p) = 1$) and therefore are parametrized by Honda's systems, formulas of Kummer's type were deduced by D. Benois and S. Vostokov (1990), for $n = 1$ and one-dimensional formal groups, and by V. Abrashkin (1997) for arbitrary n and arbitrary formal group with restriction that K contains a primitive p^n th root of unity. For one dimensional formal groups and arbitrary n without restriction that K contains a primitive p^n th root of unity in the ramified case formulas were obtained by S. Vostokov and A. Demchenko (2000). For arbitrary n and arbitrary formal group without restrictions on K Abrashkin's formula was established by Benois (2000), see subsection 6.6 of Part II.

Sen's formulas were generalized to all p-divisible groups by D. Benois (1997) using an interpretation of the Hilbert pairing in terms of an explicit construction of p-adic periods. T. Fukaya (1998) generalized the latter for higher local fields.

8.2.3. Explicit formulas for p-adic representations. The previously discussed explicit formulas can be viewed as a description of the exponential map from the tangent space of a formal group to the first cohomology group with coefficients in the Tate module. Bloch and Kato (1990) defined a generalization of the exponential map to de Rham representations. An explicit description of this map is closely related to the computation of Tamagawa numbers of motives which play an important role in the Bloch–Kato conjecture. The description of this map for the $\mathbb{Q}_p(n)$ over cyclotomic fields was given by Bloch–Kato (1990) and Kato (1993); it can be viewed as a vast generalization of Iwasawa's formula (the case $n = 1$). B. Perrin-Riou constructed an Iwasawa theory for crystalline representations over an absolutely unramified local field and conjectured an explicit description of the cup product of the cohomology groups. There are three different approaches which culminate in the proof of this conjecture by P. Colmez (1998), K. Kato–M. Kurihara–T. Tsuji (unpublished) and for crystalline representations of finite height by D. Benois (1998).

K. Kato (1999) gave generalizations of explicit formulas of Artin–Hasse, Iwasawa and Wiles type to p-adically complete discrete valuation fields and p-divisible groups which relates norm compatible sequences in the Milnor K-groups and trace compatible sequences in differential forms; these formulas are applied in his other work to give an explicit description in the case of p-adic completions of function fields of modular curves.

8.3. Explicit formulas in higher dimensional fields of characteristic 0

Let K be an n-dimensional field of characteristic 0, $\operatorname{char}(K_{n-1}) = p$, $p > 2$. Let $\zeta_{p^m} \in K$.

Let $t_1, \dots, t_n$ be a system of local parameters of K.

For an element

$$\alpha = t_n^{i_n} \dots t_1^{i_1}\theta(1 + \sum a_J t_n^{j_n} \dots t_1^{j_1}), \quad \theta \in \mathcal{R}^*, a_J \in W(K_0),$$

$(j_1, \dots, j_n) > (0, \dots, 0)$ denote by $\underline{\alpha}$ the following element

$$X_n^{i_n} \dots X_1^{i_1}\theta(1 + \sum a_J X_n^{j_n} \dots X_1^{j_1})$$

in $F\{\{X_1\}\}\dots\{\{X_n\}\}$ where F is the fraction field of $W(K_0)$. Clearly $\underline{\alpha}$ is not uniquely determined even if the choice of a system of local parameters is fixed.

Independently of class field theory define the following explicit map

$$V(\ ,\ )_m\colon (K^*)^{n+1} \to \langle \zeta_{p^m} \rangle$$

by the formula

$$V(\alpha_1, \dots, \alpha_{n+1})_m = \zeta_{p^m}^{\operatorname{Tr}\,\operatorname{res}\ \Phi(\alpha_1,\dots,\alpha_{n+1})/\underline{s}}, \quad \Phi(\alpha_1, \dots, \alpha_{n+1})$$
$$= \sum_{i=1}^{n+1} \frac{(-1)^{n-i+1}}{p^{n-i+1}} l\left(\underline{\alpha_i}\right) \frac{d\underline{\alpha_1}}{\underline{\alpha_1}} \wedge \dots \wedge \frac{d\underline{\alpha_{i-1}}}{\underline{\alpha_{i-1}}} \wedge \frac{d\underline{\alpha_{i+1}}^{\triangle}}{\underline{\alpha_{i+1}}^{\triangle}} \wedge \dots \wedge \frac{d\underline{\alpha_{n+1}}^{\triangle}}{\underline{\alpha_{n+1}}^{\triangle}}$$

where $\underline{s} = \underline{\zeta_{p^m}}^{p^m} - 1$, $\operatorname{Tr} = \operatorname{Tr}_{W(K_0)/\mathbb{Z}_p}$, $\operatorname{res} = \operatorname{res}_{X_1,\dots,X_n}$,

$$l\left(\underline{\alpha}\right) = \frac{1}{p} \log\left(\underline{\alpha}^p / \underline{\alpha}^{\triangle}\right), \quad \left(\sum a_J X_n^{j_n} \cdots X_1^{j_i}\right)^{\triangle} = \sum \operatorname{Frob}(a_J) X_n^{pj_n} \cdots X_1^{pj_1}.$$

Theorem 1. *The map* $V(\ ,\ )_m$ *is well defined, multilinear and symbolic. It induces a homomorphism*

$$K_n(K)/p^m \times K^*/K^{*\,p^m} \to \mu_{p^m}$$

and since V *is sequentially continuous, a homomorphism*

$$V(\ ,\ )_m\colon K_n^{\mathrm{top}}(K)/p^m \times K^*/K^{*\,p^m} \to \mu_{p^m}$$

which is non-degenerate.

Comment on Proof. A set of elements $t_1, \dots, t_n$, $\varepsilon_{\mathbf{j}}, \omega$ (where $\mathbf{j}$ runs over a subset of $\mathbb{Z}^n$) is called a *Shafarevich basis* of K^*/K^{*p^m} if

(1) every $\alpha \in K^*$ can be written as a convergent product $\alpha = t_1^{i_1} \dots t_n^{i_n} \prod_{\mathbf{j}} \varepsilon_{\mathbf{j}}^{b_{\mathbf{j}}} \omega^c \bmod K^{*p^m}$, $b_{\mathbf{j}}, c \in \mathbb{Z}_p$.

(2) $V\left(\{t_1, \dots, t_n\}, \varepsilon_{\mathbf{j}}\right)_m = 1$, $\quad V\left(\{t_1, \dots, t_n\}, \omega\right)_m = \zeta_{p^m}$.

An important element of a Shafarevich basis is $\omega(a) = E(as(X))|_{X_n=t_n,\dots,X_1=t_1}$ where

$$E(f(X)) = \exp\left(\left(1 + \frac{\triangle}{p} + \frac{\triangle^2}{p^2} + \cdots\right)(f(X))\right),$$

$a \in W(K_0)$.

Now take the following elements as a Shafarevich basis of K^*/K^{*p^m}:

— elements $t_1, \dots, t_n$,

— elements $\varepsilon_J = 1 + \theta t_n^{j_n} \dots t_1^{j_1}$ where $p \nmid \gcd(j_1, \dots, j_n)$, $0 < (j_1, \dots, j_n) < p(e_1, \dots, e_n)/(p-1)$, where $(e_1, \dots, e_n) = \mathbf{v}(p)$, $\mathbf{v}$ is the discrete valuation of rank n associated to $t_1, \dots, t_n$,

— $\omega = \omega(a)$ where a is an appropriate generator of $W(K_0)/(\mathbf{F} - 1)W(K_0)$.

Using this basis it is relatively easy to show that $V(\ ,\ )_m$ is non-degenerate.

In particular, for every $\theta \in \mathcal{R}^*$ there is $\theta' \in \mathcal{R}^*$ such that

$$V\left(\left\{1+\theta t_n^{i_n}\dots t_1^{i_1}, t_1, \dots, \widehat{t_l}, \dots, t_n\right\}, 1+\theta' t_n^{pe_n/(p-1)-i_n}\dots t_1^{pe_1/(p-1)-i_1}\right)_m = \zeta_{p^m}$$

where i_l is prime to p, $0 < (i_1, \dots, i_n) < p(e_1, \dots, e_n)/(p-1)$ and $(e_1, \dots, e_n) = \mathbf{v}(p)$.

Theorem 2. *Every open subgroup N of finite index in $K_n^{\mathrm{top}}(K)$ such that $N \supset p^m K_n^{\mathrm{top}}(K)$ is the orthogonal complement with respect to $V(\ ,\ )_m$ of a subgroup in K^*/K^{*p^m}.*

Remark. Given higher local class field theory one defines the Hilbert symbol for l such that l is not divisible by $\operatorname{char}(K)$, $\mu_l \leqslant K^*$ as

$$(\ ,\ )_l\colon K_n(K)/l \times K^*/K^{*\,l} \to \langle \zeta_l \rangle, \quad (x, \beta)_l = \gamma^{\Psi_K(x)-1}$$

where $\gamma^l = \beta$, $\Psi_K\colon K_n(K) \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is the reciprocity map.

If l is prime to p, then the Hilbert symbol $(\ ,\ )_l$ coincides (up to a sign) with the $(q-1)/l$ th power of the tame symbol of 6.4.2. If $l = p^m$, then the p^m th Hilbert symbol coincides (up to a sign) with the symbol $V(\ ,\ )_m$.

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9. Exponential maps and explicit formulas

Masato Kurihara

In this section we introduce an exponential homomorphism for the Milnor K-groups for a complete discrete valuation field of mixed characteristics.

In general, to work with the additive group is easier than with the multiplicative group, and the exponential map can be used to understand the structure of the multiplicative group by using that of the additive group. We would like to study the structure of $K_q(K)$ for a complete discrete valuation field K of mixed characteristics in order to obtain arithmetic information of K. Note that the Milnor K-groups can be viewed as a generalization of the multiplicative group. Our exponential map reduces some problems in the Milnor K-groups to those of the differential modules $\Omega^{\cdot}_{\mathcal{O}_K}$ which is relatively easier than the Milnor K-groups.

As an application, we study explicit formulas of certain type.

9.1. Notation and exponential homomorphisms

Let K be a complete discrete valuation field of mixed characteristics $(0, p)$. Let $\mathcal{O}_K$ be the ring of integers, and F be its the residue field. Denote by $\mathrm{ord}_p\colon K^* \longrightarrow \mathbb{Q}$ the additive valuation normalized by $\mathrm{ord}_p(p) = 1$. For $\eta \in \mathcal{O}_K$ we have an exponential homomorphism

$$\exp_\eta\colon \mathcal{O}_K \longrightarrow K^*, \qquad a \mapsto \exp(\eta a) = \sum_{n=0}^{\infty} (\eta a)^n/n!$$

if $\mathrm{ord}_p(\eta) > 1/(p-1)$.

For $q > 0$ let $K_q(K)$ be the qth Milnor K-group, and define $\widehat{K}_q(K)$ as the p-adic completion of $K_q(K)$, i.e.

$$\widehat{K}_q(K) = \varprojlim K_q(K) \otimes \mathbb{Z}/p^n.$$

For a ring A, we denote as usual by Ω^1_A the module of the absolute differentials, i.e. $\Omega^1_A = \Omega^1_{A/\mathbb{Z}}$. For a field F of characteristic p and a p-base I of F, Ω^1_F is an F-vector space with basis dt $(t \in I)$. Let K be as above, and consider the p-adic completion $\widehat{\Omega}^1_{\mathcal{O}_K}$ of $\Omega^1_{\mathcal{O}_K}$

$$\widehat{\Omega}^1_{\mathcal{O}_K} = \varprojlim \Omega^1_{\mathcal{O}_K} \otimes \mathbb{Z}/p^n.$$

We take a lifting $\widetilde{I}$ of a p-base I of F, and take a prime element π of K. Then, $\widehat{\Omega}^1_{\mathcal{O}_K}$ is an $\mathcal{O}_K$-module (topologically) generated by $d\pi$ and dT $(T \in \widetilde{I})$ ([Ku1, Lemma 1.1]). If I is finite, then $\widehat{\Omega}^1_{\mathcal{O}_K}$ is generated by $d\pi$ and dT $(T \in \widetilde{I})$ in the ordinary sense. Put

$$\widehat{\Omega}^q_{\mathcal{O}_K} = \wedge^q \widehat{\Omega}^1_{\mathcal{O}_K}.$$

Theorem ([Ku3]). *Let $\eta \in K$ be an element such that $\operatorname{ord}_p(\eta) \geqslant 2/(p-1)$. Then for $q > 0$ there exists a homomorphism*

$$\exp^{(q)}_\eta \colon \widehat{\Omega}^q_{\mathcal{O}_K} \longrightarrow \widehat{K}_q(K)$$

such that

$$\exp^{(q)}_\eta \Big(a\frac{db_1}{b_1} \wedge \dots \wedge \frac{db_{q-1}}{b_{q-1}}\Big) = \{\exp(\eta a), b_1, \dots, b_{q-1}\}$$

for any $a \in \mathcal{O}_K$ and any $b_1, \dots, b_{q-1} \in \mathcal{O}^*_K$.

Note that we have no assumption on F (F may be imperfect). For $b_1, \dots, b_{q-1} \in \mathcal{O}_K$ we have

$$\exp^{(q)}_\eta (a \cdot db_1 \wedge \dots \wedge db_{q-1}) = \{\exp(\eta a b_1 \cdot \dots \cdot b_{q-1}), b_1, \dots, b_{q-1}\}.$$

9.2. Explicit formula of Sen

Let K be a finite extension of $\mathbb{Q}_p$ and assume that a primitive p^n th root ζ_{p^n} is in K. Denote by K_0 the subfield of K such that K/K_0 is totally ramified and $K_0/\mathbb{Q}_p$ is unramified. Let π be a prime element of O_K, and $g(T)$ and $h(T) \in \mathcal{O}_{K_0}[T]$ be polynomials such that $g(\pi) = \beta$ and $h(\pi) = \zeta_{p^n}$, respectively. Assume that α satisfies $\operatorname{ord}_p(\alpha) \geqslant 2/(p-1)$ and $\beta \in \mathcal{O}^*_K$. Then, Sen's formula ([S]) is

$$(\alpha, \beta) = \zeta^c_{p^n}, \qquad c = \frac{1}{p^n} \operatorname{Tr}_{K/\mathbb{Q}_p} \Big(\frac{\zeta_{p^n}}{h'(\pi)} \frac{g'(\pi)}{\beta} \log \alpha\Big)$$

where (α, β) is the Hilbert symbol defined by $(\alpha, \beta) = \gamma^{-1}\Psi_K(\alpha)(\gamma)$ where $\gamma^{p^n} = \beta$ and Ψ_K is the reciprocity map.

The existence of our exponential homomorphism introduced in the previous subsection helps to provide a new proof of this formula by reducing it to Artin–Hasse's formula for (α, ζ_{p^n}). In fact, put $k = \mathbb{Q}_p(\zeta_{p^n})$, and let η be an element of k such that $\mathrm{ord}_p(\eta) = 2/(p-1)$. Then, the commutative diagram

$$\begin{tikzcd} \widehat{\Omega}^1_{\mathcal{O}_K} \arrow[r, "\exp_\eta"] \arrow[d, "\mathrm{Tr}"'] & \widehat{K}_2(K) \arrow[d, "N"'] \\ \widehat{\Omega}^1_{\mathcal{O}_k} \arrow[r, "\exp_\eta"] & \widehat{K}_2(k) \end{tikzcd}$$

($N\colon \widehat{K}_2(K) \longrightarrow \widehat{K}_2(k)$ is the norm map of the Milnor K-groups, and $\mathrm{Tr}\colon \widehat{\Omega}^1_{\mathcal{O}_K} \longrightarrow \widehat{\Omega}^1_{\mathcal{O}_k}$ is the trace map of differential modules) reduces the calculation of the Hilbert symbol of elements in K to that of the Hilbert symbol of elements in k (namely reduces the problem to Iwasawa's formula [I]).

Further, since any element of $\widehat{\Omega}^1_{\mathcal{O}_k}$ can be written in the form $a d\zeta_{p^n}/\zeta_{p^n}$, we can reduce the problem to the calculation of (α, ζ_{p^n}).

In the same way, we can construct a formula of Sen's type for a higher dimensional local field (see [Ku3]), using a commutative diagram

$$\begin{tikzcd} \widehat{\Omega}^q_{\mathcal{O}_K\{\{T\}\}} \arrow[r, "\exp_\eta"] \arrow[d, "\mathrm{res}"'] & \widehat{K}_{q+1}(K\{\{T\}\}) \arrow[d, "\mathrm{res}"'] \\ \widehat{\Omega}^{q-1}_{\mathcal{O}_K} \arrow[r, "\exp_\eta"] & \widehat{K}_q(K) \end{tikzcd}$$

where the right arrow is the residue homomorphism $\{\alpha, T\} \mapsto \alpha$ in [Ka], and the left arrow is the residue homomorphism $\omega dT/T \mapsto \omega$. The field $K\{\{T\}\}$ is defined in Example 3 of subsection 1.1 and $\mathcal{O}_K\{\{T\}\} = \mathcal{O}_{K\{\{T\}\}}$.

9.3. Some open problems

Problem 1. Determine the kernel of $\exp_\eta^{(q)}$ completely. Especially, in the case of a d-dimensional local field K, the knowledge of the kernel of $\exp_\eta^{(d)}$ will give a lot of information on the arithmetic of K by class field theory. Generally, one can show that

$$p d\widehat{\Omega}^{q-2}_{\mathcal{O}_K} \subset \ker(\exp_p^{(q)}\colon \widehat{\Omega}^{q-1}_{\mathcal{O}_K} \longrightarrow \widehat{K}_q(K)).$$

For example, if K is absolutely unramified (namely, p is a prime element of K) and $p > 2$, then $p d\widehat{\Omega}^{q-2}_{\mathcal{O}_K}$ coincides with the kernel of $\exp_p^{(q)}$ ([Ku2]). But in general, this is not true. For example, if $K = \mathbb{Q}_p\{\{T\}\}(\sqrt[p]{pT})$ and $p > 2$, we can show that the kernel of $\exp_p^{(2)}$ is generated by $p d\mathcal{O}_K$ and the elements of the form $\log(1 - x^p) dx/x$ for any $x \in \mathcal{M}_K$ where $\mathcal{M}_K$ is the maximal ideal of $\mathcal{O}_K$.

Problem 2. Can one generalize our exponential map to some (formal) groups? For example, let G be a p-divisible group over K with $|K:\mathbb{Q}_p|<\infty$. Assume that the $[p^n]$-torsion points $\ker[p^n]$ of $G(K^{\mathrm{alg}})$ are in $G(K)$. We define the Hilbert symbol $K^*\times G(K)\longrightarrow \ker[p^n]$ by $(\alpha,\beta)=\Psi_K(\alpha)(\gamma)-_G\gamma$ where $[p^n]\gamma=\beta$. Benois obtained an explicit formula ([B]) for this Hilbert symbol, which is a generalization of Sen's formula. Can one define a map $\exp_G\colon \Omega^1_{\mathcal{O}_K}\otimes \mathrm{Lie}(G)\longrightarrow K^*\times G(K)/\sim$ (some quotient of $K^*\times G(K)$) by which we can interpret Benois's formula? We also remark that Fukaya recently obtained some generalization ([F]) of Benois's formula for a higher dimensional local field.

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10. Explicit higher local class field theory

Ivan Fesenko

In this section we present an approach to higher local class field theory [F1-2] different from Kato's (see section 5) and Parshin's (see section 7) approaches.

Let F ($F = K_n, \dots, K_0$) be an n-dimensional local field. We use the results of section 6 and the notations of section 1.

10.1. Modified class formation axioms

Consider now an approach based on a generalization [F2] of Neukirch's approach [N].

Below is a modified system of axioms of class formations (when applied to topological K-groups) which imposes weaker restrictions than the classical axioms (cf. section 11).

(A1). *There is a $\hat{\mathbb{Z}}$-extension of F.*

In the case of higher local fields let F_{pur}/F be the extension which corresponds to K_0^{sep}/K_0: $F_{\mathrm{pur}} = \cup_{(l,p)=1} F(\mu_l)$; the extension F_{pur} is called the *maximal purely unramified extension* of F. Denote by Frob_F the lifting of the Frobenius automorphisms of K_0^{sep}/K_0. Then

$$\mathrm{Gal}(F_{\mathrm{pur}}/F) \simeq \hat{\mathbb{Z}}, \quad \mathrm{Frob}_F \mapsto 1.$$

(A2). *For every finite separable extension F of the ground field there is an abelian group A_F such that $F \to A_F$ behaves well* (*is a Mackey functor, see for instance* [D]; *in fact we shall use just topological K-groups*) *and such that there is a homomorphism $\mathfrak{v}\colon A_F \to \mathbb{Z}$ associated to the choice of the $\hat{\mathbb{Z}}$-extension in* (A1) *which satisfies*

$$\mathfrak{v}(N_{L/F} A_L) = |L \cap F_{\mathrm{pur}} : F| \; \mathfrak{v}(A_F).$$

In the case of higher local fields we use the valuation homomorphism

$$\mathfrak{v}\colon K_n^{\mathrm{top}}(F) \to \mathbb{Z}$$

of 6.4.1. From now on we write $K_n^{\mathrm{top}}(F)$ instead of A_F. The kernel of $\mathfrak{v}$ is $VK_n^{\mathrm{top}}(F)$.

Put

$$\mathfrak{v}_L = \frac{1}{|L \cap F_{\mathrm{pur}} : F|}\mathfrak{v} \circ N_{L/F}.$$

Using (A1), (A2) for an arbitrary finite Galois extension L/F define the *reciprocity map*

$$\Upsilon_{L/F}\colon \mathrm{Gal}(L/F) \to K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L), \qquad \sigma \mapsto N_{\Sigma/F}\Pi_\Sigma \mod N_{L/F}K_n^{\mathrm{top}}(L)$$

where Σ is the fixed field of $\widetilde{\sigma}$ and $\widetilde{\sigma}$ is an element of $\mathrm{Gal}(L_{\mathrm{pur}}/F)$ such that $\widetilde{\sigma}|_L = \sigma$ and $\widetilde{\sigma}|_{F_{\mathrm{pur}}} = \mathrm{Frob}_F^i$ with a positive integer i. The element Π_Σ of $K_n^{\mathrm{top}}(\Sigma)$ is any such that $\mathfrak{v}_\Sigma(\Pi_\Sigma) = 1$; it is called a *prime element* of $K_n^{\mathrm{top}}(\Sigma)$. This map doesn't depend on the choice of a prime element of $K_n^{\mathrm{top}}(\Sigma)$, since $\Sigma L/\Sigma$ is purely unramified and $VK_n^{\mathrm{top}}(\Sigma) \subset N_{\Sigma L/\Sigma}VK_n^{\mathrm{top}}(\Sigma L)$.

(A3). *For every finite subextension* L/F *of* F_{pur}/F (*which is cyclic, so its Galois group is generated by, say, a* σ)

(A3a) $|K_n^{\mathrm{top}}(F) : N_{L/F}K_n^{\mathrm{top}}(L)| = |L : F|$;

(A3b) $0 \to K_n^{\mathrm{top}}(F) \xrightarrow{i_{F/L}} K_n^{\mathrm{top}}(L) \xrightarrow{1-\sigma} K_n^{\mathrm{top}}(L)$ *is exact*;

(A3c) $K_n^{\mathrm{top}}(L) \xrightarrow{1-\sigma} K_n^{\mathrm{top}}(L) \xrightarrow{N_{L/F}} K_n^{\mathrm{top}}(F)$ *is exact.*

Using (A1), (A2), (A3) one proves that $\Upsilon_{L/F}$ is a homomorphism [F2].

(A4). *For every cyclic extensions* L/F *of prime degree with a generator* σ *and a cyclic extension* L'/F *of the same degree*

(A4a) $K_n^{\mathrm{top}}(L) \xrightarrow{1-\sigma} K_n^{\mathrm{top}}(L) \xrightarrow{N_{L/F}} K_n^{\mathrm{top}}(F)$ *is exact*;

(A4b) $|K_n^{\mathrm{top}}(F) : N_{L/F}K_n^{\mathrm{top}}(L)| = |L : F|$;

(A4c) $N_{L'/F}K_n^{\mathrm{top}}(L') = N_{L/F}K_n^{\mathrm{top}}(L) \Rightarrow L = L'$.

If all axioms (A1)–(A4) hold then the homomorphism $\Upsilon_{L/F}$ induces an isomorphism [F2]

$$\Upsilon_{L/F}^{\mathrm{ab}}\colon \mathrm{Gal}(L/F)^{\mathrm{ab}} \to K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L).$$

The method of the proof is to define explicitly (as a generalization of Hazewinkel's approach [H]) a homomorphism

$$\Psi_{L/F}^{\mathrm{ab}}\colon K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L) \to \mathrm{Gal}(L/F)^{\mathrm{ab}}$$

and then show that $\Psi_{L/F}^{\mathrm{ab}} \circ \Upsilon_{L/F}^{\mathrm{ab}}$ is the indentity.

10.2. Characteristic p case

Theorem 1 ([F1], [F2]). *In characteristic p all axioms (A1)–(A4) hold. So we get the reciprocity map $\Psi_{L/F}$ and passing to the limit the reciprocity map*

$$\Psi_F\colon K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F^{\mathrm{ab}}/F).$$

Proof. See subsection 6.8. (A4c) can be checked by a direct computation using the proposition of 6.8.1 [F2, p. 1118–1119]; (A3b) for p-extensions see in 7.5, to check it for extensions of degree prime to p is relatively easy [F2, Th. 3.3]. □

Remark. Note that in characteristic p the sequence of (A3b) is not exact for an arbitrary cyclic extension L/F (if $L \not\subset F_{\mathrm{pur}}$). The characteristic zero case is discussed below.

10.3. Characteristic zero case. I

10.3.1. prime-to-p-part.

It is relatively easy to check that all the axioms of 10.1 hold for prime-to-p extensions and for

$$K_n'(F) = K_n^{\mathrm{top}}(F)/VK_n^{\mathrm{top}}(F)$$

(note that $VK_n^{\mathrm{top}}(F) = \bigcap_{(l,p)=1} lK_n^{\mathrm{top}}(F)$). This supplies the prime-to-p-part of the reciprocity map.

10.3.2. p-part.

If $\mu_p \leqslant F^*$ then all the axioms of 10.1 hold; if $\mu_p \nleqslant F^*$ then everything with exception of the axiom (A3b) holds.

Example. Let $k = \mathbb{Q}_p(\zeta_p)$. Let $\omega \in k$ be a p-primary element of k which means that $k(\sqrt[p]{\omega})/k$ is unramified of degree p. Then due to the description of K_2 of a local field (see subsection 6.1 and [FV, Ch.IX §4]) there is a prime elements π of k such that $\{\omega, \pi\}$ is a generator of $K_2(k)/p$. Since $\alpha = i_{k/k(\sqrt[p]{\omega})}\{\omega, \pi\} \in pK_2(k(\sqrt[p]{\omega}))$, the element α lies in $\bigcap_{l\geqslant 1} lK_2(k(\sqrt[p]{\omega}))$. Let $F = k\{\{t\}\}$. Then $\{\omega, \pi\} \notin pK_2^{\mathrm{top}}(F)$ and $i_{F/F(\sqrt[p]{\omega})}\{\omega, \pi\} = 0$ in $K_2^{\mathrm{top}}(F(\sqrt[p]{\omega}))$.

Since all other axioms are satisfied, according to 10.1 we get the reciprocity map

$$\Upsilon_{L/F}\colon \mathrm{Gal}(L/F) \to K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L), \quad \sigma \mapsto N_{\Sigma/F}\Pi_\Sigma$$

for every finite Galois p-extension L/F.

To study its properties we need to introduce the notion of Artin–Schreier trees (cf. [F3]) as those extensions in characteristic zero which in a certain sense come from characteristic p. Not quite precisely, there are sufficiently many finite Galois p-extensions for which one can directly define an explicit homomorphism

$$K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L) \to \mathrm{Gal}(L/F)^{\mathrm{ab}}$$

and show that the composition of $\Upsilon^{\mathrm{ab}}_{L/F}$ with it is the identity map.

10.4. Characteristic zero case. II: Artin–Schreier trees

10.4.1.

Definition. A p-extension L/F is called an *Artin–Schreier tree* if there is a tower of subfields $F = F_0 - F_1 - \dots - F_r = L$ such that each F_i/F_{i-1} is cyclic of degree p, $F_i = F_{i-1}(\alpha)$, $\alpha^p - \alpha \in F_{i-1}$.

A p-extension L/F is called a *strong Artin–Schreier tree* if every cyclic subextension M/E of degree p, $F \subset E \subset M \subset L$, is of type $E = M(\alpha)$, $\alpha^p - \alpha \in M$.

Call an extension L/F *totally ramified* if $f(L|F) = 1$ (i.e. $L \cap F_{\mathrm{pur}} = F$).

Properties of Artin–Schreier trees.

(1) if $\mu_p \not\leqslant F^*$ then every p-extension is an Artin–Schreier tree; if $\mu_p \leqslant F^*$ then $F(\sqrt[p]{a})/F$ is an Artin–Schreier tree if and only if $aF^{*p} \leqslant V_F F^{*p}$.

(2) for every cyclic totally ramified extension L/F of degree p there is a Galois totally ramified p-extension E/F such that E/F is an Artin–Schreier tree and $E \supset L$.

For example, if $\mu_p \leqslant F^*$, F is two-dimensional and t_1, t_2 is a system of local parameters of F, then $F(\sqrt[p]{t_1})/F$ is not an Artin–Schreier tree. Find an $\varepsilon \in V_F \setminus V_F^p$ such that M/F ramifies along t_1 where $M = F(\sqrt[p]{\varepsilon})$. Let $t_{1,M}, t_2 \in F$ be a system of local parameters of M. Then $t_1 t_{1,M}^{-p}$ is a unit of M. Put $E = M\left(\sqrt[p]{t_1 t_{1,M}^{-p}}\right)$. Then $E \supset F(\sqrt[p]{t_1})$ and E/F is an Artin–Schreier tree.

(3) Let L/F be a totally ramified finite Galois p-extension. Then there is a totally ramified finite p-extension Q/F such that LQ/Q is a strong Artin–Schreier tree and $L_{\mathrm{pur}} \cap Q_{\mathrm{pur}} = F_{\mathrm{pur}}$.

(4) For every totally ramified Galois extension L/F of degree p which is an Artin–Schreier tree we have

$$\mathfrak{v}_{L_{\mathrm{pur}}}(K_n^{\mathrm{top}}(L_{\mathrm{pur}})^{\mathrm{Gal}(L/F)}) = p\mathbb{Z}$$

where $\mathfrak{v}$ is the valuation map defined in 10.1, $K_n^{\mathrm{top}}(L_{\mathrm{pur}}) = \varinjlim_M K_n^{\mathrm{top}}(M)$ where M/L runs over finite subextensions in L_{pur}/L and the limit is taken with respect to the maps $i_{M/M'}$ induced by field embeddings.

Proposition 1. *For a strong Artin–Schreier tree* L/F *the sequence*

$$1 \to \mathrm{Gal}(L/F)^{\mathrm{ab}} \xrightarrow{g} VK_n^{\mathrm{top}}(L_{\mathrm{pur}})/I(L|F) \xrightarrow{N_{L_{\mathrm{pur}}/F_{\mathrm{pur}}}} VK_n^{\mathrm{top}}(F_{\mathrm{pur}}) \to 0$$

is exact, where $g(\sigma) = \sigma\Pi - \Pi$, $\mathfrak{v}_L(\Pi) = 1$, $I(L|F) = \langle \sigma\alpha - \alpha : \alpha \in VK_n^{\mathrm{top}}(L_{\mathrm{pur}})\rangle$.

Proof. Induction on $|L : F|$ using the property $N_{L_{\mathrm{pur}}/M_{\mathrm{pur}}} I(L|F) = I(M|F)$ for a subextension M/F of L/F. □

10.4.2. As a generalization of Hazewinkel's approach [H] we have

Corollary. *For a strong Artin–Schreier tree* L/F *define a homomorphism*

$$\Psi_{L/F}: VK_n^{\mathrm{top}}(F)/N_{L/F}VK_n^{\mathrm{top}}(L) \to \mathrm{Gal}(L/F)^{\mathrm{ab}}, \quad \alpha \mapsto g^{-1}((\mathrm{Frob}_L - 1)\beta)$$

where $N_{L_{\mathrm{pur}}/F_{\mathrm{pur}}}\beta = i_{F/F_{\mathrm{pur}}}\alpha$ *and* Frob_L *is defined in 10.1.*

Proposition 2. $\Psi_{L/F} \circ \Upsilon_{L/F}^{\mathrm{ab}}: \mathrm{Gal}(L/F)^{\mathrm{ab}} \to \mathrm{Gal}(L/F)^{\mathrm{ab}}$ *is the identity map; so for a strong Artin–Schreier tree* $\Upsilon_{L/F}^{\mathrm{ab}}$ *is injective and* $\Psi_{L/F}$ *is surjective.*

Remark. As the example above shows, one cannot define $\Psi_{L/F}$ for non-strong Artin–Schreier trees.

Theorem 2. $\Upsilon_{L/F}^{\mathrm{ab}}$ *is an isomorphism.*

Proof. Use property (3) of Artin–Schreier trees to deduce from the commutative diagram

$$\begin{tikzcd}
\mathrm{Gal}(LO/Q) \arrow[r, "\Upsilon_{LQ/Q}"] \arrow[d] & K_n^{\mathrm{top}}(Q)/N_{LQ/Q}K_n^{\mathrm{top}}(LQ) \arrow[d, "N_{Q/F}"'] \\
\mathrm{Gal}(L/F) \arrow[r, "\Upsilon_{L/F}"] & K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L)
\end{tikzcd}$$

that $\Upsilon_{L/F}$ is a homomorphism and injective. Surjectivity follows by induction on degree. □

Passing to the projective limit we get the reciprocity map

$$\Psi_F: K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

whose image in dense in $\mathrm{Gal}(F^{\mathrm{ab}}/F)$.

Remark. For another slightly different approach to deduce the properties of $\Upsilon_{L/F}$ see [F1].

10.5

Theorem 3. *The following diagram is commutative*

$$\begin{array}{ccc}
K_n^{\mathrm{top}}(F) & \xrightarrow{\Psi_F} & \mathrm{Gal}(F^{\mathrm{ab}}/F) \\
{\scriptstyle\partial}\big\downarrow & & \big\downarrow \\
K_{n-1}^{\mathrm{top}}(K_{n-1}) & \xrightarrow{\Psi_{K_{n-1}}} & \mathrm{Gal}(K_{n-1}^{\mathrm{ab}}/K_{n-1}).
\end{array}$$

Proof. Follows from the explicit definition of $\Upsilon_{L/F}$, since $\partial\{t_1,\dots,t_n\}$ is a prime element of $K_{n-1}^{\mathrm{top}}(K_{n-1})$. □

Existence Theorem ([F1-2])**.** *Every open subgroup of finite index in $K_n^{\mathrm{top}}(F)$ is the norm group of a uniquely determined abelian extension L/F.*

Proof. Let N be an open subgroup of $K_n^{\mathrm{top}}(F)$ of prime index l.

If $p \neq l$, then there is an $\alpha \in F^*$ such that N is the orthogonal complement of $\langle\alpha\rangle$ with respect to $t^{(q-1)/l}$ where t is the tame symbol defined in 6.4.2.

If $\mathrm{char}(F) = p = l$, then there is an $\alpha \in F$ such that N is the orthogonal complement of $\langle\alpha\rangle$ with respect to $(\ ,\ ]_1$ defined in 6.4.3.

If $\mathrm{char}(F) = 0, l = p,\ \mu_p \leqslant F^*$, then there is an $\alpha \in F^*$ such that N is the orthogonal complement of $\langle\alpha\rangle$ with respect to V_1 defined in 6.4.4 (see the theorems in 8.3). If $\mu_p \nleqslant F^*$ then pass to $F(\mu_p)$ and then back to F using $(|F(\mu_p):F|,p)=1$.

Due to Kummer and Artin–Schreier theory, Theorem 2 and Remark of 8.3 we deduce that $N = N_{L/F}K_n^{\mathrm{top}}(L)$ for an appropriate cyclic extension L/F.

The theorem follows by induction on index. □

Remark 1. From the definition of K_n^{top} it immediately follows that open subgroups of finite index in $K_n(F)$ are in one-to-one correspondence with open subgroups in $K_n^{\mathrm{top}}(F)$. Hence the correspondence $L \mapsto N_{L/F}K_n(L)$ is a one-to-one correspondence between finite abelian extensions of F and open subgroups of finite index in $K_n(F)$.

Remark 2. If K_0 is perfect and not separably p-closed, then there is a generalization of the previous class field theory for totally ramified p-extensions of F (see Remark in 16.1). There is also a generalization of the existence theorem [F3].

Corollary 1. *The reciprocity map $\Psi_F\colon K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(L/F)$ is injective.*

Proof. Use the corollary of Theorem 1 in 6.6. □

Corollary 2. *For an element $\Pi \in K_n^{\mathrm{top}}(F)$ such that $\mathfrak{v}_F(\Pi) = 1$ there is an infinite abelian extension F_Π/F such that*

$$F^{\mathrm{ab}} = F_{\mathrm{pur}} F_\Pi, \quad F_{\mathrm{pur}} \cap F_\Pi = F$$

and $\Pi \in N_{L/F} K_n^{\mathrm{top}}(L)$ for every finite extension L/F, $L \subset F_\Pi$.

Problem. Construct (for $n > 1$) the extension F_Π explicitly?

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11. Generalized class formations and higher class field theory

Michael Spieß

Let K ($K = K_n, K_{n-1}, \dots, K_0$) be an n-dimensional local field (whose last residue field is finite of characteristic p).

The following theorem can be viewed as a generalization to higher dimensional local fields of the fact $\mathrm{Br}(F) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ for classical local fields F with finite residue field (see section 5).

Theorem (Kato). *There is a canonical isomorphism*

$$h\colon H^{n+1}(K, \mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

Kato established higher local reciprocity map (see section 5 and [K1, Th. 2 of §6] (two-dimensional case), [K2, Th. II], [K3, §4]) using in particular this theorem.

In this section we deduce the reciprocity map for higher local fields from this theorem and Bloch–Kato's theorem of section 4. Our approach which uses generalized class formations simplifies Kato's original argument.

We use the notations of section 5. For a complex $X^{\cdot}$ the shifted-by-n complex $X^{\cdot}[n]$ is defined as $(X^{\cdot}[n])^q = X^{n+q}, d_{X^{\cdot}[n]} = (-1)^n d_{X^{\cdot}}$. For a (pro-)finite group G the derived category of G-modules is denoted by $D(G)$.

11.0. Classical class formations

We begin with recalling briefly the classical theory of class formations.

A pair (G, C) consisting of a profinite group G and a discrete G-module C is called a *class formation* if

(C1) $H^1(H, C) = 0$ for every open subgroup H of G.

(C2) There exists an isomorphism $\mathrm{inv}_H\colon H^2(H, C) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ for every open subgroup H of G.

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(C3) For all pairs of open subgroups $V \leqslant U \leqslant G$ the diagram

$$\begin{array}{ccc} H^2(U,C) & \xrightarrow{\text{res}} & H^2(V,C) \\ \downarrow{\scriptstyle \mathrm{inv}_U} & & \downarrow{\scriptstyle \mathrm{inv}_V} \\ \mathbb{Q}/\mathbb{Z} & \xrightarrow{\times |U:V|} & \mathbb{Q}/\mathbb{Z} \end{array}$$

is commutative.

Then for a pair of open subgroups $V \leqslant U \leqslant G$ with V normal in U the group $H^2(U/V, C^V) \simeq \ker(H^2(U,C) \to H^2(V,C))$ is cyclic of order $|U : V|$. It has a canonical generator $u_{L/K}$ which is called the *fundamental class*; it is mapped to $1/|L : K| + \mathbb{Z}$ under the composition

$$H^2(U/V, C^V) \xrightarrow{\text{inf}} H^2(U,C) \xrightarrow{\mathrm{inv}_U} \mathbb{Q}/\mathbb{Z}.$$

Cup product with $u_{L/K}$ induces by the Tate–Nakayama lemma an isomorphism

$$\widehat{H}^{q-2}(U/V, \mathbb{Z}) \xrightarrow{\sim} \widehat{H}^q(U/V, C^V).$$

Hence for $q = 0$ we get $C^U / \mathrm{cor}_{U/V}(C^V) \xrightarrow{\sim} (U/V)^{\mathrm{ab}}$.

An example of a class formation is the pair $(G_K, \mathbb{G}_m)$ consisting of the absolute Galois group of a local field K and the G_K-module $\mathbb{G}_m = (K^{\mathrm{sep}})^*$. We get an isomorphism

$$K^*/N_{L/K}L^* \xrightarrow{\sim} \mathrm{Gal}(L/K)^{\mathrm{ab}}$$

for every finite Galois extension L/K.

In order to give an analogous proof of the reciprocity law for higher dimensional local fields one has to work with complexes of modules rather than a single module.

The concepts of the class formations and Tate's cohomology groups as well as the Tate–Nakayama lemma have a straightforward generalization to bounded complexes of modules. Let us begin with Tate's cohomology groups (see [Kn] and [Ko1]).

11.1. Tate's cohomology groups

Let G be a finite group. Recall that there is an exact sequence (called a complete resolution of G)

$$X^{\cdot} \qquad \dots \to X^{-2} \to X^{-1} \to X^0 \to X^1 \to \dots$$

of free finitely generated $\mathbb{Z}[G]$-modules together with a map $X^0 \to \mathbb{Z}$ such that the sequence

$$\dots \to X^{-1} \to X^0 \to \mathbb{Z} \to 0$$

is exact.

Definition. Let G be a finite group. For a a bounded complex

$$A^{\cdot} \qquad\qquad \dots \to A^{-1} \to A^0 \to A^1 \to \dots$$

of G-modules Tate's cohomology groups $\widehat{H}^q(G, A^{\cdot})$ are defined as the (hyper-)cohomology groups of the single complex associated to the double complex

$$Y^{i,j} = \mathrm{Hom}_G(X^{-i}, A^j)$$

with suitably determined sign rule. In other words,

$$\widehat{H}^q(G, A^{\cdot}) = H^q(\mathrm{Tot}(\mathrm{Hom}(X^{\cdot}, A^{\cdot}))^G).$$

Remark. If A is a G-module, then $\widehat{H}^q(G, A^{\cdot})$ coincides with ordinary Tate's cohomology group of G with coefficients in A where

$$A^{\cdot} \qquad\qquad \dots \to 0 \to A \to 0 \to \dots \qquad (A \text{ is at degree } 0).$$

Lemma (Tate–Nakayama–Koya, [Ko2]). *Suppose that*

(i) $\widehat{H}^1(H, A^{\cdot}) = 0$ *for every subgroup* H *of* G*;*

(ii) *there is* $a \in \widehat{H}^2(G, A^{\cdot})$ *such that* $\mathrm{res}_{G/H}(a)$ *generates* $\widehat{H}^2(H, A^{\cdot})$ *and is of order* $|H|$ *for every subgroups* H *of* G.

Then

$$\widehat{H}^{q-2}(G, \mathbb{Z}) \xrightarrow{\cup a} \widehat{H}^q(G, A^{\cdot})$$

is an isomorphism for all q.

11.2. Generalized notion of class formations

Now let G be a profinite group and $C^{\cdot}$ a bounded complex of G-modules.

Definition. The pair $(G, C^{\cdot})$ is called a *generalized class formation* if it satisfies (C1)–(C3) above (of course, we have to replace cohomology by hypercohomology).

As in the classical case the following lemma yields an abstract form of class field theory

Lemma. *If* $(G, C^{\cdot})$ *is a generalized class formation, then for every open subgroup* H *of* G *there is a canonical map*

$$\rho_H \colon H^0(H, C^{\cdot}) \to H^{\mathrm{ab}}$$

such that the image of ρ_H *is dense in* H^{ab} *and such that for every pair of open subgroups* $V \leqslant U \leqslant G$, V *normal in* U, ρ_U *induces an isomorphism*

$$H^0(U, C^{\cdot}) / \mathrm{cor}_{U/V}\, H^0(V, C^{\cdot}) \xrightarrow{\sim} (U/V)^{\mathrm{ab}}.$$

11.3. Important complexes

In order to apply these concepts to higher dimensional class field theory we need complexes which are linked to K-theory as well as to the Galois cohomology groups $H^{n+1}(K, \mathbb{Q}/\mathbb{Z}(n))$. Natural candidates are the Beilinson–Lichtenbaum complexes.

Conjecture ([Li1]). *Let K be a field. There is a sequence of bounded complexes $\mathbb{Z}(n)$, $n \geqslant 0$, of G_K-modules such that*

(a) $\mathbb{Z}(0) = \mathbb{Z}$ *concentrated in degree* 0*;* $\mathbb{Z}(1) = \mathbb{G}_m[-1]$;
(b) $\mathbb{Z}(n)$ *is acyclic outside* $[1, n]$;
(c) *there are canonical maps* $\mathbb{Z}(m) \otimes^{\mathbb{L}} \mathbb{Z}(n) \to \mathbb{Z}(m+n)$;
(d) $H^{n+1}(K, \mathbb{Z}(n)) = 0$;
(e) *for every integer* m *there is a triangle* $\mathbb{Z}(n) \xrightarrow{m} \mathbb{Z}(n) \to \mathbb{Z}/m(n) \to \mathbb{Z}(n)[1]$ *in* $D(G_K)$;
(f) $H^n(K, \mathbb{Z}(n))$ *is identified with the Milnor* K*-group* $K_n(K)$.

Remarks. 1. This conjecture is very strong. For example, (d), (e), and (f) would imply the Milnor–Bloch–Kato conjecture stated in 4.1.

2. There are several candidates for $\mathbb{Z}(n)$, but only in the case where $n = 2$ proofs have been given so far, i.e. there exists a complex $\mathbb{Z}(2)$ satisfying (b), (d), (e) and (f) (see [Li2]).

By using the complex $\mathbb{Z}(2)$ defined by Lichtenbaum, Koya proved that for 2-dimensional local field K the pair $(G_K, \mathbb{Z}(2))$ is a class formation and deduced the reciprocity map for K (see [Ko1]). Once the existence of the $\mathbb{Z}(n)$ with the properties (b), (d), (e) and (f) above is established, his proof would work for arbitrary higher dimensional local fields as well (i.e. $(G_K, \mathbb{Z}(n))$ would be a class formation for an n-dimensional local field K).

However, for the purpose of applications to local class field theory it is enough to work with the following simple complexes which was first considered by B. Kahn [Kn].

Definition. Let $\check{\mathbb{Z}}(n) \in D(G_K)$ be the complex $\mathbb{G}_m^{\overset{\mathbb{L}}{\otimes} n}[-n]$.

Properties of $\check{\mathbb{Z}}(n)$.

(a) it is acyclic outside $[1, n]$;
(b) for every m prime to the characteristic of K if the latter is non-zero, there is a triangle

$$\check{\mathbb{Z}}(n) \xrightarrow{m} \check{\mathbb{Z}}(n) \to \mathbb{Z}/m(n) \to \check{\mathbb{Z}}(n)[1]$$

in $D(G_K)$;

(c) for every m as in (b) there is a commutative diagram

$$\begin{array}{ccc} K^{*\otimes n} & \longrightarrow & H^n(K, \check{\mathbb{Z}}(n)) \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow \\ K_n(K)/m & \longrightarrow & H^n(K, \mathbb{Z}/m(n)). \end{array}$$

where the bottom horizontal arrow is the Galois symbol and the left vertical arrow is given by $x_1 \otimes \cdots \otimes x_n \mapsto \{x_1, \ldots, x_n\} \mod m$.

The first two statements are proved in [Kn], the third in [Sp].

11.4. Applications to n-dimensional local class field theory

Let K be an n-dimensional local field. For simplicity we assume that $\mathrm{char}(K) = 0$. According to sections 3 and 5 for every finite extension L of K there are isomorphisms

(1) $$K_n(L)/m \xrightarrow{\sim} H^n(L, \mathbb{Z}/m(n)), \quad H^{n+1}(L, \mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

Lemma. $(G, \check{\mathbb{Z}}(n)[n])$ *is a generalized class formation.*

The triangle (b) above yields short exact sequences

$$0 \to H^i(K, \check{\mathbb{Z}}(n))/m \to H^i(K, \mathbb{Z}/m(n)) \to {}_mH^{i+1}(K, \check{\mathbb{Z}}(n)) \to 0$$

for every integer i. (1) and the diagram (c) show that ${}_mH^{n+1}(K, \check{\mathbb{Z}}(n)) = 0$ for all $m \neq 0$. By property (a) above $H^{n+1}(K, \check{\mathbb{Z}}(n))$ is a torsion group, hence $= 0$. Therefore (C1) holds for $(G, \check{\mathbb{Z}}(n)[n])$. For (C2) note that the above exact sequence for $i = n + 1$ yields $H^{n+1}(K, \mathbb{Z}/m(n)) \to {}_mH^{n+2}(K, \check{\mathbb{Z}}(n))$. By taking the direct limit over all m and using (1) we obtain

$$H^{n+2}(K, \check{\mathbb{Z}}(n)) \xrightarrow{\sim} H^{n+1}(K, \mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

Now we can establish the reciprocity map for K: put $C^{\cdot} = \check{\mathbb{Z}}(n)[n]$ and let L/K be a finite Galois extension of degree m. By applying abstract class field theory (see the lemma of 11.2) to $(G, C^{\cdot})$ we get

$$\begin{aligned} K_n(K)/N_{L/K}K_n(L) &\xrightarrow{\sim} H^n(K, \mathbb{Z}/m(n))/\operatorname{cor} H^n(L, \mathbb{Z}/m(n)) \\ &\xrightarrow{\sim} H^0(K, C^{\cdot})/m/\operatorname{cor} H^0(L, C^{\cdot})/m \xrightarrow{\sim} \mathrm{Gal}(L/K)^{\mathrm{ab}}. \end{aligned}$$

For the existence theorem see the previous section or Kato's paper in this volume.

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12. Two types of complete discrete valuation fields

Masato Kurihara

In this section we discuss results of a paper [Ku1] which is an attempt to understand the structure of the Milnor K-groups of complete discrete valuation fields of mixed characteristics in the case of an arbitrary residue field.

12.0. Definitions

Let K be a complete discrete valuation field of mixed characteristics $(0,p)$ with the ring of integers $\mathcal{O}_K$. We consider the p-adic completion $\widehat{\Omega}^1_{\mathcal{O}_K}$ of $\Omega^1_{\mathcal{O}_K/\mathbb{Z}}$ as in section 9.

Note that

(a) If K is a finite extension of $\mathbb{Q}_p$, then

$$\widehat{\Omega}^1_{\mathcal{O}_K} = (\mathcal{O}_K/\mathcal{D}_{K/\mathbb{Q}_p})d\pi$$

where $\mathcal{D}_{K/\mathbb{Q}_p}$ is the different of $K/\mathbb{Q}_p$, and π is a prime element of K.

(b) If $K = k\{\{t_1\}\}\dots\{\{t_{n-1}\}\}$ with $|k : \mathbb{Q}_p| < \infty$ (for the definition see subsection 1.1), then

$$\widehat{\Omega}^1_{\mathcal{O}_K} = (\mathcal{O}_k/\mathcal{D}_{k/\mathbb{Q}_p})d\pi \oplus \mathcal{O}_K dt_1 \oplus \dots \oplus \mathcal{O}_K dt_{n-1}$$

where π is a prime element of $\mathcal{O}_k$.

But in general, the structure of $\widehat{\Omega}^1_{\mathcal{O}_K}$ is a little more complicated. Let F be the residue field of K, and consider a natural map

$$\varphi\colon \widehat{\Omega}^1_{\mathcal{O}_K} \longrightarrow \Omega^1_F.$$

Definition. Let $\operatorname{Tors}\widehat{\Omega}^1_{\mathcal{O}_K}$ be the torsion part of $\widehat{\Omega}^1_{\mathcal{O}_K}$. If $\varphi(\operatorname{Tors}\widehat{\Omega}^1_{\mathcal{O}_K}) = 0$, K is said to be of *type I*, and said to be of *type II* otherwise.

So if K is a field in (a) or (b) as above, K is of type I.

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Let π be a prime element and $\{t_i\}$ be a lifting of a p-base of F. Then, there is a relation

$$a d\pi + \sum b_i dt_i = 0$$

with $a,\ b_i \in \mathcal{O}_K$. The field K is of type I if and only if $v_K(a) < \min_i v_K(b_i)$, where v_K is the normalized discrete valuation of K.

Examples.

(1) If $v_K(p)$ is prime to p, or if F is perfect, then K is of type I.

(2) The field $K = \mathbb{Q}_p\{\{t\}\}(\pi)$ with $\pi^p = pt$ is of type II. In this case we have

$$\widehat{\Omega}^1_{\mathcal{O}_K} \simeq \mathcal{O}_K/p \oplus \mathcal{O}_K.$$

The torsion part is generated by $dt - \pi^{p-1}d\pi$ (we have $pdt - p\pi^{p-1}d\pi = 0$), so $\varphi(dt - \pi^{p-1}d\pi) = dt \neq 0$.

12.1. The Milnor K-groups

Let π be a prime element, and put $e = v_K(p)$. Section 4 contains the definition of the homomorphism

$$\rho_m \colon \Omega_F^{q-1} \oplus \Omega_F^{q-2} \longrightarrow \mathrm{gr}_m K_q(K).$$

Theorem. *Put* $\ell = \mathrm{length}_{\mathcal{O}_K}(\mathrm{Tors}\,\widehat{\Omega}^1_{\mathcal{O}_K})$.

(a) *If K is of type I, then for $m \geqslant \ell + 1 + 2e/(p-1)$*

$$\rho_m|_{\Omega_F^{q-1}} \colon \Omega_F^{q-1} \longrightarrow \mathrm{gr}_m K_q(K)$$

is surjective.

(b) *If K is of type II, then for $m \geqslant \ell + 2e/(p-1)$ and for $q \geqslant 2$*

$$\rho_m|_{\Omega_F^{q-2}} \colon \Omega_F^{q-2} \longrightarrow \mathrm{gr}_m K_q(K)$$

is surjective.

For the proof we used the exponential homomorphism for the Milnor K-groups defined in section 9.

Corollary. *Define the subgroup $U_iK_q(K)$ of $K_q(K)$ as in section 4, and define the subgroup $V_iK_q(K)$ as generated by $\{1 + \mathcal{M}_K^i, \mathcal{O}_K^*, \dots, \mathcal{O}_K^*\}$ where $\mathcal{M}_K$ is the maximal ideal of $\mathcal{O}_K$.*

(a) *If K is of type I, then for sufficiently large m we have $U_mK_q(K) = V_mK_q(K)$.*

(b) *If K is of type II, then for sufficiently large m, we have $V_mK_q(K) = U_{m+1}K_q(K)$. Especially, $\mathrm{gr}_m K_q(K) = 0$ for sufficiently large m prime to p.*

Example. Let $K = \mathbb{Q}_p\{\{t\}\}(\pi)$ where $\pi^p = pt$ as in Example (2) of subsection 12.0, and assume $p > 2$. Then, we can determine the structures of $\mathrm{gr}_m K_q(K)$ as follows ([Ku2]).

For $m \leqslant p+1$, $\mathrm{gr}_m K_q(K)$ is determined by Bloch and Kato ([BK]). We have an isomorphism $\mathrm{gr}_0 K_2(K) = K_2(K)/U_1K_2(K) \simeq K_2(F) \oplus F^*$, and $\mathrm{gr}_p K_q(K)$ is a certain quotient of $\Omega^1_F/dF \oplus F$ (cf. [BK]). The homomorphism ρ_m induces an isomorphism from

$$\begin{cases} \Omega^1_F & \text{if } 1 \leqslant m \leqslant p-1 \text{ or } m = p+1 \\ 0 & \text{if } i \geqslant p+2 \text{ and } i \text{ is prime to } p \\ F/F^p & \text{if } m = 2p \\ & (x \mapsto \{1 + p\pi^p x, \pi\} \text{ induces this isomorphism}) \\ F^{p^{n-2}} & \text{if } m = np \text{ with } n \geqslant 3 \\ & (x \mapsto \{1 + p^n x, \pi\} \text{ induces this isomorphism}) \end{cases}$$

onto $\mathrm{gr}_m K_2(K)$.

12.2. Cyclic extensions

For cyclic extensions of K, by the argument using higher local class field theory and the theorem of 12.1 we have (cf. [Ku1])

Theorem. *Let ℓ be as in the theorem of 12.1.*

(a) *If K is of type I and $i \geqslant 1+\ell+2e/(p-1)$, then K does not have ferociously ramified cyclic extensions of degree p^i. Here, we call an extension L/K ferociously ramified if $|L:K| = |k_L : k_K|_{\mathrm{ins}}$ where k_L (resp. k_K) is the residue field of L (resp. K).*

(b) *If K is of type II and $i \geqslant \ell + 2e/(p-1)$, then K does not have totally ramified cyclic extensions of degree p^i.*

The bounds in the theorem are not so sharp. By some consideration, we can make them more precise. For example, using this method we can give a new proof of the following result of Miki.

Theorem (Miki, [M]). *If $e < p-1$ and L/K is a cyclic extension, the extension of the residue fields is separable.*

For $K = \mathbb{Q}_p\{\{t\}\}(\sqrt[p]{pt})$ with $p > 2$, we can show that it has no cyclic extensions of degree p^3.

Miki also showed that for any K, there is a constant c depending only on K such that K has no ferociously ramified cyclic extensions of degree p^i with $i > c$.

For totally ramified extensions, we guess the following. Let F^{p^∞} be the maximal perfect subfield of F, namely $F^{p^\infty} = \bigcap F^{p^n}$. We regard the ring of Witt vectors $W(F^{p^\infty})$ as a subring of $\mathcal{O}_K$, and write k_0 for the quotient field of $W(F^{p^\infty})$, and write k for the algebraic closure of k_0 in K. Then, k is a finite extension of k_0, and is a complete discrete valuation field of mixed characteristics $(0, p)$ with residue field F^{p^∞}.

Conjecture. *Suppose that $e(K|k) > 1$, i.e. a prime element of $\mathcal{O}_k$ is not a prime element of $\mathcal{O}_K$. Then there is a constant c depending only on K such that K has no totally ramified cyclic extension of degree p^i with $i > c$.*

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13. Abelian extensions of absolutely unramified complete discrete valuation fields

Masato Kurihara

In this section we discuss results of [K]. We assume that p is an odd prime and K is an absolutely unramified complete discrete valuation field of mixed characteristics $(0,p)$, so p is a prime element of the valuation ring $\mathcal{O}_K$. We denote by F the residue field of K.

13.1. The Milnor K-groups and differential forms

For $q>0$ we consider the Milnor K-group $K_q(K)$, and its p-adic completion $\widehat{K}_q(K)$ as in section 9. Let $U_1\widehat{K}_q(K)$ be the subgroup generated by $\{1+p\mathcal{O}_K, K^*, \dots, K^*\}$. Then we have:

Theorem. *Let K be as above. Then the exponential map* $\exp_p$ *for the element p, defined in section 9, induces an isomorphism*

$$\exp_p\colon \widehat{\Omega}^{q-1}_{\mathcal{O}_K}/pd\widehat{\Omega}^{q-2}_{\mathcal{O}_K} \xrightarrow{\sim} U_1\widehat{K}_q(K).$$

The group $\widehat{K}_q(K)$ carries arithmetic information of K, and the essential part of $\widehat{K}_q(K)$ is $U_1\widehat{K}_q(K)$. Since the left hand side $\widehat{\Omega}^{q-1}_{\mathcal{O}_K}/pd\widehat{\Omega}^{q-2}_{\mathcal{O}_K}$ can be described explicitly (for example, if F has a finite p-base I, $\widehat{\Omega}^1_{\mathcal{O}_K}$ is a free $\mathcal{O}_K$-module generated by $\{dt_i\}$ where $\{t_i\}$ are a lifting of elements of I), we know the structure of $U_1\widehat{K}_q(K)$ completely from the theorem.

In particular, for subquotients of $\widehat{K}_q(K)$ we have:

Corollary. *The map* $\rho_m\colon \Omega^{q-1}_F \oplus \Omega^{q-2}_F \longrightarrow \mathrm{gr}_m K_q(K)$ *defined in section 4 induces an isomorphism*

$$\Omega^{q-1}_F/B_{m-1}\Omega^{q-1}_F \xrightarrow{\sim} \mathrm{gr}_m K_q(K)$$

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where $B_{m-1}\Omega_F^{q-1}$ *is the subgroup of* Ω_F^{q-1} *generated by the elements* $a^{p^j} d\log a \wedge d\log b_1 \wedge \dots \wedge d\log b_{q-2}$ *with* $0 \leqslant j \leqslant m-1$ *and* $a,\ b_i \in F^*$.

13.2. Cyclic p-extensions of K

As in section 12, using some class field theoretic argument we get arithmetic information from the structure of the Milnor K-groups.

Theorem. *Let* $W_n(F)$ *be the ring of Witt vectors of length* n *over* F. *Then there exists a homomorphism*

$$\Phi_n\colon H^1(K,\mathbb{Z}/p^n) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\overline{K}/K),\mathbb{Z}/p^n) \longrightarrow W_n(F)$$

for any $n \geqslant 1$ *such that:*

(1) *The sequence*

$$0 \to H^1(K_{\mathrm{ur}}/K,\mathbb{Z}/p^n) \to H^1(K,\mathbb{Z}/p^n) \xrightarrow{\Phi_n} W_n(F) \to 0$$

is exact where K_{ur} *is the maximal unramified extension of* K.

(2) *The diagram*

$$\begin{tikzcd} H^1(K,\mathbb{Z}/p^{n+1}) \arrow[r, "p"] \arrow[d, "\Phi_{n+1}"] & H^1(K,\mathbb{Z}/p^n) \arrow[d, "\Phi_n"] \\ W_{n+1}(F) \arrow[r, "\mathbf{F}"] & W_n(F) \end{tikzcd}$$

is commutative where $\mathbf{F}$ *is the Frobenius map.*

(3) *The diagram*

$$\begin{tikzcd} H^1(K,\mathbb{Z}/p^{n}) \arrow[r] \arrow[d, "\Phi_{n}"] & H^1(K,\mathbb{Z}/p^{n+1}) \arrow[d, "\Phi_{n+1}"] \\ W_{n}(F) \arrow[r, "\mathbf{V}"] & W_{n+1}(F) \end{tikzcd}$$

is commutative where $\mathbf{V}((a_0,\dots,a_{n-1})) = (0,a_0,\dots,a_{n-1})$ *is the Verschiebung map.*

(4) *Let* E *be the fraction field of the completion of the localization* $O_K[T]_{(p)}$ *(so the residue field of* E *is* $F(T)$*). Let*

$$\lambda\colon W_n(F) \times W_n(F(T)) \xrightarrow{\rho} {}_{p^n}\mathrm{Br}(F(T)) \oplus H^1(F(T),\mathbb{Z}/p^n)$$

be the map defined by $\lambda(w,w') = (i_2(p^{n-1}w\,dw'), i_1(ww'))$ *where* ${}_{p^n}\mathrm{Br}(F(T))$ *is the* p^n*-torsion of the Brauer group of* $F(T)$, *and we consider* $p^{n-1}w\,dw'$ *as an element of* $W_n\Omega^1_{F(T)}$ *(*$W_n\Omega^{\cdot}_{F(T)}$ *is the de Rham Witt complex). Let*

$$i_1\colon W_n(F(T)) \longrightarrow H^1(F(T),\mathbb{Z}/p^n)$$

be the map defined by Artin–Schreier–Witt theory, and let

$$i_2\colon W_n\Omega^1_{F(T)} \longrightarrow {}_{p^n}\operatorname{Br}(F(T))$$

be the map obtained by taking Galois cohomology from an exact sequence

$$0 \longrightarrow (F(T)^{\mathrm{sep}})^*/((F(T)^{\mathrm{sep}})^*)^{p^n} \longrightarrow W_n\Omega^1_{F(T)^{\mathrm{sep}}} \longrightarrow W_n\Omega^1_{F(T)^{\mathrm{sep}}} \longrightarrow 0.$$

Then we have a commutative diagram

$$\begin{array}{ccccc} H^1(K,\mathbb{Z}/p^n) & \times E^*/(E^*)^{p^n} & \xrightarrow{\ \cup\ } & \operatorname{Br}(E) \\ \Phi_n\big\downarrow & \big\uparrow\psi_n & & \big\uparrow i \\ W_n(F) & \times W_n(F(T)) & \xrightarrow{\ \lambda\ } & {}_{p^n}\operatorname{Br}(F(T)) \oplus H^1(F(T),\mathbb{Z}/p^n) \end{array}$$

where i *is the map in subsection 5.1, and*

$$\psi_n((a_0,\dots,a_{n-1})) = \exp\Big(\sum_{i=0}^{n-1}\sum_{j=1}^{n-i} p^{i+j}\,\widetilde{a}_i^{\,p^{n-i-j}}\Big)$$

($\widetilde{a}_i$ *is a lifting of* a_i *to* $\mathcal{O}_K$ *).*

(5) *Suppose that* $n = 1$ *and* F *is separably closed. Then we have an isomorphism*

$$\Phi_1\colon H^1(K,\mathbb{Z}/p) \simeq F.$$

Suppose that $\Phi_1(\chi) = a$*. Then the extension* L/K *which corresponds to the character* χ *can be described as follows. Let* $\widetilde{a}$ *be a lifting of* a *to* $\mathcal{O}_K$*. Then* $L = K(x)$ *where* x *is a solution of the equation*

$$X^p - X = \widetilde{a}/p.$$

The property (4) characterizes Φ_n.

Corollary (Miki)**.** *Let* $L = K(x)$ *where* $x^p - x = a/p$ *with some* $a \in \mathcal{O}_K$*.* L *is contained in a cyclic extension of* K *of degree* p^n *if and only if*

$$a \bmod p \in F^{p^{n-1}}.$$

This follows from parts (2) and (5) of the theorem. More generally:

Corollary. *Let* χ *be a character corresponding to the extension* L/K *of degree* p^n*, and* $\Phi_n(\chi) = (a_0,\dots,a_{n-1})$*. Then for* $m > n$*,* L *is contained in a cyclic extension of* K *of degree* p^m *if and only if* $a_i \in F^{p^{m-n}}$ *for all* i *such that* $0 \leqslant i \leqslant n-1$.

Remarks.

(1) Fesenko gave a new and simple proof of this theorem from his general theory on totally ramified extensions (cf. subsection 16.4).

(2) For any $q > 0$ we can construct a homomorphism

$$\Phi_n\colon H^q(K, \mathbb{Z}/p^n(q-1)) \longrightarrow W_n\Omega_F^{q-1}$$

by the same method. By using this homomorphism, we can study the Brauer group of K, for example.

Problems.

(1) Let $\chi_{Ж}$ be the character of the extension constructed in 14.1. Calculate $\Phi_n(\chi_{Ж})$.

(2) Assume that F is separably closed. Then we have an isomorphism

$$\Phi_n\colon H^1(K, \mathbb{Z}/p^n) \simeq W_n(F).$$

This isomorphism is reminiscent of the isomorphism of Artin–Schreier–Witt theory. For $w = (a_0, \dots, a_{n-1}) \in W_n(F)$, can one give an explicit equation of the corresponding extension L/K using $a_0, \dots, a_{n-1}$ for $n \geqslant 2$ (where L/K corresponds to the character χ such that $\Phi_n(\chi) = w$)?

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14. Explicit abelian extensions of complete discrete valuation fields

Igor Zhukov

14.0

For higher class field theory Witt and Kummer extensions are very important. In fact, Parshin's construction of class field theory for higher local fields of prime characteristic [P] is based on an explicit (Artin–Schreier–Witt) pairing; see [F] for a generalization to the case of a perfect residue field. Kummer extensions in the mixed characteristic case can be described by using class field theory and Vostokov's symbol [V1], [V2]; for a perfect residue field, see [V3], [F].

An explicit description of non Kummer abelian extensions for a complete discrete valuation field K of characteristic 0 with residue field k_K of prime characteristic p is an open problem. We are interested in totally ramified extensions, and, therefore, in p-extensions (tame totally ramified abelian extensions are always Kummer and their class field theory can be described by means of the higher tame symbol defined in subsection 6.4.2).

In the case of an absolutely unramified K there is a beautiful description of all abelian totally ramified p-extensions in terms of Witt vectors over k_K by Kurihara (see section 13 and [K]). Below we give another construction of some totally ramified cyclic p-extensions for such K. The construction is complicated; however, the extensions under consideration are constructed explicitly, and eventually we obtain a certain description of the whole maximal abelian extension of K. Proofs are given in [VZ].

14.1

We recall that cyclic extensions of K of degree p can be described by means of Artin–Schreier extensions, see [FV, III.2]. Namely, for a cyclic L/K of degree p we have $L = K(x)$, $x^p - x = a$, where $v_K(a) = -1$ if L/K is totally ramified, and $v_K(a) = 0$ if L/K is unramified.

Notice that if $v_K(a_1 - a_2) \geqslant 0$, then for corresponding cyclic extensions L_1/K and L_2/K we have $L_1 K_{\mathrm{ur}} = L_2 K_{\mathrm{ur}}$. (If $v_K(a_1 - a_2) \geqslant 1$, then, moreover, $L_1 = L_2$.) We obtain immediately the following description of the maximal abelian extension of K of exponent p: $K^{\mathrm{ab},p} = K_{\mathrm{ur}}^{\mathrm{ab},p} \prod_d K_d$, where $K_d = K(x)$, $x^p - x = -p^{-1}d$, and d runs over any fixed system of representatives of k_K^* in $\mathcal{O}_K$. This is a part of a more precise statement at the end of the next subsection.

14.2

It is easy to determine whether a given cyclic extension L/K of degree p can be embedded into a cyclic extension of degree p^n, $n \geqslant 2$.

Proposition. *In the above notation, let b be the residue of pa in k_K. Then there is a cyclic extension M/K of degree p^n such that $L \subset M$ if and only if $b \in {k_K}^{p^{n-1}}$.*

The proof is based on the following theorem of Miki [M]. Let F be a field of characteristic not equal to p and let $\zeta_p \in F$. Let $L = F(\alpha)$, $\alpha^p = a \in F$. Then $a \in F^{*p} N_{F(\zeta_{p^n})/F} F(\zeta_{p^n})^*$ if and only if there is a cyclic extension M/F of degree p^n such that $L \subset M$.

Corollary. *Denote by K^{ab,p^n} (respectively $K_{\mathrm{ur}}^{\mathrm{ab},p^n}$) the maximal abelian (respectively abelian unramified) extension of K of exponent p^n. Choose $A_i \subset \mathcal{O}_K$, $1 \leqslant i \leqslant n$, in such a way that $\{\overline{d} : d \in A_i\}$ is an $\mathbb{F}_p$-basis of $k_K^{p^{i-1}}/k_K^{p^i}$ for $i \leqslant n-1$ and an $\mathbb{F}_p$-basis of $k_K^{p^{n-1}}$ for $i = n$. Let $K_{i,d}$ ($d \in A_i$) be any cyclic extension of degree p^i that contains x with $x^p - x = -p^{-1}d$. Then $K^{\mathrm{ab},p^n}/K$ is the compositum of linearly disjoint extensions $K_{i,d}/K$ ($1 \leqslant i \leqslant n$; d runs over A_i) and $K_{\mathrm{ur}}^{\mathrm{ab},p^n}/K$.*

From now on, let $p > 3$. For any $n \geqslant 1$ and any $b \in k_K^{p^{n-1}}$, we shall give a construction of a cyclic extension $K_{n,d}/K$ of degree p^n such that $x \in K_{n,d}$, $x^p - x = -p^{-1}d$, where $d \in \mathcal{O}_K$ is such that its residue $\overline{d}$ is equal to b.

14.3

Denote by G the Lubin–Tate formal group over $\mathbb{Z}_p$ such that multiplication by p in it takes the form $[p]_G(X) = pX + X^p$.

Let $\mathcal{O}$ be the ring of integers of the field E defined in (2) of Theorem 13.2, and v the valuation on E.

Proposition. *There exist $g_i \in \mathcal{O}$, $i \in \mathbb{Z}$, and $R_i \in \mathcal{O}$, $i \geqslant 0$, satisfying the following conditions.*

(1) $g_0 \equiv 1 \mod p\mathcal{O}$, $g_i \equiv 0 \mod p\mathcal{O}$ *for* $i \neq 0$.

(2) $R_0 = T$.

(3) $v(g_i) \geqslant -i + 2 + \left[\frac{i}{p}\right] + \left[\frac{i-2}{p}\right]$ *for* $i \leqslant -1$.

(4) *Let* $g(X) = \sum_{-\infty}^{\infty} g_i X^{i(p-1)+1}$, $R(X,T) = \sum_{i=0}^{\infty} R_i X^{i(p-1)+1}$. *Then*

$$g(X) +_G [p]_G R(g(X),T) = g(X +_G R([p]_G X, T^p)).$$

Remark. We do not expect that the above conditions determine g_i and R_i uniquely. However, in [VZ] a certain canonical way to construct (g,R) by a process of the p-adic approximation is given.

Fix a system (g,R) satisfying the above conditions. Denote

$$S = \sum_{i=0}^{\infty} S_i(T) X^{i(p-1)+1} = T^{-1}X + \dots$$

the series which is inverse to R with respect to substitution in $\mathcal{O}[[X]]$.

Theorem. *Let $d \in \mathcal{O}_K^*$. Consider $\beta_1, \dots, \beta_n \in K^{\mathrm{sep}}$ such that*

$$\beta_1^p - \beta_1 = -p^{-1} \sum_{i\geqslant 0} S_i(d^{p^{n-1}})(-p)^i,$$

$$\beta_j^p - \beta_j = -p^{-1} \sum_{-\infty}^{+\infty} g_i(d^{p^{n-j}})(-p)^i \beta_{j-1}^{i(p-1)+1}, \quad j \geqslant 2.$$

Then $K_{n,d^{p^{n-1}}} = K(\beta_1, \dots, \beta_n)$ is a cyclic extension of K of degree p^n containing a zero of the polynomial $X^p - X + p^{-1}d^{p^{n-1}}$.

Remark. We do not know which Witt vector corresponds to $K_{n,d^{p^{n-1}}}/K$ in Kurihara's theory (cf. section 13). However, one could try to construct a parallel theory in which (the canonical character of) this extension would correspond to $(\overline{d}^{p^{n-1}}, 0, 0, \dots) \in W_n(k_K)$.

14.4

If one is interested in explicit equations for abelian extensions of K of exponent p^n for a fixed n, then it is sufficient to compute a certain p-adic approximation to g (resp. R) by polynomials in $\mathbb{Z}_{(p)}[T, T^{-1}, X, X^{-1}]$ (resp. $\mathbb{Z}_{(p)}[T, T^{-1}, X]$). Let us make this statement more precise.

In what follows we consider a fixed pair (g,R) constructed in [VZ]. Denote

$$K_{j,d^{p^{n-1}}} = K(\beta_1, \dots, \beta_j).$$

Let v be the (non-normalized) extension of the valuation of K to $K_{n,d^{p^{n-1}}}$. Then $v(\beta_j) = -p^{-1} - \dots - p^{-j}$, $j = 1, \dots, n$.

We assert that in the defining equations for $K_{n,d^{p^{n-1}}}$ the pair (g,R) can be replaced with $(\widetilde{g}, \widetilde{R})$ such that

(1) $v(\widetilde{g}_i - g_i) > n + \max_{j=1,\dots,n-1}(-j - i\cdot p^{-j} + p^{-1} + p^{-2} + \dots + p^{-j}), \quad i \in \mathbb{Z}$,

and

(2) $$v(\widetilde{R}_i - R_i) > n - i, \quad i \geqslant 0.$$

Theorem. *Assume that the pair* $(\widetilde{g}, \widetilde{R})$ *satisfies* (1) *and* (2)*. Define* $\widetilde{S}$ *as* $\widetilde{R}^{-1}$*. Let*

$$\widetilde{\beta}_1^p - \widetilde{\beta}_1 = -p^{-1} \sum_{i \geqslant 0} \widetilde{S}_i(d^{p^{n-1}})(-p)^i,$$

$$\widetilde{\beta}_j^p - \widetilde{\beta}_j = -p^{-1} \sum_{-\infty}^{+\infty} \widetilde{g}_i(d^{p^{n-j}})(-p)^i \widetilde{\beta}_{j-1}^{i(p-1)+1}, \quad j \geqslant 2.$$

Then $K(\widetilde{\beta}_1, \dots, \widetilde{\beta}_n) = K(\beta_1, \dots, \beta_n)$.

Proof. It is easy to check by induction on j that $\widetilde{\beta}_j \in K_{j,d^{p^{n-1}}}$ and $v(\widetilde{\beta}_j - \beta_j) > n - j$, $j = 1, \dots, n$.

Remark. For a fixed n, one may take $\widetilde{R}_i = 0$ for $i \geqslant n$, $\widetilde{g}_i = 0$ for all sufficiently small or sufficiently large i.

14.5

If we consider non-strict inequalities in (1) and (2), then we obtain an extension $\widetilde{K}_{n,d^{p^{n-1}}}$ such that $\widetilde{K}_{n,d^{p^{n-1}}} K_{\mathrm{ur}} = K_{n,d^{p^{n-1}}} K_{\mathrm{ur}}$. In particular, let $n = 2$. Calculation of (R,g) in [VZ] shows that

$$g_i \overset{p^2\mathcal{O}}{\equiv} \begin{cases} 0, & i < -1 \\ p \cdot \frac{T^{1-p}-1}{2}, & i = -1 \\ 1 + p \cdot \frac{T^{1-p}-1}{2}(1 - T^p), & i = 0 \end{cases}$$

Therefore, one may take $\widetilde{g}_i = 0$ for $i < -1$ or $i > 0$, $\widetilde{g}_{-1} = p \cdot \frac{T^{1-p}-1}{2}$, $\widetilde{g}_0 = 1 + p \cdot \frac{T^{1-p}-1}{2}(1 - T^p)$. Further, one may take $\widetilde{R} = TX$. Thus, we obtain the following

Theorem. *For any* $d \in \mathcal{O}_K^*$, *let* $\widetilde{K}_{1,d} = K(y)$, *where* $y^p - y = -p^{-1}d$. *Next, let* $\widetilde{K}_{2,d^p} = K(y_1, y_2)$, *where*

$$y_1^p - y_1 = -p^{-1}d^p,$$
$$y_2^p - y_2 = -p^{-1}y_1 + p^{-1} \cdot \frac{d^{1-p}-1}{2} y_1^{-p+2} - \frac{d^{1-p}-1}{2}(1-d^p)y_1.$$

Then

1. *All* $\widetilde{K}_{1,d}/K$ *are cyclic of degree* p, *and all* $\widetilde{K}_{2,d^p}/K$ *are cyclic of degree* p^2.
2. $K^{\mathrm{ab},p^2}/K$ *is the compositum of linearly disjoint extensions described below:*
   (a) $\widetilde{K}_{1,d}/K$, *where* d *runs over a system of representatives of an* $\mathbb{F}_p$*-basis of* k_K/k_K^p;
   (b) $\widetilde{K}_{2,d^p}/K$, *where* d *runs over a system of representatives of an* $\mathbb{F}_p$*-basis of* k_K;
   (c) $K_{\mathrm{ur}}^{\mathrm{ab},p^2}/K$.

14.6

One of the goals of developing explicit constructions for abelian extensions would be to write down explicit formulas for class field theory. We are very far from this goal in the case of non Kummer extensions of an absolutely unramified higher local field. However, the K-group involved in the reciprocity map can be computed for such fields in a totally explicit way.

Let K be an absolutely unramified n-dimensional local field with any perfect residue field. Then [Z, §11] gives an explicit description of

$$U(1)K_n^{\mathrm{top}}K = \langle \{\alpha, \beta_1, \ldots, \beta_{n-1}\} : \alpha, \beta_i \in K^*, v(\alpha - 1) > 0 \rangle.$$

Notice that the structure of $K_n^{\mathrm{top}}K/U(1)K_n^{\mathrm{top}}K$, i.e., the quotient group responsible for tamely ramified extensions, is well known. We cite here a result in the simplest possible case $K = \mathbb{Q}_p\{\{t\}\}$.

Theorem. *Let* $K = \mathbb{Q}_p\{\{t\}\}$.

1. *For every* $\alpha \in U_1K_2^{\mathrm{top}}(K)$ *there are* $n_j \in \mathbb{Z}_p$, $j \in \mathbb{Z} \setminus \{0\}$ *which are uniquely determined modulo* $p^{v_{\mathbb{Q}_p}(j)+1}$ *and there is* $n_0 \in \mathbb{Z}_p$ *which is uniquely determined such that*
$$\alpha = \sum_j n_j \{1 - pt^j, t\}.$$
2. *For any* $j \neq 0$ *we have*
$$p^{v_{\mathbb{Q}_p}(j)+1}\{1 - pt^j, t\} = 0.$$

Proof. Use explicit class field theory of section 10 and the above mentioned theorem of Miki.

Question. How does $\{1-pt^j, t\}$ act on $K_{n,d^{p^{n-1}}}$?

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15. On the structure of the Milnor K-groups of complete discrete valuation fields

Jinya Nakamura

15.0. Introduction

For a discrete valuation field K the unit group K^* of K has a natural decreasing filtration with respect to the valuation, and the graded quotients of this filtration are written in terms of the residue field. The Milnor K-group $K_q(K)$ is a generalization of the unit group and it also has a natural decreasing filtration defined in section 4. However, if K is of mixed characteristic and has absolute ramification index greater than one, the graded quotients of this filtration are known in some special cases only.

Let K be a complete discrete valuation field with residue field $k = k_K$; we keep the notations of section 4. Put $v_p = v_{\mathbb{Q}_p}$.

A description of $\mathrm{gr}_n K_q(K)$ is known in the following cases:

(i) (Bass and Tate [BT]) $\mathrm{gr}_0 K_q(K) \simeq K_q(k) \oplus K_{q-1}(k)$.

(ii) (Graham [G]) If the characteristic of K and k is zero, then $\mathrm{gr}_n K_q(K) \simeq \Omega_k^{q-1}$ for all $n \geqslant 1$.

(iii) (Bloch [B], Kato [Kt1]) If the characteristic of K and of k is $p > 0$ then

$$\mathrm{gr}_n K_q(K) \simeq \operatorname{coker}\left(\Omega_k^{q-2} \longrightarrow \Omega_k^{q-1}/B_s^{q-1} \oplus \Omega_k^{q-2}/B_s^{q-2}\right)$$

where $\omega \longmapsto (\mathrm{C}^{-s}(d\omega), (-1)^q m\, \mathrm{C}^{-s}(\omega)$ and where $n \geqslant 1$, $s = v_p(n)$ and $m = n/p^s$.

(iv) (Bloch–Kato [BK]) If K is of mixed characteristic $(0,p)$, then

$$\mathrm{gr}_n K_q(K) \simeq \operatorname{coker}\left(\Omega_k^{q-2} \longrightarrow \Omega_k^{q-1}/B_s^{q-1} \oplus \Omega_k^{q-2}/B_s^{q-2}\right)$$

where $\omega \longmapsto (\mathrm{C}^{-s}(d\omega), (-1)^q m\, \mathrm{C}^{-s}(\omega))$ and where $1 \leqslant n < ep/(p-1)$ for $e = v_K(p)$, $s = v_p(n)$ and $m = n/p^s$; and

$$\mathrm{gr}_{\frac{ep}{p-1}} K_q(K) \simeq \operatorname{coker}\left(\Omega_k^{q-2} \longrightarrow \Omega_k^{q-1}/(1+a\,\mathrm{C})B_s^{q-1} \oplus \Omega_k^{q-2}/(1+a\,\mathrm{C})B_s^{q-2}\right)$$

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where $\omega \longmapsto ((1+a\,\mathrm{C})\,\mathrm{C}^{-s}(d\omega), (-1)^q m(1+a\,\mathrm{C})\,\mathrm{C}^{-s}(\omega))$ and where a is the residue class of p/π^e for fixed prime element of K, $s = v_p(ep/(p-1))$ and $m = ep/(p-1)p^s$.

(v) (Kurihara [Ku1], see also section 13) If K is of mixed characteristic $(0,p)$ and absolutely unramified (i.e., $v_K(p) = 1$), then $\mathrm{gr}_n K_q(K) \simeq \Omega_k^{q-1}/B_{n-1}^{q-1}$ for $n \geqslant 1$.

(vi) (Nakamura [N2]) If K is of mixed characteristic $(0,p)$ with $p > 2$ and $p \nmid e = v_K(p)$, then

$$\mathrm{gr}_n K_q(K) \simeq \begin{cases} \text{as in (iv)} & (1 \leqslant n \leqslant ep/(p-1)) \\ \Omega_k^{q-1}/B_{l_n+s_n}^{q-1} & (n > ep/(p-1)) \end{cases}$$

where l_n is the maximal integer which satisfies $n - l_n e \geqslant e/(p-1)$ and $s_n = v_p(n - l_n e)$.

(vii) (Kurihara [Ku3]) If K_0 is the fraction field of the completion of the localization $\mathbb{Z}_p[T]_{(p)}$ and $K = K_0(\sqrt[p]{pT})$ for a prime $p \neq 2$, then

$$\mathrm{gr}_n K_2(K) \simeq \begin{cases} \text{as in (iv)} & (1 \leqslant n \leqslant p) \\ k/k^p & (n = 2p) \\ k^{p^{l-2}} & (n = lp, l \geqslant 3) \\ 0 & (\text{otherwise}). \end{cases}$$

(viii) (Nakamura [N1]) Let K_0 be an absolutely unramified complete discrete valuation field of mixed characteristic $(0,p)$ with $p > 2$. If $K = K_0(\zeta_p)(\sqrt[p]{\pi})$ where π is a prime element of $K_0(\zeta_p)$ such that $d\pi^{p-1} = 0$ in $\Omega^1_{\mathcal{O}_{K_0(\zeta_p)}}$, then $\mathrm{gr}_n K_q(K)$ are determined for all $n \geqslant 1$. This is complicated, so we omit the details.

(ix) (Kahn [Kh]) Quotients of the Milnor K-groups of a complete discrete valuation field K with perfect residue field are computed using symbols.

Recall that the group of units $U_{1,K}$ can be described as a topological $\mathbb{Z}_p$-module. As a generalization of this classical result, there is an appraoch different from (i)-(ix) for higher local fields K which uses topological convergence and

$$K_q^{\mathrm{top}}(K) = K_q(K)/\cap_{l\geqslant 1}\, lK_q(K)$$

(see section 6). It provides not only the description of $\mathrm{gr}_n K_q(K)$ but of the whole $K_q^{\mathrm{top}}(K)$ in characteristic p (Parshin [P]) and in characteristic 0 (Fesenko [F]). A complete description of the structure of $K_q^{\mathrm{top}}(K)$ of some higher local fields with small ramification is given by Zhukov [Z].

Below we discuss (vi).

15.1. Syntomic complex and Kurihara's exponential homomorphism

15.1.1. Syntomic complex. Let $A = \mathcal{O}_K$ and let A_0 be the subring of A such that A_0 is a complete discrete valuation ring with respect to the restriction of the valuation of K, the residue field of A_0 coincides with $k = k_K$ and A_0 is absolutely unramified. Let π be a fixed prime of K. Let $B = A_0[[X]]$. Define

$$\mathcal{J} = \ker[B \xrightarrow{X \mapsto \pi} A]$$
$$\mathcal{I} = \ker[B \xrightarrow{X \mapsto \pi} A \xrightarrow{\bmod p} A/p] = \mathcal{J} + pB.$$

Let D and $J \subset D$ be the PD-envelope and the PD-ideal with respect to $B \to A$, respectively. Let $I \subset D$ be the PD-ideal with respect to $B \to A/p$. Namely,

$$D = B\left[\frac{x^j}{j!}\ ;\ j \geqslant 0, x \in \mathcal{J}\right], \quad J = \ker(D \to A), \quad I = \ker(D \to A/p).$$

Let $J^{[r]}$ (resp. $I^{[r]}$) be the r-th divided power, which is the ideal of D generated by

$$\left\{\frac{x^j}{j!}\ ;\ j \geqslant r, x \in \mathcal{J}\right\}, \ \left(\text{resp. } \left\{\frac{x^i}{i!}\frac{p^j}{j!}\ ;\ i+j \geqslant r, x \in \mathcal{I}\right\}\right).$$

Notice that $I^{[0]} = J^{[0]} = D$. Let $I^{[n]} = J^{[n]} = D$ for a negative n. We define the complexes $\mathbb{J}^{[q]}$ and $\mathbb{I}^{[q]}$ as

$$\mathbb{J}^{[q]} = [J^{[q]} \xrightarrow{d} J^{[q-1]} \otimes_B \widehat{\Omega}^1_B \xrightarrow{d} J^{[q-2]} \otimes_B \widehat{\Omega}^2_B \longrightarrow \cdots]$$
$$\mathbb{I}^{[q]} = [I^{[q]} \xrightarrow{d} I^{[q-1]} \otimes_B \widehat{\Omega}^1_B \xrightarrow{d} I^{[q-2]} \otimes_B \widehat{\Omega}^2_B \longrightarrow \cdots]$$

where $\widehat{\Omega}^q_B$ is the p-adic completion of Ω^q_B. We define $\mathbb{D} = \mathbb{I}^{[0]} = \mathbb{J}^{[0]}$.

Let $\mathbb{T}$ be a fixed set of elements of A_0^* such that the residue classes of all $T \in \mathbb{T}$ in k forms a p-base of k. Let f be the Frobenius endomorphism of A_0 such that $f(T) = T^p$ for any $T \in \mathbb{T}$ and $f(x) \equiv x^p \mod p$ for any $x \in A_0$. We extend f to B by $f(X) = X^p$, and to D naturally. For $0 \leqslant r < p$ and $0 \leqslant s$, we get

$$f(J^{[r]}) \subset p^r D, \quad f(\widehat{\Omega}^s_B) \subset p^s \widehat{\Omega}^s_B,$$

since

$$f(x^{[r]}) = (x^p + py)^{[r]} = (p! x^{[p]} + py)^{[r]} = p^{[r]}((p-1)! x^{[p]} + y)^r,$$
$$f\Big(z\frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_s}{T_s}\Big) = z\frac{dT_1^p}{T_1^p} \wedge \cdots \wedge \frac{dT_s^p}{T_s^p} = zp^s\frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_s}{T_s},$$

where $x \in \mathcal{J}$, y is an element which satisfies $f(x) = x^p + py$, and $T_1, \ldots, T_s \in \mathbb{T} \cup \{X\}$. Thus we can define

$$f_q = \frac{f}{p^q}\colon J^{[r]} \otimes \widehat{\Omega}^{q-r}_B \longrightarrow D \otimes \widehat{\Omega}^{q-r}_B$$

for $0 \leqslant r < p$. Let $\mathscr{S}(q)$ and $\mathscr{S}'(q)$ be the mapping fiber complexes (cf. Appendix) of

$$\mathbb{J}^{[q]} \xrightarrow{1-f_q} \mathbb{D} \quad \text{and} \quad \mathbb{I}^{[q]} \xrightarrow{1-f_q} \mathbb{D}$$

respectively, for $q < p$. For simplicity, from now to the end, we assume p is large enough to treat $\mathscr{S}(q)$ and $\mathscr{S}'(q)$. $\mathscr{S}(q)$ is called the *syntomic complex* of A with respect to B, and $\mathscr{S}'(q)$ is also called the *syntomic complex* of A/p with respect to B (cf. [Kt2]).

Theorem 1 (Kurihara [Ku2]). *There exists a subgroup S^q of $H^q(\mathscr{S}(q))$ such that $U_X H^q(\mathscr{S}(q)) \simeq U_1 \widehat{K_q}(A)$ where $\widehat{K_q}(A) = \varprojlim K_q(A)/p^n$ is the p-adic completion of $K_q(A)$ (see subsection 9.1).*

Outline of the proof. Let $U_X(D \otimes \widehat{\Omega}_B^{q-1})$ be the subgroup of $D \otimes \widehat{\Omega}_B^{q-1}$ generated by $XD \otimes \widehat{\Omega}_B^{q-1}$, $D \otimes \widehat{\Omega}_B^{q-2} \wedge dX$ and $I \otimes \widehat{\Omega}_B^{q-1}$, and let

$$S^q = U_X(D \otimes \widehat{\Omega}_B^{q-1})/((dD \otimes \widehat{\Omega}_B^{q-2} + (1-f_q)J \otimes \widehat{\Omega}_B^{q-1}) \cap U_X(D \otimes \widehat{\Omega}_B^{q-1})).$$

The infinite sum $\sum_{n \geqslant 0} f_q^n(dx)$ converges in $D \otimes \widehat{\Omega}_B^q$ for $x \in U_X(D \otimes \widehat{\Omega}_B^{q-1})$. Thus we get a map

$$\begin{aligned} U_X(D \otimes \widehat{\Omega}_B^{q-1}) &\longrightarrow H^q(\mathscr{S}(q)) \\ x &\longmapsto \Big(x, \sum_{n=0}^{\infty} f_q^n(dx)\Big) \end{aligned}$$

and we may assume S^q is a subgroup of $H^q(\mathscr{S}(q))$. Let E_q be the map

$$\begin{aligned} E_q\colon U_X(D \otimes \widehat{\Omega}_B^{q-1}) &\longrightarrow \widehat{K}_q(A) \\ x\frac{dT_1}{T_1} \wedge \dots \wedge \frac{dT_{q-1}}{T_{q-1}} &\longmapsto \{E_1(x), T_1, \dots, T_{q-1}\}, \end{aligned}$$

where $E_1(x) = \exp \circ (\sum_{n \geqslant 0} f_1^n)(x)$ is *Artin–Hasse's exponential homomorphism*. In [Ku2] it was shown that E_q vanishes on

$$(dD \otimes \widehat{\Omega}_B^{q-2} + (1-f_q)J \otimes \widehat{\Omega}_B^{q-1}) \cap U_X(D \otimes \widehat{\Omega}_B^{q-1}),$$

hence we get the map

$$E_q\colon S^q \longrightarrow \widehat{K}_q(A).$$

The image of E_q coincides with $U_1 \widehat{K}_q(A)$ by definition.

On the other hand, define $s_q\colon \widehat{K}_q(A) \longrightarrow S^q$ by

$$\begin{aligned} &s_q(\{a_1, \dots, a_q\}) \\ &= \sum_{i=1}^{q} (-1)^{i-1} \frac{1}{p} \log\Big(\frac{f(\widetilde{a}_i)}{\widetilde{a}_i{}^p}\Big) \frac{d\widetilde{a}_1}{\widetilde{a}_1} \wedge \dots \wedge \frac{d\widetilde{a_{i-1}}}{\widetilde{a_{i-1}}} \wedge f_1\Big(\frac{d\widetilde{a_{i+1}}}{\widetilde{a_{i+1}}}\Big) \wedge \dots \wedge f_1\Big(\frac{d\widetilde{a}_q}{\widetilde{a}_q}\Big) \end{aligned}$$

(cf. [Kt2], compare with the series Φ in subsection 8.3), where $\widetilde{a}$ is a lifting of a to D. One can check that $s_q \circ E_q = -\mathrm{id}$. Hence $S^q \simeq U_1\widehat{K}_q(A)$. Note that if $\zeta_p \in K$, then one can show $U_1\widehat{K}_q(A) \simeq U_1\widehat{K}_q(K)$ (see [Ku4] or [N2]), thus we have $S^q \simeq U_1\widehat{K}_q(K)$. □

Example. We shall prove the equality $s_q \circ E_q = -\mathrm{id}$ in the following simple case. Let $q = 2$. Take an element $adT/T \in U_X(D \otimes \widehat{\Omega}_B^{q-1})$ for $T \in \mathbb{T} \cup \{X\}$. Then

$$\begin{aligned}
&s_q \circ E_q\big(a\frac{dT}{T}\big)\\
&= s_q(\{E_1(\widetilde{a}), T\})\\
&= \frac{1}{p}\log\left(\frac{f(E_1(a))}{E_1(a)^p}\right) f_1\left(\frac{dT}{T}\right)\\
&= \frac{1}{p}\left(\log\circ f\circ\exp\circ\sum_{n\geqslant 0} f_1^n(a) - p\log\circ\exp\circ\sum_{n\geqslant 0} f_1^n(a)\right)\frac{dT}{T}\\
&= \left(f_1\sum_{n\geqslant 0} f_1^n(a) - \sum_{n\geqslant 0} f_1^n(a)\right)\frac{dT}{T}\\
&= -a\frac{dT}{T}.
\end{aligned}$$

15.1.2. Exponential Homomorphism. The usual exponential homomorphism

$$\begin{aligned}
\exp_\eta\colon A &\longrightarrow A^*\\
x &\longmapsto \exp(\eta x) = \sum_{n\geqslant 0}\frac{x^n}{n!}
\end{aligned}$$

is defined for $\eta \in A$ such that $v_A(\eta) > e/(p-1)$. This map is injective. Section 9 contains a definition of the map

$$\begin{aligned}
\exp_\eta\colon \widehat{\Omega}_A^{q-1} &\longrightarrow \widehat{K}_q(A)\\
x\frac{dy_1}{y_1}\wedge\dots\wedge\frac{dy_{q-1}}{y_{q-1}} &\longmapsto \{\exp(\eta x), y_1, \dots, y_{q-1}\}
\end{aligned}$$

for $\eta \in A$ such that $v_A(\eta) \geqslant 2e/(p-1)$. This map is not injective in general. Here is a description of the kernel of $\exp_\eta$.

Theorem 2. *The following sequence is exact:*

$$(*)\qquad H^{q-1}(\mathscr{S}'(q)) \xrightarrow{\psi} \Omega_A^{q-1}/pd\widehat{\Omega}_A^{q-2} \xrightarrow{\exp_p} \widehat{K}_q(A).$$

Sketch of the proof. There is an exact sequence of complexes

$$
\begin{array}{ccccccc}
0 \to \mathrm{MF}\left(\begin{array}{c} \mathbb{J}^{[q]} \\ {\scriptstyle 1-f_q}\big\downarrow \\ \mathbb{D} \end{array}\right) & \to & \mathrm{MF}\left(\begin{array}{c} \mathbb{I}^{[q]} \\ {\scriptstyle 1-f_q}\big\downarrow \\ \mathbb{D} \end{array}\right) & \to \mathbb{I}^{[q]}/\mathbb{J}^{[q]} \to 0, \\
\| & & \| & \\
\mathscr{S}(q) & & \mathscr{S}'(q) &
\end{array}
$$

where MF means the mapping fiber complex. Thus, taking cohomologies we have the following diagram with the exact top row

$$
\begin{array}{ccccc}
H^{q-1}(\mathscr{S}'(q)) & \xrightarrow{\ \psi\ } & H^{q-1}(\mathbb{I}^{[q]}/\mathbb{J}^{[q]}) & \xrightarrow{\ \delta\ } & H^{q}(\mathscr{S}(q)) \\
& & {\scriptstyle (1)}\big\uparrow & & {\scriptstyle \mathrm{Thm.1}}\big\uparrow \\
& & \widehat{\Omega}_A^{q-1}/pd\widehat{\Omega}_A^{q-2} & \xrightarrow{\ \exp_p\ } & U_1\widehat{K}_q(A),
\end{array}
$$

where the map (1) is induced by

$$\widehat{\Omega}_A^{q-1} \ni \omega \longmapsto p\widetilde{\omega} \in I \otimes \widehat{\Omega}_B^{q-1}/J \otimes \widehat{\Omega}_B^{q-1} = (\mathbb{I}^{[q]}/\mathbb{J}^{[q]})^{q-1}.$$

We denoted the left horizontal arrow of the top row by ψ and the right horizontal arrow of the top row by δ. The right vertical arrow is injective, thus the claims are

(1) is an isomorphism,

(2) this diagram is commutative.

First we shall show (1). Recall that

$$H^{q-1}(\mathbb{I}^{[q]}/\mathbb{J}^{[q]}) = \operatorname{coker}\left(\frac{I^{[2]} \otimes \widehat{\Omega}_B^{q-2}}{J^{[2]} \otimes \widehat{\Omega}_B^{q-2}} \longrightarrow \frac{I \otimes \widehat{\Omega}_B^{q-2}}{J \otimes \widehat{\Omega}_B^{q-2}}\right).$$

From the exact sequence

$$0 \longrightarrow J \longrightarrow D \longrightarrow A \longrightarrow 0,$$

we get $D \otimes \widehat{\Omega}_B^{q-1}/J \otimes \widehat{\Omega}_B^{q-1} = A \otimes \widehat{\Omega}_B^{q-1}$ and its subgroup $I \otimes \widehat{\Omega}_B^{q-2}/J \otimes \widehat{\Omega}_B^{q-2}$ is $pA \otimes \widehat{\Omega}_B^{q-1}$ in $A \otimes \widehat{\Omega}_B^{q-1}$. The image of $I^{[2]} \otimes \widehat{\Omega}_B^{q-2}$ in $pA \otimes \widehat{\Omega}_B^{q-1}$ is equal to the image of

$$\mathfrak{J}^2 \otimes \widehat{\Omega}_B^{q-2} = \mathcal{J}^2 \otimes \widehat{\Omega}_B^{q-2} + p\mathcal{J}\widehat{\Omega}_B^{q-2} + p^2\widehat{\Omega}_B^{q-2}.$$

On the other hand, from the exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow B \longrightarrow A \longrightarrow 0,$$

we get an exact sequence

$$(\mathcal{J}/\mathcal{J}^2) \otimes \widehat{\Omega}_B^{q-2} \xrightarrow{\ d\ } A \otimes \widehat{\Omega}_B^{q-1} \longrightarrow \widehat{\Omega}_A^{q-1} \longrightarrow 0.$$

Thus $d\mathcal{J}^2 \otimes \widehat{\Omega}_B^{q-2}$ vanishes on $pA \otimes \widehat{\Omega}_B^{q-1}$, hence

$$H^{q-1}(\mathbb{I}^{[q]}/\mathbb{J}^{[q]}) = \frac{pA \otimes \widehat{\Omega}_B^{q-1}}{pd\mathcal{J}\widehat{\Omega}_B^{q-2} + p^2 d\widehat{\Omega}_B^{q-2}} \overset{p^{-1}}{\simeq} \frac{A \otimes \widehat{\Omega}_B^{q-1}}{d\mathcal{J}\widehat{\Omega}_B^{q-2} + pd\widehat{\Omega}_B^{q-2}} \simeq \widehat{\Omega}_A^{q-1}/pd\widehat{\Omega}_A^{q-2},$$

which completes the proof of (1).

Next, we shall demonstrate the commutativity of the diagram on a simple example. Consider the case where $q = 2$ and take $adT/T \in \widehat{\Omega}_A^1$ for $T \in \mathbb{T} \cup \{\pi\}$. We want to show that the composite of

$$\widehat{\Omega}_A^1/pdA \xrightarrow{(1)} H^1(\mathbb{I}^{[2]}/\mathbb{J}^{[2]}) \xrightarrow{\delta} S^q \xrightarrow{E_q} U_1\widehat{K}_2(A)$$

coincides with $\exp_p$. By (1), the lifting of adT/T in $(\mathbb{I}^{[2]}/\mathbb{J}^{[2]})^1 = I\otimes\widehat{\Omega}_B^1/J\otimes\widehat{\Omega}_B^1$ is $p\widetilde{a}\otimes dT/T$, where $\widetilde{a}$ is a lifting of a to D. Chasing the connecting homomorphism δ,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (J\otimes\widehat{\Omega}_B^1)\oplus D & \longrightarrow & (I\otimes\widehat{\Omega}_B^1)\oplus D & \longrightarrow & (I\otimes\widehat{\Omega}_B^1)/(J\otimes\widehat{\Omega}_B^1) & \longrightarrow & 0 \\
 & & {\scriptstyle d}\big\downarrow & & {\scriptstyle d}\big\downarrow & & {\scriptstyle d}\big\downarrow & & \\
0 & \longrightarrow & (D\otimes\widehat{\Omega}_B^2)\oplus(D\otimes\widehat{\Omega}_B^1) & \longrightarrow & (D\otimes\widehat{\Omega}_B^2)\oplus(D\otimes\widehat{\Omega}_B^1) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & {\scriptstyle d}\big\downarrow & & {\scriptstyle d}\big\downarrow & & {\scriptstyle d}\big\downarrow & &
\end{array}$$

(the left column is $\mathcal{S}(2)$, the middle is $\mathcal{S}'(2)$ and the right is $\mathbb{I}^{[2]}/\mathbb{J}^{[2]}$); $p\widetilde{a}dT/T$ in the upper right goes to $(pd\widetilde{a}\wedge dT/T, (1-f_2)(p\widetilde{a}\otimes dT/T))$ in the lower left. By E_2, this element goes

$$\begin{aligned}
&E_2\big((1-f_2)\big(p\widetilde{a}\otimes\frac{d}{T}\big)\big) = E_2\big((1-f_1)(p\widetilde{a})\otimes\frac{dT}{T}\big)\\
&= \{E_1((1-f_1)(p\widetilde{a})), T\} = \big\{\exp\circ\big(\sum_{n\geqslant 0} f_1^n\big)\circ(1-f_1)(p\widetilde{a}), T\big\}\\
&= \{\exp(pa), T\}.
\end{aligned}$$

in $U_1\widehat{K}_2(A)$. This is none other than the map $\exp_p$. □

By Theorem 2 we can calculate the kernel of $\exp_p$. On the other hand, even though $\exp_p$ is not surjective, the image of $\exp_p$ includes $U_{e+1}\widehat{K}_q(A)$ and we already know $\mathrm{gr}_i\widehat{K}_q(K)$ for $0\leqslant i\leqslant ep/(p-1)$. Thus it is enough to calculate the kernel of $\exp_p$ in order to know all $\mathrm{gr}_i\widehat{K}_q(K)$. Note that to know $\mathrm{gr}_i\widehat{K}_q(K)$, we may assume that $\zeta_p\in K$, and hence $\widehat{K}_q(A) = U_0\widehat{K}_q(K)$.

15.2. Computation of the kernel of the exponential homomorphism

15.2.1. Modified syntomic complex. We introduce a modification of $\mathscr{S}'(q)$ and calculate it instead of $\mathscr{S}'(q)$. Let $\mathbb{S}_q$ be the mapping fiber complex of

$$1 - f_q\colon (\mathbb{J}^{[q]})^{\geqslant q-2} \longrightarrow \mathbb{D}^{\geqslant q-2}.$$

Here, for a complex $C^{\cdot}$, we put

$$C^{\geqslant n} = (0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow C^n \longrightarrow C^{n+1} \longrightarrow \cdots).$$

By definition, we have a natural surjection $H^{q-1}(\mathbb{S}_q) \to H^{q-1}(\mathscr{S}'(q))$, hence $\psi(H^{q-1}(\mathbb{S}_q)) = \psi(H^{q-1}(\mathscr{S}'(q)))$, which is the kernel of $\exp_p$.

To calculate $H^{q-1}(\mathbb{S}_q)$, we introduce an X-filtration. Let $0 \leqslant r \leqslant 2$ and $s = q-r$. Recall that $B = A_0[[X]]$. For $i \geqslant 0$, let $\mathrm{fil}_i(I^{[r]} \otimes_B \widehat{\Omega}^s_B)$ be the subgroup of $I^{[r]} \otimes_B \widehat{\Omega}^s_B$ generated by the elements

$$\left\{ X^n \frac{(X^e)^j}{j!} \frac{p^l}{l!} a\omega : n + ej \geqslant i, n \geqslant 0, j + l \geqslant r, a \in D, \omega \in \widehat{\Omega}^s_B \right\}$$
$$\cup \left\{ X^n \frac{(X^e)^j}{j!} \frac{p^l}{l!} av \wedge \frac{dX}{X} : n + ej \geqslant i, n \geqslant 1, j + l \geqslant r, a \in D, v \in \widehat{\Omega}^{s-1}_B \right\}.$$

The map $1 - f_q\colon I^{[r]} \otimes \widehat{\Omega}^s_B \to D \otimes \widehat{\Omega}^s_B$ preserves the filtrations. By using the latter we get the following

Proposition 3. *$H^{q-1}(\mathrm{fil}_i \mathbb{S}_q)_i$ form a finite decreasing filtration of $H^{q-1}(\mathbb{S}_q)$. Denote*

$$\mathrm{fil}_i H^{q-1}(\mathbb{S}_q) = H^{q-1}(\mathrm{fil}_i \mathbb{S}_q),$$
$$\mathrm{gr}_i H^{q-1}(\mathbb{S}_q) = \mathrm{fil}_i H^{q-1}(\mathbb{S}_q) / \mathrm{fil}_{i+1} H^{q-1}(\mathbb{S}_q).$$

Then $\mathrm{gr}_i H^{q-1}(\mathbb{S}_q)$

$$= \begin{cases} 0 & (\textit{ if } i > 2e) \\ X^{2e-1}dX \wedge \left(\widehat{\Omega}^{q-3}_{A_0}/p\right) & (\textit{ if } i = 2e) \\ X^i\left(\widehat{\Omega}^{q-2}_{A_0}/p\right) \oplus X^{i-1}dX \wedge (\widehat{\Omega}^{q-3}_{A_0}/p) & (\textit{ if } e < i < 2e) \\ X^e\left(\widehat{\Omega}^{q-2}_{A_0}/p\right) \oplus X^{e-1}dX \wedge \left(\mathfrak{Z}_1\widehat{\Omega}^{q-3}_{A_0}/p^2\widehat{\Omega}^{q-3}_{A_0}\right) & (\textit{ if } i = e, p \mid e) \\ X^{e-1}dX \wedge \left(\mathfrak{Z}_1\widehat{\Omega}^{q-3}_{A_0}/p^2\widehat{\Omega}^{q-3}_{A_0}\right) & (\textit{ if } i = e, p \nmid e) \\ \left(X^i \dfrac{\left(p^{\max(\eta'_i - v_p(i),0)}\widehat{\Omega}^{q-2}_{A_0} \cap \mathfrak{Z}_{\eta_i}\widehat{\Omega}^{q-2}_{A_0}\right) + p^2\widehat{\Omega}^{q-2}_{A_0}}{p^2\widehat{\Omega}^{q-2}_{A_0}}\right) & \\ \qquad \oplus \left(X^{i-1}dX \wedge \dfrac{\mathfrak{Z}_{\eta_i}\widehat{\Omega}^{q-3}_{A_0} + p^2\widehat{\Omega}^{q-3}_{A_0}}{p^2\widehat{\Omega}^{q-3}_{A_0}}\right) & (\textit{ if } 1 \leqslant i < e) \\ 0 & (\textit{ if } i = 0). \end{cases}$$

Here η_i and η'_i are the integers which satisfy $p^{\eta_i-1}i < e \leqslant p^{\eta_i}i$ and $p^{\eta'_i-1}i - 1 < e \leqslant p^{\eta'_i}i - 1$ for each i,

$$\mathfrak{Z}_n\widehat{\Omega}^q_{A_0} = \ker\left(\widehat{\Omega}^q_{A_0} \xrightarrow{d} \widehat{\Omega}^{q+1}_{A_0}/p^n\right)$$

for positive n, and $\mathfrak{Z}_n\widehat{\Omega}^q_{A_0} = \widehat{\Omega}^q_{A_0}$ for $n \leqslant 0$.

Outline of the proof. From the definition of the filtration we have the exact sequence of complexes:

$$0 \longrightarrow \mathrm{fil}_{i+1}\mathbb{S}_q \longrightarrow \mathrm{fil}_i\mathbb{S}_q \longrightarrow \mathrm{gr}_i\mathbb{S}_q \longrightarrow 0$$

and this sequence induce a long exact sequence

$$\cdots \to H^{q-2}(\mathrm{gr}_i\mathbb{S}_q) \to H^{q-1}(\mathrm{fil}_{i+1}\mathbb{S}_q) \to H^{q-1}(\mathrm{fil}_i\mathbb{S}_q) \to H^{q-1}(\mathrm{gr}_i\mathbb{S}_q) \to \cdots .$$

The group $H^{q-2}(\mathrm{gr}_i\mathbb{S}_q)$ is

$$H^{q-2}(\mathrm{gr}_i\mathbb{S}_q) = \ker\begin{pmatrix} \mathrm{gr}_i I^{[2]} \otimes \widehat{\Omega}^{q-2}_B \longrightarrow (\mathrm{gr}_i I \otimes \widehat{\Omega}^{q-1}_B) \oplus (\mathrm{gr}_i D \otimes \widehat{\Omega}^{q-2}_B) \\ x \longmapsto (dx, (1-f_q)x) \end{pmatrix}.$$

The map $1 - f_q$ is equal to 1 if $i \geqslant 1$ and $1 - f_q \colon p^2\widehat{\Omega}^{q-2}_{A_0} \to \widehat{\Omega}^{q-2}_{A_0}$ if $i = 0$, thus they are all injective. Hence $H^{q-2}(\mathrm{gr}_i\mathbb{S}_q) = 0$ for all i and we deduce that $H^{q-1}(\mathrm{fil}_i\mathbb{S}_q)_i$ form a decreasing filtration on $H^{q-1}(\mathbb{S}_q)$.

Next, we have to calculate $H^{q-2}(\mathrm{gr}_i\mathbb{S}_q)$. The calculation is easy but there are many cases which depend on i, so we omit them. For more detail, see [N2].

Finally, we have to compute the image of the last arrow of the exact sequence

$$0 \longrightarrow H^{q-1}(\mathrm{fil}_{i+1}\mathbb{S}_q) \longrightarrow H^{q-1}(\mathrm{fil}_i\mathbb{S}_q) \longrightarrow H^{q-1}(\mathrm{gr}_i\mathbb{S}_q)$$

because it is not surjective in general. Write down the complex $\mathrm{gr}_i\mathbb{S}_q$:

$$\cdots \to (\mathrm{gr}_i I \otimes \widehat{\Omega}_B^{q-1}) \oplus (\mathrm{gr}_i D \otimes \widehat{\Omega}_B^{q-2}) \xrightarrow{d} (\mathrm{gr}_i D \otimes \widehat{\Omega}_B^{q}) \oplus (\mathrm{gr}_i D \otimes \widehat{\Omega}_B^{q-1}) \to \cdots,$$

where the first term is the degree $q-1$ part and the second term is the degree q part. An element (x,y) in the first term which is mapped to zero by d comes from $H^{q-1}(\mathrm{fil}_i\mathbb{S}_q)$ if and only if there exists $z \in \mathrm{fil}_i D \otimes \widehat{\Omega}_B^{q-2}$ such that $z \equiv y$ modulo $\mathrm{fil}_{i+1} D \otimes \widehat{\Omega}_B^{q-2}$ and

$$\sum_{n\geqslant 0} f_q^n(dz) \in \mathrm{fil}_i I \otimes \widehat{\Omega}_B^{q-1}.$$

From here one deduces Proposition 3. □

15.2.2. Differential modules. Take a prime element π of K such that $\pi^{e-1}d\pi = 0$. We assume that $p \nmid e$ in this subsection. Then we have

$$\widehat{\Omega}_A^q \simeq \left(\bigoplus_{i_1<i_2<\cdots<i_q} A\frac{dT_{i_1}}{T_{i_1}} \wedge \cdots \wedge \frac{dT_{i_q}}{T_{i_q}}\right)$$
$$\oplus \left(\bigoplus_{i_1<i_2<\cdots<i_{q-1}} A/(\pi^{e-1})\frac{dT_{i_1}}{T_{i_1}} \wedge \cdots \wedge \frac{dT_{i_{q-1}}}{T_{i_{q-1}}} \wedge d\pi\right),$$

where $\{T_i\} = \mathbb{T}$. We introduce a filtration on $\widehat{\Omega}_A^q$ as

$$\mathrm{fil}_i\widehat{\Omega}_A^q = \begin{cases} \widehat{\Omega}_A^q & (\text{ if } i = 0) \\ \pi^i\widehat{\Omega}_A^q + \pi^{i-1}d\pi \wedge \widehat{\Omega}_A^{q-1} & (\text{ if } i \geqslant 1). \end{cases}$$

The subquotients are

$$\begin{aligned} \mathrm{gr}_i\widehat{\Omega}_A^q &= \mathrm{fil}_i\widehat{\Omega}_A^q/\mathrm{fil}_{i+1}\widehat{\Omega}_A^q \\ &= \begin{cases} \Omega_F^q & (\text{ if } i = 0 \text{ or } i \geqslant e) \\ \Omega_F^q \oplus \Omega_F^{q-1} & (\text{ if } 1 \leqslant i < e), \end{cases} \end{aligned}$$

where the map is

$$\begin{aligned} &\Omega_F^q \ni \omega \longmapsto \pi^i\widetilde{\omega} \in \pi^i\widehat{\Omega}_A^q \\ &\Omega_F^{q-1} \ni \omega \longmapsto \pi^{i-1}d\pi \wedge \widetilde{\omega} \in \pi^{i-1}d\pi \wedge \widehat{\Omega}_A^{q-1}. \end{aligned}$$

Here $\widetilde{\omega}$ is the lifting of ω. Let $\mathrm{fil}_i(\widehat{\Omega}_A^q/pd\widehat{\Omega}_A^{q-1})$ be the image of $\mathrm{fil}_i\widehat{\Omega}_A^q$ in $\widehat{\Omega}_A^q/pd\widehat{\Omega}_A^{q-1}$. Then we have the following:

Proposition 4. *For $j \geqslant 0$,*

$$\mathrm{gr}_j\left(\widehat{\Omega}^q_A/pd\widehat{\Omega}^{q-1}_A\right) = \begin{cases} \Omega^q_F & (j=0) \\ \Omega^q_F \oplus \Omega^{q-1}_F & (1 \leqslant j < e) \\ \Omega^q_F/B^q_l & (e \leqslant j), \end{cases}$$

where l be the maximal integer which satisfies $j - le \geqslant 0$.

Proof. If $1 \leqslant j < e$, $\mathrm{gr}_j\widehat{\Omega}^q_A = \mathrm{gr}_j(\widehat{\Omega}^q_A/pd\widehat{\Omega}^{q-1}_A)$ because $pd\widehat{\Omega}^{q-1}_A \subset \mathrm{fil}_e\widehat{\Omega}^q_A$. Assume that $j \geqslant e$ and let l be as above. Since $\pi^{e-1}d\pi = 0$, $\widehat{\Omega}^{q-1}_A$ is generated by elements $p\pi^i d\omega$ for $0 \leqslant i < e$ and $\omega \in \widehat{\Omega}^{q-1}_{A_0}$. By [I] (Cor. 2.3.14), $p\pi^i d\omega \in \mathrm{fil}_{e(1+n)+i}\widehat{\Omega}^q_A$ if and only if the residue class of $p^{-n}d\omega$ belongs to B_{n+1}. Thus $\mathrm{gr}_j(\widehat{\Omega}^q_A/pd\widehat{\Omega}^{q-1}_A) \simeq \Omega^q_F/B^q_l$. □

By definition of the filtrations, $\exp_p$ preserves the filtrations on $\widehat{\Omega}^{q-1}_A/pd\widehat{\Omega}^{q-2}_A$ and $\widehat{K}_q(K)$. Furthermore, $\exp_p\colon \mathrm{gr}_i(\widehat{\Omega}^{q-1}_A/pd\widehat{\Omega}^{q-2}_A) \to \mathrm{gr}_{i+e}K_q(K)$ is surjective and its kernel is the image of $\psi(H^{q-1}(\mathbb{S}_q)) \cap \mathrm{fil}_i(\widehat{\Omega}^{q-1}_A/pd\widehat{\Omega}^{q-2}_A)$ in $\mathrm{gr}_i(\widehat{\Omega}^{q-1}_A/pd\widehat{\Omega}^{q-2}_A)$. Now we know both $\widehat{\Omega}^{q-1}_A/pd\widehat{\Omega}^{q-2}_A$ and $H^{q-1}(\mathbb{S}_q)$ explicitly, thus we shall get the structure of $K_q(K)$ by calculating ψ. But ψ does not preserve the filtration of $H^{q-1}(\mathbb{S}_q)$, so it is not easy to compute it. For more details, see [N2], especially sections 4-8 of that paper. After completing these calculations, we get the result in (vi) in the introduction.

Remark. Note that if $p \mid e$, the structure of $\widehat{\Omega}^{q-1}_A/pd\widehat{\Omega}^{q-2}_A$ is much more complicated. For example, if $e = p(p-1)$, and if $\pi^e = p$, then $p\pi^{e-1}d\pi = 0$. This means the torsion part of $\widehat{\Omega}^{q-1}_A$ is larger than in the the case where $p \nmid e$. Furthermore, if $\pi^{p(p-1)} = pT$ for some $T \in \mathbb{T}$, then $p\pi^{e-1}d\pi = pdT$, this means that $d\pi$ is not a torsion element. This complexity makes it difficult to describe the structure of $K_q(K)$ in the case where $p \mid e$.

Appendix. The mapping fiber complex.

This subsection is only a note on homological algebra to introduce the mapping fiber complex. The mapping fiber complex is the degree -1 shift of the mapping cone complex.

Let $C^{\cdot} \xrightarrow{f} D^{\cdot}$ be a morphism of non-negative cochain complexes. We denote the degree i term of $C^{\cdot}$ by C^i.

Then the mapping fiber complex $\mathrm{MF}(f)^{\cdot}$ is defined as follows.

$$\mathrm{MF}(f)^i = C^i \oplus D^{i-1},$$

$$\text{differential} \quad d\colon C^i \oplus D^{i-1} \longrightarrow C^{i+1} \oplus D^i$$

$$(x, y) \longmapsto (dx, f(x) - dy).$$

By definition, we get an exact sequence of complexes:

$$0 \longrightarrow D[-1]^{\cdot} \longrightarrow \mathrm{MF}(f)^{\cdot} \longrightarrow C^{\cdot} \longrightarrow 0,$$

where $D[-1]^{\cdot} = (0 \to D^0 \to D^1 \to \cdots)$ (degree -1 shift of $D^{\cdot}$.)

Taking cohomology, we get a long exact sequence

$$\cdots \to H^i(\mathrm{MF}(f)^{\cdot}) \to H^i(C^{\cdot}) \to H^{i+1}(D^{\cdot}[-1]) \to H^{i+1}(\mathrm{MF}(f)^{\cdot}) \to \cdots,$$

which is the same as the following exact sequence

$$\cdots \to H^i(\mathrm{MF}(f)^{\cdot}) \to H^i(C^{\cdot}) \xrightarrow{f} H^i(D^{\cdot}) \to H^{i+1}(\mathrm{MF}(f)^{\cdot}) \to \cdots.$$

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16. Higher class field theory without using K-groups

Ivan Fesenko

Let F be a complete discrete valuation field with residue field $k = k_F$ of characteristic p. In this section we discuss an alternative to higher local class field theory method which describes abelian totally ramified extensions of F without using K-groups. For n-dimensional local fields this gives a description of abelian totally ramified (with respect to the discrete valuation of rank one) extensions of F. Applications are sketched in 16.3 and 16.4.

16.1. p-class field theory

Suppose that k is perfect and $k \neq \wp(k)$ where $\wp\colon k \to k$, $\wp(a) = a^p - a$.

Let $\widetilde{F}$ be the maximal abelian unramified p-extension of F. Then due to Witt theory $\mathrm{Gal}(\widetilde{F}/F)$ is isomorphic to $\prod_\kappa \mathbb{Z}_p$ where $\kappa = \dim_{\mathbb{F}_p} k/\wp(k)$. The isomorphism is non-canonical unless k is finite where the canonical one is given by $\mathrm{Frob}_F \mapsto 1$.

Let L be a totally ramified Galois p-extension of F.

Let $\mathrm{Gal}(\widetilde{F}/F)$ act trivially on $\mathrm{Gal}(L/F)$.

Denote

$$\mathrm{Gal}(L/F)^\sim = H^1_{\mathrm{cont}}((\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F)) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F)).$$

Then $\mathrm{Gal}(L/F)^\sim \simeq \oplus_\kappa \mathrm{Gal}(L/F)$ non-canonically.

Put $\widetilde{L} = L\widetilde{F}$. Denote by $\varphi \in \mathrm{Gal}(\widetilde{L}/L)$ the lifting of $\varphi \in \mathrm{Gal}(\widetilde{F}/F)$.

For $\chi \in \mathrm{Gal}(L/F)^\sim$ denote

$$\Sigma_\chi = \{\alpha \in \widetilde{L} : \alpha^{\varphi\chi(\varphi)} = \alpha \quad \text{for all } \varphi \in \mathrm{Gal}(\widetilde{F}/F)\}.$$

The extension Σ_χ/F is totally ramified.

As an generalization of Neukirch's approach [N] introduce the following:

Definition. Put

$$\Upsilon_{L/F}(\chi) = N_{\Sigma_\chi/F}\pi_\chi / N_{L/F}\pi_L \mod N_{L/F}U_L$$

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where π_χ is a prime element of Σ_χ and π_L is a prime element of L.

This map is well defined. Compare with 10.1.

Theorem ([F1, Th. 1.7]). *The map $\Upsilon_{L/F}$ is a homomorphism and it induces an isomorphism*

$$\mathrm{Gal}(L \cap F^{\mathrm{ab}}/F)^{\sim} \xrightarrow{\sim} U_F/N_{L/F}U_L \xrightarrow{\sim} U_{1,F}/N_{L/F}U_{1,L}.$$

Proof. One of the easiest ways to prove the theorem is to define and use the map which goes in the reverse direction. For details see [F1, sect. 1]. □

Problem. If π is a prime element of F, then p-class field theory implies that there is a totally ramified abelian p-extension F_π of F such that $F_\pi\widetilde{F}$ coincides with the maximal abelian p-extension of F and $\pi \in N_{F_\pi/F}F_\pi^*$. Describe F_π explicitly (like Lubin–Tate theory does in the case of finite k).

Remark. Let K be an n-dimensional local field ($K = K_n, \ \dots,, \ K_0$) with K_0 satisfying the same restrictions as k above.

For a totally ramified Galois p-extension L/K (for the definition of a totally ramified extension see 10.4) put

$$\mathrm{Gal}(L/K)^{\sim} = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\widetilde{K}/K), \mathrm{Gal}(L/K))$$

where $\widetilde{K}$ is the maximal p-subextension of K_{pur}/K (for the definition of K_{pur} see (A1) of 10.1).

There is a map $\Upsilon_{L/K}$ which induces an isomorphism [F2, Th. 3.8]

$$\mathrm{Gal}(L \cap K^{\mathrm{ab}}/K)^{\sim} \xrightarrow{\sim} VK_n^t(K)/N_{L/K}VK_n^t(L)$$

where $VK_n^t(K) = \{V_K\} \cdot K_{n-1}^t(K)$ and K_n^t was defined in 2.0.

16.2. General abelian local p-class field theory

Now let k be an arbitrary field of characteristic p, $\wp(k) \neq k$.

Let $\widetilde{F}$ be the maximal abelian unramified p-extension of F.

Let L be a totally ramified Galois p-extension of F. Denote

$$\mathrm{Gal}(L/F)^{\sim} = H^1_{\mathrm{cont}}((\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F)) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F)).$$

In a similar way to the previous subsection define the map

$$\Upsilon_{L/F} \colon \mathrm{Gal}(L/F)^{\sim} \to U_{1,F}/N_{L/F}U_{1,L}.$$

In fact it lands in $U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}})/N_{L/F}U_{1,L}$ and we denote this new map by the same notation.

Definition. Let $\mathbf{F}$ be complete discrete valuation field such that $\mathbf{F} \supset \widetilde{F}$, $e(\mathbf{F}|\widetilde{F}) = 1$ and $k_{\mathbf{F}} = \cup_{n\geqslant 0} k_{\widetilde{F}}^{p^{-n}}$. Put $\mathbf{L} = L\mathbf{F}$.

Denote $I(L|F) = \langle \varepsilon^{\sigma-1} : \varepsilon \in U_{1,\mathbf{L}}, \sigma \in \mathrm{Gal}(L/F)\rangle \cap U_{1,\widetilde{L}}$.

Then the sequence

$$(*) \qquad 1 \to \mathrm{Gal}(L/F)^{\mathrm{ab}} \xrightarrow{g} U_{1,\widetilde{L}}/I(L|F) \xrightarrow{N_{\widetilde{L}/\widetilde{F}}} N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}} \to 1$$

is exact where $g(\sigma) = \pi_L^{\sigma-1}$ and π_L is a prime element of L (compare with Proposition 1 of 10.4.1).

Generalizing Hazewinkel's method [H] introduce

Definition. Define a homomorphism

$$\Psi_{L/F}\colon (U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}})/N_{L/F}U_{1,L} \to \mathrm{Gal}(L \cap F^{\mathrm{ab}}/F)^{\sim}, \quad \Psi_{L/F}(\varepsilon) = \chi$$

where $\chi(\varphi) = g^{-1}(\eta^{1-\varphi})$, $\eta \in U_{1,\widetilde{L}}$ is such that $\varepsilon = N_{\widetilde{L}/\widetilde{F}}\eta$.

Properties of $\Upsilon_{L/F}, \Psi_{L/F}$.

(1) $\Psi_{L/F} \circ \Upsilon_{L/F} = \mathrm{id}$ on $\mathrm{Gal}(L \cap F^{\mathrm{ab}}/F)^{\sim}$, so $\Psi_{L/F}$ is an epimorphism.

(2) Let $\mathcal{F}$ be a complete discrete valuation field such that $\mathcal{F} \supset F$, $e(\mathcal{F}|F) = 1$ and $k_{\mathcal{F}} = \cup_{n\geqslant 0} k_F^{p^{-n}}$. Put $\mathcal{L} = L\mathcal{F}$. Let

$$\lambda_{L/F}\colon (U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}})/N_{L/F}U_{1,L} \to U_{1,\mathcal{F}}/N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}}$$

be induced by the embedding $F \to \mathcal{F}$. Then the diagram

$$\begin{array}{ccccc}
\mathrm{Gal}(L/F)^{\sim} & \xrightarrow{\Upsilon_{L/F}} & (U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}})/N_{L/F}U_{1,L} & \xrightarrow{\Psi_{L/F}} & \mathrm{Gal}(L \cap F^{\mathrm{ab}}/F)^{\sim} \\
\downarrow & & \lambda_{L/F}\downarrow & & \mathrm{iso}\downarrow \\
\mathrm{Gal}(\mathcal{L}/\mathcal{F})^{\sim} & \xrightarrow{\Upsilon_{\mathcal{L}/\mathcal{F}}} & U_{1,\mathcal{F}}/N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}} & \xrightarrow{\Psi_{\mathcal{L}/\mathcal{F}}} & \mathrm{Gal}(\mathcal{L} \cap \mathcal{F}^{\mathrm{ab}}/\mathcal{F})^{\sim}
\end{array}$$

is commutative.

(3) Since $\Psi_{\mathcal{L}/\mathcal{F}}$ is an isomorphism (see 16.1), we deduce that $\lambda_{L/F}$ is surjective and $\ker(\Psi_{L/F}) = \ker(\lambda_{L/F})$, so

$$(U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}})/N_*(L/F) \xrightarrow{\sim} \mathrm{Gal}(L \cap F^{\mathrm{ab}}/F)^{\sim}$$

where $N_*(L/F) = U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}} \cap N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}}$.

Theorem ([F3, Th. 1.9]). *Let L/F be a cyclic totally ramified p-extension. Then*

$$\Upsilon_{L/F}\colon \mathrm{Gal}(L/F)^{\sim} \to (U_{1,F} \cap N_{\widetilde{L}/\widetilde{F}} U_{1,\widetilde{L}})/N_{L/F}U_{1,L}$$

is an isomorphism.

Proof. Since L/F is cyclic we get $I(L|F) = \{\varepsilon^{\sigma-1} : \varepsilon \in U_{1,\widetilde{L}}, \sigma \in \operatorname{Gal}(L/F)\}$, so

$$I(L|F) \cap U_{1,\widetilde{L}}^{\varphi-1} = I(L|F)^{\varphi-1}$$

for every $\varphi \in \operatorname{Gal}(\widetilde{L}/L)$.

Let $\Psi_{L/F}(\varepsilon) = 1$ for $\varepsilon = N_{\widetilde{L}/\widetilde{F}}\eta \in U_{1,F}$. Then $\eta^{\varphi-1} \in I(L|F) \cap U_{1,\widetilde{L}}^{\varphi-1}$, so $\eta \in I(L|F)L_\varphi$ where L_φ is the fixed subfield of $\widetilde{L}$ with respect to φ. Hence $\varepsilon \in N_{L_\varphi/F \cap L_\varphi} U_{1,L_\varphi}$. By induction on κ we deduce that $\varepsilon \in N_{L/F}U_{1,L}$ and $\Psi_{L/F}$ is injective. □

Remark. Miki [M] proved this theorem in a different setting which doesn't mention class field theory.

Corollary. *Let L_1/F, L_2/F be abelian totally ramified p-extensions. Assume that L_1L_2/F is totally ramified. Then*

$$N_{L_2/F}U_{1,L_2} \subset N_{L_1/F}U_{1,L_1} \Longleftrightarrow L_2 \supset L_1.$$

Proof. Let M/F be a cyclic subextension in L_1/F. Then
$N_{\mathcal{M}/\mathcal{F}}U_{1,\mathcal{M}} \supset N_{\mathcal{L}_2/\mathcal{F}}U_{1,\mathcal{L}_2}$, so $\mathcal{M} \subset \mathcal{L}_2$ and $M \subset L_2$. Thus $L_1 \subset L_2$. □

Problem. Describe $\ker(\Psi_{L/F})$ for an arbitrary L/F. It is known [F3, 1.11] that this kernel is trivial in one of the following situations:

(1) L is the compositum of cyclic extensions M_i over F, $1 \leqslant i \leqslant m$, such that all ramification breaks of $\operatorname{Gal}(M_i/F)$ with respect to the upper numbering are not greater than every break of $\operatorname{Gal}(M_{i+1}/F)$ for all $1 \leqslant i \leqslant m-1$.

(2) $\operatorname{Gal}(L/F)$ is the product of cyclic groups of order p and a cyclic group.

No example with non-trivial kernel is known.

16.3. Norm groups

Proposition ([F3, Prop. 2.1])**.** *Let F be a complete discrete valuation field with residue field of characteristic p. Let L_1/F and L_2/F be abelian totally ramified p-extensions. Let $N_{L_1/F}L_1^* \cap N_{L_2/F}L_2^*$ contain a prime element of F. Then L_1L_2/F is totally ramified.*

Proof. If k_F is perfect, then the claim follows from p-class field theory in 16.1. If k_F is imperfect then use the fact that there is a field $\mathcal{F}$ as above which satisfies $L_1\mathcal{F} \cap L_2\mathcal{F} = (L_1 \cap L_2)\mathcal{F}$. □

Theorem (uniqueness part of the existence theorem) ([F3, Th. 2.2])**.** *Let $k_F \neq \wp(k_F)$. Let L_1/F, L_2/F be totally ramified abelian p-extensions. Then*

$$N_{L_2/F}L_2^* = N_{L_1/F}L_1^* \iff L_1 = L_2.$$

Proof. Use the previous proposition and corollary in 16.2. □

16.4. Norm groups more explicitly

Let F be of characteristic 0. In general if k is imperfect it is very difficult to describe $N_{L/F}U_{1,L}$. One partial case can be handled: let the absolute ramification index $e(F)$ be equal to 1 (the description below can be extended to the case of $e(F) < p-1$).

Let π be a prime element of F.

Definition.

$$\mathcal{E}_{n,\pi}\colon W_n(k_F) \to U_{1,F}/U_{1,F}^{p^n}, \quad \mathcal{E}_{n,\pi}((a_0, \dots, a_{n-1})) = \prod_{0\leqslant i\leqslant n-1} E(\widetilde{a_i}^{p^{n-i}}\pi)^{p^i}$$

where $E(X) = \exp(X + X^p/p + X^{p^2}/p^2 + \dots)$ and $\widetilde{a_i} \in \mathcal{O}_F$ is a lifting of $a_i \in k_F$ (this map is basically the same as the map ψ_n in Theorem 13.2).

The following property is easy to deduce:

Lemma. *$\mathcal{E}_{n,\pi}$ is a monomorphism. If k_F is perfect then $\mathcal{E}_{n,\pi}$ is an isomorphism.*

Theorem ([F3, Th. 3.2])**.** *Let $k_F \neq \wp(k_F)$ and let $e(F) = 1$. Let π be a prime element of F.*

Then cyclic totally ramified extensions L/F of degree p^n such that $\pi \in N_{L/F}L^*$ are in one-to-one correspondence with subgroups

$$\mathcal{E}_{n,\pi}\big(\mathbf{F}(w)\wp(W_n(k_F))\big)U_{1,F}^{p^n}$$

of $U_{1,F}/U_{1,F}^{p^n}$ where w runs over elements of $W_n(k_F)^*$.

Hint. Use the theorem of 16.3. If k_F is perfect, the assertion follows from p-class field theory.

Remark. The correspondence in this theorem was discovered by M. Kurihara [K, Th. 0.1], see the sequence (1) of theorem 13.2. The proof here is more elementary since it doesn't use étale vanishing cycles.

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17. An approach to higher ramification theory

Igor Zhukov

We use the notation of sections 1 and 10.

17.0. Approach of Hyodo and Fesenko

Let K be an n-dimensional local field, L/K a finite abelian extension. Define a filtration on $\operatorname{Gal}(L/K)$ (cf. [H], [F, sect. 4]) by

$$\operatorname{Gal}(L/K)^{\mathbf{i}} = \Upsilon_{L/K}^{-1}(U_{\mathbf{i}}K_n^{\mathrm{top}}(K) + N_{L/K}K_n^{\mathrm{top}}(L)/N_{L/K}K_n^{\mathrm{top}}(L)), \quad \mathbf{i} \in \mathbb{Z}_+^n,$$

where $U_{\mathbf{i}}K_n^{\mathrm{top}}(K) = \{U_{\mathbf{i}}\} \cdot K_{n-1}^{\mathrm{top}}(K)$, $U_{\mathbf{i}} = 1 + P_K(\mathbf{i})$,

$$\Upsilon_{L/K}^{-1}\colon K_n^{\mathrm{top}}(K)/N_{L/K}K_n^{\mathrm{top}}(L) \xrightarrow{\sim} \operatorname{Gal}(L/K)$$

is the reciprocity map.

Then for a subextension M/K of L/K

$$\operatorname{Gal}(M/K)^{\mathbf{i}} = \operatorname{Gal}(L/K)^{\mathbf{i}} \operatorname{Gal}(L/M)/\operatorname{Gal}(L/M)$$

which is a higher dimensional analogue of Herbrand's theorem. However, if one defines a generalization of the Hasse–Herbrand function and lower ramification filtration, then for $n > 1$ the lower filtration on a subgroup does not coincide with the induced filtration in general.

Below we shall give another construction of the ramification filtration of L/K in the two-dimensional case; details can be found in [Z], see also [KZ]. This construction can be considered as a development of an approach by K. Kato and T. Saito in [KS].

Definition. Let K be a complete discrete valuation field with residue field k_K of characteristic p. A finite extension L/K is called *ferociously ramified* if $|L : K| = |k_L : k_K|_{\mathrm{ins}}$.

In addition to the nice ramification theory for totally ramified extensions, there is a nice ramification theory for ferociously ramified extensions L/K such that k_L/k_K is generated by one element; the reason is that in both cases the ring extension $\mathcal{O}_L/\mathcal{O}_K$ is monogenic, i.e., generated by one element, see section 18.

17.1. Almost constant extensions

Everywhere below K is a complete discrete valuation field with residue field k_K of characteristic p such that $|k_K : k_K^p| = p$. For instance, K can be a two-dimensional local field, or $K = \mathbb{F}_q(X_1)((X_2))$ or the quotient field of the completion of $\mathbb{Z}_p[T]_{(p)}$ with respect to the p-adic topology.

Definition. For the field K define a base (sub)field B as

$B = \mathbb{Q}_p \subset K$ if $\operatorname{char}(K) = 0$,

$B = \mathbb{F}_p((\rho)) \subset K$ if $\operatorname{char}(K) = p$, where ρ is an element of K with $v_K(\rho) > 0$.

Denote by k_0 the completion of $B(\mathcal{R}_K)$ inside K. Put $k = k_0^{\mathrm{alg}} \cap K$.

The subfield k is a maximal complete subfield of K with perfect residue field. It is called a *constant subfield* of K. A constant subfield is defined canonically if $\operatorname{char}(K) = 0$. Until the end of section 17 we assume that B (and, therefore, k) is fixed.

By v we denote the valuation $K^{\mathrm{alg}*} \to \mathbb{Q}$ normalized so that $v(B^*) = \mathbb{Z}$.

Example. If $K = F\{\{T\}\}$ where F is a mixed characteristic complete discrete valuation field with perfect residue field, then $k = F$.

Definition. An extension L/K is said to be *constant* if there is an algebraic extension l/k such that $L = Kl$.

An extension L/K is said to be *almost constant* if $L \subset L_1L_2$ for a constant extension L_1/K and an unramified extension L_2/K.

A field K is said to be *standard*, if $e(K|k) = 1$, and *almost standard*, if some finite unramified extension of K is a standard field.

Epp's theorem on elimination of wild ramification. ([E], [KZ]) *Let L be a finite extension of K. Then there is a finite extension k' of a constant subfield k of K such that $e(Lk'|Kk') = 1$.*

Corollary. *There exists a finite constant extension of K which is a standard field.*

Proof. See the proof of the Classification Theorem in 1.1.

Lemma. *The class of constant (almost constant) extensions is closed with respect to taking compositums and subextensions. If L/K and M/L are almost constant then M/K is almost constant as well.*

Definition. Denote by L_c the maximal almost constant subextension of K in L.

Properties.
(1) Every tamely ramified extension is almost constant. In other words, the (first) ramification subfield in L/K is a subfield of L_c.
(2) If L/K is normal then L_c/K is normal.
(3) There is an unramified extension L_0' of L_0 such that L_cL_0'/L_0 is a constant extension.
(4) There is a constant extension L_c'/L_c such that LL_c'/L_c' is ferociously ramified and $L_c' \cap L = L_c$. This follows immediately from Epp's theorem.

The principal idea of the proposed approach to ramification theory is to split L/K into a tower of three extensions: L_0/K, L_c/L_0, L/L_c, where L_0 is the inertia subfield in L/K. The ramification filtration for $\mathrm{Gal}(L_c/L_0)$ reflects that for the corresponding extensions of constants subfields. Next, to construct the ramification filtration for $\mathrm{Gal}(L/L_c)$, one reduces to the case of ferociously ramified extensions by means of Epp's theorem. (In the case of higher local fields one can also construct a filtration on $\mathrm{Gal}(L_0/K)$ by lifting that for the first residue fields.)

Now we give precise definitions.

17.2. Lower and upper ramification filtrations

Keep the assumption of the previous subsection. Put

$$\mathcal{A} = \{-1, 0\} \cup \{(\mathfrak{c}, s) : 0 < s \in \mathbb{Z}\} \cup \{(\mathfrak{i}, r) : 0 < r \in \mathbb{Q}\}.$$

This set is linearly ordered as follows:

$$-1 < 0 < (\mathfrak{c}, i) < (\mathfrak{i}, j) \text{ for any } i, j;$$
$$(\mathfrak{c}, i) < (\mathfrak{c}, j) \text{ for any } i < j;$$
$$(\mathfrak{i}, i) < (\mathfrak{i}, j) \text{ for any } i < j.$$

Definition. Let $G = \mathrm{Gal}(L/K)$. For any $\alpha \in \mathcal{A}$ we define a subgroup G_α in G.

Put $G_{-1} = G$, and denote by G_0 the inertia subgroup in G, i.e.,

$$G_0 = \{g \in G : v(g(a) - a) > 0 \text{ for all } a \in \mathcal{O}_L\}.$$

Let L_c/K be constant, and let it contain no unramified subextensions. Then define

$$G_{\mathfrak{c},i} = \mathrm{pr}^{-1}(\mathrm{Gal}(l/k)_i)$$

where l and k are the constant subfields in L and K respectively,

$$\mathrm{pr}\colon \ \mathrm{Gal}(L/K) \to \mathrm{Gal}(l/k) = \mathrm{Gal}(l/k)_0$$

is the natural projection and $\mathrm{Gal}(l/k)_i$ are the classical ramification subgroups. In the general case take an unramified extension K'/K such that $K'L/K'$ is constant and contains no unramified subextensions, and put $G_{\mathfrak{c},i} = \mathrm{Gal}(K'L/K')_{\mathfrak{c},i}$.

Finally, define $G_{\mathfrak{i},i}$, $i > 0$. Assume that L_c is standard and L/L_c is ferociously ramified. Let $t \in \mathcal{O}_L$, $\bar t \notin k_L^p$. Define

$$G_{\mathfrak{i},i} = \{g \in G : v(g(t) - t) \geqslant i\}$$

for all $i > 0$.

In the general case choose a finite extension l'/l such that $l'L_c$ is standard and $e(l'L|l'L_c) = 1$. Then it is clear that $\mathrm{Gal}(l'L/l'L_c) = \mathrm{Gal}(L/L_c)$, and $l'L/l'L_c$ is ferociously ramified. Define

$$G_{\mathfrak{i},i} = \mathrm{Gal}(l'L/l'L_c)_{\mathfrak{i},i}$$

for all $i > 0$.

Proposition. *For a finite Galois extension L/K the lower filtration $\{\mathrm{Gal}(L/K)_\alpha\}_{\alpha\in\mathcal{A}}$ is well defined.*

Definition. Define a generalization $h_{L/K}\colon \mathcal{A} \to \mathcal{A}$ of the Hasse–Herbrand function. First, we define

$$\Phi_{L/K}\colon \mathcal{A} \to \mathcal{A}$$

as follows:

$$\Phi_{L/K}(\alpha) = \alpha \quad \text{for } \alpha = -1, 0;$$

$$\Phi_{L/K}((\mathfrak{c}, i)) = \left(\mathfrak{c}, \frac{1}{e(L|K)} \int_0^i |\,\mathrm{Gal}(L_c/K)_{\mathfrak{c},t}|dt\right) \quad \text{for all } i > 0;$$

$$\Phi_{L/K}((\mathfrak{i}, i)) = \left(\mathfrak{i}, \int_0^i |\,\mathrm{Gal}(L/K)_{\mathfrak{i},t}|dt\right) \quad \text{for all } i > 0.$$

It is easy to see that $\Phi_{L/K}$ is bijective and increasing, and we introduce

$$h_{L/K} = \Psi_{L/K} = \Phi_{L/K}^{-1}.$$

Define the upper filtration $\mathrm{Gal}(L/K)^\alpha = \mathrm{Gal}(L/K)_{h_{L/K}(\alpha)}$.

All standard formulas for intermediate extensions take place; in particular, for a normal subgroup H in G we have $H_\alpha = H \cap G_\alpha$ and $(G/H)^\alpha = G^\alpha H/H$. The latter relation enables one to introduce the upper filtration for an infinite Galois extension as well.

Remark. The filtrations do depend on the choice of a constant subfield (in characteristic p).

Example. Let $K = \mathbb{F}_p((t))((\pi))$. Choose $k = B = \mathbb{F}_p((\pi))$ as a constant subfield. Let $L = K(b)$, $b^p - b = a \in K$. Then

if $a = \pi^{-i}$, i prime to p, then the ramification break of $\mathrm{Gal}(L/K)$ is $(\mathfrak{c}, i)$;
if $a = \pi^{-pi}t$, i prime to p, then the ramification break of $\mathrm{Gal}(L/K)$ is $(\mathfrak{i}, i)$;
if $a = \pi^{-i}t$, i prime to p, then the ramification break of $\mathrm{Gal}(L/K)$ is $(\mathfrak{i}, i/p)$;
if $a = \pi^{-i}t^p$, i prime to p, then the ramification break of $\mathrm{Gal}(L/K)$ is $(\mathfrak{i}, i/p^2)$.

Remark. A dual filtration on $K/\wp(K)$ is computed in the final version of [Z], see also [KZ].

17.3. Refinement for a two-dimensional local field

Let K be a two-dimensional local field with $\mathrm{char}\,(k_K) = p$, and let k be the constant subfield of K. Denote by

$$\mathbf{v} = (v_1, v_2)\colon (K^{\mathrm{alg}})^* \to \mathbb{Q} \times \mathbb{Q}$$

the extension of the rank 2 valuation of K, which is normalized so that:

- $v_2(a) = v(a)$ for all $a \in K^*$,
- $v_1(u) = w(\overline{u})$ for all $u \in U_{K^{\mathrm{alg}}}$, where w is a non-normalized extension of v_{k_K} on k_K^{alg}, and $\overline{u}$ is the residue of u,
- $\mathbf{v}(c) = (0, e(k|B)^{-1} v_k(c))$ for all $c \in k$.

It can be easily shown that $\mathbf{v}$ is uniquely determined by these conditions, and the value group of $\mathbf{v}|_{K^*}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Next, we introduce the index set

$$\mathcal{A}_2 = \mathcal{A} \cup \mathbb{Q}_+^2 = \mathcal{A} \cup \{(i_1, i_2) : i_1, i_2 \in \mathbb{Q}, i_2 > 0\}$$

and extend the ordering of $\mathcal{A}$ onto $\mathcal{A}_2$ assuming

$$(\mathfrak{i}, i_2) < (i_1, i_2) < (i_1', i_2) < (\mathfrak{i}, i_2')$$

for all $i_2 < i_2'$, $i_1 < i_1'$.

Now we can define G_{i_1,i_2}, where G is the Galois group of a given finite Galois extension L/K. Assume first that L_c is standard and L/L_c is ferociously ramified. Let $t \in \mathcal{O}_L$, $\overline{t} \notin k_L^p$ (e.g., a first local parameter of L). We define

$$G_{i_1,i_2} = \left\{ g \in G : \mathbf{v}\left(t^{-1}g(t) - 1\right) \geqslant (i_1, i_2) \right\}$$

for $i_1, i_2 \in \mathbb{Q}$, $i_2 > 0$. In the general case we choose l'/l (l is the constant subfield of both L and L_c) such that $l'L_c$ is standard and $l'L/l'L_c$ is ferociously ramified and put

$$G_{i_1,i_2} = \mathrm{Gal}(l'L/l'L_c)_{i_1,i_2}.$$

We obtain a well defined lower filtration $(G_\alpha)_{\alpha \in \mathcal{A}_2}$ on $G = \mathrm{Gal}(L/K)$.

In a similar way to 17.2, one constructs the Hasse–Herbrand functions $\Phi_{2,L/K}\colon \mathcal{A}_2 \to \mathcal{A}_2$ and $\Psi_{2,L/K} = \Phi_{2,L/K}^{-1}$ which extend Φ and Ψ respectively. Namely,

$$\Phi_{2,L/K}((i_1,i_2)) = \int_{(0,0)}^{(i_1,i_2)} |\operatorname{Gal}(L/K)_t|\,dt.$$

These functions have usual properties of the Hasse–Herbrand functions φ and $h=\psi$, and one can introduce an $\mathcal{A}_2$-indexed upper filtration on any finite or infinite Galois group G.

17.4. Filtration on $K^{\rm top}(K)$

In the case of a two-dimensional local field K the upper ramification filtration for $K^{\rm ab}/K$ determines a compatible filtration on $K_2^{\rm top}(K)$. In the case where $\operatorname{char}(K)=p$ this filtration has an explicit description given below.

From now on, let K be a two-dimensional local field of prime characteristic p over a quasi-finite field, and k the constant subfield of K. Introduce $\mathbf{v}$ as in 17.3. Let π_k be a prime of k.

For all $\alpha \in \mathbb{Q}_+^2$ introduce subgroups

$$\begin{aligned}
&Q_\alpha = \{\, \{\pi_k, u\} \,:\, u \in K, \mathbf{v}(u-1) \geqslant \alpha \,\} \subset VK_2^{\rm top}(K);\\
&Q_\alpha^{(n)} = \{a \in K_2^{\rm top}(K) \,:\, p^n a \in Q_\alpha\};\\
&S_\alpha = \operatorname{Cl} \bigcup_{n\geqslant 0} Q^{(n)}_{p^n\alpha}.
\end{aligned}$$

For a subgroup A, $\operatorname{Cl} A$ denotes the intersection of all open subgroups containing A.

The subgroups S_α constitute the heart of the ramification filtration on $K_2^{\rm top}(K)$. Their most important property is that they have nice behaviour in unramified, constant and ferociously ramified extensions.

Proposition 1. *Suppose that K satisfies the following property.*

(*) *The extension of constant subfields in any finite unramified extension of K is also unramified.*

Let L/K be either an unramified or a constant totally ramified extension, $\alpha \in \mathbb{Q}_+^2$. Then we have $N_{L/K} S_{\alpha,L} = S_{\alpha,K}$.

Proposition 2. *Let K be standard, L/K a cyclic ferociously ramified extension of degree p with the ramification jump h in lower numbering, $\alpha \in \mathbb{Q}_+^2$. Then:*

(1) $N_{L/K} S_{\alpha,L} = S_{\alpha+(p-1)h,K}$, *if* $\alpha > h$;

(2) $N_{L/K} S_{\alpha,L}$ *is a subgroup in* $S_{p\alpha,K}$ *of index* p, *if* $\alpha \leqslant h$.

Now we have ingredients to define a decreasing filtration $\{\mathrm{fil}_\alpha K_2^{\mathrm{top}}(K)\}_{\alpha\in\mathcal{A}_2}$ on $K_2^{\mathrm{top}}(K)$. Assume first that $\widetilde{K}$ satisfies the condition (*). It follows from [KZ, Th. 3.4.3] that for some purely inseparable constant extension K'/K the field K' is almost standard. Since K' satisfies (*) and is almost standard, it is in fact standard.

Denote

$$
\begin{aligned}
&\mathrm{fil}_{\alpha_1,\alpha_2} K_2^{\mathrm{top}}(K) = S_{\alpha_1,\alpha_2};\\
&\mathrm{fil}_{\mathfrak{i},\alpha_2} K_2^{\mathrm{top}}(K) &&= \mathrm{Cl}\bigcup_{\alpha_1\in\mathbb{Q}} \mathrm{fil}_{\alpha_1,\alpha_2} K_2^{\mathrm{top}}(K) \text{ for } \alpha_2\in\mathbb{Q}_+;\\
&T_K &&= \mathrm{Cl}\bigcup_{\alpha\in\mathbb{Q}_+^2} \mathrm{fil}_{\alpha} K_2^{\mathrm{top}}(K);\\
&\mathrm{fil}_{\mathfrak{c},i} K_2^{\mathrm{top}}(K) &&= T_K + \{\{t,u\} : u\in k,\ v_k(u-1)\geqslant i\} \text{ for all } i\in\mathbb{Q}_+,\\
& &&\quad \text{if } K=k\{\{t\}\} \text{ is standard};\\
&\mathrm{fil}_{\mathfrak{c},i} K_2^{\mathrm{top}}(K) &&= N_{K'/K}\,\mathrm{fil}_{\mathfrak{c},i} K_2^{\mathrm{top}}(K'), \text{ where } K'/K \text{ is as above};\\
&\mathrm{fil}_0 K_2^{\mathrm{top}}(K) &&= U(1)K_2^{\mathrm{top}}(K)+\{t,\mathcal{R}_K\}, \text{ where } U(1)K_2^{\mathrm{top}}(K)=\{1+P_K(1),K^*\},\\
& &&\quad t \text{ is the first local parameter};\\
&\mathrm{fil}_{-1} K_2^{\mathrm{top}}(K) &&= K_2^{\mathrm{top}}(K).
\end{aligned}
$$

It is easy to see that for some unramified extension $\widetilde{K}/K$ the field $\widetilde{K}$ satisfies the condition (*), and we define $\mathrm{fil}_\alpha K_2^{\mathrm{top}}(K)$ as $N_{\widetilde{K}/K}\,\mathrm{fil}_\alpha K_2^{\mathrm{top}}(\widetilde{K})$ for all $\alpha\geqslant 0$, and $\mathrm{fil}_{-1} K_2^{\mathrm{top}}(K)$ as $K_2^{\mathrm{top}}(K)$. It can be shown that the filtration $\{\mathrm{fil}_\alpha K_2^{\mathrm{top}}(K)\}_{\alpha\in\mathcal{A}_2}$ is well defined.

Theorem 1. *Let L/K be a finite abelian extension, $\alpha\in\mathcal{A}_2$. Then $N_{L/K}\,\mathrm{fil}_\alpha K_2^{\mathrm{top}}(L)$ is a subgroup in $\mathrm{fil}_{\Phi_{2,L/K}(\alpha)} K_2^{\mathrm{top}}(K)$ of index $|\mathrm{Gal}(L/K)_\alpha|$. Furthermore,*

$$\mathrm{fil}_{\Phi_{L/K}(\alpha)} K_2^{\mathrm{top}}(K)\cap N_{L/K}K_2^{\mathrm{top}}(L) = N_{L/K}\,\mathrm{fil}_\alpha K_2^{\mathrm{top}}(L).$$

Theorem 2. *Let L/K be a finite abelian extension, and let*

$$\Upsilon_{L/K}^{-1}\colon K_2^{\mathrm{top}}(K)/N_{L/K}K_2^{\mathrm{top}}(L)\to \mathrm{Gal}(L/K)$$

be the reciprocity map. Then

$$\Upsilon_{L/K}^{-1}(\mathrm{fil}_\alpha K_2^{\mathrm{top}}(K) \mod N_{L/K}K_2^{\mathrm{top}}(L)) = \mathrm{Gal}(L/K)^\alpha$$

for any $\alpha\in\mathcal{A}_2$.

Remarks. 1. The ramification filtration, constructed in 17.2, does not give information about the classical ramification invariants in general. Therefore, this construction can be considered only as a provisional one.

2. The filtration on $K_2^{\mathrm{top}}(K)$ constructed in 17.4 behaves with respect to the norm map much better than the usual filtration $\{U_{\mathbf{i}}K_2^{\mathrm{top}}(K)\}_{\mathbf{i}\in\mathbb{Z}_+^n}$. We hope that this filtration can be useful in the study of the structure of K^{top}-groups.

3. In the mixed characteristic case the description of "ramification" filtration on $K_2^{\mathrm{top}}(K)$ is not very nice. However, it would be interesting to try to modify the ramification filtration on $\mathrm{Gal}(L/K)$ in order to get the filtration on $K_2^{\mathrm{top}}(K)$ similar to that described in 17.4.

4. It would be interesting to compute ramification of the extensions constructed in sections 13 and 14.

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18. On ramification theory of monogenic extensions

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We discuss ramification theory for finite extensions L/K of a complete discrete valuation field K. This theory deals with quantities which measure wildness of ramification, such as different, the Artin (resp. Swan) characters and the Artin (resp. Swan) conductors. When the residue field extension k_L/k_K is separable there is a complete theory, e.g. [S], but in general it is not so. In the classical case (i.e. k_L/k_K separable) proofs of many results in ramification theory use the property that all finite extensions of valuation rings $\mathcal{O}_L/\mathcal{O}_K$ are monogenic which is not the case in general. Examples (e.g. [Sp]) show that the classical theorems do not hold in general. Waiting for a beautiful and general ramification theory, we consider a class of extensions L/K which has a good ramification theory. We describe this class and we will call its elements *well ramified extensions*. All classical results are generalizable for well ramified extensions, for example a generalization of the Hasse–Arf theorem proved by J. Borger. We also concentrate our attention on other ramification invariants, more appropriate and general; in particular, we consider two ramification invariants: the Kato conductor and Hyodo depth of ramification.

Here we comment on some works on general ramification theory.

The first direction aims to generalize classical ramification invariants to the general case working with (one dimensional) rational valued invariants. In his papers de Smit gives some properties about ramification jumps and considers the different and differential [Sm2]; he generalizes the Hilbert formula by using the monogenic conductor [Sm1]. We discuss works of Kato [K3-4] in subsection 18.2. In [K2] Kato describes ramification theory for two-dimensional local fields and he proves an analogue of the Hasse–Arf theorem for those Galois extensions in which the extension of the valuation rings (with respect to the discrete valuation of rank 2) is monogenic.

The second direction aims to extend ramification invariants from one dimensional to either higher dimensional or to more complicated objects which involve differential forms (as in Kato's works [K4], [K5]). By using higher local class field theory, Hyodo [H] defines generalized ramification invariants, like depth of ramification (see Theorem

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5 below). We discuss relations of his invariants with the (one dimensional) Kato conductor in subsection 18.3 below. Zhukov [Z] generalizes the classical ramification theory to the case where $|k_K : k_K^p| = p$ (see section 17 of this volume). From the viewpoint of this section the existence of Zhukov's theory is in particular due to the fact that in the case where $|k_K : k_K^p| = p$ one can reduce various assertions to the well ramified case.

18.0. Notations and definitions

In this section we recall some general definitions. We only consider complete discrete valuation fields K with residue fields k_K of characteristic $p > 0$. We also assume that $|k_K : k_K^p|$ is finite.

Definition. Let L/K be a finite Galois extension, $G = \mathrm{Gal}(L/K)$. Let $G_0 = \mathrm{Gal}(L/L \cap K_{\mathrm{ur}})$ be the inertia subgroup of G. Define functions

$$i_G, s_G : G \to \mathbb{Z}$$

by

$$i_G(\sigma) = \begin{cases} \inf_{x \in \mathcal{O}_L \setminus \{0\}} v_L(\sigma(x) - x) & \text{if } \sigma \neq 1 \\ +\infty & \text{if } \sigma = 1 \end{cases}$$

and

$$s_G(\sigma) = \begin{cases} \inf_{x \in \mathcal{O}_L \setminus \{0\}} v_L(\sigma(x)/x - 1) & \text{if } \sigma \neq 1, \sigma \in G_0 \\ +\infty & \text{if } \sigma = 1 \\ 0 & \text{if } \sigma \notin G_0. \end{cases}$$

Then $s_G(\sigma) \leqslant i_G(\sigma) \leqslant s_G(\sigma)+1$ and if k_L/k_K is separable, then $i_G(\sigma) = s_G(\sigma)+1$ for $\sigma \in G_0$. Note that the functions i_G, s_G depend not only on the group G, but on the extension L/K; we will denote i_G also by $i_{L/K}$.

Definition. The Swan function is defined as

$$\mathrm{Sw}_G(\sigma) = \begin{cases} -|k_L : k_K| s_G(\sigma), & \text{if } \sigma \in G_0 \setminus \{1\} \\ -\sum_{\tau \in G_0 \setminus \{1\}} \mathrm{Sw}_G(\tau), & \text{if } \sigma = 1 \\ 0 & \text{if } \sigma \notin G_0. \end{cases}$$

For a character χ of G its Swan conductor

(1) $$\mathrm{sw}(\chi) = \mathrm{sw}_G(\chi) = (\mathrm{Sw}_G, \chi) = \frac{1}{|G|} \sum_{\sigma \in G} \mathrm{Sw}_G(\sigma)\chi(\sigma)$$

is an integer if k_L/k_K is separable (Artin's Theorem) and is not an integer in general (e.g. [Sp, Ch. I]).

18.1. Well ramified extensions

Definition. Let L/K be a finite Galois p-extension. The extension L/K is called *well ramified* if $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ for some $\alpha \in L$.

18.1.1. Structure theorem for well ramified extensions.

Definition. We say that an extension L/K is in *case I* if k_L/k_K is separable; an extension L/K is in *case II* if $|L : K| = |k_L : k_K|$ (i.e. L/K is ferociously ramified in the terminology of 17.0) and $k_L = k_K(a)$ is purely inseparable over k_K.

Extensions in case I and case II are well ramified. An extension which is simultaneously in case I and case II is the trivial extension.

We characterize well ramified extensions by means of the function i_G in the following theorem.

Theorem 1 ([Sp, Prop. 1.5.2]). *Let L/K be a finite Galois p-extension. Then the following properties are equivalent:*
(i) *L/K is well ramified;*
(ii) *for every normal subgroup H of G the Herbrand property holds:*
for every $1 \neq \tau \in G/H$

$$i_{G/H}(\tau) = \frac{1}{e(L|L^H)} \sum_{\sigma \in \tau H} i_G(\sigma);$$

(iii) *the Hilbert formula holds:*

$$v_L(\mathcal{D}_{L/K}) = \sum_{\sigma \neq 1} i_G(\sigma) = \sum_{i \geqslant 0} (|G_i| - 1),$$

for the definition of G_i see subsection 18.2.

From the definition we immediately deduce that if M/K is a Galois subextension of a well ramified L/K then L/M is well ramified; from (ii) we conclude that M/K is well ramified.

Now we consider well ramified extensions L/K which are not in case I nor in case II.

Example. (Well ramified extension not in case I and not in case II). Let K be a complete discrete valuation field of characteristic zero. Let $\zeta_{p^2} \in K$. Consider a cyclic extension of degree p^2 defined by $L = K(x)$ where x a root of the polynomial $f(X) = X^{p^2} - (1 + u\pi)\alpha^p$, $\alpha \in U_K$, $\bar{\alpha} \notin k_K^p$, $u \in U_K$, π is a prime of K. Then $e(L|K) = p = f(L|K)^{\text{ins}}$, so L/K is not in case I nor in case II. Using Theorem 1, one can show that $\mathcal{O}_L = \mathcal{O}_K[x]$ by checking the Herbrand property.

Definition. A well ramified extension which is not in case I and is not in case II is said to be in *case III*.

Note that in case III we have $e(L|K) \geqslant p, f(L|K)^{\text{ins}} \geqslant p$.

Lemma 1. *If L/K is a well ramified Galois extension, then for every ferociously ramified Galois subextension E/K such that L/E is totally ramified either $E = K$ or $E = L$.*

Proof. Suppose that there exists $K \neq E \neq L$, such that E/K is ferociously ramified and L/E is totally ramified. Let π_1 be a prime of L such that $\mathcal{O}_L = \mathcal{O}_E[\pi_1]$. Let $\alpha \in E$ be such that $\mathcal{O}_E = \mathcal{O}_K[\alpha]$. Then we have $\mathcal{O}_L = \mathcal{O}_K[\alpha, \pi_1]$. Let σ be a K-automorphism of E and denote $\widetilde{\sigma}$ a lifting of σ to $G = \mathrm{Gal}(L/K)$. It is not difficult to show that $i_G(\widetilde{\sigma}) = \min\{v_L(\widetilde{\sigma}\pi_1 - \pi_1), v_L(\sigma\alpha - \alpha)\}$. We show that $i_G(\widetilde{\sigma}) = v_L(\widetilde{\sigma}\pi_1 - \pi_1)$. Suppose we had $i_G(\widetilde{\sigma}) = v_L(\sigma\alpha - \alpha)$, then

$$(*) \qquad \frac{i_G(\widetilde{\sigma})}{e(L|E)} = v_E(\sigma\alpha - \alpha) = i_{E/K}(\sigma).$$

Furthermore, by Herbrand property we have

$$i_{E/K}(\sigma) = \frac{1}{e(L|E)} \sum_{s \in \sigma\, \mathrm{Gal}(L/E)} i_G(s) = \frac{i_G(\widetilde{\sigma})}{e(L|E)} + \frac{1}{e(L|E)} \sum_{s \neq \tilde{\sigma}} i_G(s).$$

So from $(*)$ we deduce that

$$\frac{1}{e(L|E)} \sum_{s \neq \tilde{\sigma}} i_G(s) = 0,$$

but this is not possible because $i_G(s) \geqslant 1$ for all $s \in G$. We have shown that

$$(**) \qquad i_G(s) = v_L(s\pi_1 - \pi_1) \qquad \text{for all } s \in G.$$

Now note that $\alpha \notin \mathcal{O}_K[\pi_1]$. Indeed, from $\alpha = \sum a_i \pi_1^i$, $a_i \in \mathcal{O}_K$, we deduce $\alpha \equiv a_0 \pmod{\pi_1}$ which is impossible. By $(**)$ and the Hilbert formula (cf. Theorem 1) we have

$$(***) \qquad v_L(\mathcal{D}_{L/K}) = \sum_{s \neq 1} i_G(s) = \sum_{s \neq 1} v_L(s\pi_1 - \pi_1) = v_L(f'(\pi_1)),$$

where $f(X)$ denotes the minimal polynomial of π_1 over K.

Now let the ideal $\mathcal{T}_{\pi_1} = \{x \in \mathcal{O}_L \,:\, x\mathcal{O}_K[\pi_1] \subset \mathcal{O}_L\}$ be the conductor of $\mathcal{O}_K[\pi_1]$ in $\mathcal{O}_L$ (cf. [S, Ch. III, §6]). We have (cf. loc.cit.)

$$\mathcal{T}_{\pi_1}\mathcal{D}_{L/K} = f'(\pi_1)\mathcal{O}_L$$

and then $(***)$ implies $\mathcal{T}_{\pi_1} = \mathcal{O}_L$, $\mathcal{O}_L = \mathcal{O}_K[\pi_1]$, which contradicts $\alpha \notin \mathcal{O}_K[\pi_1]$. $\square$

Theorem 2 (Spriano)**.** *Let L/K be a Galois well ramified p-extension. Put $K_0 = L \cap K_{\mathrm{ur}}$. Then there is a Galois subextension T/K_0 of L/K_0 such that T/K_0 is in case I and L/T in case II.*

Proof. Induction on $|L : K_0|$.

Let M/K_0 be a Galois subextension of L/K_0 such that $|L : M| = p$. Let T/K_0 be a Galois subextension of M/K_0 such that T/K_0 is totally ramified and M/T is in case II. Applying Lemma 1 to L/T we deduce that L/M is ferociously ramified, hence in case II. □

In particular, if L/K is a Galois p-extension in case III such that $L \cap K_{\mathrm{ur}} = K$, then there is a Galois subextension T/K of L/K such that T/K is in case I, L/T in case I and $K \neq T \neq L$.

18.1.2. Modified ramification function for well ramified extensions.

In the general case one can define a filtration of ramification groups as follows. Given two integers $n, m \geqslant 0$ the (n,m)-ramification group $G_{n,m}$ of L/K is

$$G_{n,m} = \{\sigma \in G : \ v_L(\sigma(x) - x) \geqslant n + m, \text{ for all } x \in \mathcal{M}_L^m\}.$$

Put $G_n = G_{n+1,0}$ and $H_n = G_{n,1}$, so that the classical ramification groups are the G_n. It is easy to show that $H_i \geqslant G_i \geqslant H_{i+1}$ for $i \geqslant 0$.

In case I we have $G_i = H_i$ for all $i \geqslant 0$; in case II we have $G_i = H_{i+1}$ for all $i \geqslant 0$, see [Sm1]. If L/K is in case III, we leave to the reader the proof of the following equality

$$\begin{aligned} G_i &= \{\sigma \in \mathrm{Gal}(L/K) : v_L(\sigma(x) - x) \geqslant i + 1 \quad \text{for all } x \in \mathcal{O}_L\} \\ &= \{\sigma \in \mathrm{Gal}(L/K) : v_L(\sigma(x) - x) \geqslant i + 2 \quad \text{for all } x \in \mathcal{M}_L\} = H_{i+1}. \end{aligned}$$

We introduce another filtration which allows us to simultaneously deal with case I, II and III.

Definition. Let L/K be a finite Galois well ramified extension. The modified t-th *ramification group* $G[t]$ for $t \geqslant 0$ is defined by

$$G[t] = \{\sigma \in \mathrm{Gal}(L/K) \ : \ i_G(\sigma) \geqslant t\}.$$

We call an integer number m a *modified ramification jump* of L/K if $G[m] \neq G[m+1]$.

From now on we will consider only p-extensions.

Definition. For a well ramified extension L/K define the modified Hasse–Herbrand function $\mathfrak{s}_{L/K}(u)$, $u \in \mathbb{R}_{\geqslant 0}$ as

$$\mathfrak{s}_{L/K}(u) = \int_0^u \frac{|G[t]|}{e(L|K)}\, dt.$$

Put $g_i = |\mathcal{G}_i|$. If $m \leqslant u \leqslant m+1$ where m is a non-negative integer, then

$$\mathfrak{s}_{L/K}(u) = \frac{1}{e(L|K)}(g_1 + \cdots + g_m + g_{m+1}(u-m)).$$

We drop the index L/K in $\mathfrak{s}_{L/K}$ if there is no risk of confusion. One can show that the function $\mathfrak{s}$ is continuous, piecewise linear, increasing and convex. In case I, if $\varphi_{L/K}$ denotes the classical Hasse–Herbrand function as in [S, Ch. IV], then $\mathfrak{s}_{L/K}(u) = 1 + \varphi_{L/K}(u-1)$. We define a *modified upper numbering* for ramification groups by $G(\mathfrak{s}_{L/K}(u)) = G[u]$.

If m is a modified ramification jumps, then the number $\mathfrak{s}_{L/K}(m)$ is called a *modified upper ramification jump* of L/K.

For well ramified extensions we can show the Herbrand theorem as follows.

Lemma 2. *For* $u \geqslant 0$ *we have* $\mathfrak{s}_{L/K}(u) = \dfrac{1}{e(L|K)} \displaystyle\sum_{\sigma \in G} \inf(i_G(\sigma), u).$

The proof goes exactly as in [S, Lemme 3, Ch.IV, §3].

Lemma 3. *Let* H *be a normal subgroup of* G *and* $\tau \in G/H$ *and let* $j(\tau)$ *be the upper bound of the integers* $i_G(\sigma)$ *where* σ *runs over all automorphisms of* G *which are congruent to* τ *modulo* H. *Then we have*

$$i_{L^H/K}(\tau) = \mathfrak{s}_{L/L^H}(j(\tau)).$$

For the proof see Lemme 4 loc.cit. (note that Theorem 1 is fundamental in the proof). In order to show Herbrand theorem, we have to show the multiplicativity in the tower of extensions of the function $\mathfrak{s}_{L/K}$.

Lemma 4. *With the above notation, we have* $\mathfrak{s}_{L/K} = \mathfrak{s}_{L^H/K} \circ \mathfrak{s}_{L/L^H}$.

For the proof see Prop. 15 loc.cit.

Corollary. *If* L/K *is well ramified and* H *is a normal subgroup of* $G = \mathrm{Gal}(L/K)$, *then the Herbrand theorem holds:*

$$(G/H)(u) = G(u)H/H \quad \textit{for all } u \geqslant 0.$$

It is known that the upper ramification jumps (with respect the classical function φ) of an abelian extension in case I are integers. This is the Hasse–Arf theorem. Clearly the same result holds with respect the function $\mathfrak{s}$. In fact, if m is a classical ramification jump and $\varphi_{L/K}(m)$ is the upper ramification jump, then the modified ramification jump is $m+1$ and the modified upper ramification jumps is $\mathfrak{s}_{L/K}(m+1) = 1 + \varphi_{L/K}(m)$ which is an integer. In case II it is obvious that the modified upper ramification jumps are integers. For well ramified extensions we have the following theorem, for the proof see the end of 18.2.

Theorem 3 (Borger). *The modified upper ramification jumps of abelian well ramified extensions are integers.*

18.2. The Kato conductor

We have already remarked that the Swan conductor $\mathrm{sw}(\chi)$ for a character χ of the Galois group G_K is not an integer in general. In [K3] Kato defined a modified Swan conductor in case I, II for any character χ of G_K; and [K4] contains a definition of an integer valued conductor (which we will call the Kato conductor) for characters of degree 1 in the general case i.e. not only in cases I and II.

We recall its definition. The map $K^* \to H^1(K, \mathbb{Z}/n(1))$ (cf. the definition of $H^q(K)$ in subsection 5.1) induces a pairing

$$\{\ ,\ \}\colon H^q(K) \times K_r(K) \to H^{q+r}(K),$$

which we briefly explain only for K of characteristic zero, in characteristic $p > 0$ see [K4, (1.3)]. For $a \in K^*$ and a fixed $n \geqslant 0$, let $\{a\} \in H^1(K, \mathbb{Z}/n(1))$ be the image under the connecting homomorphism $K^* \to H^1(K, \mathbb{Z}/n(1))$ induced by the exact sequence of G_K-modules

$$1 \longrightarrow \mathbb{Z}/n(1) \longrightarrow K_s^* \xrightarrow{n} K_s^* \longrightarrow 1.$$

For $a_1, ..., a_r \in K^*$ the symbol $\{a_1, ..., a_r\} \in H^r(K, \mathbb{Z}/n(r))$ is the cup product $\{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_r\}$. For $\chi \in H^q(K)$ and $a_1, \dots, a_r \in K^*$ $\{\chi, a_1, ..., a_r\} \in H_n^{q+r}(K)$ is the cup product $\{\chi\} \cup \{a_1\} \cup \cdots \cup \{a_r\}$. Passing to the limit we have the element $\{\chi, a_1, ..., a_r\} \in H^{q+r}(K)$.

Definition. Following Kato, we define an increasing filtration $\{\mathrm{fil}_m H^q(K)\}_{m \geqslant 0}$ of $H^q(K)$ by

$$\mathrm{fil}_m H^q(K) = \{\chi \in H^q(K) : \{\chi|_M, U_{m+1,M}\} = 0 \quad \text{for every } M\}$$

where M runs through all complete discrete valuation fields satisfying $\mathcal{O}_K \subset \mathcal{O}_M$, $\mathcal{M}_M = \mathcal{M}_K \mathcal{O}_M$; here $\chi|_M$ denotes the image of $\chi \in H^q(K)$ in $H^q(M)$.

Then one can show $H^q(K) = \cup_{m \geqslant 0} \mathrm{fil}_m H^q(K)$ [K4, Lemma (2.2)] which allows us to give the following definition.

Definition. For $\chi \in H^q(K)$ the *Kato conductor* of χ is the integer $\mathrm{ksw}(\chi)$ defined by

$$\mathrm{ksw}(\chi) = \min\{m \geqslant 0 : \chi \in \mathrm{fil}_m H^q(K)\}.$$

This integer $\mathrm{ksw}(\chi)$ is a generalization of the classical Swan conductor as stated in the following proposition.

Proposition 1. *Let $\chi \in H^1(K)$ and let L/K be the corresponding finite cyclic extension and suppose that L/K is in case I or II. Then*
(a) $\mathrm{ksw}(\chi) = \mathrm{sw}(\chi)$ *(see formula* (1) *).*
(b) Let t be the maximal modified ramification jump. Then

$$\mathrm{ksw}(\chi) = \begin{cases} \mathfrak{s}_{L/K}(t) - 1 & \textit{case I} \\ \mathfrak{s}_{L/K}(t) & \textit{case II.} \end{cases}$$

Proof. (a) See [K4, Prop. (6.8)]. (b) This is a computation left to the reader. □

We compute the Kato conductor in case III.

Theorem 4 (Spriano)**.** *If L/K is a cyclic extension in case III and if χ is the corresponding element of $H^1(K)$, then* $\mathrm{ksw}(\chi) = \mathrm{sw}(\chi) - 1$. *If t is the maximal modified ramification jump of L/K, then* $\mathrm{ksw}(\chi) = \mathfrak{s}_{L/K}(t) - 1$.

Before the proof we explain how to compute the Kato conductor $\mathrm{ksw}(\chi)$ where $\chi \in H^1(K)$. Consider the pairing $H^1(K) \times K^* \to H^2(K)$, ($q = 1 = r$). It coincides with the symbol $(\cdot,\cdot)$ defined in [S, Ch. XIV]. In particular, if $\chi \in H^1(K)$ and $a \in K^*$, then $\{\chi, a\} = 0$ if and only if the element a is a norm of the extension L/K corresponding to χ. So we have to compute the minimal integer m such that $U_{m+1,M}$ is in the norm of the cyclic extension corresponding to $\chi|_M$ when M runs through all complete discrete valuation fields satisfying $\mathcal{M}_M = \mathcal{M}_K \mathcal{O}_M$. The minimal integer n such that $U_{n+1,K}$ is contained in the norm of L/K is not, in general, the Kato conductor (for instance if the residue field of K is algebraically closed)

Here is a characterization of the Kato conductor which helps to compute it and does not involve extensions M/K, cf. [K4, Prop. (6.5)].

Proposition 2. *Let K be a complete discrete valuation field. Suppose that $|k_K : k_K^p| = p^c < \infty$, and $H_p^{c+1}(k_K) \neq 0$. Then for $\chi \in H^q(K)$ and $n \geqslant 0$*

$$\chi \in \mathrm{fil}_n H^q(K) \iff \{\chi, U_{n+1} K^M_{c+2-q}(K)\} = 0 \quad \textit{in } H^{c+2}(K),$$

for the definition of $U_{n+1} K^M_{c+2-q}(K)$ see subsection 4.2.

In the following we will only consider characters χ such that the corresponding cyclic extensions L/K are p-extension, because $\mathrm{ksw}(\chi) = 0$ for tame characters χ, cf. [K4, Prop. (6.1)]. We can compute the Kato conductor in the following manner.

Corollary. *Let K be as in Proposition 2. Let $\chi \in H^1(K)$ and assume that the corresponding cyclic extension L/K is a p-extension. Then the minimal integer n such that*

$$U_{n+1,K} \subset N_{L/K} L^*$$

is the Kato conductor of χ.

Proof. By the hypothesis (i.e. $U_{n+1,K} \subset N_{L/K}L^*$) we have $\mathrm{ksw}(\chi) \geqslant n$. Now $U_{n+1,K} \subset N_{L/K}L^*$, implies that $U_{n+1}K_{c+1}(K)$ is contained in the norm group $N_{L/K}K_{c+1}(L)$. By [K1, II, Cor. at p. 659] we have that $\{\chi, U_{n+1}K_{c+1}(K)\} = 0$ in $H^{c+2}(K)$ and so by Proposition 2 $\mathrm{ksw}(\chi) \leqslant n$. □

Beginning of the proof of Theorem 4. Let L/K be an extension in case III and let $\chi \in H^1(K)$ be the corresponding character. We can assume that $H_p^{c+1}(k_K) \neq 0$, otherwise we consider the extension $k = \cup_{i\geqslant 0} k_K(T^{p^{-i}})$ of the residue field k_K, preserving a p-base, for which $H_p^{c+1}(k) \neq 0$ (see [K3, Lemma (3-9)]).

So by the above Corollary we have to compute the minimal integer n such that $U_{n+1,K} \subset N_{L/K}L^*$.

Let T/K be the totally ramified extension defined by Lemma 1 (here T/K is uniquely determined because the extension L/K is cyclic). Denote by $U_{v,L}$ for $v \in \mathbb{R}, v \geqslant 0$ the group $U_{n,L}$ where n is the smallest integer $\geqslant v$.

If t is the maximal modified ramification jump of L/K, then

$$U_{\mathfrak{s}_{L/T}(t)+1,T} \subset N_{L/T}L^* \tag{1}$$

because L/T is in case II and its Kato conductor is $\mathfrak{s}_{L/T}(t)$ by Proposition 1 (b). Now consider the totally ramified extension T/K. By [S, Ch. V, Cor. 3 §6] we have

$$N_{T/K}(U_{s,T}) = U_{\mathfrak{s}_{T/K}(s+1)-1,K} \quad \text{if} \quad \mathrm{Gal}(T/K)_s = \{1\}. \tag{2}$$

Let $t' = i_{T/K}(\tau)$ be the maximal modified ramification jump of T/K. Let r be the maximum of $i_{L/K}(\sigma)$ where σ runs over all representatives of the coset $\tau\,\mathrm{Gal}(L/T)$. By Lemma 3 $t' = \mathfrak{s}_{L/T}(r)$. Note that $r < t$ (we explain it in the next paragraph), so

$$t' = \mathfrak{s}_{L/T}(r) < \mathfrak{s}_{L/T}(t). \tag{3}$$

To show that $r < t$ it suffices to show that for a generator ρ of $\mathrm{Gal}(L/K)$

$$i_{L/K}(\rho^{p^m}) > i_{L/K}(\rho^{p^{m-1}})$$

for $|T:K| \leqslant p^m \leqslant |L:K|$. Write $\mathcal{O}_L = \mathcal{O}_K(a)$ then

$$\rho^{p^m}(a) - a = \rho^{p^{m-1}}(b) - b, \quad b = \sum_{i=0}^{p-1} \rho^{p^{m-1}i}(a).$$

Then $b = pa + \pi^i f(a)$ where π is a prime element of L, $f(X) \in \mathcal{O}_K[X]$ and $i = i_{L/K}(\rho^{p^{m-1}})$. Hence $i_{L/K}(\rho^{p^m}) = v_L(\rho^{p^m}(a) - a) \geqslant \min(i + v_L(p), 2i)$, so $i_{L/K}(\rho^{p^m}) > i_{L/K}(\rho^{p^{m-1}})$, as required.

Now we use the fact that the number $\mathfrak{s}_{L/K}(t)$ is an integer (by Borger's Theorem). We shall show that $U_{\mathfrak{s}_{L/K}(t),K} \subset N_{L/K}L^*$.

By (3) we have $\mathrm{Gal}(T/K)_{\mathfrak{s}_{L/T}(t)} = \{1\}$ and so we can apply (2). By (1) we have $U_{\mathfrak{s}_{L/T}(t)+1,T} \subset N_{L/T}L^*$, and by applying the norm map $N_{T/K}$ we have (by (2))

$$N_{T/K}(U_{\mathfrak{s}_{L/T}(t)+1,T}) = U_{\mathfrak{s}_{T/K}(\mathfrak{s}_{L/T}(t)+2)-1,K} \subset N_{L/K}L^*.$$

Thus it suffices to show that the smallest integer $\geqslant \mathfrak{s}_{T/K}(\mathfrak{s}_{L/T}(t)+2)-1$ is $\mathfrak{s}_{L/K}(t)$. Indeed we have

$$\mathfrak{s}_{T/K}(\mathfrak{s}_{L/T}(t)+2)-1 = \mathfrak{s}_{T/K}(\mathfrak{s}_{L/T}(t)) + \frac{2}{|T:K|} - 1 = \mathfrak{s}_{L/K}(t) - 1 + \frac{2}{p^e}$$

where we have used Lemma 4. By Borger's theorem $\mathfrak{s}_{L/K}(t)$ is an integer and thus we have shown that $\mathrm{ksw}(\chi) \leqslant \mathfrak{s}_{L/K}(t) - 1$.

Now we need a lemma which is a key ingredient to deduce Borger's theorem.

Lemma 5. *Let L/K be a Galois extension in case III. If $k_L = k_K(a^{1/f})$ then $a \in k_K \setminus k_K^p$ where $f = |L:T| = f(L/K)^{\mathrm{ins}}$. Let α be a lifting of a in K and let $M = K(\beta)$ where $\beta^f = \alpha$.*

If $\sigma \in \mathrm{Gal}(L/K)$ and $\sigma' \in \mathrm{Gal}(LM/M)$ is such that $\sigma'|_L = \sigma$ then

$$i_{LM/M}(\sigma') = e(LM|L)i_{L/K}(\sigma).$$

Proof. (After J. Borger). Note that the extension M/K is in case II and LM/M is in case I, in particular it is totally ramified. Let $x \in \mathcal{O}_L$ such that $\mathcal{O}_L = \mathcal{O}_K[x]$. One can check that $x^f - \alpha \in \mathcal{M}_L \setminus \mathcal{M}_L^2$. Let $g(X)$ be the minimal polynomial of β over K. Then $g(X+x)$ is an Eisenstein polynomial over L (because $g(X+x) \equiv X^f + x^f - \alpha \equiv X^f \mod \mathcal{M}_L$) and $\beta - x$ is a root of $g(X+x)$. So $\beta - x$ is a prime of LM and we have

$$i_{LM/M}(\sigma') = v_{LM}(\sigma'(\beta - x) - (\beta - x)) = v_{LM}(\sigma'(x) - x) = e(LM|L)i_{L/K}(\sigma).$$

□

Proof of Theorem 3 and Theorem 4. Now we deduce simultaneously the formula for the Kato conductor in case III and Borger's theorem. We compute the classical Artin conductor $A(\chi|_M)$. By the preceding lemma we have

(2)

$$\begin{aligned} A(\chi|_M) &= \frac{1}{e(LM|M)} \sum_{\sigma' \in \mathrm{Gal}(LM/M)} \chi|_M(\sigma') i_{LM/M}(\sigma') \\ &= \frac{e(LM|L)}{e(LM|M)} \sum_{\sigma' \in \mathrm{Gal}(LM/M)} \chi|_M(\sigma') i_{L/K}(\sigma) = \frac{1}{e(L|K)} \sum_{\sigma \in G} \chi(\sigma) i_{L/K}(\sigma). \end{aligned}$$

Since $A(\chi|_M)$ is an integer by Artin's theorem we deduce that the latter expression is an integer. Now by the well known arguments one deduces the Hasse–Arf property for L/K.

The above argument also shows that the Swan conductor (=Kato conductor) of LM/M is equal to $A(\chi|_M)-1$, which shows that $\mathrm{ksw}(\chi) \geqslant A(\chi|_M)-1 = \mathfrak{s}_{L/K}(t)-1$, so $\mathrm{ksw}(\chi) = \mathfrak{s}_{L/K}(t) - 1$ and Theorem 4 follows. □

18.3. More ramification invariants

18.3.1. Hyodo's depth of ramification. This ramification invariant was introduced by Hyodo in [H]. We are interested in its link with the Kato conductor.

Let K be an m-dimensional local field, $m \geqslant 1$. Let $t_1, \dots, t_m$ be a system of local parameters of K and let $\mathbf{v}$ be the corresponding valuation.

Definition. Let L/K be a finite extension. The *depth of ramification* of L/K is

$$d_K(L/K) = \inf\{\mathbf{v}(\mathrm{Tr}_{L/K}(y)/y) : y \in L^*\} \in \mathbb{Q}^m.$$

The right hand side expression exists; and, in particular, if $m = 1$ then $d_K(L/K) = v_K(\mathcal{D}_{L/K}) - (1 - v_K(\pi_L))$, see [H]. The main result about the depth is stated in the following theorem (see [H, Th. (1-5)]).

Theorem 5 (Hyodo)**.** *Let L be a finite Galois extension of an m-dimensional local field K. For $l \geqslant 1$ define*

$$\mathbf{j}(l) = \mathbf{j}_{L/K}(l) = \begin{cases} \max\{\mathbf{i} : 1 \leqslant \mathbf{i} \in \mathbb{Z}^m, \quad |\Psi_{L/K}(U_{\mathbf{i}}K_m^{\mathrm{top}}(K))| \geqslant p^l\} & \textit{if it exists} \\ 0 & \textit{otherwise} \end{cases}$$

where $\Psi_{L/K}$ is the reciprocity map; the definition of $U_{\mathbf{i}}K_m^{\mathrm{top}}(K)$ is given in 17.0. Then

$$(p-1)\sum_{l\geqslant 1}\mathbf{j}(l)/p^l \leqslant d_K(L/K) \leqslant (1-p^{-1})\sum_{l\geqslant 1}\mathbf{j}(l). \tag{3}$$

Furthermore, these inequalities are the best possible (*cf.* [H, Prop. (3-4) and Ex. (3-5)]).

For $\mathbf{i} \in \mathbb{Z}^m$, let $G^{\mathbf{i}}$ be the image of $U_{\mathbf{i}}K_m^{\mathrm{top}}(K)$ in $\mathrm{Gal}(L/K)$ under the reciprocity map $\Psi_{L/K}$. The numbers $\mathbf{j}(l)$ are called jumping number (by Hyodo) and in the classical case, i.e. $m = 1$, they coincide with the upper ramification jumps of L/K.

For local fields (i.e. 1-dimensional local fields) one can show that the first inequality in (3) is actually an equality. Hyodo stated ([H, p.292]) *"It seems that we can define nice ramification groups only when the first equality of (3) holds."*

For example, if L/K is of degree p, then the inequalities in (3) are actually equalities and in this case we actually have a nice ramification theory. For an abelian extension L/K [H, Prop. (3-4)] shows that the first equality of (3) holds if at most one diagonal component of $E(L/K)$ (for the definition see subsection 1.2) is divisible by p.

Extensions in case I or II verify the hypothesis of Hyodo's proposition, but it is not so in case III. We shall show below that the first equality does not hold in case III.

18.3.2. The Kato conductor and depth of ramification.

Consider an m-dimensional local field K, $m \geqslant 1$. Proposition 2 of 18.2 shows (if the first residue field is of characteristic $p > 0$) that for $\chi \in H^1(K)$, $\chi \in \mathrm{fil}_n H^1(K)$ if and only if the induced homomorphism $K_m(K) \to \mathbb{Q}/\mathbb{Z}$ annihilates $U_{n+1}K_m(K)$ (cf. also in [K4, Remark (6.6)]). This also means that the Kato conductor of the extension L/K corresponding to χ is the m-th component of the last ramification jump $\mathbf{j}(1)$ (recall that $\mathbf{j}(1) = \max\{\mathbf{i} : 1 \leqslant \mathbf{i} \in \mathbb{Z}^m, \quad |G^{\mathbf{i}}| \geqslant p\}$).

Example. Let L/K as in Example of 18.1.1 and assume that K is a 2-dimensional local field with the first residue field of characteristic $p > 0$ and let $\chi \in H^1(K)$ be the corresponding character. Let $\mathbf{j}(l)_i$ denotes the i-th component of $\mathbf{j}(l)$. Then by Theorem 3 and by the above discussion we have

$$\mathrm{ksw}(\chi) = \mathbf{j}(1)_2 = \mathfrak{s}_{L/K}(pe/(p-1)) - 1 = \frac{(2p-1)e}{p-1} - 1.$$

If T/K is the subextension of degree p, we have

$$d_K(T/K)_2 = p^{-1}(p-1)\mathbf{j}(2)_2 \implies \mathbf{j}(2)_2 = \frac{pe}{p-1} - 1.$$

The depth of ramification is easily computed:

$$d_K(L/K)_2 = d_K(T/K)_2 + d_K(L/T)_2 = \frac{(p-1)}{p}\left(\frac{2pe}{p-1} - 1\right).$$

The left hand side of (3) is $(p-1)(\mathbf{j}(1)/p + \mathbf{j}(2)/p^2)$, so for the second component we have

$$(p-1)\left(\frac{\mathbf{j}(1)_2}{p} + \frac{\mathbf{j}(2)_2}{p^2}\right) = 2e - \frac{(p^2-1)}{p^2} \neq d_K(L/K)_2.$$

Thus, the first equality in (3) does not hold for the extension L/K.

If K is a complete discrete valuation (of rank one) field, then in the well ramified case straightforward calculations show that

$$e(L|K)d_K(L/K) = \begin{cases} \sum_{\sigma\neq 1} s_G(\sigma) & \text{case I,II} \\ \sum_{\sigma\neq 1} s_G(\sigma) - e(L|K) + 1 & \text{case III} \end{cases}$$

Let $\chi \in H^1(K)$ and assume that the corresponding extension L/K is well ramified. Let t denote the last ramification jump of L/K; then from the previous formula and Theorem 4 we have

$$e(L|K)\,\mathrm{ksw}(\chi) = \begin{cases} d_L(L/K) + t & \text{case I,II} \\ d_L(L/K) + t - 1 & \text{case III} \end{cases}$$

In the general case, we can indicate the following relation between the Kato conductor and Hyodo's depth of ramification.

Theorem 6 (Spriano)**.** *Let $\chi \in H^1(K, \mathbb{Z}/p^n)$ and let L/K be the corresponding cyclic extension. Then*

$$\mathrm{ksw}(\chi) \leqslant d_K(L/K) + \frac{t}{e(L|K)}$$

where t is the maximal modified ramification jump.

Proof. In [Sp, Prop. 3.7.3] we show that

$$(*) \qquad \mathrm{ksw}(\chi) \leqslant \left[\frac{1}{e(L|K)} \left(\sum_{\sigma \in G} \mathrm{Sw}_G(\sigma)\chi(\sigma) - M_{L/K} \right) \right],$$

where $[x]$ indicates the integer part of $x \in \mathbb{Q}$ and the integer $M_{L/K}$ is defined by

$$(**) \qquad d_L(L/K) + M_{L/K} = \sum_{\sigma \neq 1} s_G(\sigma).$$

Thus, the inequality in the statement follows from $(*)$ and $(**)$. □

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Existence theorem for higher local fields

Kazuya Kato

0. Introduction

A field K is called an n-dimensional local field if there is a sequence of fields $k_n, \dots, k_0$ satisfying the following conditions: k_0 is a finite field, k_i is a complete discrete valuation field with residue field k_{i-1} for $i = 1, \dots, n$, and $k_n = K$.

In [9] we defined a canonical homomorphism from the nth Milnor group $K_n(K)$ (cf. [14]) of an n-dimensional local field K to the Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ of the maximal abelian extension of K and generalized the familiar results of the usual local class field theory to the case of arbitrary dimension except the "existence theorem".

An essential difficulty with the existence theorem lies in the fact that K (resp. the multiplicative group K^*) has no appropriate topology in the case where $n \geqslant 2$ (resp. $n \geqslant 3$) which would be compatible with the ring (resp. group) structure and which would take the topologies of the residue fields into account. Thus we abandon the familiar tool "topology" and define the openness of subgroups and the continuity of maps from a new point of view.

In the following main theorems the words "open" and "continuous" are not used in the topological sense. They are explained below.

Theorem 1. *Let K be an n-dimensional local field. Then the correspondence*

$$L \to N_{L/K} K_n(L)$$

is a bijection from the set of all finite abelian extensions of K to the set of all open subgroups of $K_n(K)$ of finite index.

This existence theorem is essentially contained in the following theorem which expresses certain Galois cohomology groups of K (for example the Brauer group of

K) by using the Milnor K-group of K. For a field k we define the group $H^r(k)$ ($r \geqslant 0$) as follows (cf. [9, §3.1]). In the case where $\operatorname{char}(k) = 0$ let

$$H^r(k) = \varinjlim H^r(k, \mu_m^{\otimes(r-1)})$$

(the Galois cohomology). In the case where $\operatorname{char}(k) = p > 0$ let

$$H^r(k) = \varinjlim H^r(k, \mu_m^{\otimes(r-1)}) + \varinjlim H^r_{p^i}(k).$$

Here in each case m runs over all integers invertible in k, μ_m denotes the group of all mth roots of 1 in the separable closure k^{sep} of k, and $\mu_m^{\otimes(r-1)}$ denotes its $(r-1)$th tensor power as a $\mathbb{Z}/m$-module on which $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ acts in the natural way. In the case where $\operatorname{char}(k) = p > 0$ we denote by $H^r_{p^i}(k)$ the cokernel of

$$F - 1\colon C_i^{r-1}(k) \to C_i^{r-1}(k)/\{C_i^{r-2}(k), T\}$$

where $C_i^{\cdot}$ is the group defined in [3, Ch.II,§7] (see also Milne [13, §3]). For example, $H^1(k)$ is isomorphic to the group of all continuous characters of the compact abelian group $\mathrm{Gal}(k^{\mathrm{ab}}/k)$ and $H^2(k)$ is isomorphic to the Brauer group of k.

Theorem 2. *Let K be as in Theorem 1. Then $H^r(K)$ vanishes for $r > n+1$ and is isomorphic to the group of all continuous characters of finite order of $K_{n+1-r}(K)$ in the case where $0 \leqslant r \leqslant n+1$.*

We shall explain the contents of each section.

For a category $\mathcal{C}$ the category of pro-objects $\mathrm{pro}(\mathcal{C})$ and the category of ind-objects $\mathrm{ind}(\mathcal{C})$ are defined as in Deligne [5]. Let $\mathcal{F}_0$ be the category of finite sets, and let $\mathcal{F}_1$, $\mathcal{F}_2$, $\dots$ be the categories defined by $\mathcal{F}_{n+1} = \mathrm{ind}(\mathrm{pro}(\mathcal{F}_n))$. Let $\mathcal{F}_\infty = \cup_n \mathcal{F}_n$. In section 1 we shall show that n-dimensional local fields can be viewed as ring objects of $\mathcal{F}_n$. More precisely we shall define a ring object $\underline{K}$ of $\mathcal{F}_n$ corresponding to an n-dimensional local field K such that K is identified with the ring $[e, \underline{K}]_{\mathcal{F}_\infty}$ of morphisms from the one-point set e (an object of $\mathcal{F}_0$) to $\underline{K}$, and a group object $\underline{K^*}$ such that K^* is identified with $[e, \underline{K^*}]_{\mathcal{F}_\infty}$. We call a subgroup N of $K_q(K)$ open if and only if the map

$$K^* \times \dots \times K^* \to K_q(K)/N, \quad (x_1, \dots, x_q) \mapsto \{x_1, \dots, x_q\} \mod N$$

comes from a morphism $\underline{K^*} \times \dots \times \underline{K^*} \to K_q(K)/N$ of $\mathcal{F}_\infty$ where $K_q(K)/N$ is viewed as an object of $\mathrm{ind}(\mathcal{F}_0) \subset \mathcal{F}_1$. We call a homomorphism $\varphi\colon K_q(K) \to \mathbb{Q}/\mathbb{Z}$ a continuous character if and only if the induced map

$$K^* \times \dots \times K^* \to \mathbb{Q}/\mathbb{Z}, \quad (x_1, \dots, x_q) \mapsto \varphi(\{x_1, \dots, x_q\})$$

comes from a morphism of $\mathcal{F}_\infty$ where $\mathbb{Q}/\mathbb{Z}$ is viewed as an object of $\mathrm{ind}(\mathcal{F}_0)$. In each case such a morphism of $\mathcal{F}_\infty$ is unique if it exists (cf. Lemma 1 of section 1).

In section 2 we shall generalize the self-duality of the additive group of a one-dimensional local field in the sense of Pontryagin to arbitrary dimension.

Section 3 is a preliminary one for section 4. There we shall prove some ring-theoretic properties of $[X, \underline{K}]_{\mathcal{F}_\infty}$ for objects X of $\mathcal{F}_\infty$.

In section 4 we shall treat the norm groups of cohomological objects. For a field k denote by $\mathcal{E}(k)$ the category of all finite extensions of k in a fixed algebraic closure of k with the inclusion maps as morphisms. Let H be a functor from $\mathcal{E}(k)$ to the category Ab of all abelian groups such that $\varinjlim_{k' \in \mathcal{E}(k)} H(k') = 0$. For $w_1, \dots, w_g \in H(k)$ define the K_q-norm group $N_q(w_1, \dots, w_g)$ as the subgroup of $K_q(k)$ generated by the subgroups $N_{k'/k} K_q(k')$ where k' runs over all fields in $\mathcal{E}(k)$ such that $\{w_1, \dots, w_g\} \in \ker(H(k) \to H(k'))$ and where $N_{k'/k}$ denotes the canonical norm homomorphism of the Milnor K-groups (Bass and Tate [2, §5] and [9, §1.7]). For example, if $H = H^1$ and $\chi_1, \dots, \chi_g \in H^1(k)$ then $N_q(\chi_1, \dots, \chi_g)$ is nothing but $N_{k'/k} K_q(k')$ where k' is the finite abelian extension of k corresponding to $\cap_i \ker(\chi_i\colon \mathrm{Gal}(k^{\mathrm{ab}}/k) \to \mathbb{Q}/\mathbb{Z})$. If $H = H^2$ and $w \in H^2(k)$ then $N_1(w)$ is the image of the reduced norm map $A^* \to k^*$ where A is a central simple algebra over k corresponding to w.

As it is well known for a one-dimensional local field k the group $N_1(\chi_1, \dots, \chi_g)$ is an open subgroup of k^* of finite index for any $\chi_1, \dots, \chi_g \in H^1(k)$ and the group $N_1(w) = k^*$ for any $w \in H^2(k)$. We generalize these facts as follows.

Theorem 3. *Let K be an n-dimensional local field and let $r \geqslant 1$.*

(1) *Let $w_1, \dots, w_g \in H^r(K)$. Then the norm group $N_{n+1-r}(w_1, \dots, w_g)$ is an open subgroup of $K_{n+1-r}(K)$ of finite index.*

(2) *Let M be a discrete torsion abelian group endowed with a continuous action of* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. *Let H be the Galois cohomology functor $H^r(\ ,M)$. Then for every $w \in H^r(K, M)$ the group $N_{n+1-r}(w)$ is an open subgroup of $K_{n+1-r}(K)$ of finite index.*

Let k be a field and let $q, r \geqslant 0$. We define a condition (N_q^r, k) as follows: for every $k' \in \mathcal{E}(k)$ and every discrete torsion abelian group M endowed with a continuous action of $\mathrm{Gal}(k'^{\mathrm{sep}}/k')$

$$N_q(w_1, \dots, w_g) = K_q(k')$$

for every $i > r$, $w_1, \dots, w_g \in H^i(k')$, $w_1, \dots, w_g \in H^i(k', M)$, and in addition $|k : k^p| \leqslant p^{q+r}$ in the case where $\mathrm{char}\,(k) = p > 0$.

For example, if k is a perfect field then the condition (N_0^r, k) is equivalent to $\mathrm{cd}(k) \leqslant r$ where cd denotes the cohomological dimension (Serre [16]).

Proposition 1. *Let K be a complete discrete valuation field with residue field k. Let $q \geqslant 1$ and $r \geqslant 0$. Then the two conditions (N_q^r, K) and (N_{q-1}^r, k) are equivalent.*

On the other hand by [11] the conditions (N_0^r, K) and (N_0^{r-1}, k) are equivalent for any $r \geqslant 1$. By induction on n we obtain

Corollary. *Let K be an n-dimensional local field. Then the condition (N_q^r, K) holds if and only if $q + r \geqslant n + 1$.*

We conjecture that if $q+r=q'+r'$ then the two conditions (N_q^r, k) and $(N_{q'}^{r'}, k)$ are equivalent for any field k.

Finally in section 5 we shall prove Theorem 2. Then Theorem 1 will be a corollary of Theorem 2 for $r=1$ and of [9, §3, Theorem 1] which claims that the canonical homomorphism

$$K_n(K) \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

induces an isomorphism $K_n(K)/N_{L/K}K_n(L) \xrightarrow{\sim} \mathrm{Gal}(L/K)$ for each finite abelian extension L of K.

I would like to thank Shuji Saito for helpful discussions and for the stimulation given by his research in this area (e.g. his duality theorem of Galois cohomology groups with locally compact topologies for two-dimensional local fields).

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Notation.

We follow the notation in the beginning of this volume. References to sections in this text mean references to sections of this work and not of the whole volume.

All fields and rings in this paper are assumed to be commutative.

Denote by Sets, Ab, Rings the categories of sets, of abelian groups and of rings respectively.

If $\mathcal{C}$ is a category and X, Y are objects of $\mathcal{C}$ then $[X,Y]_{\mathcal{C}}$ (or simply $[X,Y]$) denotes the set of morphisms $X \to Y$.

1. Definition of the continuity for higher local fields

1.1. Ring objects of a category corresponding to rings.

For a category $\mathcal{C}$ let $\mathcal{C}^\circ$ be the dual category of $\mathcal{C}$. If $\mathcal{C}$ has a final object we always denote it by e. Then, if $\theta: X \to Y$ is a morphism of $\mathcal{C}$, $[e,\theta]$ denotes the induced map $[e,X] \to [e,Y]$.

In this subsection we prove the following

Proposition 2. *Let $\mathcal{C}$ be a category with a final object e in which the product of any two objects exists. Let $\underline{R}$ be a ring object of $\mathcal{C}$ such that for a prime p the morphism $\underline{R} \to \underline{R}$, $x \mapsto px$ is the zero morphism, and via the morphism $\underline{R} \to \underline{R}$, $x \mapsto x^p$ the* *latter $\underline{R}$ is a free module of finite rank over the former $\underline{R}$. Let $R = [e, \underline{R}]$, and let A be a ring with a nilpotent ideal I such that $R = A/I$ and such that I^i/I^{i+1} is a free R-module of finite rank for any i.*

Then:

(1) *There exists a ring object $\underline{A}$ of $\mathcal{C}$ equipped with a ring isomorphism $j: A \xrightarrow{\sim} [e, \underline{A}]$ and with a homomorphism of ring objects $\theta: \underline{A} \to \underline{R}$ having the following properties:*
   (a) *$[e, \theta] \circ j: A \to R$ coincides with the canonical projection.*
   (b) *For any object X of $\mathcal{C}$, $[X, \underline{A}]$ is a formally etale ring over A in the sense of Grothendieck* [7, Ch. 0 §19], *and θ induces an isomorphism*

$$[X, \underline{A}]/I[X, \underline{A}] \simeq [X, \underline{R}].$$

(2) *The above triple $(\underline{A}, j, \theta)$ is unique in the following sense. If $(\underline{A}', j', \theta')$ is another triple satisfying the same condition in (1), then there exists a unique isomorphism of ring objects $\psi: \underline{A} \xrightarrow{\sim} \underline{A}'$ such that $[e, \psi] \circ j = j'$ and $\theta = \theta' \circ \psi$.*

(3) *The object $\underline{A}$ is isomorphic (if one forgets the ring-object structure) to the product of finitely many copies of $\underline{R}$.*

(4) *If $\mathcal{C}$ has finite inverse limits, the above assertions (1) and (2) are valid if conditions "free module of finite rank" on $\underline{R}$ and I^i/I^{i+1} are replaced by conditions "direct summand of a free module of finite rank".*

Example. Let R be a non-discrete locally compact field and A a local ring of finite length with residue field R. Then in the case where $\operatorname{char}(R) > 0$ Proposition 2 shows that there exists a canonical topology on A compatible with the ring structure such that A is homeomorphic to the product of finitely many copies of R. On the other hand, in the case where $\operatorname{char}(R) = 0$ it is impossible in general to define canonically such a topology on A. Of course, by taking a section $s: R \to A$ (as rings), A as a vector space over $s(R)$ has the vector space topology, but this topology depends on the choice of s in general. This reflects the fact that in the case of $\operatorname{char}(R) = 0$ the ring of R-valued continuous functions on a topological space is not in general formally smooth over R contrary to the case of $\operatorname{char}(R) > 0$.

Proof of Proposition 2. Let X be an object of $\mathcal{C}$; put $R_X = [X, \underline{R}]$. The assumptions on $\underline{R}$ show that the homomorphism

$$R^{(p)} \otimes_R R_X \to R_X, \quad x \otimes y \mapsto xy^p$$

is bijective, where $R^{(p)} = R$ as a ring and the structure homomorphism $R \to R^{(p)}$ is $x \mapsto x^p$. Hence by [10, §1 Lemma 1] there exists a formally etale ring A_X over A with a ring isomorphism $\theta_X: A_X/IA_X \simeq R_X$. The property "formally etale" shows that the correspondence $X \to A_X$ is a functor $\mathcal{C}^\circ \to \mathrm{Rings}$, and that the system θ_X forms a morphism of functors. More explicitly, let n and r be sufficiently large integers, let $W_n(R)$ be the ring of p-Witt vectors over R of length n, and let $\varphi: W_n(R) \to A$ be the homomorphism

$$(x_0, x_1, \dots) \mapsto \sum_{i=0}^{r} p^i \widetilde{x}_i^{\,p^{r-i}}$$

where $\widetilde{x}_i$ is a representative of $x_i \in R$ in A. Then A_X is defined as the tensor product

$$W_n(R_X) \otimes_{W_n(R)} A$$

induced by φ. Since $\mathrm{Tor}_1^{W_n(R)}(W_n(R_X), R) = 0$ we have

$$\mathrm{Tor}_1^{W_n(R)}(W_n(R_X), A/I^i) = 0$$

for every i. This proves that the canonical homomorphism

$$I^i/I^{i+1} \otimes_R R_X \to I^i A_X / I^{i+1} A_X$$

is bijective for every i. Hence each functor $X \to I^i A_X/I^{i+1}A_X$ is representable by a finite product of copies of $\underline{R}$, and it follows immediately that the functor A_X is represented by the product of finitely many copies of $\underline{R}$. $\square$

1.2. n-dimensional local fields as objects of $\mathcal{F}_n$.

Let K be an n-dimensional local field. In this subsection we define a ring object $\underline{K}$ and a group object $\underline{K^*}$ by induction on n.

Let $k_0, \dots, k_n = K$ be as in the introduction. For each i such that $\mathrm{char}(k_{i-1}) = 0$ (if such an i exists) choose a ring morphism $s_i\colon k_{i-1} \to \mathcal{O}_{k_i}$ such that the composite $k_{i-1} \to \mathcal{O}_{k_i} \to \mathcal{O}_{k_i}/\mathcal{M}_{k_i}$ is the indentity map. Assume $n \geqslant 1$ and let $\underline{k_{n-1}}$ be the ring object of $\mathcal{F}_{n-1}$ corresponding to k_{n-1} by induction on n.

If $\mathrm{char}(k_{n-1}) = p > 0$, the construction of $\underline{K}$ below will show by induction on n that the assumptions of Proposition 2 are satisfied when one takes $\mathcal{F}_{n-1}$, $\underline{k_{n-1}}$, k_{n-1} and $\mathcal{O}_K/\mathcal{M}_K^r$ $(r \geqslant 1)$ as $\mathcal{C}$, $\underline{R}$, R and A. Hence we obtain a ring object $\underline{\mathcal{O}_K/\mathcal{M}_K^r}$ of $\mathcal{F}_{n-1}$. We identify $\mathcal{O}_K/\mathcal{M}_K^r$ with $[e, \underline{\mathcal{O}_K/\mathcal{M}_K^r}]$ via the isomorphism j of Proposition 2.

If $\mathrm{char}(k_{n-1}) = 0$, let $\underline{\mathcal{O}_K/\mathcal{M}_K^r}$ be the ring object of $\mathcal{F}_{n-1}$ which represents the functor

$$\mathcal{F}_{n-1}^{\circ} \to \mathrm{Rings}, \quad X \mapsto \mathcal{O}_K/\mathcal{M}_K^r \otimes_{k_{n-1}} [X, \underline{k_{n-1}}],$$

where $\mathcal{O}_K/\mathcal{M}_K^r$ is viewed as a ring over k_{n-1} via s_{n-1}.

In each case let $\underline{\mathcal{O}_K}$ be the object "$\varprojlim$" $\underline{\mathcal{O}_K/\mathcal{M}_K^r}$ of $\mathrm{pro}(\mathcal{F}_{n-1})$. We define $\underline{K}$ as the ring object of $\mathcal{F}_n$ which corresponds to the functor

$$\mathrm{pro}(\mathcal{F}_{n-1})^{\circ} \to \mathrm{Rings}, \quad X \mapsto K \otimes_{\mathcal{O}_K} [X, \underline{\mathcal{O}_K}].$$

Thus, $\underline{K}$ is defined canonically in the case of $\mathrm{char}(k_{n-1}) > 0$, and it depends (and doesn't depend) on the choices of s_i in the case of $\mathrm{char}(k_{n-1}) = 0$ in the following sense. Assume that another choice of sections s_i' yields $\underline{k_i}'$ and $\underline{K}'$. Then there exists an isomorphism of ring objects $\underline{K} \xrightarrow{\sim} \underline{K}'$ which induces $\underline{k_i} \xrightarrow{\sim} \underline{k_i}'$ for each i. But in general there is no isomorphism of ring objects $\psi: \underline{K} \to \underline{K}'$ such that $[e, \psi]: K \to K$ is the indentity map.

Now let $\underline{K}^*$ be the object of $\mathcal{F}_n$ which represents the functor

$$\mathcal{F}_n^\circ \to \text{Sets}, \quad X \mapsto [X, \underline{K}]^*.$$

This functor is representable because $\mathcal{F}_n$ has finite inverse limits as can be shown by induction on n.

Definition 1. We define fine (resp. cofine) objects of $\mathcal{F}_n$ by induction on n. All objects in $\mathcal{F}_0$ are called fine (resp. cofine) objects of $\mathcal{F}_0$. An object of $\mathcal{F}_n$ ($n \geqslant 1$) is called a fine (resp. cofine) object of $\mathcal{F}_n$ if and only if it is expressed as $X =$ "$\varinjlim$" X_λ for some objects X_λ of $\text{pro}(\mathcal{F}_{n-1})$ and each X_λ is expressed as $X_\lambda =$ "$\varprojlim$" $X_{\lambda\mu}$ for some objects $X_{\lambda\mu}$ of $\mathcal{F}_{n-1}$ satisfying the condition that all $X_{\lambda\mu}$ are fine (resp. cofine) objects of $\mathcal{F}_{n-1}$ and the maps $[e, X_\lambda] \to [e, X_{\lambda\mu}]$ are surjective for all λ, μ (resp. the maps $[e, X_\lambda] \to [e, X]$ are injective for all λ).

Recall that if $i \leqslant j$ then $\mathcal{F}_i$ is a full subcategory of $\mathcal{F}_j$. Thus each $\mathcal{F}_i$ is a full subcategory of $\mathcal{F}_\infty = \cup_i \mathcal{F}_i$.

Lemma 1.

(1) *Let K be an n-dimensional local field. Then an object of $\mathcal{F}_n$ of the form*

$$\underline{K} \times \dots \underline{K} \times \underline{K}^* \times \dots \times \underline{K}^*$$

is a fine and cofine object of $\mathcal{F}_n$. Every set S viewed as an object of $\text{ind}(\mathcal{F}_0)$ *is a fine and cofine object of $\mathcal{F}_1$.*

(2) *Let X and Y be objects of $\mathcal{F}_\infty$, and assume that X is a fine object of $\mathcal{F}_n$ for some n and Y is a cofine object of $\mathcal{F}_m$ for some m. Then two morphisms $\theta, \theta': X \to Y$ coincide if $[e, \theta] = [e, \theta']$.*

As explained in 1.1 the definition of the object $\underline{K}$ depends on the sections $s_i: k_{i-1} \to \mathcal{O}_{k_i}$ chosen for each i such that $\text{char}(k_{i-1}) = 0$. Still we have the following:

Lemma 2.

(1) *Let N be a subgroup of $K_q(K)$ of finite index. Then openness of N doesn't depend on the choice of sections s_i.*

(2) *Let $\varphi: K_q(K) \to \mathbb{Q}/\mathbb{Z}$ be a homomorphism of finite order. Then the continuity of χ doesn't depend on the choice of sections s_i.*

The exact meaning of Theorems 1,2,3 is now clear.

2. Additive duality

2.1. Category of locally compact objects.

If $\mathcal{C}$ is the category of finite abelian groups, let $\widetilde{\mathcal{C}}$ be the category of topological abelian groups G which possess a totally disconnected open compact subgroup H such that G/H is a torsion group. If $\mathcal{C}$ is the category of finite dimensional vector spaces over a fixed (discrete) field k, let $\widetilde{\mathcal{C}}$ be the category of locally linearly compact vector spaces over k (cf. Lefschetz [12]). In both cases the canonical self-duality of $\widetilde{\mathcal{C}}$ is well known. These two examples are special cases of the following general construction.

Definition 2. For a category $\mathcal{C}$ define a full subcategory $\widetilde{\mathcal{C}}$ of $\mathrm{ind}(\mathrm{pro}(\mathcal{C}))$ as follows. An object X of $\mathrm{ind}(\mathrm{pro}(\mathcal{C}))$ belongs to $\widetilde{\mathcal{C}}$ if and only if it is expressed in the form $"\varinjlim"_{j\in J}"\varprojlim"_{i\in I}X(i,j)$ for some directly ordered sets I and J viewed as small categories in the usual way and for some functor $X\colon I^\circ\times J\to\mathcal{C}$ satisfying the following conditions.

(i) If $i,i'\in I$, $i\leqslant i'$ then the morphism $X(i',j)\to X(i,j)$ is surjective for every $j\in J$. If $j,j'\in J$, $j\leqslant j'$ then the morphism $X(i,j)\to X(i,j')$ is injective for every $i\in I$.

(ii) If $i,i'\in I$, $i\leqslant i'$ and $j,j'\in J$, $j\leqslant j'$ then the square

$$\begin{array}{ccc} X(i',j) & \longrightarrow & X(i',j') \\ \downarrow & & \downarrow \\ X(i,j) & \longrightarrow & X(i,j') \end{array}$$

is cartesian and cocartesian.

It is not difficult to prove that $\widetilde{\mathcal{C}}$ is equivalent to the full subcategory of $\mathrm{pro}(\mathrm{ind}(\mathcal{C}))$ (as well as $\mathrm{ind}(\mathrm{pro}(\mathcal{C}))$) consisting of all objects which are expressed in the form $"\varprojlim"_{i\in I}"\varinjlim"_{j\in J}X(i,j)$ for some triple (I,J,X) satisfying the same conditions as above. In this equivalence the object $"\varinjlim"_{j\in J}"\varprojlim"_{i\in I}X(i,j)$ corresponds to $"\varprojlim"_{i\in I}"\varinjlim"_{j\in J}X(i,j)$.

Definition 3. Let $\mathcal{A}_0$ be the category of finite abelian groups, and let $\mathcal{A}_1,\mathcal{A}_2,\dots$ be the categories defined as $\mathcal{A}_{n+1}=\widetilde{\mathcal{A}_n}$.

It is easy to check by induction on n that $\mathcal{A}_n$ is a full subcategory of the category $\mathcal{F}_n^{\mathrm{ab}}$ of all abelian group objects of $\mathcal{F}_n$ with additive morphisms.

2.2. Pontryagin duality.

The category $\mathcal{A}_0$ is equivalent to its dual via the functor

$$D_0 \colon \mathcal{A}_0^\circ \xrightarrow{\sim} \mathcal{A}_0, \quad X \mapsto \mathrm{Hom}(X, \mathbb{Q}/\mathbb{Z}).$$

By induction on n we get an equivalence

$$D_n \colon \mathcal{A}_n^\circ \xrightarrow{\sim} \mathcal{A}_n, \quad \mathcal{A}_n^\circ = (\widetilde{\mathcal{A}_{n-1}})^\circ = \widetilde{\mathcal{A}_{n-1}^\circ} \xrightarrow{D_{n-1}} \widetilde{\mathcal{A}_{n-1}} = \mathcal{A}_n$$

where we use $(\widetilde{\mathcal{C}})^\circ = \widetilde{\mathcal{C}^\circ}$. As in the case of $\mathcal{F}_n$ each $\mathcal{A}_n$ is a full subcategory of $\mathcal{A}_\infty = \cup_n \mathcal{A}_n$. The functors D_n induce an equivalence

$$D \colon \mathcal{A}_\infty^\circ \xrightarrow{\sim} \mathcal{A}_\infty$$

such that $D \circ D$ coincides with the indentity functor.

Lemma 3. *View $\mathbb{Q}/\mathbb{Z}$ as an object of $\mathrm{ind}(\mathcal{A}_0) \subset \mathcal{A}_\infty \subset \mathcal{F}_\infty^{\mathrm{ab}}$. Then:*

(1) *For every object X of $\mathcal{A}_\infty$*

$$[X, \mathbb{Q}/\mathbb{Z}]_{\mathcal{A}_\infty} \simeq [e, D(X)]_{\mathcal{F}_\infty}.$$

(2) *For all objects X, Y of $\mathcal{A}_\infty$ $[X, D(Y)]_{\mathcal{A}_\infty}$ is canonically isomorphic to the group of biadditive morphisms $X \times Y \to \mathbb{Q}/\mathbb{Z}$ in $\mathcal{F}_\infty$.*

Proof. The isomorphism of (1) is given by

$$[X, \mathbb{Q}/\mathbb{Z}]_{\mathcal{A}_\infty} \simeq [D(\mathbb{Q}/\mathbb{Z}), D(X)]_{\mathcal{A}_\infty} = [\widehat{\mathbb{Z}}, D(X)]_{\mathcal{A}_\infty} \xrightarrow{\sim} [e, D(X)]_{\mathcal{F}_\infty}$$

($\widehat{\mathbb{Z}}$ is the totally disconnected compact abelian group $\varprojlim_{n>0} \mathbb{Z}/n$ and the last arrow is the evaluation at $1 \in \widehat{\mathbb{Z}}$). The isomorphism of (2) is induced by the canonical biadditive morphism $D(Y) \times Y \to \mathbb{Q}/\mathbb{Z}$ which is defined naturally by induction on n. □

Compare the following Proposition 3 with Weil [17, Ch. II §5 Theorem 3].

Proposition 3. *Let K be an n-dimensional local field, and let V be a vector space over K of finite dimension, $V' = \mathrm{Hom}_K(V, K)$. Then*

:

(1) *The abelian group object $\underline{V}$ of $\mathcal{F}_n$ which represents the functor $X \to V \otimes_K [X, \underline{K}]$ belongs to $\mathcal{A}_n$.*

(2) *$[\underline{K}, \mathbb{Q}/\mathbb{Z}]_{\mathcal{A}_\infty}$ is one-dimensional with respect to the natural K-module structure and its non-zero element induces due to Lemma 3 (2) an isomorphism $\underline{V'} \simeq D(\underline{V})$.*

3. Properties of the ring of $\underline{K}$-valued morphisms

3.1. Multiplicative groups of certain complete rings.

Proposition 4. *Let A be a ring and let π be a non-zero element of A such that $A = \varprojlim A/\pi^n A$. Let $R = A/\pi A$ and $B = A[\pi^{-1}]$. Assume that at least one of the following two conditions is satisfied.*

(i) *R is reduced (i.e. having no nilpotent elements except zero) and there is a ring homomorphism $s: R \to A$ such that the composite $R \xrightarrow{s} A \to A/\pi A$ is the identity.*

(ii) *For a prime p the ring R is annihilated by p and via the homomorphism $R \to R$, $x \mapsto x^p$ the latter R is a finitely generated projective module over the former R.*

Then we have

$$B^* \simeq A^* \times \Gamma(\mathrm{Spec}(R), \mathbb{Z})$$

where $\Gamma(\mathrm{Spec}(R), \mathbb{Z})$ is the group of global sections of the constant sheaf $\mathbb{Z}$ on $\mathrm{Spec}(R)$ with Zariski topology. The isomorphism is given by the homomorphism of sheaves $\mathbb{Z} \to \mathcal{O}^*_{\mathrm{Spec}(R)}$, $1 \mapsto \pi$, the map

$$\Gamma(\mathrm{Spec}(R), \mathbb{Z}) \simeq \Gamma(\mathrm{Spec}(A), \mathbb{Z}) \to \Gamma(\mathrm{Spec}(B), \mathbb{Z})$$

and the inclusion map $A^* \to B^*$.

Proof. Let Aff_R be the category of affine schemes over R. In case (i) let $\mathcal{C} = \mathrm{Aff}_R$. In case (ii) let $\mathcal{C}$ be the category of all affine schemes $\mathrm{Spec}(R')$ over R such that the map

$$R^{(p)} \otimes_R R' \to R', \quad x \otimes y \mapsto xy^p$$

(cf. the proof of Proposition 2) is bijective. Then in case (ii) every finite inverse limit and finite sum exists in $\mathcal{C}$ and coincides with that taken in Aff_R. Furthermore, in this case the inclusion functor $\mathcal{C} \to \mathrm{Aff}_R$ has a right adjoint. Indeed, for any affine scheme X over R the corresponding object in $\mathcal{C}$ is $\varprojlim X_i$ where X_i is the Weil restriction of X with respect to the homomorphism $R \to R$, $x \mapsto x^{p^i}$.

Let $\underline{R}$ be the ring object of $\mathcal{C}$ which represents the functor $X \to \Gamma(X, \mathcal{O}_X)$, and let $\underline{R^*}$ be the object which represents the functor $X \to [X, \underline{R}]^*$, and $\underline{0}$ be the final object e regarded as a closed subscheme of $\underline{R}$ via the zero morphism $e \to \underline{R}$.

Lemma 4. *Let X be an object of $\mathcal{C}$ and assume that X is reduced as a scheme (this condition is always satisfied in case (ii)). Let $\theta: X \to \underline{R}$ be a morphism of $\mathcal{C}$. If $\theta^{-1}(\underline{R^*})$ is a closed subscheme of X, then X is the direct sum of $\theta^{-1}(\underline{R^*})$ and $\theta^{-1}(\underline{0})$ (where the inverse image notation are used for the fibre product).*

The group B^* is generated by elements x of A such that $\pi^n \in Ax$ for some $n \geqslant 0$. In case (i) let $\underline{A/\pi^{n+1}A}$ be the ring object of $\mathcal{C}$ which represents the functor $X \to A/\pi^{n+1}A \otimes_R [X, \underline{R}]$ where $A/\pi^{n+1}A$ is viewed as an R-ring via a fixed section s. In case (ii) we get a ring object $\underline{A/\pi^{n+1}A}$ of $\mathcal{C}$ by Proposition 2 (4).

In both cases there are morhisms $\theta_i : \underline{R} \to \underline{A/\pi^{n+1}A}$ ($0 \leqslant i \leqslant n$) in $\mathcal{C}$ such that the morphism

$$\underline{R} \times \cdots \times \underline{R} \to \underline{A/\pi^{n+1}A}, \quad (x_0, \ldots, x_n) \mapsto \sum_{i=0}^{n} \theta_i(x_i)\pi^i$$

is an isomorphism.

Now assume $xy = \pi^n$ for some $x, y \in A$ and take elements $x_i, y_i \in R = [e, \underline{R}]$ ($0 \leqslant i \leqslant n$) such that

$$x \mod \pi^{n+1} = \sum_{i=0}^{n} \theta_i(x_i)\pi^i, \quad y \mod \pi^{n+1} = \sum_{i=0}^{n} \theta_i(y_i)\pi^i.$$

An easy computation shows that for every $r = 0, \ldots, n$

$$\Big(\bigcap_{i=0}^{r-1} x_i^{-1}(\underline{0})\Big) \bigcap x_r^{-1}(\underline{R^*}) = \Big(\bigcap_{i=0}^{r-1} x_i^{-1}(\underline{0})\Big) \bigcap \Big(\bigcap_{i=0}^{n-r-1} y_i^{-1}(\underline{0})\Big).$$

By Lemma 4 and induction on r we deduce that $e = \mathrm{Spec}(R)$ is the direct sum of the closed open subschemes $\big(\cap_{i=0}^{r-1} x_i^{-1}(\underline{0})\big) \cap x_r^{-1}(\underline{R^*})$ on which the restriction of x has the form $a\pi^r$ for an invertible element $a \in A$. $\square$

3.2. Properties of the ring $[X, \underline{K}]$.

Results of this subsection will be used in section 4.

Definition 4. For an object X of $\mathcal{F}_\infty$ and a set S let

$$\mathrm{lcf}(X, S) = \varinjlim_I [X, I]$$

where I runs over all finite subsets of S (considering each I as an object of $\mathcal{F}_0 \subset \mathcal{F}_\infty$).

Lemma 5. *Let K be an n-dimensional local field and let X be an object of $\mathcal{F}_\infty$. Then:*

(1) *The ring $[X, \underline{K}]$ is reduced.*

(2) *For every set S there is a canonical bijection*

$$\mathrm{lcf}(X, S) \xrightarrow{\sim} \Gamma(\mathrm{Spec}([X, \underline{K}]), S)$$

where S on the right hand side is regarded as a constant sheaf on $\mathrm{Spec}([X, \underline{K}])$.

Proof of (2). If I is a finite set and $\theta\colon X \to I$ is a morphism of $\mathcal{F}_\infty$ then X is the direct sum of the objects $\theta^{-1}(i) = X \times_I \{i\}$ in $\mathcal{F}_\infty$ $(i \in I)$. Hence we get the canonical map of (2). To prove its bijectivity we may assume $S = \{0, 1\}$. Note that $\Gamma(\mathrm{Spec}(R), \{0, 1\})$ is the set of idempotents in R for any ring R. We may assume that X is an object of $\mathrm{pro}(\mathcal{F}_{n-1})$.

Let k_{n-1} be the residue field of $k_n = K$. Then

$$\Gamma(\mathrm{Spec}([X, \underline{K}]), \{0, 1\}) \simeq \Gamma(\mathrm{Spec}([X, \underline{k_{n-1}}]), \{0, 1\})$$

by (1) applied to the ring $[X, \underline{k_{n-1}}]$. □

Lemma 6. *Let K be an n-dimensional local field of characteristic $p > 0$. Let $k_0, \dots, k_n$ be as in the introduction. For each $i = 1, \dots, n$ let π_i be a lifting to K of a prime element of k_i. Then for each object X of $\mathcal{F}_\infty$ $[X, \underline{K}]^*$ is generated by the subgroups*

$$[X, \underline{K^p(\pi^{(s)})}]^*$$

where s runs over all functions $\{1, \dots, n\} \to \{0, 1, \dots, p-1\}$ and $\pi^{(s)}$ denotes $\pi_1^{s(1)} \dots \pi_n^{s(n)}$, $\underline{K^p(\pi^{(s)})}$ is the subring object of $\underline{K}$ corresponding to $K^p(\pi^{(s)})$, i.e.

$$[X, \underline{K^p(\pi^{(s)})}] = K^p(\pi^{(s)}) \otimes_{K^p} [X, \underline{K}].$$

Proof. Indeed, Proposition 4 and induction on n yield morphisms

$$\theta^{(s)}\colon \underline{K^*} \to \underline{K^p(\pi^{(s)})^*}$$

such that the product of all $\theta^{(s)}$ in $\underline{K^*}$ is the identity morphism $\underline{K^*} \to \underline{K^*}$. □

The following similar result is also proved by induction on n.

Lemma 7. *Let K, k_0 and $(\pi_i)_{1 \leqslant i \leqslant n}$ be as in Lemma 6. Then there exists a morphism of $\mathcal{A}_\infty$*
(cf. section 2)

$$(\theta_1, \theta_2)\colon \underline{\Omega^n_K} \to \underline{\Omega^n_K} \times \underline{k_0}$$

such that

$$x = (1 - \mathrm{C})\theta_1(x) + \theta_2(x) d\pi_1/\pi_1 \wedge \dots \wedge d\pi_n/\pi_n$$

for every object X of $\mathcal{F}_\infty$ and for every $x \in [X, \underline{\Omega^n_K}]$ where $\underline{\Omega^n_K}$ is the object which represents the functor $X \to \Omega^n_K \otimes_K [X, \underline{K}]$ and C denotes the Cartier operator ([4], *or see 4.2 in Part I for the definition*).

Generalize the Milnor K-groups as follows.

Definition 5. For a ring R let $\Gamma_0(R) = \Gamma(\mathrm{Spec}(R), \mathbb{Z})$. The morphism of sheaves

$$\mathbb{Z} \times \mathcal{O}^*_{\mathrm{Spec}(R)} \to \mathcal{O}^*_{\mathrm{Spec}(R)}, \quad (n, x) \mapsto x^n$$

determines the $\Gamma_0(R)$-module structure on R^*. Put $\Gamma_1(R) = R^*$ and for $q \geqslant 2$ put

$$\Gamma_q(R) = \otimes^q_{\Gamma_0(R)} \Gamma_1(R)/J_q$$

where $\otimes^q_{\Gamma_0(R)}\Gamma_1(R)$ is the qth tensor power of $\Gamma_1(R)$ over $\Gamma_0(R)$ and J_q is the subgroup of the tensor power generated by elements $x_1 \otimes \cdots \otimes x_q$ which satisfy $x_i + x_j = 1$ or $x_i + x_j = 0$ for some $i \neq j$. An element $x_1 \otimes \cdots \otimes x_q \mod J_q$ will be denoted by $\{x_1, \dots, x_q\}$.

Note that $\Gamma_q(k) = K_q(k)$ for each field k and $\Gamma_q(R_1 \times R_2) \simeq \Gamma_q(R_1) \times \Gamma_q(R_2)$ for rings R_1, R_2.

Lemma 8. *In one of the following two cases*

(i) A, R, B, π *as in Proposition 4*

(ii) *an n-dimensional local field K, an object X of $\mathcal{F}_\infty$, $A = [X, \underline{\mathcal{O}_K}]$, $R = [X, \underline{k_{n-1}}]$, $B = [X, \underline{K}]$,*

let $U_i\Gamma_q(B)$ be the subgroup of $\Gamma_q(B)$ generated by elements $\{1 + \pi^i x, y_1, \dots, y_{q-1}\}$ such that $x \in A$, $y_j \in B^*$, $q, i \geqslant 1$.

Then:

(1) *There is a homomorphism $\rho_0^q \colon \Gamma_q(R) \to \Gamma_q(B)/U_1\Gamma_q(B)$ such that*

$$\rho_0^q(\{x_1, \dots, x_q\}) = \{\widetilde{x_1}, \dots, \widetilde{x_q}\} \mod U_1\Gamma_q(B)$$

where $\widetilde{x_i} \in A$ is a representative of x_i. In case (i) (resp. (ii)) the induced map

$$\Gamma_q(R) + \Gamma_{q-1}(R) \to \Gamma_q(B)/U_1\Gamma_q(B), \quad (x, y) \mapsto \rho_0^q(x) + \{\rho_0^{q-1}(y), \pi\}$$

(*resp.*

$$\Gamma_q(R)/m + \Gamma_{q-1}(R)/m \to \Gamma_q(B)/(U_1\Gamma_q(B) + m\Gamma_q(B)),$$
$$(x, y) \mapsto \rho_0^q(x) + \{\rho_0^{q-1}(y), \pi\})$$

is bijective (*resp. bijective for every non-zero integer m*).

(2) *If m is an integer invertible in R then $U_1\Gamma_q(B)$ is m-divisible.*

(3) *In case (i) assume that R is additively generated by R^*. In case (ii) assume that* $\mathrm{char}\,(k_{n-1}) = p > 0$. *Then there exists a unique homomorphism*

$$\rho_i^q \colon \Omega_R^{q-1} \to U_i\Gamma_q(B)/U_{i+1}\Gamma_q(B)$$

such that

$$\rho_i^q(x dy_1/y_1 \wedge \cdots \wedge dy_{q-1}/y_{q-1}) = \{1 + \widetilde{x}\pi^i, \widetilde{y_1}, \dots, \widetilde{y_{q-1}}\} \mod U_{i+1}\Gamma_q(B)$$

for every $x \in R$, $y_1, \dots, y_{q-1} \in R^*$. The induced map

$$\Omega_R^{q-1} \oplus \Omega_R^{q-2} \to U_i\Gamma_q(B)/U_{i+1}\Gamma_q(B), \quad (x, y) \mapsto \rho_i^q(x) + \{\rho_i^{q-1}(y), \pi\}$$

is surjective. If i is invertible in R then the homomorphism ρ_i^q is surjective.

Proof. In case (i) these results follow from Proposition 4 by Bass–Tate's method [2, Proposition 4.3] for (1), Bloch's method [3, §3] for (3) and by writing down the kernel of $R \otimes R^* \to \Omega^1_R$, $x \otimes y \mapsto x dy/y$ as in [9, §1 Lemma 5].

If X is an object of $\mathrm{pro}(\mathcal{F}_{n-1})$ then case (ii) is a special case of (i) except $n = 1$ and $k_0 = \mathbb{F}_2$ where $[X, \underline{k_0}]$ is not generated by $[X, \underline{k_0}]^*$ in general. But in this exceptional case it is easy to check directly all the assertions.

For an arbitrary X we present here only the proof of (3) because the proof of (1) is rather similar.

Put $k = k_{n-1}$. For the existence of ρ_i^q it suffices to consider the cases where $X = \underline{\Omega_k^{q-1}}$ and $X = \underline{k} \times \prod^{q-1} \underline{k^*}$ ($\prod^r Y$ denotes the product of r copies of Y). Note that these objects are in $\mathrm{pro}(\mathcal{F}_{n-1})$ since $[X, \underline{\Omega_k^q}] = \Omega^q_{[X,\underline{k}]}$ for any X and q.

The uniqueness follows from the fact that $[X, \underline{\Omega_k^{q-1}}]$ is generated by elements of the form $x dc_1/c_1 \wedge \dots \wedge dc_{q-1}/c_{q-1}$ such that $x \in [X, \underline{k}]$ and $c_1, \dots, c_{q-1} \in k^*$.

To prove the surjectivity we may assume $X = (1 + \pi^i \underline{\mathcal{O}_K}) \times \prod^{q-1} \underline{K^*}$ and it suffices to prove in this case that the typical element in $U_i\Gamma_q(B)/U_{i+1}\Gamma_q(B)$ belongs to the image of the homomorphism introduced in (3). Let $\underline{U_K}$ be the object of $\mathcal{F}_n$ which represents the functor $X \to [X, \underline{\mathcal{O}_K}]^*$. By Proposition 4 there exist morphisms $\theta_1 : \underline{K^*} \to \coprod_{i=0}^{p-1} \underline{U_K}\pi^i$ (the direct sum in $\mathcal{F}_n$) and $\theta_2 : \underline{K^*} \to \underline{K^*}$ such that $x = \theta_1(x)\theta_2(x)^p$ for each X in $\mathcal{F}_\infty$ and each $x \in [X, \underline{K^*}]$ (in the proof of (1) p is replaced by m). Since $\coprod_{i=0}^{p-1} \underline{U_K}\pi^i$ belongs to $\mathrm{pro}(\mathcal{F}_{n-1})$ and $(1 + \pi^i[X, \underline{\mathcal{O}_K}])^p \subset 1 + \pi^{i+1}[X, \underline{\mathcal{O}_K}]$ we are reduced to the case where X is an object of $\mathrm{pro}(\mathcal{F}_{n-1})$. □

4. Norm groups

In this section we prove Theorem 3 and Proposition 1. In subsection 4.1 we reduce these results to Proposition 6.

4.1. Reduction steps.

Definition 6. Let k be a field and let $H : \mathcal{E}(k) \to \mathrm{Ab}$ be a functor such that $\varinjlim_{k' \in \mathcal{E}(k)} H(k') = 0$. Let $w \in H(k)$ (cf. Introduction). For a ring R over k and $q \geqslant 1$ define the subgroup $N_q(w, R)$ (resp. $L_q(w, R)$) of $\Gamma_q(R)$ as follows.

An element x belongs to $N_q(w, R)$ (resp. $L_q(w, R)$) if and only if there exist
a finite set J and element $0 \in J$,

a map $f\colon J \to J$ such that for some $n \geqslant 0$ the nth iteration f^n with respect to the composite is a constant map with value 0,
and a family $(E_j, x_j)_{j\in J}$ ($E_j \in \mathcal{E}(k)$), $x_j \in \Gamma_q(E_j \otimes_k R)$) satisfying the following conditions:

(i) $E_0 = k$ and $x_0 = x$.

(ii) $E_{f(j)} \subset E_j$ for every $j \in J$.

(iii) Let $j \in f(J)$. Then there exists a family $(y_t, z_t)_{t\in f^{-1}(j)}$ ($y_t \in (E_t \otimes_k R)^*$, $z_t \in \Gamma_{q-1}(E_j \otimes_k R)$) such that $x_t = \{y_t, z_t\}$ for all $t \in f^{-1}(j)$ and

$$x_j = \sum_{t\in f^{-1}(j)} \{N_{E_t\otimes_k R/E_j\otimes_k R}(y_t), z_t\}$$

where $N_{E_t\otimes_k R/E_j\otimes_k R}$ denotes the norm homomorphism

$$(E_t \otimes_k R)^* \to (E_j \otimes_k R)^*.$$

(iv) If $j \in J \setminus f(J)$ then w belongs to the kernel of $H(k) \to H(E_j)$ (resp. then one of the following two assertions is valid:
(a) w belongs to the kernel of $H(k) \to H(E_j)$,
(b) x_j belongs to the image of $\Gamma(\mathrm{Spec}(E_j \otimes_k R), K_q(E_j)) \to \Gamma_q(E_j \otimes_k R)$, where $K_q(E_j)$ denotes the constant sheaf on $\mathrm{Spec}(E_j \otimes_k R)$ defined by the set $K_q(E_j)$).

Remark. If the groups $\Gamma_q(E_j \otimes_k R)$ have a suitable "norm" homomorphism then x is the sum of the "norms" of x_j such that $f^{-1}(j) = \emptyset$. In particular, in the case where $R = k$ we get $N_q(w,k) \subset N_q(w)$ and $N_1(w,k) = N_1(w)$.

Definition 7. For a field k let $[\mathcal{E}(k), \mathrm{Ab}]$ be the abelian category of all functors $\mathcal{E}(k) \to \mathrm{Ab}$.

(1) For $q \geqslant 0$ let $\mathcal{N}_{q,k}$ denote the full subcategory of $[\mathcal{E}(k), \mathrm{Ab}]$ consisting of functors H such that $\varinjlim_{k'\in\mathcal{E}(k)} H(k') = 0$ and such that for every $k' \in \mathcal{E}(k)$, $w \in H(k')$ the norm group $N_q(w)$ coincides with $K_q(k')$. Here $N_q(w)$ is defined with respect to the functor $\mathcal{E}(k') \to \mathrm{Ab}$.

(2) If K is an n-dimensional local field and $q \geqslant 1$, let $\underline{\mathcal{N}}_{q,K}$ (resp. $\underline{\mathcal{L}}_{q,K}$) denote the full subcategory of $[\mathcal{E}(K), \mathrm{Ab}]$ consisting of functors H such that

$$\varinjlim_{K'\in\mathcal{E}(K)} H(K') = 0$$

and such that for every $K' \in \mathcal{E}(K)$, $w \in H(K')$ and every object X of $\mathcal{F}_\infty$ the group $N_q(w, [X, \underline{K'}])$ (resp. $L_q(w, [X, \underline{K'}])$) coincides with $\Gamma_q([X, \underline{K'}])$.

Lemma 9. *Let K be an n-dimensional local field and let H be an object of $\underline{\mathcal{L}}_{q,K}$. Then for every $w \in H(K)$ the group $N_q(w)$ is an open subgroup of $K_q(K)$ of finite index.*

Proof. Consider the case where $X = \prod^q \underline{K^*}$. We can take a system $(E_j, x_j)_{j\in J}$ as in Definition 6 such that $E_0 = K$, x_0 is the canonical element in $\Gamma_q([X, \underline{K}])$ and such that if $j \notin f(J)$ and $w \notin \ker(H(K) \to H(E_j))$ then x_j is the image of an element θ_j of $\mathrm{lcf}(X, K_q(E_j))$. Let $\theta \in \mathrm{lcf}(X, K_q(K)/N_q(w))$ be the sum of $N_{E_j/K} \circ \theta_j$ mod $N_q(w)$. Then the canonical map $[e, X] = \prod^q K^* \to K_q(K)/N_q(w)$ comes from θ. □

Definition 8. Let k be a field. A collection $\{\mathcal{C}_{k'}\}_{k'\in\mathcal{E}(k)}$ of full subcategories $\mathcal{C}_{k'}$ of $[\mathcal{E}(k'), \mathrm{Ab}]$ is called *admissible* if and only if it satisfies conditions (i) – (iii) below.

(i) Let $E \in \mathcal{E}(k)$. Then every subobject, quotient object, extension and filtered inductive limit (in the category of $[\mathcal{E}(E), \mathrm{Ab}]$) of objects of $\mathcal{C}_E$ belongs to $\mathcal{C}_E$.

(ii) Let $E, E' \in \mathcal{E}(k)$ and $E \subset E'$. If H is in $\mathcal{C}_E$ then the composite functor $\mathcal{E}(E') \to \mathcal{E}(E) \xrightarrow{H} \mathrm{Ab}$ is in $\mathcal{C}_{E'}$.

(iii) Let $E \in \mathcal{E}(k)$ and H is in $[\mathcal{E}(E), \mathrm{Ab}]$. Then H is in $\mathcal{C}_E$ if conditions (a) and (b) below are satisfied for a prime p.

  (a) For some $E' \in \mathcal{E}(E)$ such that $|E' : E|$ is prime to p the composite functor $(E') \to (E) \xrightarrow{H} \mathrm{Ab}$ is in $\mathcal{C}_{E'}$.

  (b) Let q be a prime number distinct from p and let S be a direct subordered set of $\mathcal{E}(E)$. If the degree of every finite extension of the field $\varinjlim_{E'\in S} E'$ is a power of p then $\varinjlim_{E'\in S} H(E') = 0$.

Lemma 10.

(1) *For each field k and q the collection $\{\mathcal{N}_{q,k'}\}_{k'\in\mathcal{E}(k)}$ is admissible. If K is an n-dimensional local field then the collections $\{\underline{\mathcal{N}}_{q,k'}\}_{k'\in\mathcal{E}(k)}$ and $\{\underline{\mathcal{L}}_{q,k'}\}_{k'\in\mathcal{E}(k)}$ are admissible.*

(2) *Let k be a field. Assume that a collection $\{\mathcal{C}_{k'}\}_{k'\in\mathcal{E}(k)}$ is admissible. Let $r \geqslant 1$ and for every prime p there exist $E \in \mathcal{E}(k)$ such that $|E : k|$ is prime to p and such that the functor $H^r(\ ,\mathbb{Z}/p^r)\colon \mathcal{E}(E) \to \mathrm{Ab}$ is in $\mathcal{C}_E$. Then for each $k' \in \mathcal{E}(k)$, each discrete torsion abelian group M endowed with a continuous action of $\mathrm{Gal}(k'^{\mathrm{sep}}/k')$ and each $i \geqslant r$ the functor*

$$H^i(\ , M)\colon \mathcal{E}(k') \to \mathrm{Ab}$$

is in $\mathcal{C}_{k'}$.

Definition 9. For a field k, $r \geqslant 0$ and a non-zero integer m define the group $H^r_m(k)$ as follows.

If $\mathrm{char}\,(k) = 0$ let

$$H^r_m(k) = H^r(k, \mu_m^{\otimes(r-1)}).$$

If $\mathrm{char}\,(k) = p > 0$ and $m = m'p^i$ where m' is prime to p and $i \geqslant 0$ let

$$H^r_m(k) = H^r_{m'}(k, \mu_{m'}^{\otimes(r-1)}) \oplus \mathrm{coker}(F - 1\colon C_i^{r-1}(k) \to C_i^{r-1}(k)/\{C_i^{r-2}(k), T\})$$

(where $C_i^{\cdot}$ is the group defined in [3, Ch.II,§7], $C_i^r = 0$ for $r < 0$).

By the above results it suffices for the proof of Theorem 3 to prove the following Proposition 5 in the case where m is a prime number.

Proposition 5. *Let K be an n-dimensional local field. Let $q, r \geqslant 1$ and let m be a non-zero integer. Then the functor $H_m^r\colon \mathcal{E}(K) \to \mathrm{Ab}$ is in $\mathcal{L}_{q,K}$ if $q + r = n + 1$ and in $\mathcal{N}_{q,K}$ if $q + r > n + 1$.*

Now we begin the proofs of Proposition 1 and Proposition 5.

Definition 10. Let K be a complete discrete valuation field, $r \geqslant 0$ and m be a non-zero integer.

(1) Let $H^r_{m,\mathrm{ur}}$ and $H^r_m/H^r_{m,\mathrm{ur}}$ be the functors $\mathcal{E}(K) \to \mathrm{Ab}$:

$$H^r_{m,\mathrm{ur}}(K') = \ker(H^r_m(K') \to H^r_m(K'_{\mathrm{ur}})),$$
$$(H^r_m/H^r_{m,\mathrm{ur}})(K') = H^r_m(K')/H^r_{m,\mathrm{ur}}(K')$$

where K'_{ur} is the maximal unramified extension of K'.

(2) Let I^r_m (resp. J^r_m) be the functor $\mathcal{E}(K) \to \mathrm{Ab}$ such that $I^r_m(K') = H^r_m(k')$ (resp. $J^r_m(K') = H^r_m(k')$) where k' is the residue field of K' and such that the homomorphism $I^r_m(K') \to I^r_m(K'')$ (resp. $J^r_m(K') \to J^r_m(K'')$) for $K' \subset K''$ is $j_{k''/k'}$ (resp. $e(K''|K')j_{k''/k'}$) where k'' is the residue field of K'', $j_{k''/k'}$ is the canonical homomorphism induced by the inclusion $k' \subset k''$ and $e(K''|K')$ is the index of ramification of K''/K'.

Lemma 11. *Let K and m be as in Definition 10.*

(1) *For $r \geqslant 1$ there exists an exact sequence of functors*

$$0 \to I^r_m \to H^r_{m,\mathrm{ur}} \to J^{r-1}_m \to 0.$$

(2) *J^r_m is in $\mathcal{N}_{1,K}$ for every $r \geqslant 0$.*

(3) *Let $q, r \geqslant 1$. Then I^r_m is in $\mathcal{N}_{q,K}$ if and only if $H^r_m\colon \mathcal{E}(k) \to \mathrm{Ab}$ is in $\mathcal{N}_{q-1,k}$ where k is the residue field of K.*

Proof. The assertion (1) follows from [11]. The assertion (3) follows from the facts that $1 + \mathcal{M}_K \subset N_{L/K}(L^*)$ for every unramified extension L of K and that there exists a canonical split exact sequence

$$0 \to K_q(k) \to K_q(K)/U_1K_q(K) \to K_{q-1}(k) \to 0. \qquad \square$$

The following proposition will be proved in 4.4.

Proposition 6. *Let K be a complete discrete valuation field with residue field k. Let $q, r \geqslant 1$ and m be a non-zero integer. Assume that $[k : k^p| \leqslant p^{q+r-2}$ if $\mathrm{char}(k) = p > 0$. Then:*

(1) $H^r_m/H^r_{m,\mathrm{ur}}$ *is in* $\mathcal{N}_{q,K}$.
(2) *If* K *is an* n*-dimensional local field with* $n \geqslant 1$ *then* $H^r_m/H^r_{m,\mathrm{ur}}$ *is in* $\underline{\mathcal{N}}_{q,K}$.

Proposition 1 follows from this proposition by Lemma 10 and Lemma 11 (note that if $\mathrm{char}\,(k) = p > 0$ and $i \geqslant 0$ then $H^r_{p^i}(k)$ is isomorphic to $\ker(p^i\colon H^r(k) \to H^r(k))$ as it follows from [11]).

Lemma 12. *Let* K *be an* n*-dimensional local field and let* X *be an object of* $\mathcal{F}_\infty$. *Consider the following cases.*
(i) $q > n+1$ *and* m *is a non-zero integer.*
(ii) $q = n+1$, $\mathrm{char}\,(K) = p > 0$ *and* m *is a power of* p.
(iii) $q = n+1$ *and* m *is a non-zero integer.*

Let $x \in \Gamma_q([X, \underline{K}]$. *Then in cases (i) and (ii)* (*resp. in case (iii)*) *there exist a triple* $(J, 0, f)$ *and a family* $(E_j, x_j)_{j\in J}$ *which satisfy all the conditions in Definition 6 with* $k = K$ *except condition (iv), and which satisfy the following condition:*
(iv)' *If* $j \in J \setminus f(J)$ *then* $x_j \in m\Gamma_q([X, \underline{E_j}])$
(*resp.* x_j *belongs to* $m\Gamma_q([X, \underline{E_j}])$
or to the image of $\mathrm{lcf}(X, K_q(E_j)) \to \Gamma_q([X, \underline{E_j}])$).

Corollary. *Let* K *be an* n*-dimensional local field. Then* $mK_{n+1}(K)$ *is an open subgroup of finite index of* $K_{n+1}(K)$ *for every non-zero integer* m.

This corollary follows from case (iii) above by the argument in the proof of Lemma 9.

Proof of Lemma 12. We may assume that m is a prime number.

First we consider case (ii). By Lemma 6 we may assume that there are elements $b_1, \dots, b_{n+1} \in [X, \underline{K}]^*$ and $c_1, \dots, c_{n+1} \in K^*$ such that $x = \{b_1, \dots, b_{n+1}\}$ and $b_i \in [X, \underline{K^p(c_i)}]^*$ for each i. We may assume that $|K^p(c_1, \dots, c_r) : K^p| = p^r$ and $c_{r+1} \in K^p(c_1, \dots, c_r)$ for some $r \leqslant n$. Let $J = \{0, 1, \dots, r\}$, and define $f\colon J \to J$ by $f(j) = j-1$ for $j \geqslant 1$ and $f(0) = 0$. Put $E_j = K(c_1^{1/p}, \dots, c_j^{1/p})$ and $x_j = \{b_1^{1/p}, \dots, b_j^{1/p}, b_{j+1}, \dots, b_{n+1}\}$. Then
$x_r = p\{b_1^{1/p}, \dots, b_{r+1}^{1/p}, b_{r+2}, \dots, b_{n+1}\}$ in $\Gamma_{n+1}([X, \underline{E_r}])$.

Next we consider cases (i) and (iii). If K is a finite field then the assertion for (i) follows from Lemma 13 below and the assertion for (iii) is trivial. Assume $n \geqslant 1$ and let k be the residue field of K. By induction on n Lemma 8 (1) (2) and case (ii) of Lemma 12 show that we may assume $x \in U_1\Gamma_q([X, \underline{K}])$, $\mathrm{char}\,(K) = 0$ and $m = \mathrm{char}\,(k) = p > 0$. Furthermore we may assume that K contains a primitive pth root ζ of 1. Let $e_K = v_K(p)$ and let π be a prime element of K. Then

$$U_i\Gamma_q([X, \underline{\mathcal{O}_K}]) \subset pU_1\Gamma_q([X, \underline{\mathcal{O}_K}]), \quad \text{if } i > pe_K/(p-1).$$

From this and Lemma 8 (3) (and a computation of the map $x \mapsto x^p$ on $U_1\Gamma_q([X, \underline{\mathcal{O}_K}])$) it follows that $U_1\Gamma_q([X, \underline{K}])$ is p-divisible if $q > n+1$ and that there is a surjective homomorphism

$$[X, \underline{\Omega_k^{n-1}}]/(1-\mathrm{C})[X, \underline{\Omega_k^{n-1}}] \to U_1\Gamma_{n+1}([X,\underline{K}])/pU_1\Gamma_{n+1}([X,\underline{K}]),$$
$$x dy_1/y_1 \wedge \dots \wedge dy_{n-1}/y_{n-1} \mapsto \{1+\widetilde{x}(\zeta-1)^p, \widetilde{y_1}, \dots, \widetilde{y_{n_1}}, \pi\}$$

where C is the Cartier operator. By Lemma 7

$$[X, \underline{\Omega_k^{n-1}}]/(1-\mathrm{C})[X, \underline{\Omega_k^{n-1}}] = \mathrm{lcf}(X, \Omega_k^{n-1}/(1-\mathrm{C})\Omega_k^{n-1}). \qquad \square$$

Lemma 13. *Let K be a finite field and let X be an object of $\mathcal{F}_\infty$. Then*

(1) $\Gamma_q[X,\underline{K}] = 0$ *for* $q \geqslant 2$.

(2) *For every finite extension L of K the norm homomorphism $[X,\underline{L}]^* \to [X,\underline{K}]^*$ is surjective.*

Proof. Follows from Lemma 5 (2). $\square$

Proof of Proposition 5 assuming Proposition 6. If K is a finite field, the assertion of Proposition 5 follows from Lemma 13.

Let $n \geqslant 1$. Let k be the residue field of K. Let I_m^r and J_m^r be as in Definition 10. Assume $q+r = n+1$ (resp. $q+r > n+1$). Using Lemma 8 (1) and the fact that

$$U_1\Gamma_q([X,\underline{K}]) \subset N_{L/K}\Gamma_q([X,\underline{L}])$$

for every unramified extension L/K we can deduce that I_m^r is in $\underline{\mathcal{L}}_{q,K}$ (resp. $\underline{\mathcal{N}}_{q,K}$) from the induction hypothesis $H_m^r \colon \mathcal{E}(k) \to \mathrm{Ab}$ is in $\underline{\mathcal{L}}_{q-1,k}$ (resp. $\underline{\mathcal{N}}_{q-1,k}$). We can deduce J_m^{r-1} is in $\underline{\mathcal{L}}_{q,K}$ (resp. $\underline{\mathcal{N}}_{q,K}$) from the hypothesis $H_m^{r-1} \colon \mathcal{E}(k) \to \mathrm{Ab}$ is in $\underline{\mathcal{L}}_{q,k}$ (resp. $\underline{\mathcal{N}}_{q,k}$). Thus $H_{m,\mathrm{ur}}^r$ is in $\underline{\mathcal{L}}_{q,K}$ (resp. $\underline{\mathcal{N}}_{q,K}$). $\square$

4.2. Proof of Proposition 6.

Let k be a field and let m be a non-zero integer. Then $\oplus_{r\geqslant 0} H_m^r(k)$ (cf. Definition 9) has a natural right $\oplus_{q\geqslant 0}K_q(k)$-module structure (if m is invertible in k this structure is defined by the cohomological symbol $h_{m,k}^q \colon K_q(k)/m \to H^q(k, \mu_m^{\otimes q})$ and the cup-product, cf. [9, §3.1]). We denote the product in this structure by $\{w, a\}$ ($a \in \oplus_{q\geqslant 0}K_q(k)$, $w \in \oplus_{r\geqslant 0}H_m^r(k)$).

Definition 11. Let K be a complete discrete valuation field with residue field k such that $\mathrm{char}(k) = p > 0$. Let $r \geqslant 1$. We call an element w of $H_p^r(K)$ *standard* if and only if w is in one of the following forms (i) or (ii).

(i) $w = \{\chi, a_1, \dots, a_{r-1}\}$ where χ is an element of $H_p^1(K)$ corresponding to a totally ramified cyclic extension of K of degree p, and $a_1, \dots, a_{r-1}$ are elements of $\mathcal{O}_K^*$ such that

$$|k^p(\overline{a_1}, \dots, \overline{a_{r-1}}) : k^p| = p^{r-1}$$

($\overline{a_i}$ denotes the residue of a_i).

(ii) $w = \{\chi, a_1, \dots, a_{r-2}, \pi\}$ where χ is an element of $H^1_p(K)$ corresponding to a cyclic extension of K of degree p whose residue field is an inseparable extension of k of degree p, π is a prime element of K and $a_1, \dots, a_{r-2}$ are elements of $\mathcal{O}_K^*$ such that $|k^p(\overline{a_1}, \dots, \overline{a_{r-2}}) : k^p| = p^{r-2}$.

Lemma 14. *Let K and k be as in Definition 11. Assume that $|k : k^p| = p^{r-1}$. Then for every element $w \in H^r_p(K) \setminus H^r_{p,\mathrm{ur}}(K)$ there exists a finite extension L of K such that $|L : K|$ is prime to p and such that the image of w in $H^r_p(L)$ is standard.*

Proof. If $\mathrm{char}(K) = p$ the proof goes just as in the proof of [8, §4 Lemma 5] where the case of $r = 2$ was treated.

If $\mathrm{char}(K) = 0$ we may assume that K contains a primitive pth root of 1. Then the cohomological symbol $h^r_{p,K}\colon K_r(K)/p \to H^r_p(K)$ is surjective and

$$\mathrm{coker}(h^r_{p,K}\colon U_1K_r(K) \to H^r_p(K)) \simeq \nu_{r-1}(k)$$

by [11] and $|k : k^p| = p^{r-1}$.

Here we are making the following:

Definition 12. Let K be a complete discrete valuation field. Then $U_iK_q(K)$ for $i, q \geqslant 1$ denotes $U_i\Gamma_q(K)$ of Lemma 8 case (i) (take $A = \mathcal{O}_K$ and $B = K$).

Definition 13. Let k be a field of characteristic $p > 0$. As in Milne [13] denote by $\nu_r(k)$ the kernel of the homomorphism

$$\Omega^r_k \to \Omega^r_k/d(\Omega^{r-1}_k), \quad x\,dy_1/y_1 \wedge \dots \wedge dy_r/y_r \mapsto (x^p - x)dy_1/y_1 \wedge \dots \wedge dy_r/y_r.$$

By [11, Lemma 2] for every element α of $\nu_{r-1}(k)$ there is a finite extension k' of k such that $|k' : k|$ is prime to p and the image of α in $\nu_{r-1}(k')$ is the sum of elements of type

$$dx_1/x_1 \wedge \dots \wedge dx_r/x_r.$$

Hence we can follow the method of the proof of [8, §4 Lemma 5 or §2 Proposition 2]. $\square$

Proof of Proposition 6. If m is invertible in k then $H^r_m = H^r_{m,\mathrm{ur}}$. Hence we may assume that $\mathrm{char}(k) = p > 0$ and $m = p^i$, $i \geqslant 1$. Since $\ker(p\colon H^r_{p^i}/H^r_{p^i,\mathrm{ur}} \to H^r_{p^i}/H^r_{p^i,\mathrm{ur}})$ is isomorphic to $H^r_p/H^r_{p,\mathrm{ur}}$ by [11], we may assume $m = p$.

The proof of part (1) is rather similar to the proof of part (2). So we present here only the proof of part (2), but the method is directly applicable to the proof of (1).

The proof is divided in several steps. In the following K always denotes an n-dimensional local field with $n \geqslant 1$ and with residue field k such that $\mathrm{char}(k) = p > 0$, except in Lemma 21. X denotes an object of $\mathcal{F}_\infty$.

Step 1. In this step w denotes a standard element of $H_p^r(K)$ and $\overline{w}$ is its image in $(H_p^r/H_{p,\mathrm{ur}}^r)(K)$. We shall prove here that $U_1\Gamma_q([X,\underline{K}]) \subset N(\overline{w},[X,\underline{K}])$. We fix a presentation of w as in (i) or (ii) of Definition 11. Let L be a cyclic extension of K corresponding to χ. In case (i) (resp. (ii)) let h be a prime element of L (resp. an element of $\mathcal{O}_L$ such that the residue class $\overline{h}$ is not contained in k). Let G be the subgroup of K^* generated by $a_1,\ldots,a_{r-1}$ (resp. by $a_1,\ldots,a_{r-2},\pi$), by $1+\mathcal{M}_K$ and $N_{L/K}(h)$. Let l be the subfield of k generated by the residue classes of $a_1,\ldots,a_{r-1}$ (resp. $a_1,\ldots,a_{r-2},N_{L/K}(h)$).

Let $i \geqslant 1$. Let $G_{i,q}$ be the subgroup of $U_i\Gamma_q([X,\underline{K}])$ generated by $\{U_i\Gamma_{q-1}([X,\underline{K}]),G\}$ and $U_{i+1}\Gamma_q([X,\underline{K}])$. Under these notation we have the following Lemma 15, 16 ,17.

Lemma 15.

(1) $G_{i,q} \subset N_q(w,[X,\underline{K}])+U_{i+1}\Gamma_q([X,\underline{K}])$.

(2) *The homomorphism* ρ_i^q *of Lemma 8 (3) induces the surjections*

$$[X,\underline{\Omega_k^{q-1}}] \to [X,\underline{\Omega_{k/l}^{q-1}}] \xrightarrow{\overline{\rho_i^q}} U_1\Gamma_q([X,\underline{K}])/G_{i,q}.$$

(3) *If* ρ_i^q *is defined using a prime element* π *which belongs to* G *then the above homomorphism* $\overline{\rho_i^q}$ *annihilates the image of the exterior derivation* $d\colon [X,\underline{\Omega_{k/l}^{q-2}}] \to [X,\underline{\Omega_{k/l}^{q-1}}]$.

Lemma 16. *Let* a *be an element of* K^* *such that* $v_K(a)=i$ *and* $a=a_1^{s(1)}\ldots a_{r-1}^{s(r-1)}N_{L/K}(h)^{s(r)}$ *(resp.* $a=a_1^{s(1)}\ldots a_{r-2}^{s(r-2)}\pi^{s(r-1)}N_{L/K}(h)^{s(r)}$ *) where* s *is a map* $\{0,\ldots,r\}\to\mathbb{Z}$ *such that* $p\nmid s(j)$ *for some* $j\neq r$.

Then $1-x^pa\in N_1(w,[X,\underline{K}])$ *for each* $x\in[X,\underline{\mathcal{O}_K}]$.

Proof. It follows from the fact that $w\in\{H_p^{r-1}(K),a\}$ and $1-x^pa$ is the norm of $1-xa^{1/p}\in[X,\underline{K(a^{1/p})}]^*$ ($\underline{K(a^{1/p})}$ denotes the ring object which represents the functor $X\to K(a^{1/p})\otimes_K[X,\underline{K}]$). □

Lemma 17. *Let* σ *be a generator of* $\mathrm{Gal}(L/K)$ *and let* $a=h^{-1}\sigma(h)-1$, $b=N_{L/K}(a)$, $t=v_K(b)$. *Let* $f=1$ *in case (i) and let* $f=p$ *in case (ii). Let* $N\colon[X,\underline{L}]^*\to[X,\underline{K}]^*$ *be the norm homomorphism. Then:*

(1) *If* $f|i$ *and* $1\leqslant i<t$ *then for every* $x\in\mathcal{M}_L^{i/f}[X,\underline{\mathcal{O}_L}]$

$$N(1+x)\equiv 1+N(x) \mod \mathcal{M}_K^{i+1}[X,\underline{\mathcal{O}_K}].$$

(2) *For every* $x\in[X,\underline{\mathcal{O}_K}]$

$$N(1+xa)\equiv 1+(x^p-x)b \mod \mathcal{M}_K^{t+1}[X,\underline{\mathcal{O}_K}].$$

In case (ii) for every integer r prime to p and every $x \in [X, \underline{\mathcal{O}_K}]$

$$N(1 + xh^r a) \equiv 1 + x^p N(h)^r b \mod \mathcal{M}_K^{t+1}[X, \underline{\mathcal{O}_K}].$$

(3)

$$1 + \mathcal{M}_K^{t+1}[X, \underline{\mathcal{O}_K}] \subset N(1 + \mathcal{M}_L^{t/f+1}[X, \underline{\mathcal{O}_L}]).$$

Proof. Follows from the computation of the norm homomorphism $L^* \to K^*$ in Serre [15, Ch. V §3] and [8, §1]. □

From these lemmas we have

(1) If $0 < i < t$ then

$$U_i\Gamma_q([X, \underline{K}]) \subset N_q(w, [X, \underline{K}]) + U_{i+1}\Gamma_q([X, \underline{K}]).$$

(2) $U_{t+1}\Gamma_q([X, \underline{K}]) \subset N_q(w, [X, \underline{K}])$.

(3) In case (ii) let $a_{r-1} = N_{L/K}(h)$. then in both cases (i) and (ii) the homomorphism

$$\begin{aligned}
&[X, \underline{\Omega_k^{q+r-2}}] \to U_1\Gamma_q([X, \underline{K}])/N_q(w, [X, \underline{K}]),\\
&xd\overline{a_1}/\overline{a_1} \wedge \cdots \wedge d\overline{a_{r-1}}/\overline{a_{r-1}} \wedge dy_1/y_1 \wedge \cdots \wedge dy_{q-1}/y_{q-1}\\
&\qquad\qquad \mapsto \{1 + \widetilde{x}b, \widetilde{y_1}, \ldots, \widetilde{y_{q-1}}\},
\end{aligned}$$

($x \in [X, \underline{k}], y_i \in [X, \underline{k^*}]$) annihilates $(1 - \mathrm{C})[X, \underline{\Omega_k^{q+r-2}}]$.

Lemma 7 and (1), (2), (3) imply that $U_1\Gamma_q([X, \underline{K}])$ is contained in the sum of $N_q(w, [X, \underline{K}])$ and the image of $\operatorname{lcf}(X, U_{t+1}K_q(K))$.

Lemma 18. *For each $u \in \mathcal{O}_K$ there exists an element ψ of $H^1_{p,\mathrm{ur}}(K)$ such that $(1 + ub)N_{L/K}(h)^{-1}$ is contained in the norm group $N_{L'/K}L'^*$ where L' is the cyclic extension of K corresponding to $\chi + \psi$ (χ corresponds to L/K).*

Proof. Follows from [9, §3.3 Lemma 15] (can be proved using the formula

$$N_{L_{\mathrm{ur}}/K_{\mathrm{ur}}}(1 + xa) \equiv 1 + (x^p - x)b \mod b\mathcal{M}_{K_{\mathrm{ur}}}$$

for $x \in \mathcal{O}_{K_{\mathrm{ur}}}$. □

Lemma 18 shows that $1 + ub$ is contained in the subgroup generated by $N_{L/K}L^*$ and $N_{L'/K}L'^*$, $\chi_L = 0$, $\chi_{L'} \in H^1_{p,\mathrm{ur}}(L')$.

Step 2. Next we prove that

$$U_1\Gamma_q([X, \underline{K}]) \subset N(\overline{w}, [X, \underline{K}])$$

for every $w \in H^r_p(K)$ where $\overline{w}$ is the image of w in $(H^r_p/H^r_{p,\mathrm{ur}})(K)$.

Lemma 19. *Let $q, r \geqslant 1$ and let $w \in H_p^r(K)$. Then there exists $i \geqslant 1$ such that $p^i\Gamma_q([X, \underline{K'}])$ and $U_{e(K'|K)i}\Gamma_q([X, \underline{K'}])$ are contained in $N_q(w_{K'}, [X, \underline{K'}])$ for every $K' \in \mathcal{E}(K)$ where $w_{K'}$ denotes the image of w in $H_p^r(K')$ and $e(K'|K)$ denotes the ramification index of K'/K.*

Lemma 20. *Let $i \geqslant 1$ and $x \in U_1\Gamma_q([X, \underline{K}])$; (resp. $x = \{u_1, \dots, u_q\}$ with $u_i \in [X, \mathcal{O}^*_{\underline{K}}]$; resp. $x \in \Gamma_q([X, \underline{K}])$).*

Then there exists a triple $(J, 0, f)$ and a family $(E_j, x_j)_{j \in J}$ which satisfy all the conditions of Definition 6 except (iv) and satisfy condition (iv)' below.

(iv)' *If $j \notin f(J)$ then x_j satisfy one of the following three properties:*

(a) $x_j \in p^i\Gamma_q([X, \underline{E_j}])$.

(b) $x_j \in U_{e(E_j|K)i}\Gamma_q([X, \underline{E_j}])$; *(resp. (b) $x_j \in U_1\Gamma_q([X, \underline{E_j}])$).*

(c) *Let $\overline{E_j}$ be the residue field of E_j. There are elements $c_1, \dots, c_{q-1}$ of $\mathcal{O}^*_{E_j}$ such that*
$x_j \in \{U_1\Gamma_1([X, \underline{E_j}]), c_1, \dots, c_{q-1}\}$ *and* $|\overline{E_j}^p(c_1, \dots, c_{q-1}) : \overline{E_j}^p| = p^{q-1}$;
(resp. (c) There are elements $b_1, \dots, b_q$ of $[X, \mathcal{O}^*_{\underline{E_j}}]$ and $c_1, \dots, c_q$ of $\mathcal{O}^*_{E_j}$ such that $x_j = \{b_1, \dots, b_q\}$ and such that for each m the residue class $\overline{b_m} \in [X, \overline{\underline{E_j}}]$ belongs to $[X, \overline{\underline{E_j}}]^p[\overline{c_m}]$ and $|\overline{E_j}^p(c_1, \dots, c_q) : \overline{E_j}^p| = p^q$);
(resp. (c) There are elements $b_1, \dots, b_{q-1}$ of $[X, \mathcal{O}^*_{\underline{E_j}}]$ and $c_1, \dots, c_{q-1}$ of $\mathcal{O}^*_{E_j}$ such that $x_j \in \{[X, \underline{E_j}]^*, b_1, \dots, b_{q-1}\}$ and such that for each m the residue class $\overline{b_m} \in [X, \overline{\underline{E_j}}]$ belongs to $[X, \overline{\underline{E_j}}]^p[\overline{c_m}]$ and
$|\overline{E_j}^p(c_1, \dots, c_{q-1}) : \overline{E_j}^p| = p^{q-1}$).

Using Lemma 19 and 20 it suffices for the purpose of this step to consider the following elements
$\{u, c_1, \dots, c_{q-1}\} \in U_1\Gamma_q([X, \underline{K}])$ such that $u \in U_1\Gamma_1([X, \underline{K}])$, $c_1, \dots, c_{q-1} \in \mathcal{O}^*_K$ and $|k^p(\overline{c_1}, \dots, \overline{c_{q-1}} : k^p| = p^{q-1}$.

For each $i = 1, \dots, q-1$ and each $s \geqslant 0$ take a p^s th root $c_{i,s}$ of $-c_i$ satisfying $c^p_{i,s+1} = c_{i,s}$. Note that $N_{k(c_{i,s+1})/k(c_{i,s})}(-c_{i,s+1}) = -c_{i,s}$. For each $m \geqslant 0$ write m in the form $(q-1)s + r$ ($s \geqslant 0$, $0 \leqslant r < q-1$). Let E_m be the finite extension of K of degree p^m generated by $c_{i,s+1}$ ($1 \leqslant i \leqslant r$) and $c_{i,s}$ ($r+1 \leqslant i \leqslant q-1$) and let

$$x_m = \{u, -c_{1,s+1}, \dots, -c_{r,s+1}, -c_{r+1,s}, \dots, -c_{q-1,s}\} \in \Gamma_q([X, \underline{E_m}]).$$

Then $E_\infty = \varinjlim E_m$ is a henselian discrete valuation field with residue field $\overline{E_\infty}$ satisfying $|\overline{E_\infty} : \overline{E_\infty}^p| \leqslant p^{r-1}$. Hence by Lemma 14 and Lemma 21 below there exists $m < \infty$ such that for some finite extension E'_m of E_m of degree prime to p the image of w in $H_p^r(E'_m)$ is standard. Let $J = \{0, 1, \dots, m, m'\}$, $f(j) = j-1$ for $1 \leqslant j \leqslant m$, $f(0) = 0$, $f(m') = m$, $E_{m'} = E'_m$ and

$$x_{m'} = \{u^{1/|E'_m : E_m|}, c_1, \dots, c_{q-1}\}.$$

Then from Step 1 we deduce $\{u, c_1, \dots, c_{q-1}\} \in N_q(\overline{w}, [X, \underline{K}])$.

Lemma 21. *Let K be a henselian discrete valuation field, and let $\widehat{K}$ be its completion. Then $H^r_m(K) \simeq H^r_m(\widehat{K})$ for every r and m.*

Proof. If m is invertible in K this follows from the isomorphism $\mathrm{Gal}(\widehat{K}^{\mathrm{sep}}/\widehat{K}) \simeq \mathrm{Gal}(K^{\mathrm{sep}}/K)$ (cf. [1, Lemma 2.2.1]). Assume $\mathrm{char}(K) = p > 0$ and $m = p^i$ ($i \geqslant 1$). For a field k of characteristic $p > 0$ the group $H^r_{p^i}(k)$ is isomorphic to $(H^1_{p^i}(k) \otimes k^* \otimes \cdots \otimes k^*)/J$ where J is the subgroup of the tensor product generated by elements of the form (cf. [9, §2.2 Corollary 4 to Proposition 2])

(i) $\chi \otimes a_1 \otimes \cdots \otimes a_{r-1}$ such that $a_i = a_j$ for some $i \neq j$,

(ii) $\chi \otimes a_1 \otimes \cdots \otimes a_{r-1}$ such that $a_i \in N_{k_\chi/k} k_\chi^*$ for some i where k_χ is the extension of k corresponding to χ.

By the above isomorphism of the Galois groups $H^1_{p^i}(K) \simeq H^1_{p^i}(\widehat{K})$. Furthermore if L is a cyclic extension of K then $1 + \mathcal{M}^n_K \subset N_{L/K} L^*$ and $1 + \mathcal{M}^n_{\widehat{K}} \subset N_{L\widehat{K}/\widehat{K}}(L\widehat{K})^*$ for sufficiently large n. Since $K^*/(1 + \mathcal{M}^n_K) \simeq \widehat{K}^*/(1 + \mathcal{M}^n_{\widehat{K}})$, the lemma follows. □

Step 3. In this step we prove that the subgroup of $\Gamma_q([X, \underline{K}])$ generated by $U_1\Gamma_q([X, \underline{K}])$ and elements of the form $\{u_1, \dots, u_q\}$ ($u_i \in [X, \underline{\mathcal{O}^*_K}]$) is contained in $N_q(\overline{w}, [X, \underline{K}])$. By Lemma 20 it suffices to consider elements $\{b_1, \dots, b_q\}$ such that $b_i \in [X, \underline{\mathcal{O}^*_K}]$ and such that there are elements $c_j \in \mathcal{O}^*_K$ satisfying

$$|k^p(\overline{c_1}, \dots, \overline{c_q}) : k^p| = p^q$$

and $\overline{b_i} \in [X, \underline{k}]^p[\overline{c_i}]$ for each i. Define fields E_m as in Step 2 replacing $q-1$ by q. Then $E_\infty = \varinjlim E_m$ is a henselian discrete valuation field with residue field $\overline{E_\infty}$ satisfying $|\overline{E_\infty} : \overline{E_\infty}^p| \leqslant p^{r-2}$. Hence $H^r_p(\widehat{E_\infty}) = H^r_{p,\mathrm{ur}}(\widehat{E_\infty})$. By Lemma 21 there exists $m < \infty$ such that $w_{E_m} \in H^r_{p,\mathrm{ur}}(E_m)$.

Step 4. Let w be a standard element. Then there exists a prime element π of K such that $\pi \in N_1(w, [X, \underline{K}]) = \Gamma_q([X, \underline{K}])$.

Step 5. Let w be any element of $H^r_p(K)$. To show that $\Gamma_q([X, \underline{K}]) = N_q(\overline{w}, [X, \underline{K}])$ it suffices using Lemma 20 to consider elements of $\Gamma_q([X, \underline{K}])$ of the form $\{x, b_1, \dots, b_{q-1}\}$ ($x \in [X, \underline{K}]^*$, $b_i \in [X, \underline{\mathcal{O}^*_K}]$) such that there are elements $c_1, \dots, c_{q-1} \in \mathcal{O}^*_K$ satisfying $|k^p(\overline{c_1}, \dots, \overline{c_{q-1}}) : k^p| = p^{q-1}$ and $\overline{b_i} \in [X, \underline{k}]^p[\overline{c_i}]$ for each i. The fields E_m are defined again as in Step 2, and we are reduced to the case where w is standard. □

5. Proof of Theorem 2

Let K be an n-dimensional local field. By [9, §3 Proposition 1] $H^r(K) = 0$ for $r > n+1$ and there exists a canonical isomorphism $H^{n+1}(K) \simeq \mathbb{Q}/\mathbb{Z}$.
For $0 \leqslant r \leqslant n+1$ the canonical pairing

$$\{\ ,\ \}: H^r(K) \times K_{n+1-r}(K) \to H^{n+1}(K)$$

(see subsection 4.2) induces a homomorphism

$$\Phi_K^r: H^r(K) \to \mathrm{Hom}(K_{n+1-r}(K), \mathbb{Q}/\mathbb{Z}).$$

if $w \in H^r(K)$ with $r \geqslant 1$ (resp. $r = 0$) then $\Phi_K^r(w)$ annihilates the norm group $N_{n+1-r}(w)$ (resp. $\Phi_K^r(w)$ annihilates $mK_{n+1}(K)$ where m is the order of w). Since $N_{n+1-r}(w)$ (resp. $mK_{n+1}(K)$) is open in $K_{n+1-r}(K)$ by Theorem 3 (resp. Corollary to Lemma 12), $\Phi_K^r(w)$ is a continuous character of $K_{n+1-r}(K)$ of finite order.

5.1. Continuous characters of prime order.

In this subsection we shall prove that for every prime p the map Φ_K^r ($0 \leqslant r \leqslant n+1$) induces a bijection between $H_p^r(K)$ (cf. Definition 10) and the group of all continuous characters of order p of $K_{n+1-r}(K)$. We may assume that $n \geqslant 1$ and $1 \leqslant r \leqslant n$. Let k be the residue field of K. In the case where $\mathrm{char}\,(k) \neq p$ the above assertion follows by induction on n from the isomorphisms

$$H_p^r(k) \oplus H_p^{r-1}(k) \simeq H_p^r(K), \quad K_q(k)/p \oplus K_q(k)/p \simeq K_q(K)/p.$$

Now we consider the case of $\mathrm{char}\,(k) = p$.

Definition 14. Let K be a complete discrete valuation field with residue field k of characteristic $p > 0$. For $r \geqslant 1$ and $i \geqslant 0$ we define the subgroup $T_i H_p^r(K)$ of $H_p^r(K)$ as follows.

(1) If $\mathrm{char}\,(K) = p$ then let $\delta_K^r: \Omega_K^{r-1} = C_1^{r-1}(K) \to H_p^r(K)$ be the canonical projection. Then $T_i H_p^r(K)$ is the subgroup of $H_p^r(K)$ generated by elements of the form

$$\delta_K^r(x\,dy_1/y_1 \wedge \dots \wedge dy_{r-1}/y_{r-1}), \quad x \in K, y_1, \dots, y_{r-1} \in K^*, v_K(x) \geqslant -i.$$

(2) If $\mathrm{char}\,(K) = 0$ then let ζ be a primitive pth root of 1, and let $L = K(\zeta)$.
Let $j = (pe_K/(p-1) - i)e(L|K)$ where $e_K = v_K(p)$ and $e(L|K)$ is the ramification index of L/K. If $j \geqslant 1$ let $U_j H_p^r(L)$ be the image of $U_j K_r(L)$ (cf. Definition 12) under the cohomological symbol $K_r(L)/p \to H_p^r(L)$. If $j \leqslant 0$, let $U_j H_p^r(L) = H_p^r(L)$. Then $T_i H_p^r(K)$ is the inverse image of $U_j H_p^r(L)$ under the canonical injection $H_p^r(K) \to H_p^r(L)$.

Remark. $T_i H_p^1(K)$ coincides with the subgroup consisting of elements which corresponds to cyclic extensions of K of degree p with ramification number $\leqslant i$ (the ramification number is defined as t of Lemma 17).

Let K be as in Definition 14, and assume that $|k : k^p| < \infty$. Fix $q, r \geqslant 1$ such that $|k : k^p| = p^{q+r-2}$. Let $T_i = T_i H_p^r(K)$, for $i \geqslant 0$; let U_i be the image of $U_i K_q(K)$ in $K_q(K)/p$ for $i \geqslant 1$, and let $U_0 = K_q(K)/p$. Let $e = v_K(p)$ ($= \infty$ if $\mathrm{char}(K) = p$). Fix a prime element π of K. Via the homomorphism

$$(x, y) \mapsto \rho_i^q(x) + \{\rho_i^{q-1}(y), \pi\}$$

of Lemma 8 whose kernel is known by [11], we identify U_i/U_{i+1} with the following groups:

(1) $K_q(k)/p \oplus K_{q-1}(k)/p$ if $i = 0$.

(2) Ω_k^{q-1} if $0 < i < pe/(p-1)$ and i is prime to p.

(3) $\Omega_k^{q-1}/\Omega_{k,d=0}^{q-1} \oplus \Omega_k^{q-2}/\Omega_{k,d=0}^{q-2}$ if $0 < i < pe/(p-1)$ and $p|i$.

(4) $\Omega_k^{q-1}/D_{a,k}^{q-1} \oplus \Omega_k^{q-2}/D_{a,k}^{q-2}$ if $\mathrm{char}(K) = 0$, $pe/(p-1)$ is an integer and $i = pe/(p-1)$.

(5) 0 if $i > pe/(p-1)$.

Here in (3) $\Omega_{k,d=0}^q$ ($q \geqslant 0$) denotes the kernel of the exterior derivation $d\colon \Omega_k^q \to \Omega_k^{q+1}$. In (4) a denotes the residue class of $p\pi^{-e}$ where $e = v_K(p)$ and $D_{a,k}$ denotes the subgroup of Ω_k^q generated by $d(\Omega_k^{q-1})$ and elements of the form

$$(x^p + ax) dy_1/y_1 \wedge \cdots \wedge dy_q/y_q.$$

Note that $H_p^{r+1}(K) \simeq H_p^{q+r-1}(k)$ by [11]. Let $\delta = \delta_k^{q+r-1}\colon \Omega_k^{q+r-2} \to H_p^{q+r-1}(k)$ (Definition 14).

Lemma 22. *In the canonical pairing*

$$H_p^r(K) \times K_q(K)/p \to H_p^{q+r}(K) \simeq H_p^{q+r-1}(k)$$

T_i *annihilates* U_{i+1} *for each* $i \geqslant 0$. *Furthermore,*

(1) $T_0 = H_{p,\mathrm{ur}}^r(k) \simeq H_p^r(k) \oplus H_p^{r-1}(k)$, *and the induced pairing*

$$T_0 \times U_0/U_1 \to H_p^{q+r-1}(k)$$

is identified with the direct sum of the canonical pairings

$$H_p^r(k) \times K_{q-1}(k)/p \to H_p^{q+r-1}(k), \quad H_p^{r-1}(k) \times K_q(k)/p \to H_p^{q+r-1}(k).$$

(2) *If* $0 < i < pe/(p-1)$ *and* i *is prime to* p *then there exists an isomorphism*

$$T_i/T_{i-1} \simeq \Omega_k^{r-1}$$

such that the induced pairing $T_i/T_{i-1} \times U_i/U_{i+1} \to H_p^{q+r-1}(k)$ *is identified with*

$$\Omega_k^{r-1} \times \Omega_k^{q-1} \to H_p^{q+r-1}(k), \quad (w, v) \mapsto \delta(w \wedge v).$$

(3) *If* $0 < i < pe/(p-1)$ *and* $p|i$ *then there exists an isomorphism*

$$T_i/T_{i-1} \simeq \Omega_k^{r-1}/\Omega_{k,d=0}^{r-1} \oplus \Omega_k^{r-2}/\Omega_{k,d=0}^{r-2}$$

such that the induced pairing is identified with

$$(w_1 \oplus w_2, v_1 \oplus v_2) \mapsto \delta(dw_1 \wedge v_2 + dw_2 \wedge v_1).$$

(4) *If* $\operatorname{char}(K) = 0$ *and* $pe/(p-1)$ *is not an integer, then* $H_p^r(K) = T_i$ *for the maximal integer* i *smaller than* $pe/(p-1)$. *Assume that* $\operatorname{char}(K) = 0$ *and* $pe/(p-1)$ *is an integer. Let* a *be the residue element of* $p\pi^{-e}$ *and let for* $s \geqslant 0$

$$\nu_s(a, F) = \ker(\Omega_{k,d=0}^s \to \Omega_k^s, \quad w \mapsto \mathrm{C}(w) + aw)$$

(C *denotes the Cartier operator*). *Then there exists an isomorphism*

$$T_{pe/(p-1)}/T_{pe/(p-1)-1} \simeq \nu_r(a,k) \oplus \nu_{r-1}(a,k)$$

such that the induced pairing is identified with

$$(w_1 \oplus w_2, v_1 \oplus v_2) \mapsto \delta(w_1 \wedge v_2 + w_2 \wedge v_1).$$

Proof. If $\operatorname{char}(K) = p$ the lemma follows from a computation in the differential modules Ω_K^s ($s = r-1, q+r-1$). In the case where $\operatorname{char}(K) = 0$ let ζ be a primitive pth root of 1 and let $L = K(\zeta)$. Then the cohomological symbol $K_r(L)/p \to H_p^r(L)$ is surjective and the structure of $H_p^r(L)$ is explicitly given in [11]. Since

$$H_p^r(K) \simeq \{x \in H_p^r(L) : \sigma(x) = x \quad \text{for all } \sigma \in \operatorname{Gal}(L/K)\},$$

the structure of $H_p^r(K)$ is deduced from that of $H_p^r(L)$ and the description of the pairing

$$H_p^r(K) \times K_q(K)/p \to H_p^{q+r}(K)$$

follows from a computation of the pairing

$$K_r(L)/p \times K_q(L)/p \to K_{q+r}(L)/p. \qquad \square$$

Lemma 23. *Let* K *be an* n*-dimensional local field such that* $\operatorname{char}(K) = p > 0$. *Then the canonical map* $\delta_K^n \colon \Omega_K^n \to H_p^{n+1}(K) \simeq \mathbb{Z}/p$ (*cf. Definition 14*) *comes from a morphism* $\underline{\Omega_K^n} \to \mathbb{Z}/p$ *of* $\mathcal{A}_\infty$.

Proof. Indeed it comes from the composite morphism of $\mathcal{F}_\infty$

$$\underline{\Omega_K^n} \xrightarrow{\theta_2} k_0 \xrightarrow{\operatorname{Tr}_{k_0/\mathbb{F}_p}} \mathbb{F}_p$$

defined by Lemma 7. $\square$

Now let K be an n-dimensional local field ($n \geqslant 1$) with residue field k such that $\operatorname{char}(k) = p > 0$. Let $1 \leqslant r \leqslant n$, $q = n+1-r$, and let T_i and U_i ($i \geqslant 0$) be as in Lemma 22.

The injectivity of the map induced by Φ_K^r

$$H_p^r(K) \to \operatorname{Hom}(K_{n+1-r}(K)/p, \mathbb{Z}/p)$$

follows by induction on n from the injectivity of $T_i/T_{i-1} \to \operatorname{Hom}(U_i/U_{i+1}, \mathbb{Z}/p)$, $i \geqslant 1$. Note that this injectivity for all prime p implies the injectivity of Φ_K^r.

Now let $\varphi\colon K_{n+1-r}(K) \to \mathbb{Z}/p$ be a continuous character of order p. We prove that there is an element w of $H_p^r(K)$ such that $\varphi = \Phi_K^r(w)$.

The continuity of φ implies that there exists $i \geqslant 1$ such that

$$\varphi(\{x_1, \dots, x_{n+1-r}\}) = 0 \quad \text{for all } x_1, \dots, x_{n+1-r} \in 1 + \mathcal{M}_K^i .$$

Using Graham's method [6, Lemma 6] we deduce that $\varphi(U_i) = 0$ for some $i \geqslant 1$. We prove the following assertion (A_i) ($i \geqslant 0$) by downward induction on i.

(A_i) The restriction of φ to U_i coincides with the restriction of $\Phi_K^r(w)$ for some $w \in H_p^r(K)$.

Indeed, by induction on i there exists $w \in H_p^r(K)$ such that the continuous character $\varphi' = \varphi - \Phi_K^r(w)$ annihilates U_{i+1}.

In the case where $i \geqslant 1$ the continuity of φ' implies that the map

$$\Omega_k^{n-r} \oplus \Omega_k^{n-r-1} \xrightarrow{\text{Lemma 8}} U_i/U_{i+1} \xrightarrow{\varphi'} \mathbb{Z}/p$$

comes from a morphism of $\mathcal{F}_\infty$. By additive duality of Proposition 3 and Lemma 23 applied to k the above map is expressed in the form

$$(v_1, v_2) \mapsto \delta_k^n(w_1 \wedge v_2 + w_2 \wedge v_1)$$

for some $w_1 \in \Omega_k^n, w_2 \in \Omega_k^{r-1}$. By the following argument the restriction of φ' to U_i/U_{i+1} is induced by an element of T_i/T_{i-1}. For example, assume $\operatorname{char}(K) = 0$ and $i = pe/(p-1)$ (the other cases are treated similarly and more easily). Since φ' annihilates $d(\Omega_k^{n-r-1}) \oplus d(\Omega_k^{n-r-2})$ and δ_k^n annihilates $d(\Omega_k^{n-2})$ we get

$$\delta_k^n(dw_1 \wedge v_2) = \pm\delta_k^n(w_2 \wedge dv_2) = 0 \quad \text{for all } v_2 .$$

Therefore $dw_1 = 0$. For every $x \in F$, $y_1, \dots, y_{n-r-1} \in F^*$ we have

$$\delta_k^n\big((\mathrm{C}(w_1)+aw_1)\wedge x\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{n-r-1}}{y_{n-r-1}}\big) = \delta_k^n\big(w_1\wedge(x^p+ax)\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{n-r-1}}{y_{n-r-1}}\big) = 0$$

(where a is as in Lemma 22 (4)). Hence $w_1 \in \nu_r(a,k)$ and similarly $w_2 \in \nu_{r-1}(a,k)$.

In the case where $i = 0$ Lemma 22 (1) and induction on n imply that there is an element $w \in T_0$ such that $\varphi' = \Phi_K^r(w)$.

5.2. Continuous characters of higher orders.

In treatment of continuous characters of higher order the following proposition will play a key role.

Proposition 7. *Let K be an n-dimensional local field. Let p be a prime number distinct from the characteristic of K. Assume that K contains a primitive pth root ζ of 1. Let $r \geqslant 0$ and $w \in H^r(K)$. Then the following two conditions are equivalent.*
(1) $w = pw'$ *for some* $w' \in H^r(K)$.
(2) $\{w, \zeta\} = 0$ *in* $H^{r+1}(K)$.

Proof. We may assume that $0 \leqslant r \leqslant n$. Let $\delta_r\colon H^r(K) \to H^{r+1}(K, \mathbb{Z}/p)$ be the connecting homomorphism induced by the exact sequence of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-modules

$$0 \to \mathbb{Z}/p \to \varinjlim_i \mu_{p^i}^{\otimes(r-1)} \xrightarrow{p} \varinjlim_i \mu_{p^i}^{\otimes(r-1)} \to 0.$$

Condition (1) is clearly equivalent to $\delta_r(w) = 0$.

First we prove the proposition in the case where $r = n$. Since the kernel of

$$\delta_n\colon H^n(K) \to H^{n+1}(K, \mathbb{Z}/p) \simeq \mathbb{Z}/p$$

is contained in the kernel of the homomorphism $\{\ , \zeta\}\colon H^n(K) \to H^{n+1}(K)$ it suffices to prove that the latter homomorphism is not a zero map. Let i be the maximal natural number such that K contains a primitive p^ith root of 1. Since the image χ of a primitive p^ith root of 1 under the composite map

$$K^*/K^{*p} \simeq H^1(K, \mu_p) \simeq H^1(K, \mathbb{Z}/p) \to H^1(K)$$

is not zero, the injectivity of Φ_K^1 shows that there is an element a of $K_n(K)$ such that $\{\chi, a\} \neq 0$. Let w be the image of a under the composite map induced by the cohomological symbol

$$K_n(K)/p^i \to H^n(K, \mu_{p^i}^{\otimes n}) \simeq H^n(K, \mu_{p^i}^{\otimes(n-1)}) \to H^n(K).$$

Then $\{\chi, a\} = \pm\{w, \zeta\}$.

Next we consider the general case of $0 \leqslant r \leqslant n$. Let w be an element of $H^r(K)$ such that $\{w, \zeta\} = 0$. Since the proposition holds for $r = n$ we get $\{\delta_r(w), a\} = \delta_n(\{w, a\}) = 0$ for all $a \in K_{n-r}(K)$. The injectivity of Φ_K^{r+1} implies $\delta_r(w) = 0$. $\square$

Remark. We conjecture that condition (1) is equivalent to condition (2) for every field K.

This conjecture is true if $\oplus_{r\geqslant 1} H^r(K)$ is generated by $H^1(K)$ as a $\oplus_{q\geqslant 0} K_q(K)$-module.

Completion of the proof of Theorem 2. Let φ be a non-zero continuous character of $K_{n+1-r}(K)$ of finite order, and let p be a prime divisor of the order of φ. By induction on the order there exists an element w of $H^r(K)$ such that $p\varphi = \Phi_K^r(w)$.

If $\operatorname{char}(K) = p$ then $H^r(K)$ is p-divisible. If $\operatorname{char}(K) \neq p$, let $L = K(\zeta)$ where ζ is a primitive pth root of 1 and let w_L be the image of w in $H^r(L)$. Then $\Phi_L^r(w_L)\colon K_{n+1-r}(L) \to \mathbb{Q}/\mathbb{Z}$ coincides with the composite

$$K_{n+1-r}(L) \xrightarrow{N_{L/K}} K_{n+1-r}(K) \xrightarrow{p\varphi} \mathbb{Q}/\mathbb{Z}$$

and hence $\{w_L, \zeta, a\} = 0$ in $H^{n+1}(L)$ for all $a \in K_{n-r}(L)$. The injectivity of Φ_L^{r+1} and Proposition 7 imply that $w_L \in pH^r(L)$. Since $|L : K|$ is prime to p, w belongs to $pH^r(K)$.

Thus there is an element w' of $H^r(K)$ such that $w = pw'$. Then $\varphi - \Phi_K^r(w')$ is a continuous character annihilated by p. □

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Part II

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1. Higher dimensional local fields and L-functions

A. N. Parshin

1.0. Introduction

1.0.1. Recall [P1], [FP] that if X is a scheme of dimension n and

$$X_0 \subset X_1 \subset \dots X_{n-1} \subset X_n = X$$

is a flag of irreducible subschemes ($\dim(X_i) = i$), then one can define a ring

$$K_{X_0,\dots,X_{n-1}}$$

associated to the flag. In the case where everything is regularly embedded, the ring is an n-dimensional local field. Then one can form an adelic object

$$\mathbb{A}_X = \prod{}' K_{X_0,\dots,X_{n-1}}$$

where the product is taken over all the flags with respect to certain restrictions on components of adeles [P1], [Be], [Hu], [FP].

Example. Let X be an algebraic projective irreducible surface over a field k and let P be a closed point of X, $C \subset X$ be an irreducible curve such that $P \in C$.

If X and C are smooth at P, then we let $t \in \mathcal{O}_{X,P}$ be a local equation of C at P and $u \in \mathcal{O}_{X,P}$ be such that $u|_C \in \mathcal{O}_{C,P}$ is a local parameter at P. Denote by $\mathcal{C}$ the ideal defining the curve C near P. Now we can introduce a two-dimensional local field $K_{P,C}$ attached to the pair P, C by the following procedure including completions and localizations:

$$\begin{array}{ccc}
\widehat{\mathcal{O}}_{X,P} & = & k(P)[[u,t]] \supset \mathcal{C} = (t) \\
| & & \\
(\widehat{\mathcal{O}}_{X,P})_{\mathcal{C}} & = & \text{discrete valuation ring with residue field } k(P)((u)) \\
| & & \\
\widehat{\mathcal{O}}_{P,C} := \widehat{(\widehat{\mathcal{O}}_{X,P})_{\mathcal{C}}} & = & k(P)((u))[[t]] \\
| & & \\
K_{P,C} := \mathrm{Frac}\,(\widehat{\mathcal{O}}_{P,C}) & = & k(P)((u))((t))
\end{array}$$

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Note that the left hand side construction is meaningful *without* any smoothness condition.

Let K_P be the minimal subring of $K_{P,C}$ which contains $k(X)$ and $\widehat{\mathcal{O}}_{X,P}$. The ring K_P is not a field in general. Then $K \subset K_P \subset K_{P,C}$ and there is another intermediate subring $K_C = \mathrm{Frac}\,(\mathcal{O}_C) \subset K_{P,C}$. Note that in dimension 2 there is a duality between points P and curves C (generalizing the classical duality between points and lines in projective geometry). We can compare the structure of adelic components in dimension one and two:

$$\begin{array}{c} K_P \\ | \\ K \end{array} \qquad\qquad \begin{array}{ccc} & K_{P,C} & \\ \diagup & & \diagdown \\ K_P & & K_C \\ \diagdown & & \diagup \\ & K & \end{array}$$

1.0.2. In the one-dimensional case for every character $\chi\colon \mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathbb{C}^*$ we have the composite

$$\chi'\colon \mathbb{A}^* = \prod{}' K_x^* \xrightarrow{\text{reciprocity map}} \mathrm{Gal}(K^{\mathrm{ab}}/K) \xrightarrow{\chi} \mathbb{C}^*.$$

J. Tate [T] and independently K. Iwasawa introduced an analytically defined L-function

$$L(s,\chi,f) = \int_{\mathbb{A}^*} f(a)\chi'(a)|a|^s d^*a,$$

where d^* is a Haar measure on $\mathbb{A}^*$ and the function f belongs to the Bruhat–Schwartz space of functions on $\mathbb{A}$ (for the definition of this space see for instance [W1, Ch. VII]). For a special choice of f and $\chi = 1$ we get the ζ-function of the scheme X

$$\zeta_X(s) = \prod_{x \in X} (1 - N(x)^{-s})^{-1},$$

if $\dim(X) = 1$ (adding the archimedean multipliers if necessary). Here x runs through the closed points of the scheme X and $N(x) = |k(x)|$. The product converges for $\mathrm{Re}(s) > \dim X$. For $L(s,\chi,f)$ they proved the analytical continuation to the whole s-plane and the functional equation

$$L(s,\chi,f) = L(1-s,\chi^{-1},\widehat{f}),$$

using Fourier transformation ($f \mapsto \widehat{f}$) on the space $\mathbb{A}_X$ (cf. [T], [W1], [W2]).

1.0.3. Schemes can be classified according to their dimension

dim (X)	geometric case	arithmetic case
...	...	...
2	algebraic surface $/\mathbb{F}_q$	arithmetic surface
1	algebraic curve $/\mathbb{F}_q$	arithmetic curve
0	Spec$(\mathbb{F}_q)$	Spec$(\mathbb{F}_1)$

where $\mathbb{F}_1$ is the "field of one element".

The analytical method works for the row of the diagram corresponding to dimension one. The problem to prove analytical continuation and functional equation for the ζ-function of arbitrary scheme X (Hasse–Weil conjecture) was formulated by A. Weil [W2] as a generalization of the previous Hasse conjecture for algebraic curves over fields of algebraic numbers, see [S1],[S2]. It was solved in the geometric situation by A. Grothendieck who employed cohomological methods [G]. Up to now there is no extension of this method to arithmetic schemes (see, however, [D]). On the other hand, a remarkable property of the Tate–Iwasawa method is that it can be simultaneously applied to the fields of algebraic numbers (arithmetic situation) and to the algebraic curves over a finite field (algebraic situation).

For a long time the author has been advocating (see, in particular, [P4], [FP]) the following:

Problem. *Extend Tate–Iwasawa's analytic method to higher dimensions.*

The higher adeles were introduced exactly for this purpose. In dimension one the adelic groups $\mathbb{A}_X$ and $\mathbb{A}_X^*$ are locally compact groups and thus we can apply the classical harmonic analysis. The starting point for that is the measure theory on locally compact local fields such as K_P for the schemes X of dimension 1. So we have the following:

Problem. *Develop a measure theory and harmonic analysis on n-dimensional local fields.*

Note that n-dimensional local fields are not locally compact topological spaces for $n > 1$ and by Weil's theorem the existence of the Haar measure on a topological group implies its locally compactness [W3, Appendix 1].

In this work several first steps in answering these problems are described.

1.1. Riemann–Hecke method

When one tries to write the ζ-function of a scheme X as a product over local fields attached to the flags of subvarieties one meets the following obstacle. For dimension greater than one the local fields are parametrized by flags and not by the closed points itself as in the Euler product. This problem is primary to any problems with the measure and integration. I think we have to return to the case of dimension one and reformulate the Tate–Iwasawa method. Actually, it means that we have to return to the Riemann–Hecke approach [He] known long before the work of Tate and Iwasawa. Of course, it was the starting point for their approach.

The main point is a reduction of the integration over ideles to integration over a single (or finitely many) local field.

Let C be a smooth irreducible complete curve defined over a field $k = \mathbb{F}_q$.

Put $K = k(C)$. For a closed point $x \in C$ denote by K_x the fraction field of the completion $\widehat{\mathcal{O}}_x$ of the local ring $\mathcal{O}_x$.

Let P be a fixed smooth k-rational point of C. Put $U = C \setminus P$, $A = \Gamma(U, \mathcal{O}_C)$. Note that A is a discrete subgroup of K_P.

A classical method to calculate ζ-function is to write it as a Dirichlet series instead of the Euler product:

$$\zeta_C(s) = \sum_{I \in \mathrm{Div}\,(\mathcal{O}_C)} |I|_C^s$$

where $\mathrm{Div}\,(\mathcal{O}_C)$ is the semigroup of effective divisors, $I = \sum_{x \in X} n_x x$, $n_x \in \mathbb{Z}$ and $n_x = 0$ for almost all $x \in C$,

$$|I|_C = \prod_{x \in X} q^{-\sum n_x |k(x):k|}.$$

Rewrite $\zeta_C(s)$ as

$$\zeta_U(s)\zeta_P(s) = \left(\sum_{I \subset U} |I|_U^s\right)\left(\sum_{\mathrm{supp}(I)=P} |I|_P^s\right).$$

Denote $A' = A \setminus \{0\}$. For the sake of simplicity assume that $\mathrm{Pic}(U) = (0)$ and introduce A'' such that $A'' \cap k^* = (1)$ and $A' = A''k^*$. Then for every $I \subset U$ there is a unique $b \in A''$ such that $I = (b)$. We also write $|b|_* = |(b)|_*$ for $* = P, U$. Then from the product formula $|b|_C = 1$ we get $|b|_U = |b|_P^{-1}$. Hence

$$\zeta_C(s) = \left(\sum_{b \in A''} |b|_U^s\right)\left(\sum_{m \geqslant 0} q^{-ms}\right) = \left(\sum_{b \in A''} |b|_P^{-s}\right) \int_{a \in K_P^*} |a|_P^s f_+(a) d^* a$$

where in the last equality we have used local Tate's calculation, $f_+ = i^*\delta_{\widehat{\mathcal{O}}_P}$, $i: K_P^* \to K_P$, $\delta_{\widehat{\mathcal{O}}_P}$ is the characteristic function of the subgroup $\widehat{\mathcal{O}}_P$, $d^*(\widehat{\mathcal{O}}_P^*) = 1$. Therefore

$$\begin{aligned}\zeta_C(s) &= \sum_{b\in A''}\int_{a\in K_P^*} |ab^{-1}|_P^s f_+(a)d^*a \\ &= \sum_{b\in A''}\int_{c=ab^{-1}} |c|_P^s f_+(bc)d^*c = \int_{K_P^*} |c|_P^s F(c)d^*c,\end{aligned}$$

where $F(c) = \sum_{b\in A'} f_+(bc)$.

Thus, the calculation of $\zeta_C(s)$ is reduced to integration over the single local field K_P. Then we can proceed further using the Poisson summation formula applied to the function F.

This computation can be rewritten in a more functorial way as follows

$$\zeta_C(s) = \langle |\ |^s, f_0\rangle_G \cdot \langle |\ |^s, f_1\rangle_G = \langle |\ |^s, i^*(F)\rangle_{G\times G} = \langle |\ |^s, j_* \circ i^*(F)\rangle_G,$$

where $G = K_P^*$, $\langle f, f'\rangle_G = \int_G ff'dg$ and we introduced the functions $f_0 = \delta_{A''} =$ sum of Dirac's δ_a over all $a \in A''$ and $f_1 = \delta_{\mathcal{O}_P}$ on K_P and the function $F = f_0 \otimes f_1$ on $K_P \times K_P$. We also have the norm map $|\ |: G \to \mathbb{C}^*$, the convolution map $j: G\times G \to G$, $j(x,y) = x^{-1}y$ and the inclusion $i: G\times G \to K_P \times K_P$.

For the appropriate classes of functions f_0 and f_1 there are ζ-functions with a functional equation of the following kind

$$\zeta(s, f_0, f_1) = \zeta(1-s, \widehat{f}_0, \widehat{f}_1),$$

where $\widehat{f}$ is a Fourier transformation of f. We will study the corresponding spaces of functions and operations like j_* or i^* in subsection 1.3.

Remark 1. We assumed that $\mathrm{Pic}(U)$ is trivial. To handle the general case one has to consider the curve C with several points removed. Finiteness of the $\mathrm{Pic}^0(C)$ implies that we can get an open subset U with this property.

1.2. Restricted adeles for dimension 2

1.2.1. Let us discuss the situation for dimension one once more. We consider the case of the algebraic curve C as above.

One-dimensional adelic complex

$$K \oplus \prod_{x\in C} \widehat{\mathcal{O}}_x \to \prod\nolimits'_{x\in C} K_x$$

can be included into the following commutative diagram

$$\begin{array}{ccc} K \oplus \prod_{x \in C} \widehat{\mathcal{O}}_x & \longrightarrow & \prod'_{x\in C} K_x \\ \downarrow & & \downarrow \\ K \oplus \widehat{\mathcal{O}}_P & \longrightarrow & \prod'_{x\neq P} K_x/\widehat{\mathcal{O}}_x \oplus K_P \end{array}$$

where the vertical map induces an isomorphism of cohomologies of the horizontal complexes. Next, we have a commutative diagram

$$\begin{array}{ccc} K \oplus \widehat{\mathcal{O}}_P & \longrightarrow & \prod'_{x\neq P} K_x/\widehat{\mathcal{O}}_x \oplus K_P \\ \downarrow & & \downarrow \\ K/A & \longrightarrow & \prod'_{x\neq P} K_x/\widehat{\mathcal{O}}_x \end{array}$$

where the bottom horizontal arrow is an isomorphism (the surjectivity follows from the strong approximation theorem). This shows that the complex $A \oplus \widehat{\mathcal{O}}_P \to K_P$ is quasi-isomorphic to the full adelic complex. The construction can be extended to an arbitrary locally free sheaf $\mathcal{F}$ on C and we obtain that the complex

$$W \oplus \widehat{\mathcal{F}}_P \to \widehat{\mathcal{F}}_P \otimes_{\widehat{\mathcal{O}}_P} K_P,$$

where $W = \Gamma(\mathcal{F}, C \setminus P) \subset K$, computes the cohomology of the sheaf $\mathcal{F}$.

This fact is essential for the analytical approach to the ζ-function of the curve C. To understand how to generalize it to higher dimensions we have to recall another applications of this diagram, in particular, the so called Krichever correspondence from the theory of integrable systems.

Let z be a local parameter at P, so $\widehat{\mathcal{O}}_P = k[[z]]$. The Krichever correspondence assigns points of infinite dimensional Grassmanians to (C, P, z) and a torsion free coherent sheaf of $\mathcal{O}_C$-modules on C. In particular, there is an injective map from classes of triples (C, P, z) to $A \subset k((z))$. In [P5] it was generalized to the case of algebraic surfaces using the higher adelic language.

1.2.2. Let X be a projective irreducible algebraic surface over a field k, $C \subset X$ be an irreducible projective curve, and $P \in C$ be a smooth point on both C and X.

In dimension two we start with the adelic complex

$$\mathbb{A}_0 \oplus \mathbb{A}_1 \oplus \mathbb{A}_2 \to \mathbb{A}_{01} \oplus \mathbb{A}_{02} \oplus \mathbb{A}_{12} \to A_{012},$$

where

$$A_0 = K = k(X), \qquad A_1 = \prod_{C \subset X} \widehat{\mathcal{O}}_C, \qquad A_2 = \prod_{x \in X} \widehat{\mathcal{O}}_x,$$

$$A_{01} = \prod\nolimits'_{C\subset X} K_C, A_{02} = \prod\nolimits'_{x\in X} K_x, A_{12} = \prod\nolimits'_{x\in C} \widehat{\mathcal{O}}_{x,C}, A_{012} = \mathbb{A}_X = \prod\nolimits' K_{x,C}.$$

In fact one can pass to another complex whose cohomologies are the same as of the adelic complex and which is a generalization of the construction for dimension one. We have to make the following assumptions: $P \in C$ is a smooth point on both C and X, and the surface $X \setminus C$ is affine. The desired complex is

$$A \oplus A_C \oplus \widehat{\mathcal{O}}_P \to B_C \oplus B_P \oplus \widehat{\mathcal{O}}_{P,C} \to K_{P,C}$$

where the rings B_x, B_C, A_C and A have the following meaning. Let $x \in C$. Let

$B_x = \bigcap\limits_{D \neq C} (K_x \cap \widehat{\mathcal{O}}_{x,D})$ where the intersection is taken inside K_x;

$B_C = K_C \cap (\bigcap\limits_{x \neq P} B_x)$ where the intersection is taken inside $K_{x,C}$;

$A_C = B_C \cap \widehat{\mathcal{O}}_C$, $A = K \cap (\bigcap_{x \in X \setminus C} \widehat{\mathcal{O}}_x)$.

This can be easily extended to the case of an arbitrary torsion free coherent sheaf $\mathcal{F}$ on X.

1.2.3. Returning back to the question about the ζ-function of the surface X over $k = \mathbb{F}_q$ we suggest to write it as the product of three Dirichlet series

$$\zeta_X(s) = \zeta_{X\setminus C}(s)\zeta_{C\setminus P}(s)\zeta_P(s) = \Bigg(\sum_{I \subset X \setminus C} |I|_X^s\Bigg)\Bigg(\sum_{I \subset C\setminus P} |I|_X^s\Bigg)\Bigg(\sum_{I \subset \mathrm{Spec}(\widehat{\mathcal{O}}_{P,C})} |I|_X^s\Bigg).$$

Again we can assume that the surface $U = X \setminus C$ has the most trivial possible structure. Namely, $\mathrm{Pic}(U) = (0)$ and $\mathrm{Ch}(U) = (0)$. Then every rank 2 vector bundle on U is trivial. In the general case one can remove finitely many curves C from X to pass to the surface U satisfying these properties (the same idea was used in the construction of the higher Bruhat–Tits buildings attached to an algebraic surface [P3, sect. 3]).

Therefore any zero-ideal I with support in $X \setminus C$, $C \setminus P$ or P can be defined by functions from the rings A, A_C and $\mathcal{O}_P$, respectively. The fundamental difference between the case of dimension one and the case of surfaces is that zero-cycles I and ideals of finite colength on X are not in one-to-one correspondence.

Remark 2. In [P2], [FP] we show that the functional equation for the L-function on an algebraic surface over a finite field can be rewritten using the K_2-adeles. Then it has the same shape as the functional equation for algebraic curves written in terms of $\mathbb{A}^*$-adeles (as in [W1]).

1.3. Types for dimension 1

We again discuss the case of dimension one. If D is a divisor on the curve C then the Riemann–Roch theorem says

$$l(D) - l(K_C - D) = \deg(D) + \chi(\mathcal{O}_C),$$

where as usual $l(D) = \dim \ \Gamma(C, \mathcal{O}_X(D))$ and K_C is the canonical divisor. If $\mathbb{A} = \mathbb{A}_C$ and $\mathbb{A}_1 = \mathbb{A}(D)$ then

$$H^1(C, \mathcal{O}_X(D)) = \mathbb{A}/(\mathbb{A}(D) + K), \qquad H^0(C, \mathcal{O}_X(D)) = \mathbb{A}(D) \cap K$$

where $K = \mathbb{F}_q(C)$. We have the following topological properties of the groups:

$\mathbb{A}$	locally compact group,
$\mathbb{A}(D)$	compact group,
K	discrete group,
$\mathbb{A}(D) \cap K$	finite group,
$\mathbb{A}(D) + K$	group of finite index of $\mathbb{A}$.

The group $\mathbb{A}$ is dual to itself. Fix a rational differential form $\omega \in \Omega^1_K$, $\omega \neq 0$ and an additive character ψ of $\mathbb{F}_q$. The following bilinear form

$$\langle (f_x), (g_x) \rangle = \sum_x \operatorname{res}_x(f_x g_x \omega), \ (f_x), (g_x) \in \mathbb{A}$$

is non-degenerate and defines an auto-duality of $\mathbb{A}$.

If we fix a Haar measure dx on $\mathbb{A}$ then we also have the Fourier transform

$$f(x) \mapsto \widehat{f}(x) = \int_{\mathbb{A}} \psi(\langle x, y \rangle) f(y) dy$$

for functions on $\mathbb{A}$ and for distributions F defined by the Parseval equality

$$(\widehat{F}, \widehat{\phi}) = (F, \phi).$$

One can attach some functions and/or distributions to the subgroups introduced above

$$\begin{aligned}
\delta_D \quad &= \text{the characteristic function of } \mathbb{A}(D) \\
\delta_{H^1} \quad &= \text{the characteristic function of } \mathbb{A}(D) + K \\
\delta_K \quad &= \sum_{\gamma \in K} \delta_\gamma \quad \text{where } \delta_\gamma \text{ is the delta-function at the point } \gamma \\
\delta_{H^0} \quad &= \sum_{\gamma \in \mathbb{A}(D) \cap K} \delta_\gamma.
\end{aligned}$$

There are two fundamental rules for the Fourier transform of these functions

$$\widehat{\delta}_D = \operatorname{vol}(\mathbb{A}(D)) \delta_{\mathbb{A}(D)^\perp},$$

where

$$\mathbb{A}(D)^{\perp} = \mathbb{A}((\omega) - D),$$

and

$$\widehat{\delta}_{\Gamma} = \mathrm{vol}(\mathbb{A}/\Gamma)^{-1}\delta_{\Gamma^{\perp}}$$

for any discrete co-compact group Γ. In particular, we can apply that to $\Gamma = K = \Gamma^{\perp}$. We have

$$(\delta_K, \delta_D) = \#(K \cap \mathbb{A}(D)) = q^{l(D)},$$
$$(\widehat{\delta}_K, \widehat{\delta}_D) = \mathrm{vol}(\mathbb{A}(D))\mathrm{vol}(\mathbb{A}/K)^{-1}(\delta_K, \delta_{K_C - D}) = q^{\deg D} q^{\chi(\mathcal{O}_C)} q^{l(K_C - D)}$$

and the Parseval equality gives us the Riemann–Roch theorem.

The functions in these computations can be classified according to their types. There are four types of functions which were introduced by F. Bruhat in 1961 [Br].

Let V be a finite dimensional vector space over the adelic ring $\mathbb{A}$ (or over an one-dimensional local field K with finite residue field $\mathbb{F}_q$). We put

$$\begin{aligned}
\mathcal{D} &= \{\text{locally constant functions with compact support}\},\\
\mathcal{E} &= \{\text{locally constant functions}\},\\
\mathcal{D}' &= \{\text{dual to } \mathcal{D} \ = \ \text{all distributions}\},\\
\mathcal{E}' &= \{\text{dual to } \mathcal{E} \ = \ \text{distributions with compact support}\}.
\end{aligned}$$

Every V has a filtration $P \supset Q \supset R$ by compact open subgroups such that all quotients P/Q are finite dimensional vector spaces over $\mathbb{F}_q$.

If V, V' are the vector spaces over $\mathbb{F}_q$ of finite dimension then for every homomorphism $i: V \to V'$ there are two maps

$$\mathcal{F}(V) \xrightarrow{i_*} \mathcal{F}(V'), \qquad \mathcal{F}(V') \xrightarrow{i^*} \mathcal{F}(V),$$

of the spaces $\mathcal{F}(V)$ of all functions on V (or V') with values in $\mathbb{C}$. Here i^* is the standard inverse image and i_* is defined by

$$i_* f(v') = \begin{cases} 0, & \text{if } v' \notin \mathrm{im}(i) \\ \sum_{v \mapsto v'} f(v), & \text{otherwise.} \end{cases}$$

The maps i_* and i^* are dual to each other.

We apply these constructions to give a more functorial definition of the Bruhat spaces. For any triple P, Q, R as above we have an epimorphism $i: P/R \to P/Q$ with the corresponding map for functions $\mathcal{F}(P/Q) \xrightarrow{i^*} \mathcal{F}(P/R)$ and a monomorphism $j: Q/R \to P/R$ with the map for functions $\mathcal{F}(Q/R) \xrightarrow{j_*} \mathcal{F}(P/R)$.

Now the Bruhat spaces can be defined as follows

$$\begin{aligned}
\mathcal{D} &= \varinjlim_{j_*} \varinjlim_{i^*} \mathcal{F}(P/Q),\\
\mathcal{E} &= \varprojlim_{j^*} \varinjlim_{i^*} \mathcal{F}(P/Q),\\
\mathcal{D}' &= \varprojlim_{j^*} \varprojlim_{i_*} \mathcal{F}(P/Q),\\
\mathcal{E}' &= \varinjlim_{j_*} \varprojlim_{i_*} \mathcal{F}(P/Q).
\end{aligned}$$

The spaces don't depend on the choice of the chain of subspaces P, Q, R. Clearly we have

$$\begin{aligned}
&\delta_D \in \mathcal{D}(\mathbb{A}),\\
&\delta_K \in \mathcal{D}'(\mathbb{A}),\\
&\delta_{H^0} \in \mathcal{E}'(\mathbb{A}),\\
&\delta_{H^1} \in \mathcal{E}(\mathbb{A}).
\end{aligned}$$

We have the same relations for the functions $\delta_{\mathcal{O}_P}$ and $\delta_{A''}$ on the group K_P considered in section 1.

The Fourier transform preserves the spaces $\mathcal{D}$ and $\mathcal{D}'$ but interchanges the spaces $\mathcal{E}$ and $\mathcal{E}'$. Recalling the origin of the subgroups from the adelic complex we can say that, in dimension one the types of the functions have the following structure

$$\begin{array}{ccccc}
& & \mathcal{E} & & \\
& / & & \backslash & \\
\mathcal{D} & & & & \mathcal{D}' \\
& \backslash & & / & \\
& & \mathcal{E}' & &
\end{array}
\qquad
\begin{array}{ccccc}
& & \mathbf{01} & & \\
& / & & \backslash & \\
\mathbf{1} & & & & \mathbf{0} \\
& \backslash & & / & \\
& & \emptyset & &
\end{array}$$

corresponding to the full simplicial division of an edge. The Fourier transform is a reflection of the diagram with respect to the middle horizontal axis.

The main properties of the Fourier transform we need in the proof of the Riemann-Roch theorem (and of the functional equation of the ζ-function) can be summarized as the commutativity of the following cube diagram

$$\begin{tikzcd}
& \mathcal{D}\otimes\mathcal{D}' \arrow[rr, "j_*"] \arrow[dl, leftrightarrow] \arrow[dd, "i^*"] & & \mathcal{E} \arrow[dl, leftrightarrow] \arrow[dd, "\alpha^*"] \\
\mathcal{D}\otimes\mathcal{D}' \arrow[rr, "i^*"] \arrow[dd, "j_*"] & & \mathcal{E}' \arrow[dd, "\beta_*"] & \\
& \mathcal{E}' \arrow[rr, "\beta_*"] \arrow[dl, leftrightarrow] & & \mathcal{F}(\mathbb{F}_1) \arrow[dl, dash] \\
\mathcal{E} \arrow[rr, "\alpha^*"] & & \mathcal{F}(\mathbb{F}_1) &
\end{tikzcd}$$

coming from the exact sequence

$$\mathbb{A} \xrightarrow{i} \mathbb{A} \oplus \mathbb{A} \xrightarrow{j} \mathbb{A},$$

with $i(a) = (a, a)$, $j(a, b) = a - b$, and the maps

$$\mathbb{F}_1 \xrightarrow{\alpha} \mathbb{A} \xrightarrow{\beta} \mathbb{F}_1$$

with $\alpha(0) = 0$, $\beta(a) = 0$. Here $\mathbb{F}_1$ is the field of one element, $\mathcal{F}(\mathbb{F}_1) = \mathbb{C}$ and the arrows with heads on both ends are the Fourier transforms.

In particular, the commutativity of the diagram implies the Parseval equality used above:

$$\begin{aligned}
&\langle \widehat{F}, \widehat{G} \rangle = \beta_* \circ i^*(\widehat{F} \otimes \widehat{G}) \\
&= \beta_* \circ i^*(\widehat{F \otimes G}) = \beta_* j_* \widehat{(F \otimes G)} \\
&= \alpha^* \circ j_*(F \otimes G) = \beta_* \circ i^*(F \otimes G) \\
&= \langle F, G \rangle.
\end{aligned}$$

Remark 3. These constructions can be extended to the function spaces on the groups $G(\mathbb{A})$ or $G(K)$ for a local field K and a group scheme G.

1.4. Types for dimension 2

In order to understand the types of functions in the case of dimension 2 we have to look at the adelic complex of an algebraic surface. We will use physical notations and denote a space by the discrete index which corresponds to it. Thus the adelic complex can be written as

$$\emptyset \to \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2} \to \mathbf{01} \oplus \mathbf{02} \oplus \mathbf{12} \to \mathbf{012},$$

where $\emptyset$ stands for the augmentation map corresponding to the inclusion of H^0. Just as in the case of dimension one we have a duality of $\mathbb{A} = \mathbb{A}_{012} = \mathbf{012}$ with itself defined by a bilinear form

$$\langle (f_{x,C}), (g_{x,C}) \rangle = \sum_{x,C} \operatorname{res}_{x,C}(f_{x,C} g_{x,C}\, \omega), \qquad (f_{x,C}), (g_{x,C}) \in \mathbb{A}$$

which is also non-degenerate and defines the autoduality of $\mathbb{A}$.

It can be shown that

$$\mathbb{A}_0 = \mathbb{A}_{01} \cap \mathbb{A}_{02}, \quad \mathbb{A}_{01}^{\perp} = \mathbb{A}_{01}, \quad \mathbb{A}_{02}^{\perp} = \mathbb{A}_{02}, \quad \mathbb{A}_0^{\perp} = \mathbb{A}_{01} \oplus \mathbb{A}_{02},$$

and so on. The proofs depend on the following residue relations for a rational differential form $\omega \in \Omega^2_{k(X)}$

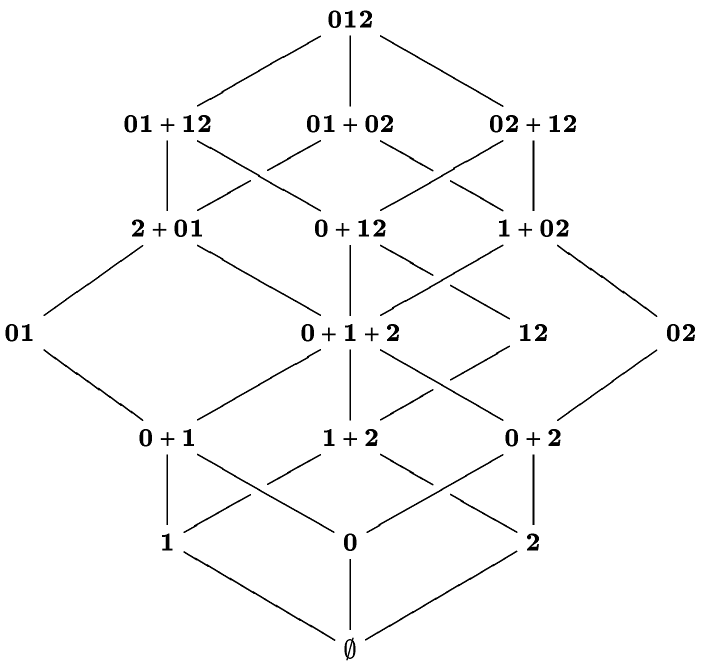
$$\text{for all } x \in X \qquad \sum_{C \ni x} \operatorname{res}_{x,C}(\omega) = 0,$$

$$\text{for all } C \subset X \qquad \sum_{x \in C} \operatorname{res}_{x,C}(\omega) = 0.$$

We see that the subgroups appearing in the adelic complex are not closed under the duality. It means that the set of types in dimension two will be greater then the set of types coming from the components of the adelic complex. Namely, we have:

Theorem 1 ([P4]). *Fix a divisor D on an algebraic surface X and let $\mathbb{A}_{12} = \mathbb{A}(D)$. Consider the lattice $\mathfrak{L}$ of the commensurability classes of subspaces in $\mathbb{A}_X$ generated by subspaces $\mathbb{A}_{01}, \mathbb{A}_{02}, \mathbb{A}_{12}$.*

The lattice $\mathfrak{L}$ is isomorphic to a free distributive lattice in three generators and has the structure shown in the diagram.



Remark 4. Two subspaces V, V' are called commensurable if $(V + V')/V \cap V'$ is of finite dimension. In the one-dimensional case *all* the subspaces of the adelic complex are commensurable (even the subspaces corresponding to different divisors). In this case we get a free distributive lattice in two generators (for the theory of lattices see [Bi]).

Just as in the case of curves we can attach to every node some space of functions (or distributions) on $\mathbb{A}$. We describe here a particular case of the construction, namely, the space $\mathcal{F}_{02}$ corresponding to the node $\mathbf{02}$. Also we will consider not the full adelic group but a single two-dimensional local field $K = \mathbb{F}_q((u))((t))$.

In order to define the space we use the filtration in K by the powers $\mathcal{M}^n$ of the maximal ideal $\mathcal{M} = \mathbb{F}_q((u))[[t]]t$ of K as a discrete valuation (of rank 1) field. Then we try to use the same procedure as for the local field of dimension 1 (see above).

If $P \supset Q \supset R$ are the elements of the filtration then we need to define the maps

$$\mathcal{D}(P/R) \xrightarrow{i_*} \mathcal{D}(P/Q), \qquad \mathcal{D}(P/R) \xrightarrow{j^*} \mathcal{D}(Q/R)$$

corresponding to an epimorphism $i: P/R \to P/Q$ and a monomorphism $j: Q/R \to P/R$. The map j^* is a restriction of the locally constant functions with compact support and it is well defined. To define the direct image i_* one needs to integrate along the fibers of the projection i. To do that we have to choose a Haar measure on the fibers for *all* $P,\ Q,\ R$ in a consistent way. In other words, we need a system of Haar measures on all quotients P/Q and by transitivity of the Haar measures in exact sequences it is enough to do that on all quotients $\mathcal{M}^n/\mathcal{M}^{n+1}$.

Since $\mathcal{O}_K/\mathcal{M} = \mathbb{F}_q((u)) = K_1$ we can first choose a Haar measure on the residue field K_1. It will depend on the choice of a fractional ideal $\mathcal{M}^i_{K_1}$ normalizing the Haar measure. Next, we have to extend the measure on all $\mathcal{M}^n/\mathcal{M}^{n+1}$. Again, it is enough to choose a second local parameter t which gives an isomorphism

$$t^n: \mathcal{O}_K/\mathcal{M} \to \mathcal{M}^n/\mathcal{M}^{n+1}.$$

Having made these choices we can put as above

$$\mathcal{F}_{02} = \varinjlim{}_{j^*} \varprojlim{}_{i_*} \mathcal{D}(P/Q)$$

where the space $\mathcal{D}$ was introduced in the previous section.

We see that contrary to the one-dimensional case the space $\mathcal{F}_{02}$ is not intrinsically defined. But the choice of all additional data can be easily controlled.

Theorem 2 ([P4]). *The set of the spaces $\mathcal{F}_{02}$ is canonically a principal homogeneous space over the valuation group Γ_K of the field K.*

Recall that Γ_K is non-canonically isomorphic to the lexicographically ordered group $\mathbb{Z} \oplus \mathbb{Z}$.

One can extend this procedure to other nodes of the diagram of types. In particular, for $\mathbf{012}$ we get the space which does not depend on the choice of the Haar measures.

The standard subgroup of the type $\mathbf{02}$ is $B_P = \mathbb{F}_p[[u]]((t))$ and it is clear that

$$\delta_{B_P} \in \mathcal{F}_{02}.$$

The functions δ_{B_C} and $\delta_{\widehat{\mathcal{O}}_{P,C}}$ have the types $\mathbf{01},\ \mathbf{12}$ respectively.

Remark 5. Note that the whole structure of all subspaces in $\mathbb{A}$ or K corresponding to different divisors or coherent sheaves is more complicated. The spaces $\mathbb{A}(D)$ of type $\mathbf{12}$ are no more commensurable. To describe the whole lattice one has to introduce several equivalence relations (commensurability up to compact subspace, a locally compact subspace and so on).

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2. Adelic constructions for direct images of differentials and symbols

Denis Osipov

2.0. Introduction

Let X be a smooth algebraic surface over a perfect field k.

Consider pairs $x \in C$, x is a closed point of X, C is either an irreducible curve on X which is smooth at x, or an irreducible analytic branch near x of an irreducible curve on X. As in the previous section 1 for every such pair $x \in C$ we get a two-dimensional local field $K_{x,C}$.

If X is a projective surface, then from the adelic description of Serre duality on X there is a local decomposition for the trace map $H^2(X, \Omega^2_X) \to k$ by using a two-dimensional residue map $\mathrm{res}_{K_{x,C}/k(x)}\colon \Omega^2_{K_{x,C}/k(x)} \to k(x)$ (see [P1]).

From the adelic interpretation of the divisors intersection index on X there is a similar local decomposition for the global degree map from the group $CH^2(X)$ of algebraic cycles of codimension 2 on X modulo the rational equivalence to $\mathbb{Z}$ by means of explicit maps from $K_2(K_{x,C})$ to $\mathbb{Z}$ (see [P3]).

Now we pass to the relative situation. Further assume that X is any smooth surface, but there are a smooth curve S over k and a smooth projective morphism $f\colon X \to S$ with connected fibres. Using two-dimensional local fields and explicit maps we describe in this section a local decomposition for the maps

$$f_*\colon H^n(X, \Omega^2_X) \to H^{n-1}(S, \Omega^1_S), \quad f_*\colon H^n(X, \mathcal{K}_2(X)) \to H^{n-1}(S, \mathcal{K}_1(S))$$

where $\mathcal{K}$ is the Zariski sheaf associated to the presheaf $U \to K(U)$. The last two groups have the following geometric interpretation:

$$H^n(X, \mathcal{K}_2(X)) = CH^2(X, 2-n), \quad H^{n-1}(S, \mathcal{K}_1(S)) = CH^1(S, 2-n)$$

where $CH^2(X, 2-n)$ and $CH^1(S, 1-n)$ are higher Chow groups on X and S (see [B]). Note also that $CH^2(X, 0) = CH^2(X)$, $CH^1(S, 0) = CH^1(S) = \mathrm{Pic}(S)$, $CH^1(S, 1) = H^0(S, \mathcal{O}^*_S)$.

Let $s = f(x) \in S$. There is a canonical embedding $f^*: K_s \to K_{x,C}$ where K_s is the quotient of the completion of the local ring of S at s.

Consider two cases:

(1) $C \neq f^{-1}(s)$. Then $K_{x,C}$ is non-canonically isomorphic to $k(C)_x((t_C))$ where $k(C)_x$ is the completion of $k(C)$ at x and t_C is a local equation of C near x.

(2) $C = f^{-1}(s)$. Then $K_{x,C}$ is non-canonically isomorphic to $k(x)((u))((t_s))$ where $\{u = 0\}$ is a transversal curve at x to $f^{-1}(s)$ and $t_s \in K_s$ is a local parameter at s, i.e. $k(s)((t_s)) = K_s$.

2.1. Local constructions for differentials

Definition. For $K = k((u))((t))$ let $U = u^i k[[u,t]]dk[[u,t]] + t^j k((u))[[t]]dk((u))[[t]]$ be a basis of neighbourhoods of zero in $\Omega^1_{k((u))[[t]]/k}$ (compare with 1.4.1 of Part I). Let $\widetilde\Omega^1_K = \Omega^1_{K/k}/(K \cdot \cap U)$ and $\widetilde\Omega^n_K = \wedge^n \widetilde\Omega^1_K$. Similarly define $\widetilde\Omega^n_{K_s}$.

Note that $\widetilde\Omega^2_{K_{x,C}}$ is a one-dimensional space over $K_{x,C}$; and $\widetilde\Omega^n_{K_{x,C}}$ does not depend on the choice of a system of local parameters of $\widehat{\mathcal{O}}_x$, where $\widehat{\mathcal{O}}_x$ is the completion of the local ring of X at x.

Definition. For $K = k((u))((t))$ and $\omega = \sum_i \omega_i(u) \wedge t^i dt = \sum_i u^i du \wedge \omega'_i(t) \in \widetilde\Omega^2_K$ put

$$\mathrm{res}_t(\omega) = \omega_{-1}(u) \in \widetilde\Omega^1_{k((u))},$$
$$\mathrm{res}_u(\omega) = \omega'_{-1}(t) \in \widetilde\Omega^1_{k((t))}.$$

Define a relative residue map

$$f_*^{x,C}: \widetilde\Omega^2_{K_{x,C}} \to \widetilde\Omega^1_{K_s}$$

as

$$f_*^{x,C}(\omega) = \begin{cases} \mathrm{Tr}_{\,k(C)_x/K_s} \mathrm{res}_{t_C}(\omega) & \text{if } C \neq f^{-1}(s) \\ \mathrm{Tr}_{\,k(x)((t_s))/K_s} \mathrm{res}_u(\omega) & \text{if } C = f^{-1}(s). \end{cases}$$

The relative residue map doesn't depend on the choice of local parameters.

Theorem (reciprocity laws for relative residues)**.** *Fix $x \in X$. Let $\omega \in \widetilde\Omega^2_{K_x}$ where K_x is the minimal subring of $K_{x,C}$ which contains $k(X)$ and $\widehat{\mathcal{O}}_x$. Then*

$$\sum_{C \ni x} f_*^{x,C}(\omega) = 0.$$

Fix $s \in S$. Let $\omega \in \widetilde{\Omega}^2_{K_F}$ where K_F is the completion of $k(X)$ with respect to the discrete valuation associated with the curve $F = f^{-1}(s)$. Then

$$\sum_{x \in F} f_*^{x,F}(\omega) = 0.$$

See [O].

2.2. The Gysin map for differentials

Definition. In the notations of subsection 1.2.1 in the previous section put

$$\Omega^1_{\mathbb{A}_S} = \{(f_s dt_s) \in \prod_{s \in S} \widetilde{\Omega}^1_{K_s}, \quad v_s(f_s) \geqslant 0 \text{ for almost all } s \in S\}$$

where t_s is a local parameter at s, v_s is the discrete valuation associated to t_s and K_s is the quotient of the completion of the local ring of S at s. For a divisor I on S define

$$\Omega_{\mathbb{A}_S}(I) = \{(f_s) \in \Omega^1_{\mathbb{A}_S} : v_s(f_s) \geqslant -v_s(I) \quad \text{for all } s \in S\}.$$

Recall that the n-th cohomology group of the following complex

$$\begin{array}{ccc} \Omega^1_{k(S)/k} \oplus \Omega^1_{\mathbb{A}_S}(0) & \longrightarrow & \Omega^1_{\mathbb{A}_S} \\ (f_0, f_1) & \longmapsto & f_0 + f_1. \end{array}$$

is canonically isomorphic to $H^n(S, \Omega^1_S)$ (see [S, Ch.II]).

The sheaf Ω^2_X is invertible on X. Therefore, Parshin's theorem (see [P1]) shows that similarly to the previous definition and definition in 1.2.2 of the previous section for the complex $\Omega^2(\mathcal{A}_X)$

$$\begin{array}{ccccc} \Omega^2_{A_0} \oplus \Omega^2_{A_1} \oplus \Omega^2_{A_2} & \longrightarrow & \Omega^2_{A_{01}} \oplus \Omega^2_{A_{02}} \oplus \Omega^2_{A_{12}} & \longrightarrow & \Omega^2_{A_{012}} \\ (f_0, f_1, f_2) & \longmapsto & (f_0 + f_1, f_2 - f_0, -f_1 - f_2) & & \\ & & (g_1, g_2, g_3) & \longmapsto & g_1 + g_2 + g_3 \end{array}$$

where

$$\Omega^2_{A_i} \subset \Omega^2_{A_{ij}} \subset \Omega^2_{A_{012}} = \Omega^2_{\mathbb{A}_X} = \prod_{x \in C}{}' \widetilde{\Omega}^2_{K_{x,C}} \subset \prod_{x \in C} \widetilde{\Omega}^2_{K_{x,C}}$$

there is a canonical isomorphism

$$H^n(\Omega^2(\mathcal{A}_X)) \simeq H^n(X, \Omega^2_X).$$

Using the reciprocity laws above one can deduce:

Theorem. *The map* $f_* = \sum_{C\ni x, f(x)=s} f_*^{x,C}$ *from* $\Omega^2_{\mathbb{A}_X}$ *to* $\Omega^1_{\mathbb{A}_S}$ *is well defined. It maps the complex* $\Omega^2(\mathcal{A}_X)$ *to the complex*

$$0 \longrightarrow \Omega^1_{k(S)/k} \oplus \Omega^1_{\mathbb{A}_S}(0) \longrightarrow \Omega^1_{\mathbb{A}_S}.$$

It induces the map $f_*\colon H^n(X,\Omega^2_X) \to H^{n-1}(S,\Omega^1_S)$ *of 2.0.*

See [O].

2.3. Local constructions for symbols

Assume that k is of characteristic 0.

Theorem. *There is an explicitly defined symbolic map*

$$f_*(\ ,\ )_{x,C}\colon K^*_{x,C} \times K^*_{x,C} \to K^*_s$$

(*see remark below*) *which is uniquely determined by the following properties*

$$N_{k(x)/k(s)}\ t_{K_{x,C}}(\alpha,\beta,f^*\gamma) = t_{K_s}(f_*(\alpha,\beta)_{x,C},\gamma) \quad \textit{for all } \alpha,\beta\in K^*_{x,C},\ \gamma\in K^*_s$$

where $t_{K_{x,C}}$ *is the tame symbol of the two-dimensional local field* $K_{x,C}$ *and* t_{K_s} *is the tame symbol of the one-dimensional local field* K_s (*see 6.4.2 of Part I*);

$$\mathrm{Tr}_{\,k(x)/k(s)}(\alpha,\beta,f^*(\gamma)]_{K_{x,C}} = (f_*(\alpha,\beta)_{x,C},\gamma]_{K_s} \quad \textit{for all } \alpha,\beta\in K^*_{x,C},\ \gamma\in K_s$$

where $(\alpha,\beta,\gamma]_{K_{x,C}} = \mathrm{res}_{K_{x,C}/k(x)}(\gamma d\alpha/\alpha \wedge d\beta/\beta)$ *and*
$(\alpha,\beta]_{K_s} = \mathrm{res}_{K_s/k(s)}(\alpha d\beta/\beta)$.

The map $f_*(\ ,\ )_{x,C}$ induces the map

$$f_*(\ ,\ )_{x,C}\colon K_2(K_{x,C}) \to K_1(K_s).$$

Corollary (reciprocity laws). *Fix a point* $s\in S$. *Let* $F = f^{-1}(s)$.
Let $\alpha,\beta\in K^*_F$. *Then*

$$\prod_{x\in F} f_*(\alpha,\beta)_{x,F} = 1.$$

Fix a point $x\in F$. *Let* $\alpha,\beta\in K^*_x$. *Then*

$$\prod_{C\ni x} f_*(\alpha,\beta)_{x,C} = 1.$$

Remark. If $C \neq f^{-1}(s)$ then $f_*(\ ,\ )_{x,C} = N_{k(C)_x/K_s}\, t_{K_{x,C}}$ where $t_{K_{x,C}}$ is the tame symbol with respect to the discrete valuation of rank 1 on $K_{x,C}$.

If $C = f^{-1}(s)$ then $f_*(\ ,\ )_{x,C} = N_{k(x)((t_s))/K_s}(\ ,\ )_f$ where $(\ ,\ )_f^{-1}$ coincides with Kato's residue homomorphism [K, §1]. An explicit formula for $(\ ,\ )_f$ is constructed in [O, Th.2].

2.4. The Gysin map for Chow groups

Assume that k is of arbitrary characteristic.

Definition. Let $K_2'(\mathbb{A}_X)$ be the subset of all $(f_{x,C}) \in K_2(K_{x,C})$, $x \in C$ such that

(a) $f_{x,C} \in K_2(\mathcal{O}_{x,C})$ for almost all irreducible curves C where $\mathcal{O}_{x,C}$ is the ring of integers of $K_{x,C}$ with respect to the discrete valuation of rank 1 on it;

(b) for all irreducible curves $C \subset X$, all integers $r \geqslant 1$ and almost all points $x \in C$

$$f_{x,C} \in K_2(\mathcal{O}_{x,C}, \mathcal{M}_C^r) + K_2(\widehat{\mathcal{O}}_x[t_C^{-1}]) \subset K_2(K_{x,C})$$

where $\mathcal{M}_C$ is the maximal ideal of $\mathcal{O}_{x,C}$ and $K_2(A, J) = \ker(K_2(A) \to K_2(A/J))$.

This definition is similar to the definition of [P2].

Definition. Using the diagonal map of $K_2(K_C)$ to $\prod_{x\in C} K_2(K_{x\in C})$ and of $K_2(K_x)$ to $\prod_{C\ni x} K_2(K_{x\in C})$ put

$$K_2'(A_{01}) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{C\subset X} K_2(K_C),$$
$$K_2'(A_{02}) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{x\in X} K_2(K_x),$$
$$K_2'(A_{12}) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{x\in C} K_2(\mathcal{O}_{x,C}),$$
$$K_2'(A_0) = K_2(k(X)),$$
$$K_2'(A_1) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{C\subset X} K_2(\mathcal{O}_C),$$
$$K_2'(A_2) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{x\in X} K_2(\widehat{\mathcal{O}}_x)$$

where $\mathcal{O}_C$ is the ring of integers of K_C.

Define the complex $K_2(\mathcal{A}_X)$:

$$K_2'(A_0) \oplus K_2'(A_1) \oplus K_2'(A_2) \to K_2'(A_{01}) \oplus K_2'(A_{02}) \oplus K_2'(A_{12}) \to K_2'(A_{012})$$
$$(f_0, f_1, f_2) \mapsto (f_0 + f_1, f_2 - f_0, -f_1 - f_2)$$
$$(g_1, g_2, g_3) \quad \mapsto \quad g_1 + g_2 + g_3$$

where $K_2'(A_{012}) = K_2'(\mathbb{A}_X)$.

Using the Gersten resolution from K-theory (see [Q, §7]) one can deduce:

Theorem. *There is a canonical isomorphism*

$$H^n(K_2(\mathcal{A}_X)) \simeq H^n(X, \mathcal{K}_2(X)).$$

Similarly one defines $K_1'(\mathbb{A}_S)$. From $H^1(S, \mathcal{K}_1(S)) = H^1(S, \mathcal{O}_S^*) = \mathrm{Pic}(S)$ (or from the approximation theorem) it is easy to see that the n-th cohomology group of the following complex

$$\begin{array}{ccc} K_1(k(S)) \oplus \sum_{s\in S} K_1(\widehat{\mathcal{O}}_s) & \longrightarrow & K_1'(\mathbb{A}_S) \\ (f_0, f_1) & \longmapsto & f_0 + f_1. \end{array}$$

is canonically isomorphic to $H^n(S, \mathcal{K}_1(S))$ (here $\widehat{\mathcal{O}}_s$ is the completion of the local ring of C at s).

Assume that k is of characteristic 0.

Using the reciprocity law above and the previous theorem one can deduce:

Theorem. *The map $f_* = \sum_{C\ni x, f(x)=s} f_*(\ ,\ )_{x,C}$ from $K_2'(\mathbb{A}_X)$ to $K_1'(\mathbb{A}_S)$ is well defined. It maps the complex $K_2(\mathcal{A}_X)$ to the complex*

$$0 \longrightarrow K_1(k(S)) \oplus \sum_{s\in S} K_1(\widehat{\mathcal{O}}_s) \longrightarrow K_1'(\mathbb{A}_S).$$

It induces the map $f_*\colon H^n(X, \mathcal{K}_2(X)) \to H^{n-1}(S, \mathcal{K}_1(S))$ of 2.0.

If $n = 2$, then the last map is the direct image morphism (Gysin map) from $CH^2(X)$ to $CH^1(S)$.

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3. The Bruhat–Tits buildings over higher dimensional local fields

A. N. Parshin

3.0. Introduction

A generalization of the Bruhat–Tits buildings for the groups $PGL(V)$ over n-dimensional local fields was introduced in [P1]. The main object of the classical Bruhat–Tits theory is a simplicial complex attached to any reductive algebraic group G defined over a field K. There are two parallel theories in the case where K has no additional structure or K is a local (or more generally, complete discrete valuation) field. They are known as the *spherical* and *euclidean* buildings correspondingly (see subsection 3.2 for a brief introduction, [BT1], [BT2] for original papers and [R], [T1] for the surveys).

In the generalized theory of buildings they correspond to local fields of dimension zero and of dimension one. The construction of the Bruhat–Tits building for the group $PGL(2)$ over two-dimensional local field was described in detail in [P2]. Later V. Ginzburg and M. Kapranov extended the theory to arbitrary reductive groups over a two-dimensional local fields [GK]. Their definition coincides with ours for $PGL(2)$ and is different for higher ranks. But it seems that they are closely related (in the case of the groups of type A_l). It remains to develop the theory for arbitrary reductive groups over local fields of dimension greater than two.

In this work we describe the structure of the higher building for the group $PGL(3)$ over a two-dimensional local field. We refer to [P1], [P2] for the motivation of these constructions.

This work contains four subsections. In 3.1 we collect facts about the Weyl group. Then in 3.2 we briefly describe the building for $PGL(2)$ over a local field of dimension not greater than two; for details see [P1], [P2]. In 3.3 we study the building for $PGL(3)$ over a local field F of dimension one and in 3.4 we describe the building over a two-dimensional local field.

We use the notations of section 1 of Part I.

If K is an n-dimensional local field, let Γ_K be the valuation group of the discrete valuation of rank n on K^*; the choice of a system of local parameters $t_1, \dots, t_n$ of K induces an isomorphism of Γ_K and the lexicographically ordered group $\mathbb{Z}^{\oplus n}$.

Let K ($K = K_2$, K_1, $K_0 = k$) be a two-dimensional local field. Let $O = O_K$, $M = M_K$, $\mathcal{O} = \mathcal{O}_K$, $\mathcal{M} = \mathcal{M}_K$ (see subsection 1.1 of Part I). Then $O = \mathrm{pr}^{-1}(\mathcal{O}_{K_1})$, $M = \mathrm{pr}^{-1}(\mathcal{M}_{K_1})$ where $\mathrm{pr}\colon \mathcal{O}_K \to K_1$ is the residue map. Let t_1, t_2 be a system of local parameters of K.

If $K \supset \mathcal{O}$ is the fraction field of a ring $\mathcal{O}$ we call $\mathcal{O}$-submodules $J \subset K$ fractional $\mathcal{O}$-ideals (or simply fractional ideals).

The ring O has the following properties:

(i) $O/M \simeq k, \quad K^* \simeq \langle t_1 \rangle \times \langle t_2 \rangle \times O^*, \quad O^* \simeq k^* \times (1 + M)$;

(ii) every finitely generated fractional O-ideal is principal and equal to

$$P(i,j) = (t_1^i t_2^j) \quad \text{for some } i, j \in \mathbb{Z}$$

(for the notation $P(i,j)$ see loc.cit.);

(iii) every infinitely generated fractional O-ideal is equal to

$$P(j) = \mathcal{M}_K^j = \langle t_1^i t_2^j : i \in \mathbb{Z} \rangle \quad \text{for some } j \in \mathbb{Z}$$

(see [FP], [P2] or section 1 of Part I). The set of these ideals is totally ordered with respect to the inclusion.

3.1. The Weyl group

Let B be the image of

$$\begin{pmatrix} O & O & \dots & O \\ M & O & \dots & O \\ & & \dots & \\ M & M & \dots & O \end{pmatrix}$$

in $PGL(m, K)$. Let N be the subgroup of monomial matrices.

Definition 1. Let $T = B \bigcap N$ be the image of

$$\begin{pmatrix} O^* & \dots & 0 \\ & \ddots & \\ 0 & \dots & O^* \end{pmatrix}$$

in G.

The group

$$W = W_{K/K_1/k} = N/T$$

is called the *Weyl group*.

There is a rich structure of subgroups in G which have many common properties with the theory of BN-pairs. In particular, there are Bruhat, Cartan and Iwasawa decompositions (see [P2]).

The Weyl group W contains the following elements of order two

$$s_i = \begin{pmatrix} 1 & \dots & 0 & & & 0 & \dots & 0 \\ & \ddots & & & & & & \\ 0 & \dots & 1 & & & & & 0 \\ 0 & \dots & & 0 & 1 & & \dots & 0 \\ 0 & & & 1 & 0 & & \dots & 0 \\ 0 & \dots & & & & 1 & \dots & 0 \\ & & & & & & \ddots & \\ 0 & \dots & 0 & & & 0 & \dots & 1 \end{pmatrix}, \quad i = 1, ..., m-1;$$

$$w_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & t_1 \\ 0 & 1 & \dots & 0 & 0 \\ & & \dots & & \\ & & \dots & & \\ & & \dots & 1 & 0 \\ t_1^{-1} & 0 & \dots & 0 & 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 & 0 & \dots & 0 & t_2 \\ 0 & 1 & \dots & 0 & 0 \\ & & \dots & & \\ & & \dots & & \\ & & \dots & 1 & 0 \\ t_2^{-1} & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The group W has the following properties:

(i) W is generated by the set S of its elements of order two,

(ii) there is an exact sequence

$$0 \to E \to W_{K/K_1/k} \to W_K \to 1,$$

where E is the kernel of the addition map

$$\underbrace{\Gamma_K \oplus \dots \oplus \Gamma_K}_{m \text{ times}} \to \Gamma_K$$

and W_K is isomorphic to the symmetric group S_m;

(iii) the elements s_i, $i = 1, \dots, m-1$ define a splitting of the exact sequence and the subgroup $\langle s_1, \dots, s_{m-1} \rangle$ acts on E by permutations.

In contrast with the situation in the theory of BN-pairs the pair (W, S) is not a Coxeter group and furthermore there is no subset S of involutions in W such that (W, S) is a Coxeter group (see [P2]).

3.2. Bruhat–Tits building for $PGL(2)$ over a local field of dimension $\leqslant 2$

In this subsection we briefly recall the main constructions. For more details see [BT1], [BT2], [P1], [P2].

3.2.1. Let k be a field (which can be viewed as a 0-dimensional local field). Let V be a vector space over k of dimension two.

Definition 2. The *spherical building* of $PGL(2)$ over k is a zero-dimensional complex

$$\Delta(k) = \Delta(PGL(V), k)$$

whose vertices are lines in V.

The group $PGL(2,k)$ acts on $\Delta(k)$ transitively. The Weyl group (in this case it is of order two) acts on $\Delta(k)$ and its orbits are *apartments* of the building.

3.2.2. Let F be a complete discrete valuation field with residue field k. Let V be a vector space over F of dimension two. We say that $L \subset V$ is a lattice if L is an $\mathcal{O}_F$-module. Two submodules L and L' belong to the same class $\langle L\rangle \sim \langle L'\rangle$ if and only if $L = aL'$, with $a \in F^*$.

Definition 3. The *euclidean building* of $PGL(2)$ over F is a one-dimensional complex $\Delta(F/k)$ whose vertices are equivalence classes $\langle L\rangle$ of lattices. Two classes $\langle L\rangle$ and $\langle L'\rangle$ are connected by an edge if and only if for some choice of L, L' there is an exact sequence

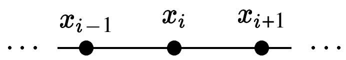
$$0 \to L' \to L \to k \to 0.$$

Denote by $\Delta_i(F/k)$ the set of i-dimensional simplices of the building $\Delta(F/k)$.

The following *link property* is important:

Let $P \in \Delta_0(F/k)$ be represented by a lattice L. Then the link of P (= the set of edges of $\Delta(F/k)$ going from P) is in one-to-one correspondence with the set of lines in the vector space $V_P = L/\mathcal{M}_F L$ (which is $\Delta(PGL(V_P), k)$).

The orbits of the Weyl group W (which is in this case an infinite group with two generators of order two) are infinite sets consisting of $x_i = \langle L_i\rangle$, $L_i = \mathcal{O}_F \oplus \mathcal{M}_F^i$.



An element w of the Weyl group acts in the following way: if $w \in E = \mathbb{Z}$ then w acts by translation of even length; if $w \notin E$ then w acts as an involution with a unique fixed point x_{i_0}: $w(x_{i+i_0}) = x_{i_0-i}$.

To formalize the connection of $\Delta(F/k)$ with $\Delta(F)$ we define a *boundary* point of $\Delta(F/k)$ as a class of half-lines such that the intersection of every two half-lines from the class is a half-line in both of them. The set of the boundary points is called the *boundary* of $\Delta(F/k)$.

There is an isomorphism between $PGL(2,F)$-sets $\Delta(F)$ and the boundary of $\Delta(F/k)$: if a half-line is represented by $L_i = \mathcal{O}_F \oplus \mathcal{M}^i_F,\ i > 0$, then the corresponding vertex of $\Delta(F)$ is the line $F \oplus (0)$ in V.

It seems reasonable to slightly change the notations to make the latter isomorphisms more transparent.

Definition 4 ([P1])**.** Put $\Delta_.[0](F/k) =$ the complex of classes of $\mathcal{O}_F$-submodules in V isomorphic to $F \oplus \mathcal{O}_F$ (so $\Delta_.[0](F/k)$ is isomorphic to $\Delta(F)$) and put

$$\Delta_.[1](F/k) = \Delta(F/k).$$

Define the *building of* $PGL(2)$ *over* F as the union

$$\Delta_.(F/k) = \Delta_.[1](F/k) \bigcup \Delta_.[0](F/k)$$

and call the subcomplex $\Delta_.[0](F/k)$ the *boundary* of the building. The discrete topology on the boundary can be extended to the whole building.

3.2.3. Let K be a two-dimensional local field.

Let V be a vector space over K of dimension two. We say that $L \subset V$ is a lattice if L is an O-module. Two submodules L and L' belong to the same class $\langle L\rangle \sim \langle L'\rangle$ if and only if $L = aL'$, with $a \in K^*$.

Definition 5 ([P1])**.** Define the vertices of the building of $PGL(2)$ over K as

$$\Delta_0[2](K/K_1/k) = \text{classes of } O\text{-submodules } L \subset V\colon\ L \simeq O \oplus O$$
$$\Delta_0[1](K/K_1/k) = \text{classes of } O\text{-submodules } L \subset V\colon\ L \simeq O \oplus \mathcal{O}$$
$$\Delta_0[0](K/K_1/k) = \text{classes of } O\text{-submodules } L \subset V\colon\ L \simeq O \oplus K.$$

Put

$$\Delta_0(K/K_1/k) = \Delta_0[2](K/K_1/k) \bigcup \Delta_0[1](K/K_1/k) \bigcup \Delta_0[0](K/K_1/k).$$

A set of $\{L_\alpha\},\ \alpha \in I$, of O-submodules in V is called a *chain* if

(i) for every $\alpha \in I$ and for every $a \in K^*$ there exists an $\alpha' \in I$ such that $aL_\alpha = L_{\alpha'}$,

(ii) the set $\{L_\alpha, \alpha \in I\}$ is totally ordered by the inclusion.

A chain $\{L_\alpha, \alpha \in I\}$ is called a *maximal chain* if it cannot be included in a strictly larger set satisfying the same conditions (i) and (ii).

We say that $\langle L_0\rangle, \langle L_1\rangle, \ldots, \langle L_m\rangle$ belong to a *simplex* of dimension m if and only if the $L_i,\ i = 0, 1, \ldots, m$ belong to a maximal chain of $\mathcal{O}_F$-submodules in V. The faces and the degeneracies can be defined in a standard way (as a deletion or repetition of a vertex). See [BT2].

Let $\{L_\alpha\}$ be a maximal chain of O-submodules in the space V. There are exactly three types of maximal chains ([P2]):

(i) if the chain contains a module L isomorphic to $O \oplus O$ then all the modules of the chain are of that type and the chain is uniquely determined by its segment

$$\dots \supset O \oplus O \supset M \oplus O \supset M \oplus M \supset \dots .$$

(ii) if the chain contains a module L isomorphic to $O \oplus \mathcal{O}$ then the chain can be restored from the segment:

$$\dots \supset O \oplus \mathcal{O} \supset O \oplus P(1,0) \supset O \oplus P(2,0) \supset \dots \supset O \oplus \mathcal{M} \supset \dots$$

(recall that $P(1,0) = M$).

(iii) if the chain contains a module L isomorphic to $\mathcal{O} \oplus \mathcal{O}$ then the chain can be restored from the segment:

$$\dots \supset \mathcal{O} \oplus \mathcal{O} \supset P(1,0) \oplus \mathcal{O} \supset P(2,0) \oplus \mathcal{O} \supset \dots \supset \mathcal{M} \oplus \mathcal{O} \supset \dots .$$

3.3. Bruhat–Tits building for $PGL(3)$ over a local field F of dimension 1

Let $G = PGL(3)$.

Let F be a one-dimensional local field, $F \supset \mathcal{O}_F \supset \mathcal{M}_F$, $\mathcal{O}_F/\mathcal{M}_F \simeq k$.

Let V be a vector space over F of dimension three. Define lattices in V and their equivalence similarly to the definition of 3.2.2.

First we define the vertices of the building and then the simplices. The result will be a simplicial set $\Delta_\cdot(G, F/k)$.

Definition 6. The *vertices* of the Bruhat–Tits building:

$$\Delta_0[1](G,F/k) = \{\text{classes of } \mathcal{O}_F\text{-submodules } L \subset V : L \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F\},$$

$$\Delta_0[0](G,F/k) = \{\text{classes of } \mathcal{O}_F\text{-submodules } L \subset V : L \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus F \text{ or } L \simeq \mathcal{O}_F \oplus F \oplus F\},$$

$$\Delta_0(G,F/k) = \Delta_0[1](G,F/k) \cup \Delta_0[0](G,F/k).$$

We say that the points of $\Delta_0[1]$ are *inner* points, the points of $\Delta_0[0]$ are *boundary* points. Sometimes we delete G and F/k from the notation if this does not lead to confusion.

We have defined the vertices only. For the simplices of higher dimension we have the following:

Definition 7. Let $\{L_\alpha, \alpha \in I\}$ be a set of $\mathcal{O}_F$-submodules in V. We say that $\{L_\alpha, \alpha \in I\}$ is a *chain* if

(i) for every $\alpha \in I$ and for every $a \in K^*$ there exists an $\alpha' \in I$ such that $aL_\alpha = L_{\alpha'}$,
(ii) the set $\{L_\alpha, \alpha \in I\}$ is totally ordered by the inclusion.

A chain $\{L_\alpha, \alpha \in I\}$ is called a *maximal chain* if it cannot be included in a strictly larger set satisfying the same conditions (i) and (ii).

We say that $\langle L_0\rangle, \langle L_1\rangle, \dots, \langle L_m\rangle$ belong to a *simplex* of dimension m if and only if the L_i, $i = 0, 1, \dots, m$ belong to a maximal chain of $\mathcal{O}_F$-submodules in V. The faces and the degeneracies can be defined in a standard way (as a deletion or repetition of a vertex). See [BT2].

To describe the structure of the building we first need to determine all types of the maximal chains. Proceeding as in [P2] (for $PGL(2)$) we get the following result.

Proposition 1. *There are exactly three types of maximal chains of $\mathcal{O}_F$-submodules in the space V:*

(i) *the chain contains a module isomorphic to $\mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F$. Then all the modules from the chain are of that type and the chain has the following structure:*

$$\dots \supset \mathcal{M}_F^i L \supset \mathcal{M}_F^i L' \supset \mathcal{M}_F^i L'' \supset \mathcal{M}_F^{i+1} L \supset \mathcal{M}_F^{i+1} L' \supset \mathcal{M}_F^{i+1} L'' \supset \dots$$

where $\langle L\rangle, \langle L'\rangle, \langle L''\rangle \in \Delta_0(G, F/k)[1]$ and $L \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F$, $L' \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{M}_F$, $L'' \simeq \mathcal{O}_F \oplus \mathcal{M}_F \oplus \mathcal{M}_F$.

(ii) *the chain contains a module isomorphic to $\mathcal{O}_F \oplus \mathcal{O}_F \oplus F$. Then the chain has the following structure:*

$$\dots \supset \mathcal{M}_F^i L \supset \mathcal{M}_F^i L' \supset \mathcal{M}_F^{i+1} L \supset \dots$$

where $\langle L\rangle, \langle L'\rangle \in \Delta_0(G, F/k)[0]$ and $L \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus F$, $L' \simeq \mathcal{M}_F \oplus \mathcal{O}_F \oplus F$.

(iii) *the chain contains a module isomorphic to $\mathcal{O}_F \oplus F \oplus F$. Then the chain has the following structure:*

$$\dots \supset \mathcal{M}_F^i L \supset \mathcal{M}_F^{i+1} L \supset \dots$$

where $\langle L\rangle \in \Delta_0(G, F/k)[0]$.

We see that the chains of the first type correspond to two-simplices, of the second type — to edges and the last type represent some vertices. It means that the simplicial set $\Delta_.$ is a disconnected union of its subsets $\Delta_.[m]$, $m = 0, 1$. The dimension of the subset $\Delta_.[m]$ is equal to one for $m = 0$ and to two for $m = 1$.

Usually the buildings are defined as combinatorial complexes having a system of subcomplexes called apartments (see, for example, [R], [T1], [T2]). We show how to introduce them for the higher building.

Definition 8. Fix a basis $e_1, e_2, e_3 \in V$. The *apartment* defined by this basis is the following set

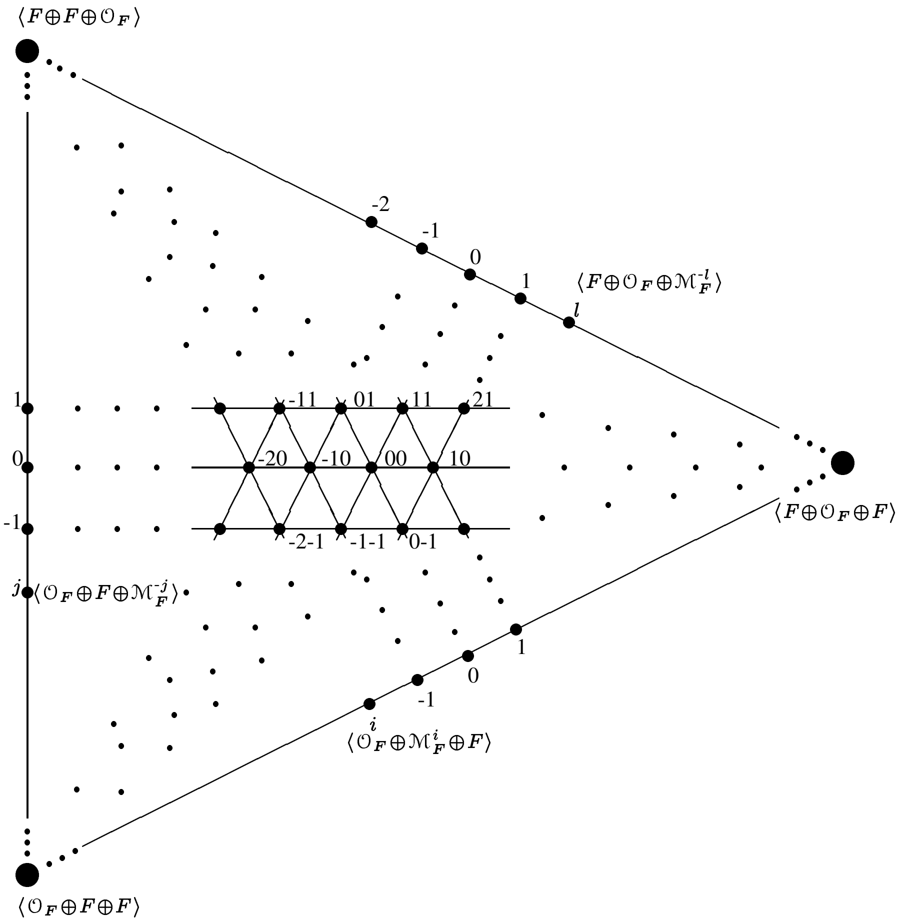
$$\Sigma_. = \Sigma_.[1] \cup \Sigma_.[0],$$

where

$$\Sigma_0[1] = \{\langle L\rangle : L = a_1e_1 \oplus a_2e_2 \oplus a_3e_3, \\ \text{where } a_1, a_2, a_3 \text{ are } \mathcal{O}_F\text{-submodules in } F \text{ isomorphic to } \mathcal{O}_F\}$$

$$\Sigma_0[0] = \{\langle L\rangle : L = a_1e_1 \oplus a_2e_2 \oplus a_3e_3, \\ \text{where } a_1, a_2, a_3 \text{ are } \mathcal{O}_F\text{-submodules in } F \text{ isomorphic either} \\ \text{to } \mathcal{O}_F \text{ or to } F \\ \text{and at least one } a_i \text{ is isomorphic to } F\}.$$

$\Sigma_.[m]$ is the minimal subcomplex having $\Sigma_0[m]$ as vertices.



It can be shown that the building $\Delta_.(G, F/k)$ is glued from the apartments, namely

$$\Delta_.(G, F/k) = \bigsqcup_{\text{all bases of } V} \Sigma_. \,/\, \text{an equivalence relation}$$

(see [T2]).

We can make this description more transparent by drawing all that in the picture above where the dots of different kinds belong to the different parts of the building. In contrast with the case of the group $PGL(2)$ it is not easy to draw the whole building and we restrict ourselves to an apartment.

Here the inner vertices are represented by the lattices

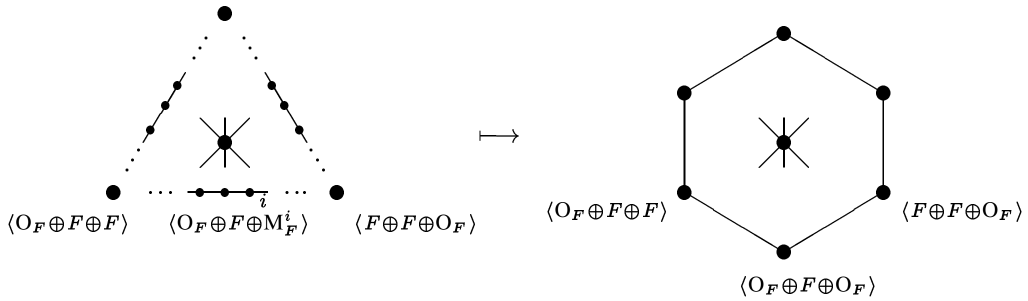
$$ij = \langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus \mathcal{M}_F^j \rangle, \quad i, j \in \mathbb{Z}.$$

The definition of the boundary gives a topology on $\Delta_0(G, F/k)$ which is discrete on both subsets $\Delta_0[1]$ and $\Delta_0[0]$. The convergence of the inner points to the boundary points is given by the following rules:

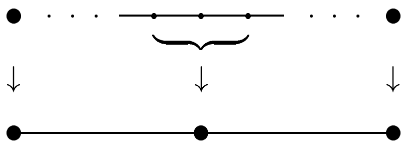
$$\langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus \mathcal{M}_F^j \rangle \xrightarrow{j \to -\infty} \langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus F \rangle,$$
$$\langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus \mathcal{M}_F^j \rangle \xrightarrow{j \to \infty} \langle F \oplus F \oplus \mathcal{O}_F \rangle,$$

because $\langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus \mathcal{M}_F^j \rangle = \langle \mathcal{M}_F^{-j} \oplus \mathcal{M}_F^{-j+i} \oplus \mathcal{O}_F \rangle$. The convergence in the other two directions can be defined along the same line (and it is shown on the picture). It is easy to extend it to the higher simplices.

Thus, there is the structure of a simplicial topological space on the apartment and then we define it on the whole building using the gluing procedure. This topology is stronger than the topology usually introduced to connect the inner part and the boundary together. The connection with standard "compactification" of the building is given by the following map:



This map is bijective on the inner simplices and on a part of the boundary can be described as



We note that the complex is not a CW-complex but only a closure finite complex. This "compactification" was used by G. Mustafin [M].

We have two kinds of connections with the buildings for other fields and groups. First, for the local field F there are two local fields of dimension 0, namely F and k. Then for every $P \in \Delta_0[1](PGL(V), F/k)$ the Link(P) is equal to $\Delta_.(PGL(V_P), k)$ where $V_P = L/\mathcal{M}_F L$ if $P = \langle L \rangle$ and the Link(P) is the boundary of the Star(P). Since the apartments for the $PGL(3,k)$ are hexagons, we can also observe this property on the picture. The analogous relation with the building of $PGL(3,K)$ is more complicated. It is shown on the picture above.

The other relations work if we change the group G but not the field. We see that three different lines go out from every inner point in the apartment. They represent the apartments of the group $PGL(2,F/k)$. They correspond to different embeddings of the $PGL(2)$ into $PGL(3)$.

Also we can describe the action of the Weyl group W on an apartment. If we fix a basis, the extension

$$0 \to \Gamma_F \oplus \Gamma_F \to W \to S_3 \to 1$$

splits. The elements from $S_3 \subset W$ act either as rotations around the point 00 or as reflections. The elements of $\mathbb{Z} \oplus \mathbb{Z} \subset W$ can be represented as triples of integers (according to property (ii) in the previous subsection). Then they correspond to translations of the lattice of inner points along the three directions going from the point 00.

If we fix an embedding $PGL(2) \subset PGL(3)$ then the apartments and the Weyl groups are connected as follows:

$$\Sigma_.(PGL(2)) \subset \Sigma_.(PGL(3)),$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & W' & \longrightarrow & S_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & W & \longrightarrow & S_3 & \longrightarrow & 1
\end{array}$$

where W' is a Weyl group of the group $PGL(2)$ over the field F/k.

3.4. Bruhat–Tits building for $PGL(3)$ over a local field of dimension 2

Let K be a two-dimensional local field. Denote by V a vector space over K of dimension three. Define lattices in V and their equivalence in a similar way to 3.2.3. We shall consider the following *types* of lattices:

$\Delta_0[2]$	222	$\langle O \oplus O \oplus O \rangle$
$\Delta_0[1]$	221	$\langle O \oplus O \oplus \mathcal{O} \rangle$
	211	$\langle O \oplus \mathcal{O} \oplus \mathcal{O} \rangle$
$\Delta_0[0]$	220	$\langle O \oplus O \oplus K \rangle$
	200	$\langle O \oplus K \oplus K \rangle$

To define the buildings we repeat the procedure from the previous subsection.

Definition 10. The *vertices* of the Bruhat–Tits building are the elements of the following set:

$$\Delta_0(G, K/K_1/k) = \Delta_0[2] \cup \Delta_0[1] \cup \Delta_0[0].$$

To define the simplices of higher dimension we can repeat word by word Definitions 7 and 8 of the previous subsection replacing the ring $\mathcal{O}_F$ by the ring O (note that we work only with the types of lattices listed above). We call the subset $\Delta[1]$ the *inner boundary* of the building and the subset $\Delta[0]$ the *external boundary*. The points in $\Delta[2]$ are the *inner* points.

To describe the structure of the building we first need to determine all types of the maximal chains. Proceeding as in [P2] for $PGL(2)$ we get the following result.

Proposition 2. *Let $\{L_\alpha\}$ be a maximal chain of O-submodules in the space V. There are exactly five types of maximal chains:*

(i) *If the chain contains a module L isomorphic to $O \oplus O \oplus O$ then all the modules of the chain are of that type and the chain is uniquely determined by its segment*

$$\dots \supset O \oplus O \oplus O \supset M \oplus O \oplus O \supset M \oplus M \oplus O \supset M \oplus M \oplus M \supset \dots$$

(ii) *If the chain contains a module L isomorphic to $O \oplus O \oplus \mathcal{O}$ then the chain can be restored from the segment:*

$$\dots \supset \text{“}\mathcal{O} \oplus \underbrace{\mathcal{O} \oplus \mathcal{O}\text{”} \supset \dots \supset O \oplus O \oplus \mathcal{O} \supset M \oplus O \oplus \mathcal{O} \supset M \oplus M \oplus \mathcal{O} \supset \dots \supset \mathcal{M} \oplus \mathcal{M}}_{\text{quotient } \simeq K_1 \oplus K_1} \oplus \mathcal{O}$$

$$= \mathcal{M} \oplus \underbrace{\mathcal{M} \oplus \mathcal{O} \supset \dots \supset \mathcal{M} \oplus \mathcal{M} \oplus O \supset \mathcal{M} \oplus \mathcal{M} \oplus M \supset \dots \supset \mathcal{M}}_{\text{quotient } \simeq K_1} \oplus \mathcal{M} \oplus \mathcal{M} \supset \dots$$

Here the modules isomorphic to $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ do not belong to this chain and are inserted as in the proof of Proposition 1 of [P2].

(iii) *All the modules $L_\alpha \simeq O \oplus \mathcal{O} \oplus \mathcal{O}$. Then the chain contains a piece*

$$\dots \supset \text{“}\mathcal{O} \oplus \mathcal{O} \underbrace{\oplus \mathcal{O}\text{”} \supset \dots \supset O \oplus \mathcal{O} \oplus \mathcal{O} \supset M \oplus \mathcal{O} \oplus \mathcal{O} \supset \dots \supset \mathcal{M} \oplus \mathcal{O} \oplus}_{\text{quotient } \simeq K_1} \mathcal{O}$$

$$= \mathcal{M} \oplus \mathcal{O} \underbrace{\oplus \mathcal{O} \supset \dots \supset \mathcal{M} \oplus O \oplus \mathcal{O} \supset \dots \supset \mathcal{M} \oplus \mathcal{M} \oplus}_{\text{quotient } \simeq K_1} \mathcal{O}$$

$$= \mathcal{M} \oplus \mathcal{M} \underbrace{\oplus \mathcal{O} \supset \dots \supset \mathcal{M} \oplus \mathcal{M} \oplus O \supset \dots \supset \mathcal{M} \oplus \mathcal{M} \oplus}_{\text{quotient } \simeq K_1} \mathcal{M} \supset \dots$$

and can also be restored from it. Here the modules isomorphic to $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ do not belong to this chain and are inserted as in the proof of Proposition 1 of [P2].

(iv) *If there is an $L_\alpha \simeq O \oplus O \oplus K$ then one can restore the chain from*

$$\dots \supset O \oplus O \oplus K \supset M \oplus O \oplus K \supset M \oplus M \oplus K \supset \dots$$

(v) *If there is an* $L_\alpha \simeq O \oplus K \oplus K$ *then the chain can be written down as*

$$\dots \supset M^i \oplus K \oplus K \supset M^{i+1} \oplus K \oplus K \supset \dots$$

We see that the chains of the first three types correspond to two-simplices, of the fourth type — to edges of the external boundary and the last type represents a vertex of the external boundary. As above we can glue the building from apartments. To introduce them we can again repeat the corresponding definition for the building over a local field of dimension one (see Definition 4 of the previous subsection). Then the apartment $\Sigma_.$ is a union

$$\Sigma_. = \Sigma_.[2] \cup \Sigma_.[1] \cup \Sigma_.[0]$$

where the pieces $\Sigma_.[i]$ contain the lattices of the types from $\Delta_.[i]$.

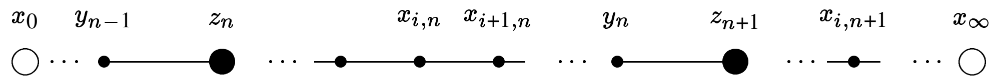
The combinatorial structure of the apartment can be seen from two pictures at the end of the subsection. There we removed the external boundary $\Sigma_.[0]$ which is simplicially isomorphic to the external boundary of an apartment of the building $\Delta_.(PGL(3), K/K_1/k)$. The dots in the first picture show a convergence of the vertices inside the apartment. As a result the building is a simplicial topological space.

We can also describe the relations of the building with buildings of the same group G over the complete discrete valuation fields K and K_1. In the first case there is a projection map

$$\pi: \Delta_.(G, K/K_1/k) \to \Delta_.(G, K/K_1).$$

Under this map the big triangles containing the simplices of type (i) are contracted into points, the triangles containing the simplices of type (ii) go to edges and the simplices of type (iii) are mapped isomorphically to simplices in the target space. The external boundary don't change.

The lines

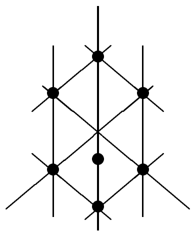


can easily be visualized inside the apartment. Only the big white dots corresponding to the external boundary are missing. We have three types of lines going from the inner points under the angle $2\pi/3$. They correspond to different embeddings of $PGL(2)$ into $PGL(3)$.

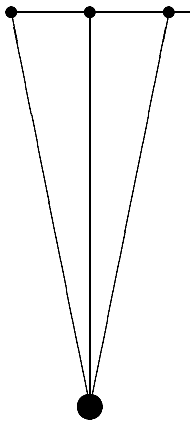
Using the lines we can understand the action of the Weyl group W on an apartment. The subgroup S_3 acts in the same way as in 3.2. The free subgroup E (see 3.1) has six types of translations along these three directions. Along each line we have two opportunities which were introduced for $PGL(2)$.

Namely, if $w \in \Gamma_K \simeq \mathbb{Z} \oplus \mathbb{Z} \subset W$ then $w = (0,1)$ acts as a shift of the whole structure to the right: $w(x_{i,n}) = x_{i,n+2}$, $w(y_n) = y_{n+2}$, $w(z_n) = z_{n+2}, w(x_0) = x_0$, $w(x_\infty) = x_\infty$.

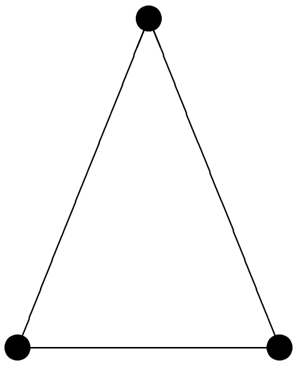
The element $w = (1,0)$ acts as a shift on the points $x_{i,n}$ but leaves fixed the points in the inner boundary $w(x_{i,n}) = x_{i+2,n}$, $w(y_n) = y_n$, $w(z_n) = z_n$, $w(x_0) = x_0$, $w(x_\infty) = x_\infty$, (see [P2, Theorem 5, v]).



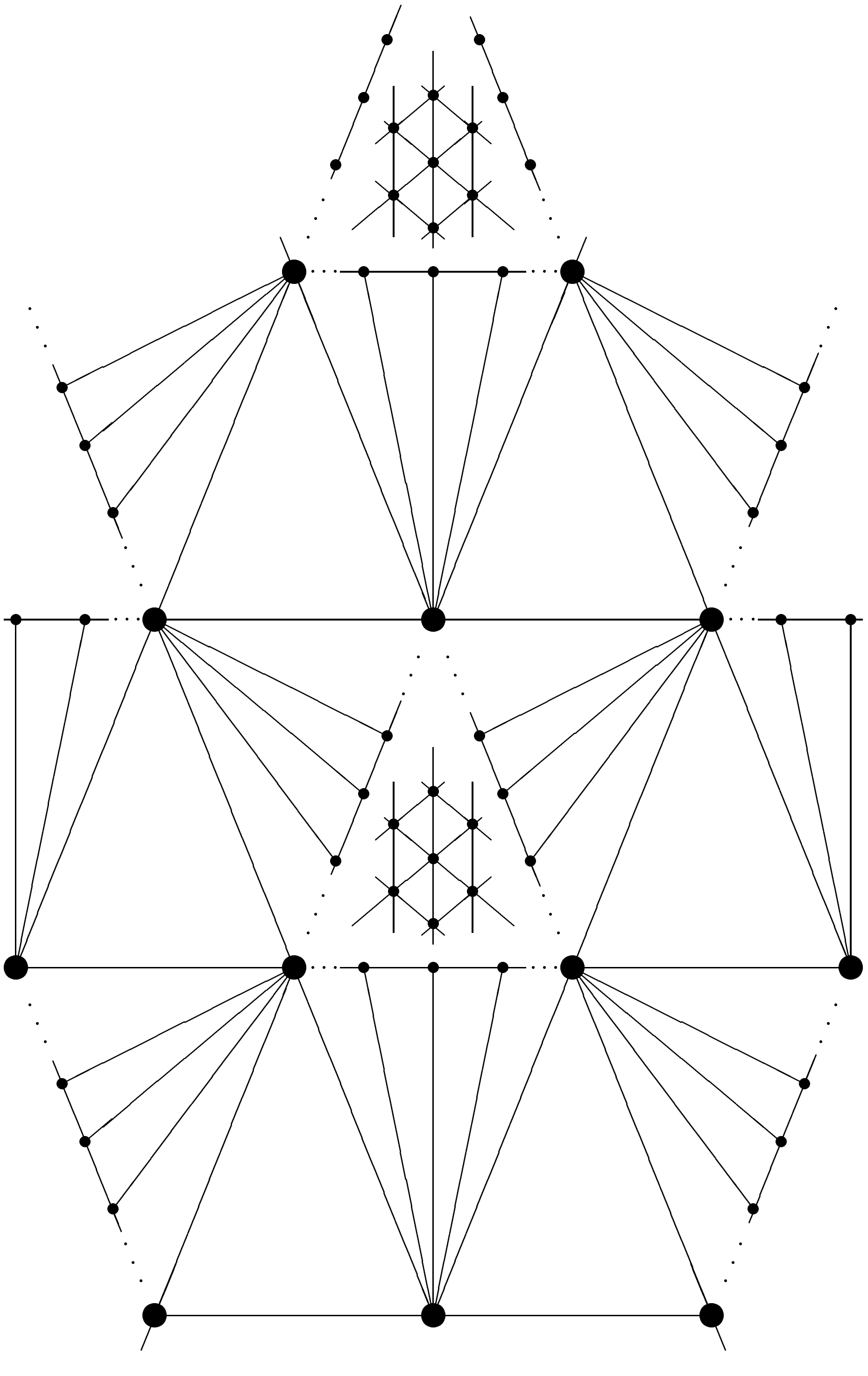
simplices of type (i)



simplices of type (ii)



simplices of type (iii)



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4. Drinfeld modules and local fields of positive characteristic

Ernst–Ulrich Gekeler

The relationship between local fields and Drinfeld modules is twofold. Drinfeld modules allow explicit construction of abelian and nonabelian extensions with prescribed properties of local and global fields of positive characteristic. On the other hand, n-dimensional local fields arise in the construction of (the compactification of) moduli schemes X for Drinfeld modules, such schemes being provided with a natural stratification $X_0 \subset X_1 \subset \cdots X_i \cdots \subset X_n = X$ through smooth subvarieties X_i of dimension i.

We will survey that correspondence, but refer to the literature for detailed proofs (provided these exist so far). An important remark is in order: The contents of this article take place in characteristic $p > 0$, and are in fact locked up in the characteristic p world. No lift to characteristic zero nor even to schemes over $\mathbb{Z}/p^2$ is known!

4.1. Drinfeld modules

Let L be a field of characteristic p containing the field $\mathbb{F}_q$, and denote by $\tau = \tau_q$ raising to the qth power map $x \mapsto x^q$. If "a" denotes multiplication by $a \in L$, then $\tau a = a^q \tau$. The ring $\mathrm{End}(\mathbb{G}_{a\,/L})$ of endomorphisms of the additive group $\mathbb{G}_{a\,/L}$ equals $L\{\tau_p\} = \{\sum a_i \tau_p^i : a_i \in L\}$, the non-commutative polynomial ring in $\tau_p = (x \mapsto x^p)$ with the above commutation rule $\tau_p a = a^p \tau$. Similarly, the subring $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a\,/L})$ of $\mathbb{F}_q$-endomorphisms is $L\{\tau\}$ with $\tau = \tau_p^n$ if $q = p^n$. Note that $L\{\tau\}$ is an $\mathbb{F}_q$-algebra since $\mathbb{F}_q \hookrightarrow L\{\tau\}$ is central.

Definition 1. Let $\mathcal{C}$ be a smooth geometrically connected projective curve over $\mathbb{F}_q$. Fix a closed (but not necessarily $\mathbb{F}_q$-rational) point ∞ of $\mathcal{C}$. The ring $A = \Gamma(\mathcal{C} - \{\infty\}, \mathcal{O}_{\mathcal{C}})$ is called a *Drinfeld ring*. Note that $A^* = \mathbb{F}_q^*$.

Example 1. If $\mathcal{C}$ is the projective line $\mathbb{P}^1{}_{/\mathbb{F}_q}$ and ∞ is the usual point at infinity then $A = \mathbb{F}_q[T]$.

Example 2. Suppose that $p \neq 2$, that $\mathcal{C}$ is given by an affine equation $Y^2 = f(X)$ with a separable polynomial $f(X)$ of even positive degree with leading coefficient a non-square in $\mathbb{F}_q$, and that ∞ is the point above $X = \infty$. Then $A = \mathbb{F}_q[X, Y]$ is a Drinfeld ring with $\deg_{\mathbb{F}_q}(\infty) = 2$.

Definition 2. An A-*structure* on a field L is a homomorphism of $\mathbb{F}_q$-algebras (in brief: an $\mathbb{F}_q$-ring homomorphism) $\gamma\colon A \to L$. Its A-*characteristic* $\mathrm{char}_A(L)$ is the maximal ideal $\ker(\gamma)$, if γ fails to be injective, and ∞ otherwise. A *Drinfeld module structure* on such a field L is given by an $\mathbb{F}_q$-ring homomorphism $\phi\colon A \to L\{\tau\}$ such that $\partial \circ \phi = \gamma$, where $\partial\colon L\{\tau\} \to L$ is the L-homomorphism sending τ to 0.

Denote $\phi(a)$ by $\phi_a \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a\,/L})$; ϕ_a induces on the additive group over L (and on each L-algebra M) a *new* structure as an A-module:

$$a * x := \phi_a(x) \quad (a \in A, x \in M). \tag{4.1.1}$$

We briefly call ϕ a Drinfeld module over L, usually omitting reference to A.

Definition 3. Let ϕ and ψ be Drinfeld modules over the A-field L. A *homomorphism* $u\colon \phi \to \psi$ is an element of $L\{\tau\}$ such that $u \circ \phi_a = \psi_a \circ u$ for all $a \in A$. Hence an *endomorphism* of ϕ is an element of the centralizer of $\phi(A)$ in $L\{\tau\}$, and u is an *isomorphism* if $u \in L^* \hookrightarrow L\{\tau\}$ is subject to $u \circ \phi_a = \psi_a \circ u$.

Define $\deg\colon a \to \mathbb{Z} \cup \{-\infty\}$ and $\deg_\tau\colon L\{\tau\} \to \mathbb{Z} \cup \{-\infty\}$ by $\deg(a) = \log_q |A/a|$ ($a \neq 0$; we write A/a for A/aA), $\deg(0) = -\infty$, and $\deg_\tau(f) =$ the well defined degree of f as a "polynomial" in τ. It is an easy exercise in Dedekind rings to prove the following

Proposition 1. *If ϕ is a Drinfeld module over L, there exists a non-negative integer r such that* $\deg_\tau(\phi_a) = r \deg(a)$ *for all* $a \in A$; r is called the *rank* $\mathrm{rk}(\phi)$ of ϕ.

Obviously, $\mathrm{rk}(\phi) = 0$ means that $\phi = \gamma$, i.e., the A-module structure on $\mathbb{G}_{a\,/L}$ is the tautological one.

Definition 4. Denote by $\mathcal{M}^r(1)(L)$ the set of isomorphism classes of Drinfeld modules of rank r over L.

Example 3. Let $A = \mathbb{F}_q[T]$ be as in Example 1 and let $K = \mathbb{F}_q(T)$ be its fraction field. Defining a Drinfeld module ϕ over K or an extension field L of K is equivalent to specifying $\phi_T = T + g_1\tau + \dots + g_r\tau^r \in L\{T\}$, where $g_r \neq 0$ and $r = \mathrm{rk}(\phi)$. In the special case where $\phi_T = T + \tau$, ϕ is called the *Carlitz module*. Two such Drinfeld modules ϕ and ϕ' are isomorphic over the algebraic closure L^{alg} of L if and only if there is some $u \in L^{\mathrm{alg}\,*}$ such that $g_i' = u^{q^i-1} g_i$ for all $i \geqslant 1$. Hence $\mathcal{M}^r(1)(L^{\mathrm{alg}})$ can be described (for $r \geqslant 1$) as an open dense subvariety of a weighted projective space of dimension $r-1$ over L^{alg}.

4.2. Division points

Definition 5. For $a \in A$ and a Drinfeld module ϕ over L, write ${}_a\phi$ for the subscheme of *a-division points* of $\mathbb{G}_{a\,/L}$ endowed with its structure of an A-module. Thus for any L-algebra M,

$$ {}_a\phi(M) = \{x \in M \colon \phi_a(x) = 0\}. $$

More generally, we put ${}_{\mathfrak{a}}\phi = \bigcap_{a\in\mathfrak{a}} \phi_a$ for an arbitrary (not necessarily principal) ideal $\mathfrak{a}$ of A. It is a finite flat group scheme of degree $\mathrm{rk}(\phi)\cdot\deg(\mathfrak{a})$, whose structure is described in the next result.

Proposition 2 ([Dr], [DH, I, Thm. 3.3 and Remark 3.4])**.** *Let the Drinfeld module ϕ over L have rank $r \geqslant 1$.*

(i) *If* $\mathrm{char}_A(L) = \infty$, *${}_{\mathfrak{a}}\phi$ is reduced for each ideal $\mathfrak{a}$ of A, and ${}_{\mathfrak{a}}\phi(L^{\mathrm{sep}}) = {}_{\mathfrak{a}}\phi(L^{\mathrm{alg}})$ is isomorphic with $(A/\mathfrak{a})^r$ as an A-module.*

(ii) *If $\mathfrak{p} = \mathrm{char}_A(L)$ is a maximal ideal, then there exists an integer h, the height* $\mathrm{ht}(\phi)$ *of ϕ, satisfying $1 \leqslant h \leqslant r$, and such that ${}_{\mathfrak{a}}\phi(L^{\mathrm{alg}}) \simeq (A/\mathfrak{a})^{r-h}$ whenever $\mathfrak{a}$ is a power of $\mathfrak{p}$, and ${}_{\mathfrak{a}}\phi(L^{\mathrm{alg}}) \simeq (A/\mathfrak{a})^r$ if $(\mathfrak{a},\mathfrak{p}) = 1$.*

The absolute Galois group G_L of L acts on ${}_{\mathfrak{a}}\phi(L^{\mathrm{sep}})$ through A-linear automorphisms. Therefore, any Drinfeld module gives rise to Galois representations on its division points. These representations tend to be “as large as possible”.

The prototype of result is the following theorem, due to Carlitz and Hayes [H1].

Theorem 1. *Let A be the polynomial ring $\mathbb{F}_q[T]$ with field of fractions K. Let $\rho\colon A \to K\{\tau\}$ be the Carlitz module, $\rho_T = T + \tau$. For any non-constant monic polynomial $a \in A$, let $K(a) := K({}_a\rho(K^{\mathrm{alg}}))$ be the field extension generated by the a-division points.*

(i) *$K(a)/K$ is abelian with group $(A/a)^*$. If σ_b is the automorphism corresponding to the residue class of b* mod *a and $x \in {}_a\rho(K^{\mathrm{alg}})$ then $\sigma_b(x) = \rho_b(x)$.*

(ii) *If $(a) = \mathfrak{p}^t$ is primary with some prime ideal $\mathfrak{p}$ then $K(a)/K$ is completely ramified at $\mathfrak{p}$ and unramified at the other finite primes.*

(iii) *If $(a) = \prod a_i$ ($1 \leqslant i \leqslant s$) with primary and mutually coprime a_i, the fields $K(a_i)$ are mutually linearly disjoint and $K = \otimes_{i\leqslant i\leqslant s} K(a_i)$.*

(iv) *Let $K_+(a)$ be the fixed field of $\mathbb{F}_q^* \hookrightarrow (A/a)^*$. Then ∞ is completely split in $K_+(a)/K$ and completely ramified in $K(a)/K_+(a)$.*

(v) *Let $\mathfrak{p}$ be a prime ideal generated by the monic polynomial $\pi \in A$ and coprime with a. Under the identification* $\mathrm{Gal}(K(a)/K) = (A/a)^*$, *the Frobenius element* $\mathrm{Frob}_{\mathfrak{p}}$ *equals the residue class of π* mod *a.*

Letting $a \to \infty$ with respect to divisibility, we obtain the field $K(\infty)$ generated over K by all the division points of ρ, with group $\mathrm{Gal}(K(\infty)/K) = \varinjlim_a (A/a)^*$, which almost agrees with the group of finite idele classes of K. It turns out that $K(\infty)$ is the maximal abelian extension of K that is tamely ramified at ∞, i.e., we get a constructive version of the class field theory of K. Hence the theorem may be seen both as a global variant of Lubin–Tate's theory and as an analogue in characteristic p of the Kronecker–Weber theorem on cyclotomic extensions of $\mathbb{Q}$.

There are vast generalizations into two directions:

(a) abelian class field theory of arbitrary global function fields $K = \mathrm{Frac}(A)$, where A is a Drinfeld ring.

(b) systems of nonabelian Galois representations derived from Drinfeld modules.

As to (a), the first problem is to find the proper analogue of the Carlitz module for an arbitrary Drinfeld ring A. As will result e.g. from Theorem 2 (see also (4.3.4)), the isomorphism classes of rank-one Drinfeld modules over the algebraic closure K^{alg} of K correspond bijectively to the (finite!) class group $\mathrm{Pic}(A)$ of A. Moreover, these Drinfeld modules $\rho^{(\mathfrak{a})}$ ($\mathfrak{a} \in \mathrm{Pic}(A)$) may be defined with coefficients in the ring $\mathcal{O}_{H_+}$ of A-integers of a certain abelian extension H_+ of K, and such that the leading coefficients of all $\rho_a^{(\mathfrak{a})}$ are units of $\mathcal{O}_{H_+}$. Using these data along with the identification of H_+ in the dictionary of class field theory yields a generalization of Theorem 1 to the case of arbitrary A. In particular, we again find an explicit construction of the class fields of K (subject to a tameness condition at ∞). However, in view of class number problems, the theory (due to D. Hayes [H2], and superbly presented in [Go2, Ch.VII]) has more of the flavour of complex multiplication theory than of classical cyclotomic theory.

Generalization (b) is as follows. Suppose that L is a finite extension of $K = \mathrm{Frac}(A)$, where A is a general Drinfeld ring, and let the Drinfeld module ϕ over L have rank r. For each power $\mathfrak{p}^t$ of a prime $\mathfrak{p}$ of A, $G_L = \mathrm{Gal}(L^{\mathrm{sep}}/L)$ acts on ${}_{\mathfrak{p}^t}\phi \simeq (A/\mathfrak{p}^t)^r$. We thus get an action of G_L on the $\mathfrak{p}$-adic *Tate module* $T_{\mathfrak{p}}(\phi) \simeq (A_{\mathfrak{p}})^r$ of ϕ (see [DH, I sect. 4]. Here of course $A_{\mathfrak{p}} = \varprojlim A/\mathfrak{p}^t$ is the $\mathfrak{p}$-adic completion of A with field of fractions $K_{\mathfrak{p}}$. Let on the other hand $\mathrm{End}(\phi)$ be the endomorphism ring of ϕ, which also acts on $T_{\mathfrak{p}}(\phi)$. It is straightforward to show that (i) $\mathrm{End}(\phi)$ acts faithfully and (ii) the two actions commute. In other words, we get an inclusion

$$i\colon \mathrm{End}(\phi) \otimes_A A_{\mathfrak{p}} \hookrightarrow \mathrm{End}_{G_L}(T_{\mathfrak{p}}(\phi)) \tag{4.2.1}$$

of finitely generated free $A_{\mathfrak{p}}$-modules. The plain analogue of the classical Tate conjecture for abelian varieties, proved 1983 by Faltings, suggests that i is in fact bijective. This has been shown by Taguchi [Tag] and Tamagawa. Taking $\mathrm{End}(T_{\mathfrak{p}}(\phi)) \simeq \mathrm{Mat}(r, A_{\mathfrak{p}})$ and the known structure of subalgebras of matrix algebras over a field into account, this means that the subalgebra

$$K_{\mathfrak{p}}[G_L] \hookrightarrow \mathrm{End}(T_{\mathfrak{p}}(\phi) \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}}) \simeq \mathrm{Mat}(r, K_{\mathfrak{p}})$$

generated by the Galois operators is as large as possible. A much stronger statement is obtained by R. Pink [P1, Thm. 0.2], who shows that the image of G_L in $\mathrm{Aut}(T_\mathfrak{p}(\phi))$ has finite index in the centralizer group of $\mathrm{End}(\phi) \otimes A_\mathfrak{p}$. Hence if e.g. ϕ has no "complex multiplications" over L^{alg} (i.e., $\mathrm{End}_{L^{\mathrm{alg}}}(\phi) = A$; this is the generic case for a Drinfeld module in characteristic ∞), then the image of G_L has finite index in $\mathrm{Aut}(T_\mathfrak{p}(\phi)) \simeq GL(r, A_\mathfrak{p})$. This is quite satisfactory, on the one hand, since we may use the Drinfeld module ϕ to construct large nonabelian Galois extensions of L with prescribed ramification properties. On the other hand, the important (and difficult) problem of estimating the index in question remains.

4.3. Weierstrass theory

Let A be a Drinfeld ring with field of fractions K, whose completion at ∞ is denoted by K_∞. We normalize the corresponding absolute value $|\ | = |\ |_\infty$ as $|a| = |A/a|$ for $0 \neq a \in A$ and let C_∞ be the completed algebraic closure of K_∞, i.e., the completion of the algebraic closure K_∞^{alg} with respect to the unique extension of $|\ |$ to K_∞^{alg}. By Krasner's theorem, C_∞ is again algebraically closed ([BGS, p. 146], where also other facts on function theory in C_∞ may be found). It is customary to indicate the strong analogies between $A, K, K_\infty, C_\infty, \ldots$ and $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$, e.g. A is a discrete and cocompact subring of K_∞. But note that C_∞ fails to be locally compact since $|C_\infty : K_\infty| = \infty$.

Definition 6. A *lattice of rank* r (an r-lattice in brief) in C_∞ is a finitely generated (hence projective) discrete A-submodule Λ of C_∞ of projective rank r, where the discreteness means that Λ has finite intersection with each ball in C_∞. The *lattice function* $e_\Lambda : C_\infty \to C_\infty$ of Λ is defined as the product

$$e_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} (1 - z/\lambda). \tag{4.3.1}$$

It is entire (defined through an everywhere convergent power series), Λ-periodic and $\mathbb{F}_q$-linear. For a non-zero $a \in A$ consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & C_\infty & \xrightarrow{e_\Lambda} & C_\infty & \longrightarrow & 0 \\ & & {\scriptstyle a}\big\downarrow & & {\scriptstyle a}\big\downarrow & & {\scriptstyle \phi_a^\Lambda}\big\downarrow & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & C_\infty & \xrightarrow{e_\Lambda} & C_\infty & \longrightarrow & 0 \end{array} \tag{4.3.2}$$

with exact lines, where the left and middle arrows are multiplications by a and ϕ_a^Λ is defined through commutativity. It is easy to verify that

(i) $\phi_a^\Lambda \in C_\infty\{\tau\}$,

(ii) $\deg_\tau(\phi_a^\Lambda) = r \cdot \deg(a)$,

(iii) $a \mapsto \phi_a^\Lambda$ is a ring homomorphism $\phi^\Lambda\colon A \to C_\infty\{\tau\}$, in fact, a Drinfeld module of rank r. Moreover, all the Drinfeld modules over C_∞ are so obtained.

Theorem 2 (Drinfeld [Dr, Prop. 3.1])**.**

(i) *Each rank-* r *Drinfeld module* ϕ *over* C_∞ *comes via* $\Lambda \mapsto \phi^\Lambda$ *from some* r*-lattice* Λ *in* C_∞.

(ii) *Two Drinfeld modules* ϕ^Λ, $\phi^{\Lambda'}$ *are isomorphic if and only if there exists* $0 \neq c \in C_\infty$ *such that* $\Lambda' = c \cdot \Lambda$.

We may thus describe $\mathcal{M}^r(1)(C_\infty)$ (see Definition 4) as the space of r-lattices modulo similarities, i.e., as some generalized upper half-plane modulo the action of an arithmetic group. Let us make this more precise.

Definition 7. For $r \geqslant 1$ let $\mathbb{P}^{r-1}(C_\infty)$ be the C_∞-points of projective $r-1$-space and $\Omega^r := \mathbb{P}^{r-1}(C_\infty) - \bigcup H(C_\infty)$, where H runs through the K_∞-rational hyperplanes of $\mathbb{P}^{r-1}$. That is, $\underline{\omega} = (\omega_1 : \ldots : \omega_r)$ belongs to *Drinfeld's half-plane* Ω^r if and only if there is no non-trivial relation $\sum a_i\omega_i = 0$ with coefficients $a_i \in K_\infty$.

Both point sets $\mathbb{P}^{r-1}(C_\infty)$ and Ω^r carry structures of analytic spaces over C_∞ (even over K_∞), and so we can speak of holomorphic functions on Ω^r. We will not give the details (see for example [GPRV, in particular lecture 6]); suffice it to say that locally uniform limits of rational functions (e.g. Eisenstein series, see below) will be holomorphic.

Suppose for the moment that the class number $h(A) = |\mathrm{Pic}(A)|$ of A equals one, i.e., A is a principal ideal domain. Then each r-lattice Λ in C_∞ is free, $\Lambda = \sum_{1\leqslant i\leqslant r} A\omega_i$, and the discreteness of Λ is equivalent with $\underline{\omega} := (\omega_1 : \ldots : \omega_r)$ belonging to $\Omega^r \hookrightarrow \mathbb{P}^{r-1}(C_\infty)$. Further, two points $\underline{\omega}$ and $\underline{\omega}'$ describe similar lattices (and therefore isomorphic Drinfeld modules) if and only if they are conjugate under $\Gamma := GL(r, A)$, which acts on $\mathbb{P}^{r-1}(C_\infty)$ and its subspace Ω^r. Therefore, we get a canonical bijection

(4.3.3) $$\Gamma \setminus \Omega^r \xrightarrow{\sim} \mathcal{M}^r(1)(C_\infty)$$

from the quotient space $\Gamma \setminus \Omega^r$ to the set of isomorphism classes $\mathcal{M}^r(1)(C_\infty)$.

In the general case of arbitrary $h(A) \in \mathbb{N}$, we let $\Gamma_i := GL(Y_i) \hookrightarrow GL(r, k)$, where $Y_i \hookrightarrow K^r$ ($1 \leqslant i \leqslant h(A)$) runs through representatives of the $h(A)$ isomorphism classes of projective A-modules of rank r. In a similar fashion (see e.g. [G1, II sect.1], [G3]), we get a bijection

(4.3.4) $$\dot{\bigcup}_{1\leqslant i\leqslant h(A)} \Gamma_i \setminus \Omega^r \xrightarrow{\sim} \mathcal{M}^r(1)(C_\infty),$$

which can be made independent of choices if we use the canonical adelic description of the Y_i.

Example 4. If $r = 2$ then $\Omega = \Omega^2 = \mathbb{P}^1(C_\infty) - -\mathbb{P}^1(K_\infty) = C_\infty - K_\infty$, which rather corresponds to $\mathbb{C} - \mathbb{R} = H^+ \bigcup H^-$ (upper and lower complex half-planes) than to H^+ alone. The group $\Gamma := GL(2, A)$ acts via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = \frac{az+b}{cz+d}$, and thus gives rise to *Drinfeld modular forms* on Ω (see [G1]). Suppose moreover that $A = \mathbb{F}_q[T]$ as in Examples 1 and 3. We define *ad hoc* a modular form of weight k for Γ as a holomorphic function $f: \Omega \to C_\infty$ that satisfies

(i) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and

(ii) $f(z)$ is bounded on the subspace $\{z \in \Omega : \inf_{x \in K_\infty} |z - x| > 1\}$ of Ω.

Further, we put M_k for the C_∞-vector space of modular forms of weight k. (In the special case under consideration, (ii) is equivalent to the usual "holomorphy at cusps" condition. For more general groups Γ, e.g. congruence subgroups of $GL(2, A)$, general Drinfeld rings A, and higher ranks $r \geqslant 2$, condition (ii) is considerably more costly to state, see [G1].) Let

$$E_k(z) := \sum_{(0,0) \neq (a,b) \in A \times A} \frac{1}{(az+b)^k} \tag{4.3.5}$$

be the *Eisenstein series* of weight k. Due to the non-archimedean situation, the sum converges for $k \geqslant 1$ and yields a modular form $0 \neq E_k \in M_k$ if $k \equiv 0\ (q-1)$. Moreover, the various M_k are linearly independent and

$$M(\Gamma) := \bigoplus_{k \geqslant 0} M_k = C_\infty[E_{q-1}, E_{q^2-1}] \tag{4.3.6}$$

is a polynomial ring in the two algebraically independent Eisenstein series of weights $q-1$ and q^2-1. There is an *a priori* different method of constructing modular forms via Drinfeld modules. With each $z \in \Omega$, associate the 2-lattice $\Lambda_z := Az + A \hookrightarrow C_\infty$ and the Drinfeld module $\phi^{(z)} = \phi^{(\Lambda_z)}$. Writing $\phi_T^{(z)} = T + g(z)\tau + \Delta(z)\tau^2$, the coefficients g and Δ become functions in z, in fact, modular forms of respective weights $q-1$ and q^2-1. We have ([Go1], [G1, II 2.10])

$$g = (T^g - T)E_{q-1}, : \Delta = (T^{q^2} - T)E_{q^2-1} + (T^{q^2} - T^q)E_{q-1}^{q+1}. \tag{4.3.7}$$

The crucial fact is that $\Delta(z) \neq 0$ for $z \in \Omega$, but Δ vanishes "at infinity". Letting $j(z) := g(z)^{q+1}/\Delta(z)$ (which is a function on Ω invariant under Γ), the considerations of Example 3 show that j is a complete invariant for Drinfeld modules of rank two. Therefore, the composite map

$$j: \Gamma \setminus \Omega \xrightarrow{\sim} \mathcal{M}^2(1)(C_\infty) \xrightarrow{\sim} C_\infty \tag{4.3.8}$$

is bijective, in fact, biholomorphic.

4.4. Moduli schemes

We want to give a similar description of $\mathcal{M}^r(1)(C_\infty)$ for $r \geqslant 2$ and arbitrary A, that is, to convert (4.3.4) into an isomorphism of analytic spaces. One proceeds as follows (see [Dr], [DH], [G3]):
(a) Generalize the notion of "Drinfeld A-module over an A-field L" to "Drinfeld A-module over an A-scheme $S \to \operatorname{Spec} A$". This is quite straightforward. Intuitively, a Drinfeld module over S is a continuously varying family of Drinfeld modules over the residue fields of S.
(b) Consider the functor on A-schemes:

$$\mathcal{M}^r : S \longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of rank-}r \\ \text{Drinfeld modules over } S \end{array} \right\}.$$

The naive initial question is to represent this functor by an S-scheme $M^r(1)$. This is impossible in view of the existence of automorphisms of Drinfeld modules even over algebraically closed A-fields.
(c) As a remedy, introduce rigidifying level structures on Drinfeld modules. Fix some ideal $0 \neq \mathfrak{n}$ of A. An $\mathfrak{n}$-*level structure* on the Drinfeld module ϕ over the A-field L whose A-characteristic doesn't divide $\mathfrak{n}$ is the choice of an isomorphism of A-modules

$$\alpha : (A/\mathfrak{n})^r \xrightarrow{\sim} {}_{\mathfrak{n}}\phi(L)$$

(compare Proposition 2). Appropriate modifications apply to the cases where $\operatorname{char}_A(L)$ divides $\mathfrak{n}$ and where the definition field L is replaced by an A-scheme S. Let $\mathcal{M}^r(\mathfrak{n})$ be the functor

$$\mathcal{M}^r(\mathfrak{n}) : S \longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of rank-}r \\ \text{Drinfeld modules over } S \text{ endowed} \\ \text{with an } \mathfrak{n}\text{-level structure} \end{array} \right\}.$$

Theorem 3 (Drinfeld [Dr, Cor. to Prop. 5.4])**.** *Suppose that $\mathfrak{n}$ is divisible by at least two different prime ideals. Then $\mathcal{M}^r(\mathfrak{n})$ is representable by a smooth affine A-scheme $M^r(\mathfrak{n})$ of relative dimension $r-1$.*

In other words, the scheme $M^r(\mathfrak{n})$ carries a "tautological" Drinfeld module ϕ of rank r endowed with a level-$\mathfrak{n}$ structure such that pull-back induces for each A-scheme S a bijection

(4.4.1) $$M^r(\mathfrak{n})(S) = \{\text{morphisms } (S, M^r(\mathfrak{n}))\} \xrightarrow{\sim} \mathcal{M}^r(\mathfrak{n})(S), \quad f \longmapsto f^*(\phi).$$

$M^r(\mathfrak{n})$ is called the (fine) *moduli scheme* for the moduli problem $\mathcal{M}^r(\mathfrak{n})$. Now the finite group $G(\mathfrak{n}) := GL(r, A/\mathfrak{n})$ acts on $\mathcal{M}^r(\mathfrak{n})$ by permutations of the level structures. By functoriality, it also acts on $M^r(\mathfrak{n})$. We let $M^r(1)$ be the quotient of $M^r(\mathfrak{n})$ by $G(\mathfrak{n})$ (which does not depend on the choice of $\mathfrak{n}$). It has the property that at least its L-valued points for algebraically closed A-fields L correspond bijectively and functorially to $\mathcal{M}^r(1)(L)$. It is therefore called a *coarse moduli scheme* for $\mathcal{M}^r(1)$. Combining the above with (4.3.4) yields a bijection

$$\dot{\bigcup}_{1 \leqslant i \leqslant h(A)} \Gamma_i \setminus \Omega^r \xrightarrow{\sim} M^r(1)(C_\infty), \tag{4.4.2}$$

which even is an isomorphism of the underlying analytic spaces [Dr, Prop. 6.6]. The most simple special case is the one dealt with in Example 4, where $M^2(1) = \mathbb{A}^1{}_{/A}$, the affine line over A.

4.5. Compactification

It is a fundamental question to construct and study a "compactification" of the affine A-scheme $M^r(\mathfrak{n})$, relevant for example for the Langlands conjectures over K, the cohomology of arithmetic subgroups of $GL(r, A)$, or the K-theory of A and K. This means that we are seeking a proper A-scheme $\overline{M}^r(\mathfrak{n})$ with an A-embedding $M^r(\mathfrak{n}) \hookrightarrow \overline{M}^r(\mathfrak{n})$ as an open dense subscheme, and which behaves functorially with respect to the forgetful morphisms $M^r(\mathfrak{n}) \to M^r(\mathfrak{m})$ if $\mathfrak{m}$ is a divisor of $\mathfrak{n}$. For many purposes it suffices to solve the apparently easier problem of constructing similar compactifications of the generic fiber $M^r(\mathfrak{n}) \times_A K$ or even of $M^r(\mathfrak{n}) \times_A C_\infty$. Note that varieties over C_∞ may be studied by analytic means, using the GAGA principle.

There are presently three approaches towards the problem of compactification:
(a) a (sketchy) construction of the present author [G2] of a compactification $\overline{M}_\Gamma$ of M_Γ, the C_∞-variety corresponding to an arithmetic subgroup Γ of $GL(r, A)$ (see (4.3.4) and (4.4.2)). We will return to this below;
(b) an analytic compactification similar to (a), restricted to the case of a polynomial ring $A = \mathbb{F}_q[T]$, but with the advantage of presenting complete proofs, by M. M. Kapranov [K];
(c) R. Pink's idea of a modular compactification of $M^r(\mathfrak{n})$ over A through a generalization of the underlying moduli problem [P2].

Approaches (a) and (b) agree essentially in their common domain, up to notation and some other choices. Let us briefly describe how one proceeds in (a). Since there is nothing to show for $r = 1$, we suppose that $r \geqslant 2$.

We let A be any Drinfeld ring. If Γ is a subgroup of $GL(r, K)$ commensurable with $GL(r, A)$ (we call such Γ *arithmetic subgroups*), the point set $\Gamma \setminus \Omega$ is the set of C_∞-points of an affine variety M_Γ over C_∞, as results from the discussion of subsection 4.4. If Γ is the congruence subgroup $\Gamma(\mathfrak{n}) = \{\gamma \in GL(r, A) : \gamma \equiv 1 \bmod \mathfrak{n}\}$, then M_Γ is one of the irreducible components of $M^r(\mathfrak{n}) \times_A C_\infty$.

Definition 8. For $\underline{\omega} = (\omega_1, : \ldots : \omega_r) \in \mathbb{P}^{r-1}(C_\infty)$ put

$$r(\underline{\omega}) := \dim_K(K\omega_1 + \cdots + K\omega_r) \quad \text{and} \quad r_\infty(\underline{\omega}) := \dim_{K_\infty}(K_\infty\omega_1 + \cdots + K_\infty\omega_r).$$

Then $1 \leqslant r_\infty(\underline{\omega}) \leqslant r(\underline{\omega}) \leqslant r$ and $\Omega^r = \{\underline{\omega} \mid r_\infty(\underline{\omega}) = r\}$. More generally, for $1 \leqslant i \leqslant r$ let

$$\Omega^{r,i} := \{\underline{\omega}: r_\infty(\underline{\omega}) = r(\underline{\omega}) = i\}.$$

Then $\Omega^{r,i} = \dot{\bigcup}\Omega_V$, where V runs through the K-subspaces of dimension i of K^r and Ω_V is constructed from V in a similar way as is $\Omega^r = \Omega_{K^r}$ from $C_\infty^r = (K^r) \otimes C_\infty$. That is, $\Omega_V = \{\underline{\omega} \in \mathbb{P}(V \otimes C_\infty) \hookrightarrow \mathbb{P}^{r-1}(C_\infty): r_\infty(\underline{\omega}) = r(\underline{\omega}) = i\}$, which has a natural structure as analytic space of dimension $\dim(V) - 1$ isomorphic with $\Omega^{\dim(V)}$. Finally, we let $\overline{\Omega}^r := \{\underline{\omega}: r_\infty(\underline{\omega}) = r(\underline{\omega})\} = \dot{\bigcup}_{1 \leqslant i \leqslant r}\Omega^{r,i}$.

$\overline{\Omega}^r$ along with its stratification through the $\Omega^{r,i}$ is stable under $GL(r, K)$, so this also holds for the arithmetic group Γ in question. The quotient $\Gamma \setminus \overline{\Omega}^r$ turns out to be the C_∞-points of the wanted compactification $\overline{M}_\Gamma$.

Definition 9. Let $P_i \hookrightarrow G := GL(r)$ be the maximal parabolic subgroup of matrices with first i columns being zero. Let H_i be the obvious factor group isomorphic $GL(r-i)$. Then $P_i(K)$ acts via $H_i(K)$ on K^{r-i} and thus on Ω^{r-i}. From

$$G(K)/P_i(K) \xrightarrow{\sim} \{\text{subspaces } V \text{ of dimension } r-i \text{ of } K^r\}$$

we get bijections

$$\begin{aligned} &G(K) \times_{P_i(K)} \Omega^{r-i} \xrightarrow{\sim} \Omega^{r,r-i}, \\ &(g, \omega_{i+1} : \ldots : \omega_r) \longmapsto (0 : \cdots : 0 : \omega_{i+1} : \ldots : \omega_r)g^{-1} \end{aligned} \tag{4.5.1}$$

and

$$\Gamma \setminus \Omega^{r,r-i} \xrightarrow{\sim} \dot{\bigcup}_{g \in \Gamma \setminus G(K)/P_i(K)} \Gamma(i,g) \setminus \Omega^{r-i}, \tag{4.5.2}$$

where $\Gamma(i,g) := P_i \cap g^{-1}\Gamma_g$, and the double quotient $\Gamma \setminus G(K)/P_i(K)$ is finite by elementary lattice theory. Note that the image of $\Gamma(i,g)$ in $H_i(K)$ (the group that effectively acts on Ω^{r-i}) is again an arithmetic subgroup of $H_i(K) = GL(r-i, K)$, and so the right hand side of (4.5.2) is the disjoint union of analytic spaces of the same type $\Gamma' \setminus \Omega^{r'}$.

Example 5. Let $\Gamma = \Gamma(1) = GL(r, A)$ and $i = 1$. Then $\Gamma \setminus G(K)/P_1(K)$ equals the set of isomorphism classes of projective A-modules of rank $r-1$, which in turn (through the determinant map) is in one-to-one correspondence with the class group $\mathrm{Pic}(A)$.

Let F_V be the image of Ω_V in $\Gamma \setminus \overline{\Omega}^r$. The different analytic spaces F_V, corresponding to locally closed subvarieties of $\overline{M}_\Gamma$, are glued together in such a way that F_U lies in the Zariski closure $\overline{F}_V$ of F_V if and only if U is Γ-conjugate to a K-subspace of V. Taking into account that $F_V \simeq \Gamma' \setminus \Omega^{\dim(V)} = M_{\Gamma'}(C_\infty)$ for some arithmetic subgroup Γ' of $GL(\dim(V), K)$, $\overline{F}_V$ corresponds to the compactification $\overline{M}_{\Gamma'}$ of $M_{\Gamma'}$.

The details of the gluing procedure are quite technical and complicated and cannot be presented here (see [G2] and [K] for some special cases). Suffice it to say that for each boundary component F_V of codimension one, a vertical coordinate t_V may be specified such that F_V is locally given by $t_V = 0$. The result (we refrain from stating a "theorem" since proofs of the assertions below in full strength and generality are published neither in [G2] nor in [K]) will be a normal projective C_∞-variety $\overline{M}_\Gamma$ provided with an open dense embedding $i: M_\Gamma \hookrightarrow \overline{M}_\Gamma$ with the following properties:

- $\overline{M}_\Gamma(C_\infty) = \Gamma \setminus \overline{\Omega}^r$, and the inclusion $\Gamma \setminus \Omega^r \hookrightarrow \Gamma \setminus \overline{\Omega}^r$ corresponds to i;
- $\overline{M}_\Gamma$ is defined over the same finite abelian extension of K as is M_Γ;
- for $\Gamma' \hookrightarrow \Gamma$, the natural map $M_{\Gamma'} \to M_\Gamma$ extends to $\overline{M}_{\Gamma'} \to \overline{M}_\Gamma$;
- the F_V correspond to locally closed subvarieties, and $\overline{F}_V = \cup F_U$, where U runs through the K-subspaces of V contained up to the action of Γ in V;
- $\overline{M}_\Gamma$ is "virtually non-singular", i.e., Γ contains a subgroup Γ' of finite index such that $\overline{M}_{\Gamma'}$ is non-singular; in that case, the boundary components of codimension one present normal crossings.

Now suppose that $\overline{M}_\Gamma$ is non-singular and that $x \in \overline{M}_\Gamma(C_\infty) = \bigcup_{1 \leqslant i \leqslant r} \Omega^{r,i}$ belongs to $\Omega^{r,1}$. Then we can find a sequence $\{x\} = X_0 \subset \cdots X_i \cdots \subset X_{r-1} = \overline{M}_\Gamma$ of smooth subvarieties $X_i = \overline{F}_{V_i}$ of dimension i. Any holomorphic function around x (or more generally, any modular form for Γ) may thus be expanded as a series in t_V with coefficients in the function field of $\overline{F}_{V_{r-1}}$, etc. Hence $\overline{M}_\Gamma$ (or rather its completion at the X_i) may be described through $(r-1)$-dimensional local fields with residue field C_∞. The expansion of some standard modular forms can be explicitly calculated, see [G1, VI] for the case of $r = 2$. In the last section we shall present at least the vanishing orders of some of these forms.

Example 6. Let A be the polynomial ring $\mathbb{F}_q[T]$ and $\Gamma = GL(r, A)$. As results from Example 3, (4.3.3) and (4.4.2),

$$M_\Gamma(C_\infty) = M^r(1)(C_\infty) = \{(g_1, \dots, g_r) \in C_\infty^r : g_r \neq 0\}/C_\infty^*,$$

where C_∞^* acts diagonally through $c(g_1, \dots, g_r) = (\dots, c^{q^i-1} g_i, \dots)$, which is the open subspace of weighted projective space $\mathbb{P}^{r-1}(q-1, \dots, q^r-1)$ with non-vanishing last coordinate. The construction yields

$$\overline{M}_\Gamma(C_\infty) = \mathbb{P}^{r-1}(q-1, \dots, q^r-1)(C_\infty) = \dot{\bigcup}_{1 \leqslant i \leqslant r} M^i(1)(C_\infty).$$

Its singularities are rather mild and may be removed upon replacing Γ by a congruence subgroup.

4.6. Vanishing orders of modular forms

In this final section we state some results about the vanishing orders of certain modular forms along the boundary divisors of $\overline{M}_\Gamma$, in the case where Γ is either $\Gamma(1) = GL(r, A)$ or a full congruence subgroup $\Gamma(\mathfrak{n})$ of $\Gamma(1)$. These are relevant for the determination of K- and Chow groups, and for standard conjectures about the arithmetic interpretation of partial zeta values.

In what follows, we suppose that $r \geqslant 2$, and put $z_i := \frac{\omega_i}{\omega_r}$ $(1 \leqslant i \leqslant r)$ for the coordinates $(\omega_1 : \dots : \omega_r)$ of $\underline{\omega} \in \Omega^r$. Quite generally, $\underline{a} = (a_1, \dots, a_r)$ denotes a vector with r components.

Definition 10. The *Eisenstein series* E_k of weight k on Ω^r is defined as

$$E_k(\underline{\omega}) := \sum_{0 \neq \underline{a} \in A^r} \frac{1}{(a_1 z_1 + \dots + a_r z_r)^k}.$$

Similarly, we define for $\underline{u} \in \mathfrak{n}^{-1} \times \dots \times \mathfrak{n}^{-1} \subset K^r$

$$E_{k,\underline{u}}(\underline{\omega}) = \sum_{\substack{0 \neq \underline{a} \in K^r \\ \underline{a} \equiv \underline{u} \bmod A^r}} \frac{1}{(a_1 z_1 + \dots + a_r z_r)^k}.$$

These are modular forms for $\Gamma(1)$ and $\Gamma(\mathfrak{n})$, respectively, that is, they are holomorphic, satisfy the obvious transformation values under $\Gamma(1)$ (resp. $\Gamma(\mathfrak{n})$), and extend to sections of a line bundle on $\overline{M}_\Gamma$. As in Example 4, there is a second type of modular forms coming directly from Drinfeld modules.

Definition 11. For $\underline{\omega} \in \Omega^r$ write $\Lambda_{\underline{\omega}} = Az_1 + \dots + Az_r$ and $e_{\underline{\omega}}$, $\phi^{\underline{\omega}}$ for the lattice function and Drinfeld module associated with $\Lambda_{\underline{\omega}}$, respectively. If $a \in A$ has degree $d = \deg(a)$,

$$\phi^{\underline{\omega}}_a = a + \sum_{1 \leqslant i \leqslant r \cdot d} \ell_i(a, \underline{\omega}) \tau^i.$$

The $\ell_i(a, \underline{\omega})$ are modular forms of weight $q^i - 1$ for Γ. This holds in particular for

$$\Delta_a(\underline{\omega}) := \ell_{rd}(a, \underline{\omega}),$$

which has weight $q^{rd} - 1$ and vanishes nowhere on Ω^r. The functions g and Δ in Example 4 merely constitute a very special instance of this construction. We further let, for $\underline{u} \in (\mathfrak{n}^{-1})^r$,

$$e_{\underline{u}}(\underline{\omega}) := e_{\underline{\omega}}(u_1 z_1 + \dots + u_r z_r),$$

the $\mathfrak{n}$-division point of type $\underline{u}$ of $\phi^{\underline{\omega}}$. If $\underline{u} \notin A^r$, $e_{\underline{u}}(\underline{\omega})$ vanishes nowhere on Ω^r, and it can be shown that in this case,

$$e_{\underline{u}}^{-1} = E_{1,\underline{u}}. \tag{4.6.1}$$

We are interested in the behavior around the boundary of $\overline{M}_\Gamma$ of these forms. Let us first describe the set $\{\overline{F}_V\}$ of boundary divisors, i.e., of irreducible components, all of codimension one, of $\overline{M}_\Gamma - M_\Gamma$. For $\Gamma = \Gamma(1) = GL(r, A)$, there is a natural bijection

$$\{\overline{F}_V\} \xrightarrow{\sim} \mathrm{Pic}(A) \tag{4.6.2}$$

described in detail in [G1, VI 5.1]. It is induced from $V \mapsto$ inverse of $\Lambda^{r-1}(V \cap A^r)$. (Recall that V is a K-subspace of dimension $r-1$ of K^r, thus $V \cap A^r$ a projective module of rank $r-1$, whose $(r-1)$-th exterior power $\Lambda^{r-1}(V \cap A^r)$ determines an element of $\mathrm{Pic}(A)$.) We denote the component corresponding to the class $(\mathfrak{a})$ of an ideal $\mathfrak{a}$ by $\overline{F}_{(\mathfrak{a})}$. Similarly, the boundary divisors of $\overline{M}_\Gamma$ for $\Gamma = \Gamma(\mathfrak{n})$ could be described via generalized class groups. We simply use (4.5.1) and (4.5.2), which now give

$$\{\overline{F}_V\} \xrightarrow{\sim} \Gamma(\mathfrak{n}) \setminus GL(r, K)/P_1(K). \tag{4.6.3}$$

We denote the class of $\nu \in GL(r, K)$ by $[\nu]$. For the description of the behavior of our modular forms along the $\overline{F}_V$, we need the partial zeta functions of A and K. For more about these, see [W] and [G1, III].

Definition 12. We let

$$\zeta_K(s) = \sum |\mathfrak{a}|^{-s} = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}$$

be the zeta function of K with numerator polynomial $P(X) \in \mathbb{Z}[X]$. Here the sum is taken over the positive divisors $\mathfrak{a}$ of K (i.e., of the curve $\mathcal{C}$ with function field K). Extending the sum only over divisors with support in $\mathrm{Spec}(A)$, we get

$$\zeta_A(s) = \sum_{0 \neq \mathfrak{a} \subset A \text{ ideal}} |\mathfrak{a}|^{-s} = \zeta_K(s)(1 - q^{-d_\infty s}),$$

where $d_\infty = \deg_{\mathbb{F}_q}(\infty)$. For a class $\mathfrak{c} \in \mathrm{Pic}(A)$ we put

$$\zeta_{\mathfrak{c}}(s) = \sum_{\mathfrak{a} \in \mathfrak{c}} |\mathfrak{a}|^{-s}.$$

If finally $\mathfrak{n} \subset K$ is a fractional A-ideal and $t \in K$, we define

$$\zeta_{t \bmod \mathfrak{n}}(s) = \sum_{\substack{a \in K \\ a \equiv t \bmod \mathfrak{n}}} |a|^{-s}.$$

Among the obvious distribution relations [G1, III sect.1] between these, we only mention

$$\zeta_{(\mathfrak{n}^{-1})}(s) = \frac{|\mathfrak{n}|^s}{q-1} \zeta_{0 \bmod \mathfrak{n}}(s). \tag{4.6.4}$$

We are now in a position to state the following theorems, which may be proved following the method of [G1, VI].

Theorem 4. *Let* $a \in A$ *be non-constant and* $\mathfrak{c}$ *a class in* $\mathrm{Pic}(A)$. *The modular form* Δ_a *for* $GL(r, A)$ *has vanishing order*

$$\mathrm{ord}_{\mathfrak{c}}(\Delta_a) = -(|a|^r - 1)\zeta_{\mathfrak{c}}(1 - r)$$

at the boundary component $\overline{F}_{\mathfrak{c}}$ *corresponding to* $\mathfrak{c}$.

Theorem 5. *Fix an ideal* $\mathfrak{n}$ *of* A *and* $\underline{u} \in K^r - A^r$ *such that* $\underline{u} \cdot \mathfrak{n} \subset A^r$, *and let* $e_{\underline{u}}^{-1} = E_{1,\underline{u}}$ *be the modular form for* $\Gamma(\mathfrak{n})$ *determined by these data. The vanishing order* $\mathrm{ord}_{[\nu]}$ *of* $E_{1,\underline{u}}(\underline{\omega})$ *at the component corresponding to* $\nu \in GL(r, K)$ (*see* (4.6.2)) *is given as follows*: *let* $\pi_1 \colon K^r \to K$ *be the projection to the first coordinate and let* $\mathfrak{a}$ *be the fractional ideal* $\pi_1(A^r \cdot \nu)$. *Write further* $\underline{u} \cdot \nu = (v_1, \dots, v_r)$. *Then*

$$\mathrm{ord}_{[\nu]} E_{1,\underline{u}}(\omega) = \frac{|\mathfrak{n}|^{r-1}}{|\mathfrak{a}|^{r-1}}(\zeta_{v_1 \bmod \mathfrak{a}}(1 - r) - \zeta_{0 \bmod \mathfrak{a}}(1 - r)).$$

Note that the two theorems do not depend on the full strength of properties of $\overline{M}_\Gamma$ as stated without proofs in the last section, but only on the *normality* of $\overline{M}_\Gamma$, which is *proved* in [K] for $A = \mathbb{F}_q[T]$, and whose generalization to arbitrary Drinfeld rings is straightforward (even though technical).

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5. Harmonic analysis on algebraic groups over two-dimensional local fields of equal characteristic

Mikhail Kapranov

In this section we review the main parts of a recent work [4] on harmonic analysis on algebraic groups over two-dimensional local fields.

5.1. Groups and buildings

Let K ($K = K_2$ whose residue field is K_1 whose residue field is K_0, see the notation in section 1 of Part I) be a two-dimensional local field of equal characteristic. Thus K_2 is isomorphic to the Laurent series field $K_1((t_2))$ over K_1. It is convenient to think of elements of K_2 as (formal) loops over K_1. Even in the case where $\mathrm{char}(K_1) = 0$, it is still convenient to think of elements of K_1 as (generalized) loops over K_0 so that K_2 consists of double loops.

Denote the residue map $\mathcal{O}_{K_2} \to K_1$ by p_2 and the residue map $\mathcal{O}_{K_1} \to K_0$ by p_1. Then the ring of integers O_K of K as of a two-dimensional local field (see subsection 1.1 of Part I) coincides with $p_2^{-1}(\mathcal{O}_{K_1})$.

Let G be a split simple simply connected algebraic group over $\mathbb{Z}$ (e.g. $G = SL_2$). Let $T \subset B \subset G$ be a fixed maximal torus and Borel subgroup of G; put $N = [B, B]$, and let W be the Weyl group of G. All of them are viewed as group schemes.

Let $L = \mathrm{Hom}(\mathbb{G}_m, T)$ be the coweight lattice of G; the Weyl group acts on L.

Recall that $I(K_1) = p_1^{-1}(B(\mathbb{F}_q))$ is called an Iwahori subgroup of $G(K_1)$ and $T(\mathcal{O}_{K_1})N(K_1)$ can be seen as the "connected component of unity" in $B(K_1)$. The latter name is explained naturally if we think of elements of $B(K_1)$ as being loops with values in B.

Definition. Put

$$D_0 = p_2^{-1}p_1^{-1}(B(\mathbb{F}_q)) \subset G(O_K),$$
$$D_1 = p_2^{-1}(T(\mathcal{O}_{K_1})N(K_1)) \subset G(O_K),$$
$$D_2 = T(\mathcal{O}_{K_2})N(K_2) \subset G(K).$$

Then D_2 can be seen as the "connected component of unity" of $B(K)$ when K is viewed as a two-dimensional local field, D_1 is a (similarly understood) connected component of an Iwahori subgroup of $G(K_2)$, and D_0 is called a double Iwahori subgroup of $G(K)$.

A choice of a system of local parameters t_1, t_2 of K determines the identification of the group K^*/O_K^* with $\mathbb{Z} \oplus \mathbb{Z}$ and identification $L \oplus L$ with $L \otimes (K^*/O_K^*)$.

We have an embedding of $L \otimes (K^*/O_K^*)$ into $T(K)$ which takes $a \otimes (t_1^j t_2^j)$, $i, j \in \mathbb{Z}$, to the value on $t_1^i t_2^j$ of the 1-parameter subgroup in T corresponding to a.

Define the action of W on $L \otimes (K^*/O_K^*)$ as the product of the standard action on L and the trivial action on K^*/O_K^*. The semidirect product

$$\widehat{\widehat{W}} = (L \otimes K^*/O_K^*) \rtimes W$$

is called the *double affine Weyl group* of G.

A (set-theoretical) lifting of W into $G(O_K)$ determines a lifting of $\widehat{\widehat{W}}$ into $G(K)$.

Proposition. *For every $i, j \in \{0, 1, 2\}$ there is a disjoint decomposition*

$$G(K) = \dot{\bigcup}_{w \in \widehat{\widehat{W}}} D_i w D_j .$$

The identification $D_i \backslash G(K)/D_j$ with $\widehat{\widehat{W}}$ doesn't depend on the choice of liftings.

Proof. Iterated application of the Bruhat, Bruhat–Tits and Iwasawa decompositions to the local fields K_2, K_1.

For the Iwahori subgroup $I(K_2) = p_2^{-1}(B(K_1))$ of $G(K_2)$ the homogeneous space $G(K)/I(K_2)$ is the "affine flag variety" of G, see [5]. It has a canonical structure of an ind-scheme, in fact, it is an inductive limit of projective algebraic varieties over K_1 (the closures of the affine Schubert cells).

Let $B(G, K_2/K_1)$ be the *Bruhat–Tits building* associated to G and the field K_2. Then the space $G(K)/I(K_2)$ is a $G(K)$-orbit on the set of flags of type (vertex, maximal cell) in the building. For every vertex v of $B(G, K_2/K_1)$ its locally finite Bruhat–Tits building β_v isomorphic to $B(G, K_1/K_0)$ can be viewed as a "microbuilding" of the *double Bruhat–Tits building* $B(G, K_2/K_1/K_0)$ of K as a two-dimensional local field constructed by Parshin ([7], see also section 3 of Part II). Then the set $G(K)/D_1$ is identified naturally with the set of all the horocycles $\{w \in \beta_v : d(z, w) = r\}$, $z \in \partial\beta_v$ of the microbuildings β_v (where the "distance" $d(z, \ )$ is viewed as an element of a natural L-torsor). The fibres of the projection $G(K)/D_1 \to G(K)/I(K_2)$ are L-torsors.

5.2. The central extension and the affine Heisenberg–Weyl group

According to the work of Steinberg, Moore and Matsumoto [6] developed by Brylinski and Deligne [1] there is a central extension

$$1 \to K_1^* \to \Gamma \to G(K_2) \to 1$$

associated to the tame symbol $K_2^* \times K_2^* \to K_1^*$ for the couple (K_2, K_1) (see subsection 6.4.2 of Part I for the general definition of the tame symbol).

Proposition. *This extension splits over every* D_i, $0 \leqslant i \leqslant 2$.

Proof. Use Matsumoto's explicit construction of the central extension.

Thus, there are identifications of every D_i with a subgroup of Γ. Put

$$\Delta_i = \mathcal{O}_{K_1}^* D_i \subset \Gamma, \qquad \Xi = \Gamma/\Delta_1.$$

The minimal integer scalar product Ψ on L and the composite of the tame symbol $K_2^* \times K_2^* \to K_1^*$ and the discrete valuation $v_{K_1}\colon K^* \to \mathbb{Z}$ induces a W-invariant skew-symmetric pairing $L \otimes K^*/O_K^* \times L \otimes K^*/O_K^* \to \mathbb{Z}$. Let

$$1 \to \mathbb{Z} \to \mathcal{L} \to L \otimes K^*/O_K^* \to 1$$

be the central extension whose commutator pairing corresponds to the latter skew-symmetric pairing. The group $\mathcal{L}$ is called the *Heisenberg group*.

Definition. The semidirect product

$$\widetilde{W} = \mathcal{L} \rtimes W$$

is called the *double affine Heisenberg–Weyl* group of G.

Theorem. *The group* $\widetilde{W}$ *is isomorphic to* $L_{\mathrm{aff}} \rtimes \widehat{W}$ *where* $L_{\mathrm{aff}} = \mathbb{Z} \oplus L$, $\widehat{W} = L \rtimes W$ *and*

$$w \circ (a, l') = (a, w(l)), \quad l \circ (a, l') = (a + \Psi(l, l'), l'), \qquad w \in W, \quad l, l' \in L, \quad a \in \mathbb{Z}.$$

For every $i, j \in \{0, 1, 2\}$ *there is a disjoint union*

$$\Gamma = \dot{\bigcup}_{w \in \widetilde{W}} \Delta_i w \Delta_j$$

and the identification $\Delta_i \backslash \Gamma / \Delta_j$ *with* $\widetilde{W}$ *is canonical.*

5.3. Hecke algebras in the classical setting

Recall that for a locally compact group Γ and its compact subgroup Δ the Hecke algebra $\mathcal{H}(\Gamma, \Delta)$ can be defined as the algebra of compactly supported double Δ-invariant continuous functions of Γ with the operation given by the convolution with respect to the Haar measure on Γ. For $C = \Delta\gamma\Delta \in \Delta\backslash\Gamma/\Delta$ the Hecke correspondence $\Sigma_C = \{(\alpha\Delta, \beta\Delta) : \alpha\beta^{-1} \in C\}$ is a Γ-orbit of $(\Gamma/\Delta) \times (\Gamma/\Delta)$.

For $x \in \Gamma/\Delta$ put $\Sigma_C(x) = \Sigma_C \cap (\Gamma/\Delta) \times \{x\}$. Denote the projections of Σ_C to the first and second component by π_1 and π_2.

Let $\mathcal{F}(\Gamma/\Delta)$ be the space of continuous functions $\Gamma/\Delta \to \mathbb{C}$. The operator

$$\tau_C \colon \mathcal{F}(\Gamma/\Delta) \to \mathcal{F}(\Gamma/\Delta), \quad f \to \pi_{2*}\pi_1^*(f)$$

is called the *Hecke operator* associated to C. Explicitly,

$$(\tau_C f)(x) = \int_{y \in \Sigma_C(x)} f(y) d\mu_{C,x},$$

where $\mu_{C,x}$ is the $\mathrm{Stab}(x)$-invariant measure induced by the Haar measure. Elements of the Hecke algebra $\mathcal{H}(\Gamma, \Delta)$ can be viewed as "continuous" linear combinations of the operators τ_C, i.e., integrals of the form $\int \phi(C)\tau_C dC$ where dC is some measure on $\Delta\backslash\Gamma/\Delta$ and ϕ is a continuous function with compact support. If the group Δ is also open (as is usually the case in the p-adic situation), then $\Delta\backslash\Gamma/\Delta$ is discrete and $\mathcal{H}(\Gamma, \Delta)$ consists of finite linear combinations of the τ_C.

5.4. The regularized Hecke algebra $\mathcal{H}(\Gamma, \Delta_1)$

Since the two-dimensional local field K and the ring O_K are not locally compact, the approach of the previous subsection would work only after a new appropriate integration theory is available.

The aim of this subsection is to make sense of the Hecke algebra $\mathcal{H}(\Gamma, \Delta_1)$.

Note that the fibres of the projection $\Xi = \Gamma/\Delta_1 \to G(K)/I(K_2)$ are L_{aff}-torsors and $G(K)/I(K_2)$ is the inductive limit of compact (profinite) spaces, so Ξ can be considered as an object of the category $\mathcal{F}_1$ defined in subsection 1.2 of the paper of Kato in this volume.

Using Theorem of 5.2 for $i = j = 1$ we introduce:

Definition. For $(w, l) \in \widetilde{W} = L_{\mathrm{aff}} \rtimes \widehat{W}$ denote by $\Sigma_{w,l}$ the Hecke correspondence (i.e., the Γ-orbit of $\Xi \times \Xi$) associated to (w, l). For $\xi \in \Xi$ put

$$\Sigma_{w,l}(\xi) = \{\xi' : (\xi, \xi') \in \Sigma_{w,l}\}.$$

The stabilizer $\mathrm{Stab}(\xi) \leqslant \Gamma$ acts transitively on $\Sigma_{w,l}(\xi)$.

Proposition. *$\Sigma_{w,l}(\xi)$ is an affine space over K_1 of dimension equal to the length of $w \in \widehat{W}$. The space of compex valued Borel measures on $\Sigma_{w,l}(\xi)$ is 1-dimensional. A choice of a* $\mathrm{Stab}(\xi)$*-invariant measure $\mu_{w,l,\xi}$ on $\Sigma_{w,l}(\xi)$ determines a measure $\mu_{w,l,\xi'}$ on $\Sigma_{w,l}(\xi')$ for every ξ'.*

Definition. For a continuous function $f\colon \Xi \to \mathbb{C}$ put

$$(\tau_{w,l}f)(\xi) = \int_{\eta \in \Sigma_{w,l}(\xi)} f(\eta) d\mu_{w,l,\xi}.$$

Since the domain of the integration is not compact, the integral may diverge. As a first step, we define the space of functions on which the integral makes sense. Note that Ξ can be regarded as an L_{aff}-torsor over the ind-object $G(K)/I(K_2)$ in the category $\mathrm{pro}(C_0)$, i.e., a compatible system of L_{aff}-torsors Ξ_ν over the affine Schubert varieties Z_ν forming an exhaustion of $G(K)/I(K_1)$. Each Ξ_ν is a locally compact space and Z_ν is a compact space. In particular, we can form the space $\mathcal{F}_0(\Xi_\nu)$ of locally constant complex valued functions on Ξ_ν whose support is compact (or, what is the same, proper with respect to the projection to Z_ν). Let $\mathcal{F}(\Xi_\nu)$ be the space of all locally constant complex functions on Ξ_ν. Then we define $\mathcal{F}_0(\Xi) = \text{"}\varprojlim\text{"}\, \mathcal{F}_0(\Xi_\nu)$ and $\mathcal{F}(\Xi) = \text{"}\varprojlim\text{"}\, \mathcal{F}(\Xi_\nu)$. They are pro-objects in the category of vector spaces. In fact, because of the action of L_{aff} and its group algebra $\mathbb{C}[L_{\mathrm{aff}}]$ on Ξ, the spaces $\mathcal{F}_0(\Xi), \mathcal{F}(\Xi)$ are naturally pro-objects in the category of $\mathbb{C}[L_{\mathrm{aff}}]$-modules.

Proposition. *If $f = (f_\nu) \in \mathcal{F}_0(X)$ then* $\mathrm{Supp}(f_\nu) \cap \Sigma_{w,l}(\xi)$ *is compact for every w, l, ξ, ν and the integral above converges. Thus, there is a well defined Hecke operator*

$$\tau_{w,l}\colon \mathcal{F}_0(\Xi) \to \mathcal{F}(\Xi)$$

which is an element of $\mathrm{Mor}(\mathrm{pro}(\mathrm{Mod}_{\mathbb{C}[L_{\mathrm{aff}}]}))$*. In particular, $\tau_{w,l}$ is the shift by l and $\tau_{w,l+l'} = \tau_{w,l'}\tau_{e,l}$.*

Thus we get Hecke operators as operators from one (pro-)vector space to another, bigger one. This does not yet allow to compose the $\tau_{w,l}$. Our next step is to consider certain infinite linear combinations of the $\tau_{w,l}$.

Let $T^\vee_{\mathrm{aff}} = \mathrm{Spec}(\mathbb{C}[L_{\mathrm{aff}}])$ be the "dual affine torus" of G. A function with finite support on L_{aff} can be viewed as the collection of coefficients of a polynomial, i.e., of an element of $\mathbb{C}[L_{\mathrm{aff}}]$ as a regular function on $T^\vee_{\mathrm{aff}}$. Further, let $Q \subset L_{\mathrm{aff}} \otimes \mathbb{R}$ be a strictly convex cone with apex 0. A function on L_{aff} with support in Q can be viewed as the collection of coefficients of a formal power series, and such series form a ring containing $\mathbb{C}[L_{\mathrm{aff}}]$. On the level of functions the ring operation is the convolution. Let $\mathcal{F}_Q(L_{\mathrm{aff}})$ be the space of functions whose support is contained in some translation of Q. It is a ring with respect to convolution.

Let $\mathbb{C}(L_{\rm aff})$ be the field of rational functions on $T^\vee_{\rm aff}$. Denote by $F^{\rm rat}_Q(L_{\rm aff})$ the subspace in $F_Q(L_{\rm aff})$ consisting of functions whose corresponding formal power series are expansions of rational functions on $T^\vee_{\rm aff}$.

If A is any $L_{\rm aff}$-torsor (over a point), then $\mathcal{F}_0(A)$ is an (invertible) module over $\mathcal{F}_0(L_{\rm aff}) = \mathbb{C}[L_{\rm aff}]$ and we can define the spaces $\mathcal{F}_Q(A)$ and $\mathcal{F}^{\rm rat}_Q(A)$ which will be modules over the corresponding rings for $L_{\rm aff}$. We also write $\mathcal{F}^{\rm rat}(A) = \mathcal{F}_0(A) \otimes_{\mathbb{C}[L_{\rm aff}]} \mathbb{C}(L_{\rm aff})$.

We then extend the above concepts "fiberwise" to torsors over compact spaces (objects of $\mathrm{pro}(C_0)$) and to torsors over objects of $\mathrm{ind}(\mathrm{pro}(C_0))$ such as Ξ.

Let $w \in \widehat{W}$. We denote by $Q(w)$ the image under w of the cone of dominant affine coweights in $L_{\rm aff}$.

Theorem. *The action of the Hecke operator $\tau_{w,l}$ takes $\mathcal{F}_0(\Xi)$ into $\mathcal{F}^{\rm rat}_{Q(w)}(\Xi)$. These operators extend to operators*

$$\tau^{\rm rat}_{w,l} : \mathcal{F}^{\rm rat}(\Xi) \to \mathcal{F}^{\rm rat}(\Xi).$$

Note that the action of $\tau^{\rm rat}_{w,l}$ involves a kind of regularization procedure, which is hidden in the identification of the $\mathcal{F}^{\rm rat}_{Q(w)}(\Xi)$ for different w, with subspaces of the same space $\mathcal{F}^{\rm rat}(\Xi)$. In practical terms, this involves summation of a series to a rational function and re-expansion in a different domain.

Let $\mathcal{H}_{\rm pre}$ be the space of finite linear combinations $\sum_{w,l} a_{w,l}\tau_{w,l}$. This is not yet an algebra, but only a $\mathbb{C}[L_{\rm aff}]$-module. Note that elements of $\mathcal{H}_{\rm pre}$ can be written as finite linear combinations $\sum_{w\in\widehat{W}} f_w(t)\tau_w$ where $f_w(t) = \sum_l a_{w,l}t^l$, $t \in T^\vee_{\rm aff}$, is the polynomial in $\mathbb{C}[L_{\rm aff}]$ corresponding to the collection of the $a_{w,l}$. This makes the $\mathbb{C}[L_{\rm aff}]$-module structure clear. Consider the tensor product

$$\mathcal{H}_{\rm rat} = \mathcal{H}_{\rm pre} \otimes_{\mathbb{C}[L_{\rm aff}]} \mathbb{C}(L_{\rm aff}).$$

Elements of this space can be considered as finite linear combinations $\sum_{w\in\widehat{W}} f_w(t)\tau_w$ where $f_w(t)$ are now rational functions. By expanding rational functions in power series, we can consider the above elements as certain infinite linear combinations of the $\tau_{w,l}$.

Theorem. *The space $\mathcal{H}_{\rm rat}$ has a natural algebra structure and this algebra acts in the space $\mathcal{F}^{\rm rat}(\Xi)$, extending the action of the $\tau_{w,l}$ defined above.*

The operators associated to $\mathcal{H}_{\rm rat}$ can be viewed as certain integro-difference operators, because their action involves integration (as in the definition of the $\tau_{w,l}$) as well as inverses of linear combinations of shifts by elements of L (these combinations act as difference operators).

Definition. The regularized Hecke algebra $\mathcal{H}(\Gamma, \Delta_1)$ is, by definition, the subalgebra in $\mathcal{H}_{\rm rat}$ consisting of elements whose action in $\mathcal{F}_{\rm rat}(\Xi)$ preserves the subspace $\mathcal{F}_0(\Xi)$.

5.5. The Hecke algebra and the Cherednik algebra

In [2] I. Cherednik introduced the so-called double affine Hecke algebra Cher_q associated to the root system of G. As shown by V. Ginzburg, E. Vasserot and the author [3], Cher_q can be thought as consisting of finite linear combinations $\sum_{w\in\widehat{W}_{\mathrm{ad}}} f_w(t)[w]$ where W_{ad} is the affine Weyl group of the adjoint quotient G_{ad} of G (it contains $\widehat{W}$) and $f_w(t)$ are rational functions on $T^\vee_{\mathrm{aff}}$ satisfying certain residue conditions. We define the modified Cherednik algebra $\ddot{H}_q$ to be the subalgebra in Cher_q consisting of linear combinations as above, but going over $\widehat{W}\subset\widehat{W}_{\mathrm{ad}}$.

Theorem. *The regularized Hecke algebra $\mathcal{H}(\Gamma,\Delta_1)$ is isomorphic to the modified Cherednik algebra $\ddot{H}_q$. In particular, there is a natural action of $\ddot{H}_q$ on $\mathcal{F}_0(\Xi)$ by integro-difference operators.*

Proof. Use the principal series intertwiners and a version of Mellin transform. The information on the poles of the intertwiners matches exactly the residue conditions introduced in [3].

Remark. The only reason we needed to assume that the 2-dimensional local field K has equal characteristic was because we used the fact that the quotient $G(K)/I(K_2)$ has a structure of an inductive limit of projective algebraic varieties over K_1. In fact, we really use only a weaker structure: that of an inductive limit of profinite topological spaces (which are, in this case, the sets of K_1-points of affine Schubert varieties over K_1). This structure is available for any 2-dimensional local field, although there seems to be no reference for it in the literature. Once this foundational matter is established, all the constructions will go through for any 2-dimensional local field.

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6. Φ-Γ-modules and Galois cohomology

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6.0. Introduction

Let G be a profinite group and p a prime number.

Definition. A finitely generated $\mathbb{Z}_p$-module V endowed with a continuous G-action is called a $\mathbb{Z}_p$-adic representation of G. Such representations form a category denoted by $\mathrm{Rep}_{\mathbb{Z}_p}(G)$; its subcategory $\mathrm{Rep}_{\mathbb{F}_p}(G)$ (respectively $\mathrm{Rep}_{p\text{-tor}}(G)$) of mod p representations (respectively p-torsion representations) consists of the V annihilated by p (respectively a power of p).

Problem. To calculate in a simple explicit way the cohomology groups $H^i(G,V)$ of the representation V.

A method to solve it for $G = G_K$ (K is a local field) is to use Fontaine's theory of Φ-Γ-modules and pass to a simpler Galois representation, paying the price of enlarging $\mathbb{Z}_p$ to the ring of integers of a two-dimensional local field. In doing this we have to replace linear with semi-linear actions.

In this paper we give an overview of the applications of such techniques in different situations. We begin with a simple example.

6.1. The case of a field of positive characteristic

Let E be a field of characteristic p, $G = G_E$ and $\sigma: E^{\mathrm{sep}} \to E^{\mathrm{sep}}$, $\sigma(x) = x^p$ the absolute Frobenius map.

Definition. For $V \in \mathrm{Rep}_{\mathbb{F}_p}(G_E)$ put $D(V) := (E^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V)^{G_E}$; σ acts on $D(V)$ by acting on E^{sep}.

Properties.

(1) $\dim_E D(V) = \dim_{\mathbb{F}_p} V$;

(2) the "Frobenius" map $\varphi: D(V) \to D(V)$ induced by $\sigma \otimes \mathrm{id}_V$ satisfies:

a) $\varphi(\lambda x) = \sigma(\lambda)\varphi(x)$ for all $\lambda \in E$, $x \in D(V)$ (so φ is σ-semilinear);

b) $\varphi(D(V))$ generates $D(V)$ as an E-vector space.

Definition. A finite dimensional vector space M over E is called an *étale Φ-module* over E if there is a σ-semilinear map $\varphi: M \to M$ such that $\varphi(M)$ generates M as an E-vector space.

Étale Φ-modules form an abelian category $\Phi M_E^{\text{ét}}$ (the morphisms are the linear maps commuting with the Frobenius φ).

Theorem 1 (Fontaine, [F]). *The functor $V \to D(V)$ is an equivalence of the categories $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$ and $\Phi M_E^{\text{ét}}$.*

We see immediately that $H^0(G_E, V) = V^{G_E} \simeq D(V)^\varphi$.

So in order to obtain an explicit description of the Galois cohomology of mod p representations of G_E, we should try to derive in a simple manner the functor associating to an étale Φ-module the group of points fixed under φ. This is indeed a much simpler problem because there is only one operator acting.

For $(M, \varphi) \in \Phi M_E^{\text{ét}}$ define the following complex of abelian groups:

$$C_1(M): \qquad 0 \to M \xrightarrow{\varphi - 1} M \to 0$$

(M stands at degree 0 and 1).

This is a functorial construction, so by taking the cohomology of the complex, we obtain a cohomological functor $(\mathcal{H}^i := H^i \circ C_1)_{i \in \mathbb{N}}$ from $\Phi M_E^{\text{ét}}$ to the category of abelian groups.

Theorem 2. *The cohomological functor $(\mathcal{H}^i \circ D)_{i \in \mathbb{N}}$ can be identified with the Galois cohomology functor $(H^i(G_E, .))_{i \in \mathbb{N}}$ for the category $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$. So, if $M = D(V)$ then $\mathcal{H}^i(M)$ provides a simple explicit description of $H^i(G_E, V)$.*

Proof of Theorem 2. We need to check that the cohomological functor $(\mathcal{H}^i)_{i \in \mathbb{N}}$ is universal; therefore it suffices to verify that for every $i \geqslant 1$ the functor $\mathcal{H}^i$ is effaceable: this means that for every $(M, \varphi_M) \in \Phi M_E^{\text{ét}}$ and every $x \in \mathcal{H}^i(M)$ there exists an embedding u of (M, φ_M) in $(N, \varphi_N) \in \Phi M_E^{\text{ét}}$ such that $\mathcal{H}^i(u)(x)$ is zero in $\mathcal{H}^i(N)$. But this is easy: it is trivial for $i \geqslant 2$; for $i = 1$ choose an element m belonging to the class $x \in M/(\varphi - 1)(M)$, put $N := M \oplus Et$ and extend φ_M to the σ-semi-linear map φ_N determined by $\varphi_N(t) := t + m$. $\square$

6.2. Φ-Γ-modules and $\mathbb{Z}_p$-adic representations

Definition. Recall that a Cohen ring is an absolutely unramified complete discrete valuation ring of mixed characteristic $(0, p > 0)$, so its maximal ideal is generated by p.

We describe a general formalism, explained by Fontaine in [F], which lifts the equivalence of categories of Theorem 1 in characteristic 0 and relates the $\mathbb{Z}_p$-adic representations of G to a category of modules over a Cohen ring, endowed with a "Frobenius" map and a group action.

Let R be an algebraically closed complete valuation (of rank 1) field of characteristic p and let H be a normal closed subgroup of G. Suppose that G acts continuously on R by ring automorphisms. Then $F := R^H$ is a perfect closed subfield of R.

For every integer $n \geq 1$, the ring $W_n(R)$ of Witt vectors of length n is endowed with the product of the topology on R defined by the valuation and then $W(R)$ with the inverse limit topology. Then the componentwise action of the group G is continuous and commutes with the natural Frobenius σ on $W(R)$. We also have $W(R)^H = W(F)$.

Let E be a closed subfield of F such that F is the completion of the p-radical closure of E in R. Suppose there exists a Cohen subring $\mathcal{O}_{\mathcal{E}}$ of $W(R)$ with residue field E and which is stable under the actions of σ and of G. Denote by $\mathcal{O}_{\widehat{\mathcal{E}}_{\mathrm{ur}}}$ the completion of the integral closure of $\mathcal{O}_{\mathcal{E}}$ in $W(R)$: it is a Cohen ring which is stable by σ and G, its residue field is the separable closure of E in R and $(\mathcal{O}_{\widehat{\mathcal{E}}_{\mathrm{ur}}})^H = \mathcal{O}_{\mathcal{E}}$.

The natural map from H to G_E is an isomorphism if and only if the action of H on R induces an isomorphism from H to G_F. We suppose that this is the case.

Definition. Let Γ be the quotient group G/H. An étale Φ-Γ-module over $\mathcal{O}_{\mathcal{E}}$ is a finitely generated $\mathcal{O}_{\mathcal{E}}$-module endowed with a σ-semi-linear Frobenius map $\varphi: M \to M$ and a continuous Γ-semi-linear action of Γ commuting with φ such that the image of φ generates the module M.

Étale Φ-Γ-modules over $\mathcal{O}_{\mathcal{E}}$ form an abelian category $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$ (the morphisms are the linear maps commuting with φ). There is a tensor product of Φ-Γ-modules, the natural one. For two objects M and N of $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$ the $\mathcal{O}_{\mathcal{E}}$-module $\mathrm{Hom}_{\mathcal{O}_{\mathcal{E}}}(M, N)$ can be endowed with an étale Φ-Γ-module structure (see [F]).

For every $\mathbb{Z}_p$-adic representation V of G, let $D_H(V)$ be the $\mathcal{O}_{\mathcal{E}}$-module $(\mathcal{O}_{\widehat{\mathcal{E}}_{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} V)^H$. It is naturally an étale Φ-Γ-module, with φ induced by the map $\sigma \otimes \mathrm{id}_V$ and Γ acting on both sides of the tensor product. From Theorem 2 one deduces the following fundamental result:

Theorem 3 (Fontaine, [F]). *The functor $V \to D_H(V)$ is an equivalence of the categories $\mathrm{Rep}_{\mathbb{Z}_p}(G)$ and $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$.*

Remark. If E is a field of positive characteristic, $\mathcal{O}_{\mathcal{E}}$ is a Cohen ring with residue field E endowed with a Frobenius σ, then we can easily extend the results of the whole subsection 6.1 to $\mathbb{Z}_p$-adic representations of G by using Theorem 3 for $G = G_E$ and $H = \{1\}$.

6.3. A brief survey of the theory of the field of norms

For the details we refer to [W], [FV] or [F].

Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic p. Put $G = G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$.

Let $\mathbb{C}$ be the completion of K^{sep}, denote the extension of the discrete valuation v_K of K to $\mathbb{C}$ by v_K. Let $R^* = \varprojlim \mathbb{C}_n^*$ where $\mathbb{C}_n = \mathbb{C}$ and the morphism from $\mathbb{C}_{n+1}$ to $\mathbb{C}_n$ is raising to the pth power. Put $R := R^* \cup \{0\}$ and define $v_R((x_n)) = v_K(x_0)$. For $(x_n), (y_n) \in R$ define

$$(x_n) + (y_n) = (z_n) \qquad \text{where } z_n = \lim_m (x_{n+m} + y_{n+m})^{p^m} .$$

Then R is an algebraically closed field of characteristic p complete with respect to v_R (cf. [W]). Its residue field is isomorphic to the algebraic closure of k and there is a natural continuous action of G on R. (Note that Fontaine denotes this field by $\mathrm{Fr}\, R$ in [F]).

Let L be a Galois extension of K in K^{sep}. Recall that one can always define the ramification filtration on $\mathrm{Gal}(L/K)$ in the upper numbering. Roughly speaking, L/K is an arithmetically profinite extension if one can define the lower ramification subgroups of G so that the classical relations between the two filtrations for finite extensions are preserved. This is in particular possible if $\mathrm{Gal}(L/K)$ is a p-adic Lie group with finite residue field extension.

The field R contains in a natural way the field of norms $N(L/K)$ of every arithmetically profinite extension L of K and the restriction of v to $N(L/K)$ is a discrete valuation. The residue field of $N(L/K)$ is isomorphic to that of L and $N(L/K)$ is stable under the action of G. The construction is functorial: if L' is a finite extension of L contained in K^{sep}, then L'/K is still arithmetically profinite and $N(L'/K)$ is a separable extension of $N(L/K)$. The direct limit of the fields $N(L'/K)$ where L' goes through all the finite extensions of L contained in K^{sep} is the separable closure E^{sep} of $E = N(L/K)$. It is stable under the action of G and the subgroup G_L identifies with G_E. The field E^{sep} is dense in R.

Fontaine described how to lift these constructions in characteristic 0 when L is the cyclotomic $\mathbb{Z}_p$-extension K_∞ of K. Consider the ring of Witt vectors $W(R)$ endowed with the Frobenius map σ and the natural componentwise action of G. Define the topology of $W(R)$ as the product of the topology defined by the valuation on R. Then one can construct a Cohen ring $\mathcal{O}_{\widehat{\mathcal{E}_{\mathrm{ur}}}}$ with residue field E^{sep} ($E = N(L/K)$) such that:

(i) $\mathcal{O}_{\widehat{\mathcal{E}}_{\mathrm{ur}}}$ is stable by σ and the action of G,

(ii) for every finite extension L of K_∞ the ring $(\mathcal{O}_{\widehat{\mathcal{E}}_{\mathrm{ur}}})^{G_L}$ is a Cohen ring with residue field E.

Denote by $\mathcal{O}_{\mathcal{E}(K)}$ the ring $(\mathcal{O}_{\widehat{\mathcal{E}}_{\mathrm{ur}}})^{G_{K_\infty}}$. It is stable by σ and the quotient $\Gamma = G/G_{K_\infty}$ acts continuously on $\mathcal{O}_{\mathcal{E}(K)}$ with respect to the induced topology. Fix a topological generator γ of Γ: it is a continuous ring automorphism commuting with σ. The fraction field of $\mathcal{O}_{\mathcal{E}(K)}$ is a two-dimensional standard local field (as defined in section 1 of Part I). If π is a lifting of a prime element of $N(K_\infty/K)$ in $\mathcal{O}_{\mathcal{E}(K)}$ then the elements of $\mathcal{O}_{\mathcal{E}(K)}$ are the series $\sum_{i\in\mathbb{Z}} a_i\pi^i$, where the coefficients a_i are in $W(k_{K_\infty})$ and converge p-adically to 0 when $i \to -\infty$.

6.4. Application of $\mathbb{Z}_p$-adic representations of G to the Galois cohomology

If we put together Fontaine's construction and the general formalism of subsection 6.2 we obtain the following important result:

Theorem 3' (Fontaine, [F]). *The functor $V \to D(V) := (\mathcal{O}_{\widehat{\mathcal{E}}_{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} V)^{G_{K_\infty}}$ defines an equivalence of the categories* $\mathrm{Rep}_{\mathbb{Z}_p}(G)$ *and* $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}(K)}}$.

Since for every $\mathbb{Z}_p$-adic representation of G we have $H^0(G,V) = V^G \simeq D(V)^\varphi$, we want now, as in paragraph 6.1, compute explicitly the cohomology of the representation using the Φ-Γ-module associated to V.

For every étale Φ-Γ-module (M,φ) define the following complex of abelian groups:

$$C_2(M): \qquad 0 \to M \xrightarrow{\alpha} M \oplus M \xrightarrow{\beta} M \to 0$$

where M stands at degree 0 and 2,

$$\alpha(x) = ((\varphi-1)x, (\gamma-1)x), \quad \beta((y,z)) = (\gamma-1)y - (\varphi-1)z.$$

By functoriality, we obtain a cohomological functor $(\mathcal{H}^i := H^i \circ C_2)_{i\in\mathbb{N}}$ from $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}(K)}}$ to the category of abelian groups.

Theorem 4 (Herr, [H]). *The cohomological functor $(\mathcal{H}^i \circ D)_{i\in\mathbb{N}}$ can be identified with the Galois cohomology functor $(H^i(G, .))_{i\in\mathbb{N}}$ for the category* $\mathrm{Rep}_{p\text{-tor}}(G)$. *So, if $M = D(V)$ then $\mathcal{H}^i(M)$ provides a simple explicit description of $H^i(G,V)$ in the p-torsion case.*

Idea of the proof of Theorem 4. We have to check that for every $i \geqslant 1$ the functor $\mathcal{H}^i$ is effaceable. For every p-torsion object $(M,\varphi_M) \in \Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}(K)}}$ and every $x \in \mathcal{H}^i(M)$ we construct an explicit embedding u of (M,φ_M) in a certain $(N,\varphi_N) \in \Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}(K)}}$

such that $\mathcal{H}^i(u)(x)$ is zero in $\mathcal{H}^i(N)$. For details see [H]. The key point is of topological nature: we prove, following an idea of Fontaine in [F], that there exists an open neighbourhood of 0 in M on which $(\varphi - 1)$ is bijective and use then the continuity of the action of Γ. □

As an application of theorem 4 we can prove the following result (due to Tate):

Theorem 5. *Assume that k_K is finite and let V be in* $\mathrm{Rep}_{p\text{-tor}}(G)$*. Without using class field theory the previous theorem implies that $H^i(G,V)$ are finite, $H^i(G,V) = 0$ for $i \geqslant 3$ and*

$$\sum_{i=0}^{2} l(H^i(G,V)) = -|K:\mathbb{Q}_p|\ l(V),$$

where $l(\ )$ denotes the length over $\mathbb{Z}_p$.

See [H].

Remark. Because the finiteness results imply that the Mittag–Leffler conditions are satisfied, it is possible to generalize the explicit construction of the cohomology and to prove analogous results for $\mathbb{Z}_p$ (or $\mathbb{Q}_p$)-adic representations by passing to the inverse limits.

6.5. A new approach to local class field theory

The results of the preceding paragraph allow us to prove without using class field theory the following:

Theorem 6 (Tate's local duality)**.** *Let V be in* $\mathrm{Rep}_{p\text{-tor}}(G)$ *and $n \in \mathbb{N}$ such that $p^n V = 0$. Put $V^*(1) :=$* $\mathrm{Hom}(V, \mu_{p^n})$*. Then there is a canonical isomorphism from $H^2(G, \mu_{p^n})$ to $\mathbb{Z}/p^n$ and the cup product*

$$H^i(G,V) \times H^{2-i}(G,V^*(1)) \xrightarrow{\cup} H^2(G,\mu_{p^n}) \simeq \mathbb{Z}/p^n$$

is a perfect pairing.

It is well known that a proof of the local duality theorem of Tate without using class field theory gives a construction of the reciprocity map. For every $n \geqslant 1$ we have by duality a functorial isomorphism between the finite groups $\mathrm{Hom}(G,\mathbb{Z}/p^n) = H^1(G,\mathbb{Z}/p^n)$ and $H^1(G,\mu_{p^n})$ which is isomorphic to $K^*/(K^*)^{p^n}$ by Kummer theory. Taking the inverse limits gives us the p-part of the reciprocity map, the most difficult part.

Sketch of the proof of Theorem 6. ([H2]).

a) Introduction of differentials:

Let us denote by Ω_c^1 the $\mathcal{O}_{\mathcal{E}(K)}$-module of continuous differential forms of $\mathcal{O}_{\mathcal{E}}$ over $W(k_{K_\infty})$. If π is a fixed lifting of a prime element of $E(K_\infty/K)$ in $\mathcal{O}_{\mathcal{E}(K)}$, then this module is free and generated by $d\pi$. Define the residue map from Ω_c^1 to $W(k_{K_\infty})$ by $\operatorname{res}\left(\sum_{i\in\mathbb{Z}} a_i\pi^i d\pi\right) := a_{-1}$; it is independent of the choice of π.

b) Calculation of some Φ-Γ-modules:

The $\mathcal{O}_{\mathcal{E}(K)}$-module Ω_c^1 is endowed with an étale Φ-Γ-module structure by the following formulas: for every $\lambda \in \mathcal{O}_{\mathcal{E}(K)}$ we put:

$$p\varphi(\lambda d\pi) = \sigma(\lambda)d(\sigma(\pi)) \ \ , \ \ \gamma(\lambda d\pi) = \gamma(\lambda)d(\gamma(\pi)).$$

The fundamental fact is that there is a natural isomorphism of Φ-Γ-modules over $\mathcal{O}_{\mathcal{E}(K)}$ between $D(\mu_{p^n})$ and the reduction $\Omega_{c,n}^1$ of Ω_c^1 modulo p^n.

The étale Φ-Γ-module associated to the representation $V^*(1)$ is $\widetilde{M} := \operatorname{Hom}(M, \Omega_{c,n}^1)$, where $M = D(V)$. By composing the residue with the trace we obtain a surjective and continuous map Tr_n from M to $\mathbb{Z}/p^n$. For every $f \in \widetilde{M}$, the map $\operatorname{Tr}_n \circ f$ is an element of the group $M^\vee$ of continuous group homomorphisms from M to $\mathbb{Z}/p^n$. This gives in fact a group isomorphism from $\widetilde{M}$ to $M^\vee$ and we can therefore transfer the Φ-Γ-module structure from $\widetilde{M}$ to $M^\vee$. But, since k is finite, M is locally compact and $M^\vee$ is in fact the Pontryagin dual of M.

c) Pontryagin duality implies local duality:

We simply dualize the complex $C_2(M)$ using Pontryagin duality (all arrows are strict morphisms in the category of topological groups) and obtain a complex:

$$C_2(M)^\vee : \qquad 0 \to M^\vee \xrightarrow{\beta^\vee} M^\vee \oplus M^\vee \xrightarrow{\alpha^\vee} M^\vee \to 0,$$

where the two $M^\vee$ are in degrees 0 and 2. Since we can construct an explicit quasi-isomorphism between $C_2(M^\vee)$ and $C_2(M)^\vee$, we easily obtain a duality between $\mathcal{H}^i(M)$ and $\mathcal{H}^{2-i}(M^\vee)$ for every $i \in \{0,1,2\}$.

d) The canonical isomorphism from $\mathcal{H}^2(\Omega_{c,n}^1)$ to $\mathbb{Z}/p^n$:

The map Tr_n from $\Omega_{c,n}^1$ to $\mathbb{Z}/p^n$ factors through the group $\mathcal{H}^2(\Omega_{c,n}^1)$ and this gives an isomorphism. But it is not canonical! In fact the construction of the complex $C_2(M)$ depends on the choice of γ. Fortunately, if we take another γ, we get a quasi-isomorphic complex and if we normalize the map Tr_n by multiplying it by the unit $-p^{v_p(\log\chi(\gamma))}/\log\chi(\gamma)$ of $\mathbb{Z}_p$, where $\log$ is the p-adic logarithm, χ the cyclotomic character and $v_p = v_{\mathbb{Q}_p}$, then everything is compatible with the change of γ.

e) The duality is given by the cup product:

We can construct explicit formulas for the cup product:

$$\mathcal{H}^i(M) \times \mathcal{H}^{2-i}(M^\vee) \xrightarrow{\cup} \mathcal{H}^2(\Omega_{c,n}^1)$$

associated with the cohomological functor $(\mathcal{H}^i)_{i\in\mathbb{N}}$ and we compose them with the preceding normalized isomorphism from $\mathcal{H}^2(\Omega^1_{c,n})$ to $\mathbb{Z}/p^n$. Since everything is explicit, we can compare with the pairing obtained in c) and verify that it is the same up to a unit of $\mathbb{Z}_p$. □

Remark. Benois, using the previous theorem, deduced an explicit formula of Coleman's type for the Hilbert symbol and proved Perrin-Riou's formula for crystalline representations ([B]).

6.6. Explicit formulas for the generalized Hilbert symbol on formal groups

Let K_0 be the fraction field of the ring W_0 of Witt vectors with coefficients in a finite field of characteristic $p > 2$ and $\mathcal{F}$ a commutative formal group of finite height h defined over W_0.

For every integer $n \geqslant 1$, denote by $\mathcal{F}[p^n]$ the p^n-torsion points in $\mathcal{F}(\mathcal{M}_C)$, where $\mathcal{M}_C$ is the maximal ideal of the completion C of an algebraic closure of K_0. The group $\mathcal{F}[p^n]$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^h$.

Let K be a finite extension of K_0 contained in K^{sep} and assume that the points of $\mathcal{F}[p^n]$ are defined over K. We then have a bilinear pairing:

$$(\ , \ ]_{\mathcal{F},n} \colon G_K^{\text{ab}} \times \mathcal{F}(\mathcal{M}_K) \to \mathcal{F}[p^n]$$

(see section 8 of Part I).

When the field K contains a primitive p^nth root of unity ζ_{p^n}, Abrashkin gives an explicit description for this pairing generalizing the classical Brückner–Vostokov formula for the Hilbert symbol ([A]). In his paper he notices that the formula makes sense even if K does not contain ζ_{p^n} and he asks whether it holds without this assumption. In a recent unpublished work, Benois proves that this is true.

Suppose for simplicity that K contains only ζ_p. Abrashkin considers in his paper the extension $\widetilde{K} := K(\pi^{p^{-r}}, r \geqslant 1)$, where π is a fixed prime element of K. It is not a Galois extension of K but is arithmetically profinite, so by [W] one can consider the field of norms for it. In order not to loose information given by the roots of unity of order a power of p, Benois uses the composite Galois extension $L := K_\infty \widetilde{K}/K$ which is arithmetically profinite. There are several problems with the field of norms $N(L/K)$, especially it is not clear that one can lift it in characteristic 0 with its Galois action. So, Benois simply considers the completion F of the p-radical closure of $E = N(L/K)$ and its separable closure F^{sep} in R. If we apply what was explained in subsection 6.2 for $\Gamma = \text{Gal}(L/K)$, we get:

Theorem 7. *The functor* $V \to D(V) := (W(F^{\text{sep}}) \otimes_{\mathbb{Z}_p} V)^{G_L}$ *defines an equivalence of the categories* $\text{Rep}_{\mathbb{Z}_p}(G)$ *and* $\Phi\Gamma M^{\text{ét}}_{W(F)}$.

Choose a topological generator γ' of $\mathrm{Gal}(L/K_\infty)$ and lift γ to an element of $\mathrm{Gal}(L/\widetilde{K})$. Then Γ is topologically generated by γ and γ', with the relation $\gamma\gamma' = (\gamma')^a\gamma$, where $a = \chi(\gamma)$ (χ is the cyclotomic character). For $(M,\varphi) \in \Phi\Gamma M^{\text{ét}}_{W(F)}$ the continuous action of $\mathrm{Gal}(L/K_\infty)$ on M makes it a module over the Iwasawa algebra $\mathbb{Z}_p[[\gamma' - 1]]$. So we can define the following complex of abelian groups:

$$C_3(M): \qquad 0 \to M_0 \xrightarrow{\alpha\mapsto A_0\alpha} M_1 \xrightarrow{\alpha\mapsto A_1\alpha} M_2 \xrightarrow{\alpha\mapsto A_2\alpha} M_3 \to 0$$

where M_0 is in degree 0, $M_0 = M_3 = M$, $M_1 = M_2 = M^3$,

$$A_0 = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \\ \gamma' - 1 \end{pmatrix}, A_1 = \begin{pmatrix} \gamma - 1 & 1-\varphi & 0 \\ \gamma' - 1 & 0 & 1 - \varphi \\ 0 & {\gamma'}^a - 1 & \delta - \gamma \end{pmatrix}, A_2 = ((\gamma')^a - 1 \;\; \delta - \gamma \;\; \varphi - 1)$$

and $\delta = ((\gamma')^a - 1)(\gamma' - 1)^{-1} \in \mathbb{Z}_p[[\gamma' - 1]]$.

As usual, by taking the cohomology of this complex, one defines a cohomological functor $(\mathcal{H}^i)_{i\in\mathbb{N}}$ from $\Phi\Gamma M^{\text{ét}}_{W(F)}$ in the category of abelian groups. Benois proves only that the cohomology of a p-torsion representation V of G injects in the groups $\mathcal{H}^i(D(V))$ which is enough to get the explicit formula. But in fact a stronger statement is true:

Theorem 8. *The cohomological functor $(\mathcal{H}^i \circ D)_{i\in\mathbb{N}}$ can be identified with the Galois cohomology functor $(H^i(G,.))_{i\in\mathbb{N}}$ for the category* $\mathrm{Rep}_{p\text{-tor}}(G)$.

Idea of the proof. Use the same method as in the proof of Theorem 4. It is only more technically complicated because of the structure of Γ. □

Finally, one can explicitly construct the cup products in terms of the groups $\mathcal{H}^i$ and, as in [B], Benois uses them to calculate the Hilbert symbol.

Remark. Analogous constructions (equivalence of category, explicit construction of the cohomology by a complex) seem to work for higher dimensional local fields. In particular, in the two-dimensional case, the formalism is similar to that of this paragraph; the group Γ acting on the Φ-Γ-modules has the same structure as here and thus the complex is of the same form. This work is still in progress.

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7. Recovering higher global and local fields from Galois groups — an algebraic approach

Ido Efrat

7.0. Introduction

We consider the following general problem: let F be a known field with absolute Galois group G_F. Let K be a field with $G_K \simeq G_F$. What can be deduced about the arithmetic structure of K?

As a prototype of this kind of questions we recall the celebrated Artin–Schreier theorem: $G_K \simeq G_{\mathbb{R}}$ if and only if K is real closed. Likewise, the fields K with $G_K \simeq G_E$ for some finite extension E of $\mathbb{Q}_p$ are the p-adically closed fields (see [Ne], [P1], [E1], [K]). Here we discuss the following two cases:

1. *K is a higher global field*
2. *K is a higher local field*

7.1. Higher global fields

We call a field *finitely generated* (or a *higher global field*) if it is finitely generated over its prime subfield. The (proven) 0-dimensional case of Grothendieck's anabelian conjecture ([G1], [G2]) can be stated as follows:

Let K, F be finitely generated infinite fields. Any isomorphism $G_K \simeq G_F$ is induced in a functorial way by an (essentially unique) isomorphism of the algebraic closures of K and F.

This statement was proven:
- by Neukirch [Ne] for finite normal extensions of $\mathbb{Q}$;
- by Iwasawa (unpublished) and Uchida [U1–3] (following Ikeda [I]) for all global fields;
- by Pop [P2] and Spiess [S] for function fields in one variable over $\mathbb{Q}$;

- by Pop ([P3–5]) in general.

For recent results on the 1-*dimensional* anabelian conjecture – see the works of Mochizuki [M], Nakamura [N] and Tamagawa [T].

7.2. Earlier approaches

Roughly speaking, the above proofs in the 0-dimensional case are divided into a local part and a global part. To explain the local part, define the Kronecker dimension $\dim(K)$ of a field K as $\mathrm{trdeg}(K/\mathbb{F}_p)$ if $\mathrm{char}(K) = p > 0$, and as $\mathrm{trdeg}(K/\mathbb{Q}) + 1$ if $\mathrm{char}(K) = 0$. Now let v be a Krull valuation on K (not necessarily discrete or of rank 1) with residue field $\overline{K}_v$. It is called 1-*defectless* if $\dim K = \dim \overline{K}_v + 1$. The main result of the local theory is the following *local correspondence*: given an isomorphism $\varphi: G_K \xrightarrow{\sim} G_F$, a closed subgroup Z of G_K is the decomposition group of some 1-defectless valuation v on K if and only if $\varphi(Z)$ is the decomposition group of some 1-defectless valuation v' on F. The 'global theory' then combines the isomorphisms between the corresponding decomposition fields to construct the desired isomorphism of the algebraic closures (see [P5] for more details).

The essence of the local correspondence is clearly the detection of valuations on a field K just from the knowledge of the group-theoretic structure of G_K. In the earlier approaches this was done by means of various *Hasse principles*; i.e., using the injectivity of the map

$$H(K) \to \prod_{v \in S} H(K_v^h)$$

for some cohomological functor H and some set S of non-trivial valuations on K, where K_v^h is the henselization of K with respect to v. Indeed, if this map is injective and $H(K) \neq 0$ then $H(K_v^h) \neq 0$ for at least one $v \in S$. In this way one finds "arithmetically interesting" valuations on K.

In the above-mentioned works the local correspondence was proved using known Hasse principles for:

(1) Brauer groups over global fields (Brauer, Hasse, Noether);
(2) Brauer groups over function fields in one variable over local fields (Witt, Tate, Lichtenbaum, Roquette, Sh. Saito, Pop);
(3) $H^3(G_K, \mathbb{Q}/\mathbb{Z}(2))$ over function fields in one variable over $\mathbb{Q}$ (Kato, Jannsen).

Furthermore, in his proof of the 0-dimensional anabelian conjecture in its general case, Pop uses a *model-theoretic* technique to transfer the Hasse principles in (2) to a more general context of conservative function fields in one variable over certain henselian valued fields. More specifically, by a deep result of Kiesler–Shelah, a property is elementary in a certain language (in the sense of the first-order predicate calculus) if and only if it is preserved by isomorphisms of models in the language, and both the property and its negation are preserved by nonprincipal ultrapowers. It turns out that in an appropriate setting, the Hasse principle for the Brauer groups satisfies these conditions, hence has an elementary nature. One can now apply model-completeness results on tame valued fields by F.-V. Kuhlmann [Ku].

This led one to the problem of finding an *algebraic* proof of the local correspondence, i.e., a proof which does not use non-standard arguments (see [S, p. 115]; other model-theoretic techniques which were earlier used in the global theory of [P2] were replaced by Spiess in [S] by algebraic ones).

We next explain how this can indeed be done (see [E3] for details and proofs).

7.3. Construction of valuations from K-theory

Our algebraic approach to the local correspondence is based on a K-theoretic (yet elementary) construction of valuations, which emerged in the early 1980's in the context of quadratic form theory (in works of Jacob [J], Ware [W], Arason–Elman–Jacob [AEJ], Hwang–Jacob [HJ]; see the survey [E2]). We also mention here the alternative approaches to such constructions by Bogomolov [B] and Koenigsmann [K]. The main result of (the first series of) these constructions is:

Theorem 1. *Let p be a prime number and let E be a field. Assume that* $\operatorname{char}(E) \neq p$ *and that $\langle -1, E^{*p}\rangle \leqslant T \leqslant E^*$ is an intermediate group such that:*
(a) *for all $x \in E^* \setminus T$ and $y \in T \setminus E^{*p}$ one has $\{x, y\} \neq 0$ in $K_2(E)$*
(b) *for all $x, y \in E^*$ which are $\mathbb{F}_p$-linearly independent mod T one has $\{x, y\} \neq 0$ in $K_2(E)$.*

Then there exists a valuation v on E with value group Γ_v such that:
(i) $\operatorname{char}(\overline{E}_v) \neq p$*;*
(ii) $\dim_{\mathbb{F}_p}(\Gamma_v/p) \geqslant \dim_{\mathbb{F}_p}(E^*/T) - 1$*;*
(iii) *either* $\dim_{\mathbb{F}_p}(\Gamma_v/p) = \dim_{\mathbb{F}_p}(E^*/T)$ *or* $\overline{E}_v \neq \overline{E}_v^p$.

In particular we have:

Corollary. *Let p be a prime number and let E be a field. Suppose that* $\operatorname{char}(E) \neq p$, $-1 \in E^{*p}$, *and that the natural symbolic map induces an isomorphism*

$$\wedge^2(E^*/E^{*p}) \xrightarrow{\sim} K_2(E)/p.$$

Then there is a valuation v on E such that
(i) $\operatorname{char}(\overline{E}_v) \neq p$*;*
(ii) $\dim_{\mathbb{F}_p}(\Gamma_v/p) \geqslant \dim_{\mathbb{F}_p}(E^*/E^{*p}) - 1$*;*
(iii) *either* $\dim_{\mathbb{F}_p}(\Gamma_v/p) = \dim_{\mathbb{F}_p}(E^*/E^{*p})$ *or* $\overline{E}_v \neq \overline{E}_v^p$.

We remark that the construction used in the proof of Theorem 1 is of a completely explicit and elementary nature. Namely, one chooses a certain intermediate group $T \leqslant H \leqslant E^*$ with $(H:T)|p$ and denotes

$$O^- = \{x \in E \setminus H \ : \ 1 - x \in T\}, \qquad O^+ = \{x \in H \ : \ xO^- \subset O^-\}.$$

It turns out that $O = O^- \cup O^+$ is a valuation ring on E, and the corresponding valuation v is as desired.

The second main ingredient is the following henselianity criterion proven in [E1]:

Proposition 1. *Let p be a prime number and let (E, v) be a valued field with* $\operatorname{char}(\overline{E}_v) \neq p$, *such that the maximal pro-p Galois group $G_{\overline{E}_v}(p)$ of $\overline{E}_v$ is infinite. Suppose that*

$$\sup_{E'} \operatorname{rk}(G_{E'}(p)) < \infty$$

with E' ranging over all finite separable extensions of E. Then v is henselian.

Here the *rank* $\operatorname{rk}(G)$ of a profinite group G is its minimal number of (topological) generators.

After translating the Corollary to the Galois-theoretic language using Kummer theory and the Merkur'ev–Suslin theorem and using Proposition 1 we obtain:

Proposition 2. *Let p be a prime number and let E be a field such that* $\operatorname{char}(E) \neq p$. *Suppose that for every finite separable extension E' of E one has*
(1) $H^1(G_{E'}, \mathbb{Z}/p) \simeq (\mathbb{Z}/p)^{n+1}$;
(2) $H^2(G_{E'}, \mathbb{Z}/p) \simeq \wedge^2 H^1(G_{E'}, \mathbb{Z}/p)$ *via the cup product;*
(3) $\dim_{\mathbb{F}_p}(\Gamma_u/p) \leqslant n$ *for every valuation u on E'.*

Then there exists a henselian valuation v on E such that $\operatorname{char}(\overline{E}_v) \neq p$ *and* $\dim_{\mathbb{F}_p}(\Gamma_v/p) = n$.

7.4. A Galois characterization of 1-defectless valuations

For a field L and a prime number p, we recall that the virtual p-cohomological dimension $\operatorname{vcd}_p(G_L)$ is the usual p-cohomological dimension $\operatorname{cd}_p(G_L)$ if $\operatorname{char}(L) \neq 0$ and is $\operatorname{vcd}_p(G_{L(\sqrt{-1})})$ if $\operatorname{char}(L) = 0$.

Definition. Let p be a prime number and let L be a field with $n = \dim L < \infty$ and $\operatorname{char}(L) \neq p$. We say that L is *p-divisorial* if there exist subfields $L \subset E \subset M \subset L^{\mathrm{sep}}$ such that
(a) M/L is Galois;
(b) every p-Sylow subgroup of G_M is isomorphic to $\mathbb{Z}_p$;
(c) the virtual p-cohomological dimension $\operatorname{vcd}_p(G_L)$ of G_L is $n+1$;

(d) either $n = 1$ or $\mathrm{Gal}(M/L)$ has no non-trivial closed normal pro-soluble subgroups;
(e) for every finite separable extension E'/E one has

$$H^1(G_{E'}, \mathbb{Z}/p) \simeq (\mathbb{Z}/p)^{n+1}, \quad H^2(G_{E'}, \mathbb{Z}/p) \simeq \wedge^2 H^1(G_{E'}, \mathbb{Z}/p)$$

via the cup product.

The main result is now:

Theorem 2 ([E3]). *Let p be a prime number and let K be a finitely generated field of characteristic different from p. Let L be an algebraic extension of K. Then the following conditions are equivalent:*
(i) *there exists a 1-defectless valuation v on K such that $L = K_v^h$;*
(ii) *L is a minimal p-divisorial separable algebraic extension of K.*

Idea of proof. Suppose first that v is a 1-defectless valuation on K. Take $L = K_v^h$ and let M be a maximal unramified extension of L. Also let w be a valuation on K such that $\Gamma_w \simeq \mathbb{Z}^{\dim(K)}$, $\mathrm{char}(\overline{K}_w) \neq p$, and such that the corresponding valuation rings satisfy $\mathcal{O}_w \subset \mathcal{O}_v$. Let K_w^h be a henselization of (K, w) containing L and take $E = K_w^h(\mu_p)$ ($E = K_w^h(\mu_4)$ if $p = 2$). One shows that L is p-divisorial with respect to this tower of of extensions.

Conversely, suppose that L is p-divisorial, and let $L \subset E \subset M \subset L^{\mathrm{sep}}$ be a tower of extensions as in the definition above. Proposition 2 gives rise to a henselian valuation w on E such that $\mathrm{char}(\overline{E}_w) \neq p$ and $\dim_{\mathbb{F}_p}(\Gamma_w/p) = \dim(K)$. Let w_0 be the unique valuation on E of rank 1 such that $\mathcal{O}_{w_0} \supset \mathcal{O}_w$, and let u be its restriction to L. The unique extension u_M of w_0 to M is henselian. Since M/L is normal, every extension of u to M is conjugate to u_M, hence is also henselian. By a classical result of F.-K. Schmidt, the non-separably closed field M can be henselian with respect to at most one valuation of rank 1. Conclude that u is henselian as well. One then shows that it is 1-defectless.

The equivalence of (i) and (ii) now follows from these two remarks, and a further application of F.-K. Schmidt's theorem.

The local correspondence now follows from the observation that condition (ii) of the Theorem is actually a condition on the closed subgroup G_L of the profinite group G_K (note that $\dim(L) = \mathrm{vcd}(G_K) - 1$).

7.5. Higher local fields

Here we report on a joint work with Fesenko [EF].

An analysis similar to the one sketched in the case of higher global fields yields:

Theorem 3 ([EF]). *Let F be an n-dimensional local field. Suppose that the canonical valuation on F of rank n has residue characteristic p. Let K be a field such that $G_K \simeq G_F$. Then there is a henselian valuation v on K such that $\Gamma_v/l \simeq (\mathbb{Z}/l)^n$ for every prime number $l \neq p$ and such that $\operatorname{char}(\overline{K}_v) = p$ or 0.*

Theorem 4 ([EF]). *Let $q = p^r$ be a prime power and let K be a field with $G_K \simeq G_{\mathbb{F}_q((t))}$. Then there is a henselian valuation v on K such that*
(1) *$\Gamma_v/l \simeq \mathbb{Z}/l$ for every prime number $l \neq p$;*
(2) *$\operatorname{char}(\overline{K}_v) = p$;*
(3) *the maximal prime-to-p Galois group $G_{\overline{K}_v}(p')$ of $\overline{K}_v$ is isomorphic to $\prod_{l\neq p}\mathbb{Z}_l$;*
(4) *if $\operatorname{char}(K) = 0$ then $\Gamma_v = p\Gamma_v$ and $\overline{K}_v$ is perfect.*

Moreover, for every positive integer d there exist valued fields (K, v) as in Theorem 4 with characteristic p and for which $\Gamma_v/p \simeq (\mathbb{Z}/p)^d$. Likewise there exist examples with $\Gamma_v \simeq \mathbb{Z}$, $G_{\overline{K}_v} \not\simeq \hat{\mathbb{Z}}$ and $\overline{K}_v$ imperfect, as well as examples with $\operatorname{char}(K) = 0$.

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8. Higher local skew fields

Alexander Zheglov

n-dimensional local skew fields are a natural generalization of n-dimensional local fields. The latter have numerous applications to problems of algebraic geometry, both arithmetical and geometrical, as it is shown in this volume. From this viewpoint, it would be reasonable to restrict oneself to commutative fields only. Nevertheless, already in class field theory one meets non-commutative rings which are skew fields finite-dimensional over their center K. For example, K is a (commutative) local field and the skew field represents elements of the Brauer group of the field K (see also an example below). In [Pa] A.N. Parshin pointed out another class of non-commutative local fields arising in differential equations and showed that these skew fields possess many features of commutative fields. He defined a skew field of formal pseudo-differential operators in n variables and studied some of their properties. He raised a problem of classifying non-commutative local skew fields.

In this section we treat the case of $n = 2$ and list a number of results, in particular a classification of certain types of 2-dimensional local skew fields.

8.1. Basic definitions

Definition. A skew field K is called a *complete discrete valuation skew field* if K is complete with respect to a discrete valuation (the residue skew field is not necessarily commutative). A field K is called an *n-dimensional local skew field* if there are skew fields $K = K_n, K_{n-1}, \dots, K_0$ such that each K_i for $i > 0$ is a complete discrete valuation skew field with residue skew field K_{i-1}.

Examples.

(1) Let k be a field. Formal pseudo-differential operators over $k((X))$ form a 2-dimensional local skew field $K = k((X))((\partial_X^{-1}))$, $\partial_X X = X\partial_X + 1$. If $\mathrm{char}(k) = 0$ we get an example of a skew field which is an infinite dimensional vector space over its centre.

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(2) Let L be a local field of equal characteristic (of any dimension). Then an element of $\mathrm{Br}(L)$ is an example of a skew field which is finite dimensional over its centre.

From now on *let* K be a two-dimensional local skew field. Let t_2 be a generator of $\mathcal{M}_{K_2}$ and t_1' be a generator of $\mathcal{M}_{K_1}$. If $t_1 \in K$ is a lifting of t_1' then t_1, t_2 is called a *system of local parameters* of K. We denote by v_{K_2} and v_{K_1} the (surjective) discrete valuations of K_2 and K_1 associated with t_2 and t_1'.

Definition. A two-dimensional local skew field K is said to *split* if there is a section of the homomorphism $\mathcal{O}_{K_2} \to K_1$ where $\mathcal{O}_{K_2}$ is the ring of integers of K_2.

Example (N. Dubrovin). Let $\mathbb{Q}((u))\langle x, y\rangle$ be a free associative algebra over $\mathbb{Q}((u))$ with generators x, y. Let $I = \langle [x,[x,y]], [y,[x,y]]\rangle$. Then the quotient

$$A = \mathbb{Q}((u))\langle x, y\rangle / I$$

is a $\mathbb{Q}$-algebra which has no non-trivial zero divisors, and in which $z = [x,y] + I$ is a central element. Any element of A can be uniquely represented in the form

$$f_0 + f_1 z + \ldots + f_m z^m$$

where $f_0, \ldots, f_m$ are polynomials in the variables x, y.

One can define a discrete valuation w on A such that $w(x) = w(y) = w(\mathbb{Q}((u))) = 0$, $w([x,y]) = 1$, $w(a) = k$ if $a = f_k z^k + \ldots + f_m z^m$, $f_k \neq 0$. The skew field B of fractions of A has a discrete valuation v which is a unique extension of w. The completion of B with respect to v is a two-dimensional local skew field which does not split (for details see [Zh, Lemma 9]).

Definition. Assume that K_1 is a field. The homomorphism

$$\varphi_0 \colon K^* \to \mathrm{Int}(K), \quad \varphi_0(x)(y) = x^{-1} y x$$

induces a homomorphism $\varphi \colon K_2^*/\mathcal{O}_{K_2}^* \to \mathrm{Aut}(K_1)$. The *canonical automorphism* of K_1 is $\alpha = \varphi(t_2)$ where t_2 is an arbitrary prime element of K_2.

Definition. Two two-dimensional local skew fields K and K' are *isomorphic* if there is an isomorphism $K \to K'$ which maps $\mathcal{O}_K$ onto $\mathcal{O}_{K'}$, $\mathcal{M}_K$ onto $\mathcal{M}_{K'}$ and $\mathcal{O}_{K_1}$ onto $\mathcal{O}_{K_1'}$, $\mathcal{M}_{K_1}$ onto $\mathcal{M}_{K_1'}$.

8.2. Canonical automorphisms of infinite order

Theorem.

(1) *Let K be a two-dimensional local skew field. If $\alpha^n \neq \mathrm{id}$ for all $n \geqslant 1$ then* $\mathrm{char}(K_2) = \mathrm{char}(K_1)$*, K splits and K is isomorphic to a two-dimensional local skew field $K_1((t_2))$ where $t_2 a = \alpha(a) t_2$ for all $a \in K_1$.*

(2) *Let K, K' be two-dimensional local skew fields and let K_1, K_1' be fields. Let $\alpha^n \neq \mathrm{id}$, $\alpha'^n \neq \mathrm{id}$ for all $n \geqslant 1$. Then K is isomorphic to K' if and only if there is an isomorphism $f\colon K_1 \to K_1'$ such that $\alpha = f^{-1}\alpha' f$ where α, α' are the canonical automorphisms of K_1 and K_1'.*

Remarks.

1. This theorem is true for any higher local skew field.
2. There are examples (similar to Dubrovin's example) of local skew fields which do not split and in which $\alpha^n = \mathrm{id}$ for some positive integer n.

Proof. (2) follows from (1). We sketch the proof of (1). For details see [Zh, Th.1].

If $\mathrm{char}(K) \neq \mathrm{char}(K_1)$ then $\mathrm{char}(K_1) = p > 0$. Hence $v(p) = r > 0$. Then for any element $t \in K$ with $v(t) = 0$ we have $ptp^{-1} \equiv \alpha^r(\bar t) \bmod \mathcal{M}_K$ where $\bar t$ is the image of t in K_1. But on the other hand, $pt = tp$, a contradiction.

Let F be the prime field in K. Since $\mathrm{char}(K) = \mathrm{char}(K_1)$ the field F is a subring of $\mathcal{O} = \mathcal{O}_{K_2}$. One can easily show that there exists an element $c \in K_1$ such that $\alpha^n(c) \neq c$ for every $n \geqslant 1$ [Zh, Lemma 5].

Then any lifting c' in $\mathcal{O}$ of c is transcendental over F. Hence we can embed the field $F(c')$ in $\mathcal{O}$. Let $\bar L$ be a maximal field extension of $F(c')$ which can be embedded in $\mathcal{O}$. Denote by L its image in $\mathcal{O}$. Take $\bar a \in K_1 \setminus \bar L$. We claim that there exists a lifting $a' \in \mathcal{O}$ of $\bar a$ such that a' commutes with every element in L. To prove this fact we use the completeness of $\mathcal{O}$ in the following argument.

Take any lifting a in $\mathcal{O}$ of $\bar a$. For every element $x \in L$ we have $axa^{-1} \equiv x \bmod \mathcal{M}_K$. If t_2 is a prime element of K_2 we can write

$$axa^{-1} = x + \delta_1(x) t_2$$

where $\delta_1(x) \in \mathcal{O}$. The map $\overline{\delta_1}\colon L \ni x \to \overline{\delta_1(x)} \in K_1$ is an α-derivation, i.e.

$$\overline{\delta_1}(ef) = \overline{\delta_1}(e)\alpha(f) + e\overline{\delta_1}(f)$$

for all $e, f \in L$. Take an element h such that $\alpha(h) \neq h$, then $\overline{\delta_1}(a) = g\alpha(a) - ag$ where $g = \overline{\delta_1}(h)/(\alpha(h) - h)$. Therefore there is $a_1 \in K_1$ such that

$$(1 + a_1 t_2) axa^{-1} (1 + a_1 t_2)^{-1} \equiv x \bmod \mathcal{M}_K^2.$$

By induction we can find an element $a' = \ldots \cdot (1 + a_1 t_2) a$ such that $a' x a'^{-1} = x$.

Now, if $\overline{a}$ is not algebraic over $\overline{L}$, then for its lifting $a' \in \mathcal{O}$ which commutes with L we would deduce that $L(a')$ is a field extension of $F(c')$ which can be embedded in $\mathcal{O}$, which contradicts the maximality of L.

Hence $\overline{a}$ is algebraic and separable over $\overline{L}$. Using a generalization of Hensel's Lemma [Zh, Prop.4] we can find a lifting a' of $\overline{a}$ such that a' commutes with elements of L and a' is algebraic over L, which again leads to a contradiction.

Finally let $\overline{a}$ be purely inseparable over $\overline{L}$, $\overline{a}^{p^k} = \overline{x}$, $x \in L$. Let a' be its lifting which commutes with every element of L. Then ${a'}^{p^k} - x$ commutes with every element of L. If $v_K({a'}^{p^k} - x) = r \neq \infty$ then similarly to the beginning of this proof we deduce that the image of $({a'}^{p^k} - x)c({a'}^{p^k} - x)^{-1}$ in K_1 is equal to $\alpha^r(c)$ (which is distinct from c), a contradiction. Therefore, ${a'}^{p^k} = x$ and the field $L(a')$ is a field extension of $F(c')$ which can be embedded in $\mathcal{O}$, which contradicts the maximality of L.

Thus, $\overline{L} = K_1$.

To prove that K is isomorphic to a skew field $K_1((t_2))$ where $t_2 a = \alpha(a) t_2$ one can apply similar arguments as in the proof of the existence of an element a' such that $a' x {a'}^{-1} = x$ (see above). So, one can find a parameter t_2 with a given property. $\square$

In some cases we have a complete classification of local skew fields.

Proposition ([Zh]). *Assume that K_1 is isomorphic to $k((t_1))$. Put*

$$\zeta = \alpha(t_1)t_1^{-1} \bmod \mathcal{M}_{K_1}.$$

Put $i_\alpha = 1$ if ζ is not a root of unity in k and $i_\alpha = v_{K_1}(\alpha^n(t_1) - t_1)$ if ζ is a primitive nth root. Assume that k is of characteristic zero. Then there is an automorphism $f \in \mathrm{Aut}_k(K_1)$ such that $f^{-1}\alpha f = \beta$ where

$$\beta(t_1) = \zeta t_1 + x t_1^{i_\alpha} + x^2 y t_1^{2i_\alpha - 1}$$

for some $x \in k^*/k^{*(i_\alpha - 1)}$, $y \in k$.

Two automorphisms α and β are conjugate if and only if

$$(\zeta(\alpha), i_\alpha, x(\alpha), y(\alpha)) = (\zeta(\beta), i_\beta, x(\beta), y(\beta)).$$

Proof. First we prove that $\alpha = f\beta' f^{-1}$ where

$$\beta'(t_1) = \zeta t_1 + x t_1^{in+1} + y t_1^{2in+1}$$

for some natural i. Then we prove that $i_\alpha = i_{\beta'}$.

Consider a set $\{\alpha_i : i \in \mathbb{N}\}$ where $\alpha_i = f_i \alpha_{i-1} f_i^{-1}$, $f_i(t_1) = t_1 + x_i t_1^i$ for some $x_i \in k$, $\alpha_1 = \alpha$. Write

$$\alpha_i(t_1) = \zeta t_1 + a_{2,i} t_1^2 + a_{3,i} t_1^3 + \dots .$$

One can check that $a_{2,2} = x_2(\zeta^2 - \zeta) + a_{2,1}$ and hence there exists an element $x_2 \in k$ such that $a_{2,2} = 0$. Since $a_{j,i+1} = a_{j,i}$, we have $a_{2,j} = 0$ for all $j \geqslant 2$. Further, $a_{3,3} = x_3(\zeta^3 - \zeta) + a_{3,2}$ and hence there exists an element $x_3 \in k$ such that $a_{3,3} = 0$. Then $a_{3,j} = 0$ for all $j \geqslant 3$. Thus, any element $a_{k,k}$ can be made equal to zero if $n \nmid (k-1)$, and therefore $\alpha = f\tilde{\alpha}f^{-1}$ where

$$\tilde{\alpha}(t_1) = \zeta t_1 + \tilde{a}_{in+1}t_1^{in+1} + \tilde{a}_{in+n+1}t_1^{in+n+1} + \dots$$

for some i, $\tilde{a}_j \in k$. Notice that $\tilde{a}_{in+1}$ does not depend on x_i. Put $x = x(\alpha) = \tilde{a}_{in+1}$.

Now we replace α by $\tilde{\alpha}$. One can check that if $n|(k-1)$ then

$$a_{j,k} = a_{j,k-1} \qquad \text{for } 2 \leqslant j < k + in$$

and

$$a_{k+in,k} = x_k x(k - in - 1) + a_{k+in} + \text{ some polynomial which does not depend on } x_k.$$

From this fact it immediately follows that $a_{2in+1,in+1}$ does not depend on x_i and for all $k \neq in+1$ $a_{k+in,k}$ can be made equal to zero. Then $y = y(\alpha) = a_{2in+1,in+1}$.

Now we prove that $i_\alpha = i_{\beta'}$. Using the formula

$$\beta'^n(t_1) = t_1 + nx(\alpha)\zeta^{-1}t_1^{in+1} + \dots$$

we get $i_{\beta'} = in+1$. Then one can check that $v_{K_1}(f^{-1}(\alpha^n - \mathrm{id}\,)f) = v_{K_1}(\alpha^n - \mathrm{id}\,) = i_\alpha$. Since $\beta'^n - \mathrm{id} = f^{-1}(\alpha^n - \mathrm{id}\,)f$, we get the identity $i_\alpha = i_{\beta'}$.

The rest of the proof is clear. For details see [Zh, Lemma 6 and Prop.5]. □

8.3. Canonical automorphisms of finite order

8.3.1. Characteristic zero case.

Assume that

a two-dimensional local skew field K splits,
K_1 is a field, $K_0 \subset Z(K)$,
$\mathrm{char}\,(K) = \mathrm{char}\,(K_0) = 0$,
$\alpha^n = \mathrm{id}$ for some $n \geqslant 1$,
for any convergent sequence (a_j) in K_1 the sequence $(t_2 a_j t_2^{-1})$ converges in K.

Lemma. *K is isomorphic to a two-dimensional local skew field $K_1((t_2))$ where*

$$t_2 a t_2^{-1} = \alpha(a) + \delta_i(a)t_2^i + \delta_{2i}(a)t_2^{2i} + \delta_{2i+n}(a)t_2^{2i+n} + \dots \qquad \textit{for all } a \in K_1$$

where $n|i$ and $\delta_j : K_1 \to K_1$ are linear maps and

$$\delta_i(ab) = \delta_i(a)\alpha(b) + \alpha(a)\delta_i(b) \qquad \textit{for every } a, b \in K_1.$$

Moreover

$$t_2^n a t_2^{-n} = a + \delta'_i(a)t_2^i + \delta'_{2i}(a)t_2^{2i} + \delta'_{2i+n}(a)t_2^{2i+n} + \dots$$

where δ'_j *are linear maps and* δ'_i *and* $\delta := \delta'_{2i} - ((i+1)/2){\delta'_i}^2$ *are derivations.*

Remark. The following fact holds for the field K of any characteristic: K is isomorphic to a two-dimensional local skew field $K_1((t_2))$ where

$$t_2 a t_2^{-1} = \alpha(a) + \delta_i(a)t_2^i + \delta_{i+1}(a)t_2^{i+1} + \dots$$

where δ_j are linear maps which satisfy some identity. For explicit formulas see [Zh, Prop.2 and Cor.1].

Proof. It is clear that K is isomorphic to a two-dimensional local skew field $K_1((t_2))$ where

$$t_2 a t_2^{-1} = \alpha(a) + \delta_1(a)t_2 + \delta_2(a)t_2^2 + \dots \qquad \text{for all } a$$

and δ_j are linear maps. Then δ_1 is a (α^2,α)-derivation, that is $\delta_1(ab) = \delta_1(a)\alpha^2(b) + \alpha(a)\delta_1(b)$.

Indeed,

$$\begin{aligned} t_2 ab t_2^{-1} &= t_2 a t_2^{-1} t_2 b t_2^{-1} = (\alpha(a) + \delta_1(a)t_2 + \dots)(\alpha(b) + \delta_1(b)t_2 + \dots) \\ &= \alpha(a)\alpha(b) + (\delta_1(a)\alpha^2(b) + \alpha(a)\delta_1(b))t_2 + \dots = \alpha(ab) + \delta_1(ab)t_2 + \dots . \end{aligned}$$

From the proof of Theorem 8.2 it follows that δ_1 is an inner derivation, i.e. $\delta_1(a) = g\alpha^2(a) - \alpha(a)g$ for some $g \in K_1$, and that there exists a $t_{2,2} = (1 + x_1 t_2)t_2$ such that

$$t_{2,2} a t_{2,2}^{-1} = \alpha(a) + \delta_{2,2}(a)t_{2,2}^2 + \dots .$$

One can easily check that $\delta_{2,2}$ is a (α^3,α)-derivation. Then it is an inner derivation and there exists $t_{2,3}$ such that

$$t_{2,3} a t_{2,3}^{-1} = \alpha(a) + \delta_{3,3}(a)t_{2,3}^3 + \dots .$$

By induction one deduces that if

$$t_{2,j} a t_{2,j}^{-1} = \alpha(a) + \delta_{n,j}(a)t_{2,j}^n + \dots + \delta_{kn,j}(a)t_{2,j}^{kn} + \delta_{j,j}(a)t_{2,j}^j + \dots$$

then $\delta_{j,j}$ is a (α^{j+1},α)-derivation and there exists $t_{2,j+1}$ such that

$$t_{2,j+1} a t_{2,j+1}^{-1} = \alpha(a) + \delta_{n,j}(a)t_{2,j+1}^n + \dots + \delta_{kn,j}(a)t_{2,j+1}^{kn} + \delta_{j+1,j+1}(a)t_{2,j+1}^{j+1} + \dots .$$

The rest of the proof is clear. For details see [Zh, Prop.2, Cor.1, Lemmas 10, 3]. □

Definition. Let $i = v_{K_2}(\varphi(t_2^n)(t_1) - t_1) \in n\mathbb{N} \cup \infty$, ($\varphi$ is defined in subsection 8.1) and let $r \in \mathbb{Z}/i$ be $v_{K_1}(x) \bmod i$ where x is the residue of $(\varphi(t_2^n)(t_1) - t_1)t_2^{-i}$. Put

$$a = \operatorname{res}_{t_1}\left(\frac{(\delta'_{2i} - \frac{i+1}{2}{\delta'_i}^2)(t_1)}{\delta'_i(t_1)^2}dt_1\right) \in K_0.$$

(δ'_i, δ'_{2i} are the maps from the preceding lemma).

Proposition. *If $n = 1$ then i, r don't depend on the choice of a system of local parameters; if $i = 1$ then a does not depend on the choice of a system of local parameters; if $n \neq 1$ then a depends only on the maps $\delta_{i+1}, \dots, \delta_{2i-1}$, i, r depend only on the maps δ_j, $j \notin n\mathbb{N}$, $j < i$.*

Proof. We comment on the statement first. The maps δ_j are uniquely defined by parameters t_1, t_2 and they depend on the choice of these parameters. So the claim that i, r depend only on the maps δ_j, $j \notin n\mathbb{N}$, $j < i$ means that i, r don't depend on the choice of parameters t_1, t_2 which preserve the maps δ_j, $j \notin n\mathbb{N}$, $j < i$.

Note that r depends only on i. Hence it is sufficient to prove the proposition only for i and a. Moreover it suffices to prove it for the case where $n \neq 1$, $i \neq 1$, because if $n = 1$ then the sets $\{\delta_j : j \notin n\mathbb{N}\}$ and $\{\delta_{i+1} : \dots, \delta_{2i-1}\}$ are empty.

It is clear that i depends on δ_j, $j \notin n\mathbb{N}$. Indeed, it is known that δ_1 is an inner (α^2, α)-derivation (see the proof of the lemma). By [Zh, Lemma 3] we can change a parameter t_2 such that δ_1 can be made equal $\delta_1(t_1) = t_1$. Then one can see that $i = 1$. From the other hand we can change a parameter t_2 such that δ_1 can be made equal to 0. In this case $i > 1$. This means that i depends on δ_1. By [Zh, Cor.3] any map δ_j is uniquely determined by the maps δ_q, $q < j$ and by an element $\delta_j(t_1)$. Then using similar arguments and induction one deduces that i depends on other maps δ_j, $j \notin n\mathbb{N}$, $j < i$.

Now we prove that i does not depend on the choice of parameters t_1, t_2 which preserve the maps δ_j, $j \notin n\mathbb{N}$, $j < i$.

Note that i does not depend on the choice of t_1: indeed, if $t'_1 = t_1 + bz^j$, $b \in K_1$ then $z^n t'_1 z^{-n} = z^n t_1 z^{-n} + (z^n b z^{-n}) z^j = t'_1 + r$, where $r \in \mathcal{M}_K^i \setminus \mathcal{M}_K^{i+1}$. One can see that the same is true for $t'_1 = c_1 t_1 + c_2 t_2^2 + \dots$, $c_j \in K_0$.

Let δ_q be the first non-zero map for given t_1, t_2. If $q \neq i$ then by [Zh, Lemma 8, (ii)] there exists a parameter t'_1 such that $z t'_1 z^{-1} = {t'_1}^{\alpha} + \delta_{q+1}(t'_1) z^{q+1} + \dots$. Using this fact and Proposition 8.2 we can reduce the proof to the case where $q = i$, $\alpha(t_1) = \xi t_1$, $\alpha(\delta_i(t_1)) = \xi \delta_i(t_1)$ (this case is equivalent to the case of $n = 1$). Then we apply [Zh, Lemma 3] to show that

$$v_{K_2}((\phi(t'_2) - 1)(t_1)) = v_{K_2}((\phi(t_2) - 1)(t_1)),$$

for any parameters t_2, t'_2, i.e. i does not depend on the choice of a parameter t_2. For details see [Zh, Prop.6].

To prove that a depends only on $\delta_{i+1}, \dots, \delta_{2i-1}$ we use the fact that for any pair of parameters t'_1, t'_2 we can find parameters $t''_1 = t_1 + r$, where $r \in \mathcal{M}_K^i$, t''_2 such that corresponding maps δ_j are equal for all j. Then by [Zh, Lemma 8] a does not depend on t''_1 and by [Zh, Lemma 3] a depends on $t''_2 = t_2 + a_1 t_2^2 + \dots$, $a_j \in K_1$ if and only if $a_1 = \dots = a_{i-1}$. Using direct calculations one can check that a doesn't depend on $t''_2 = a_0 t_2$, $a_0 \in K_1^*$.

To prove the fact it is sufficient to prove it for $t_1'' = t_1 + ct_1^h z^j$ for any $j < i$, $c \in K_0$. Using [Zh, Lemma 8] one can reduce the proof to the assertion that some identity holds. The identity is, in fact, some equation on residue elements. One can check it by direct calculations. For details see [Zh, Prop.7]. □

Remark. The numbers i, r, a can be defined only for local skew fields which splits. One can check that the definition can not be extended to the skew field in Dubrovin's example.

Theorem.

(1) *K is isomorphic to a two-dimensional local skew field $K_0((t_1))((t_2))$ such that*

$$t_2 t_1 t_2^{-1} = \xi t_1 + x t_2^i + y t_2^{2i}$$

where ξ is a primitive n th root, $x = ct_1^r$, $c \in K_0^*/(K_0^*)^d$,

$$y = (a + r(i+1)/2) t_1^{-1} x^2, \quad d = \gcd(r-1, i).$$

If $n = 1$, $i = \infty$, then K is a field.

(2) *Let K, K' be two-dimensional local skew fields of characteristic zero which splits; and let K_1, K_1' be fields. Let $\alpha^n = \mathrm{id}$, $\alpha'^{n'} = \mathrm{id}$ for some $n, n' \geqslant 1$. Then K is isomorphic to K' if and only if K_0 is isomorphic to K_0' and the ordered sets (n, ξ, i, r, c, a) and $(n', \xi', i', r', c', a')$ coincide.*

Proof. (2) follows from the Proposition of 8.2 and (1). We sketch the proof of (1).

From Proposition 8.2 it follows that there exists t_1 such that $\alpha(t_1) = \xi t_1$; $\delta_i(t_1)$ can be represented as $ct_1^r a^i$. Hence there exists t_2 such that

$$t_2 t_1 t_2^{-1} = \xi t_1 + x t_2^i + \delta_{2i}(t_1) t_2^{2i} + \dots$$

Using [Zh, Lemma 8] we can find a parameter $t_1' = t_1 \bmod \mathcal{M}_K$ such that

$$t_2 t_1' t_2^{-1} = \xi t_1 + x t_2^i + y t_2^{2i} + \dots$$

The rest of the proof is similar to the proof of the lemma. Using [Zh, Lemma 3] one can find a parameter $t_2' = t_2 \bmod \mathcal{M}_K^2$ such that $\delta_j(t_1) = 0$, $j > 2i$. □

Corollary. *Every two-dimensional local skew field K with the ordered set*

$$(n, \xi, i, r, c, a)$$

is a finite-dimensional extension of a skew field with the ordered set $(1, 1, 1, 0, 1, a)$.

Remark. There is a construction of a two-dimensional local skew field with a given set (n, ξ, i, r, c, a).

Examples.

(1) The ring of formal pseudo-differential equations is the skew field with the set $(n = 1, \xi = 1, i = 1, r = 0, c = 1, a = 0)$.

(2) The elements of $\mathrm{Br}(L)$ where L is a two-dimensional local field of equal characteristic are local skew fields. If, for example, L is a C_2- field, they split and $i = \infty$. Hence any division algebra in $\mathrm{Br}(L)$ is cyclic.

8.3.2. Characteristic p case.

Theorem. *Suppose that a two-dimensional local skew field K splits, K_1 is a field, $K_0 \subset Z(K)$, $\mathrm{char}\,(K) = \mathrm{char}\,(K_0) = p > 2$ and $\alpha = \mathrm{id}$.*

Then K is a finite dimensional vector space over its center if and only if K is isomorphic to a two-dimensional local skew field $K_0((t_1))((t_2))$ where

$$t_2^{-1} t_1 t_2 = t_1 + x t_2^i$$

with $x \in K_1^p$, $(i, p) = 1$.

Proof. The "if" part is obvious. We sketch the proof of the "only if" part.

If K is a finite dimensional vector space over its center then K is a division algebra over a henselian field. In fact, the center of K is a two-dimensional local field $k((u))((t))$. Then by [JW, Prop.1.7] $K_1/(Z(K))_1$ is a purely inseparable extension. Hence there exists t_1 such that $t_1^{p^k} \in Z(K)$ for some $k \in \mathbb{N}$ and $K \simeq K_0((t_1))((t_2))$ as a vector space with the relation

$$t_2 t_1 t_2^{-1} = t_1 + \delta_i(t_1) t_2^i + \dots$$

(see Remark 8.3.1). Then it is sufficient to show that i is prime to p and there exist parameters $t_1 \in K_1$, t_2 such that the maps δ_j satisfy the following property:

(*) If j is not divisible by i then $\delta_j = 0$. If j is divisible by i then $\delta_j = c_{j/i} \delta_i^{j/i}$ with some $c_{j/i} \in K_1$.

Indeed, if this property holds then by induction one deduces that $c_{j/i} \in K_0$, $c_{j/i} = ((i+1)\dots(i(j/i-1)+1))/(j/i)!$. Then one can find a parameter $t_2' = bt_2$, $b \in K_1$ such that δ_j' satisfies the same property and $\delta_i^2 = 0$. Then

$$t_2'^{\,-1} t_1 t_2' = t_1 - \delta_i'(t_1) t_2^i.$$

First we prove that $(i, p) = 1$. To show it we prove that if $p|i$ then there exists a map δ_j such that $\delta_j(t_1^{p^k}) \ne 0$. To find this map one can use [Zh, Cor.1] to show that $\delta_{ip}(t_1^p) \ne 0$, $\delta_{ip^2}(t_1^{p^2}) \ne 0$, $\dots$, $\delta_{ip^k}(t_1^{p^k}) \ne 0$.

Then we prove that for some t_2 property (*) holds. To show it we prove that if property (*) does not hold then there exists a map δ_j such that $\delta_j(t_1^{p^k}) \ne 0$. To find this map we reduce the proof to the case of $i \equiv 1 \bmod p$. Then we apply the following idea.

Let $j \equiv 1 \bmod p$ be the minimal positive integer such that δ_j is not equal to zero on $K_1^{p^l}$. Then one can prove that the maps $\delta_m,\ kj \leqslant m < (k+1)j,\ k \in \{1, \dots, p-1\}$ satisfy the following property:

there exist elements $c_{m,k} \in K_1$ such that

$$(\delta_m - c_{m,1}\delta - \dots - c_{m,k}\delta^k)|_{K_1^{p^l}} = 0$$

where $\delta\colon K_1 \to K_1$ is a linear map, $\delta|_{K_1^{p^l}}$ is a derivation, $\delta(t_1^j) = 0$ for $j \notin p^l\mathbb{N}$, $\delta(t_1^{p^l}) = 1,\ c_{kj,k} = c(\delta_j(t_1^{p^l}))^k,\ c \in K_0$.

Now consider maps $\widetilde{\delta_q}$ which are defined by the following formula

$$t_2^{-1}at_2 = a + \widetilde{\delta_i}(a)t_2^i + \widetilde{\delta_{i+1}}(a)t_2^{i+1} + \dots, \quad a \in K_1.$$

Then $\widetilde{\delta_q} + \delta_q + \sum_{k=1}^{q-1}\delta_k\widetilde{\delta_{q-k}} = 0$ for any q. In fact, $\widetilde{\delta_q}$ satisfy some identity which is similar to the identity in [Zh, Cor.1]. Using that identity one can deduce that if

$j \equiv 1 \bmod p$ and there exists the minimal m ($m \in \mathbb{Z}$) such that $\delta_{mp+2i}|_{K_1^{p^l}} \not\equiv 0$ if $j \not| (mp+2i)$ and $\delta_{mp+2i}|_{K_1^{p^l}} \not\equiv s\delta_j^{(2i+mp)/j}|_{K_1^{p^l}}$ for any $s \in K_1$ otherwise, and $\delta_q(t_1^{p^l}) = 0$ for $q < mp+2i,\ q \not\equiv 1 \bmod p$,

then

$(mp+2i)+(p-1)j$ is the minimal integer such that $\delta_{(mp+2i)+(p-1)j}|_{K_1^{p^{l+1}}} \not\equiv 0$.

To complete the proof we use induction and [Zh, Lemma 3] to show that there exist parameters $t_1 \in K_1,\ t_2$ such that $\delta_q(t_1^{p^l}) = 0$ for $q \not\equiv 1, 2 \bmod p$ and $\delta_j^2 = 0$ on $K_1^{p^l}$. □

Corollary 1. *If K is a finite dimensional division algebra over its center then its index is equal to p.*

Corollary 2. *Suppose that a two-dimensional local skew field K splits, K_1 is a field, $K_0 \subset Z(K)$, $\mathrm{char}(K) = \mathrm{char}(K_0) = p > 2$, K is a finite dimensional division algebra over its center of index p^k.*

Then either K is a cyclic division algebra or has index p.

Proof. By [JW, Prop. 1.7] $K_1/\overline{Z(K)}$ is the compositum of a purely inseparable extension and a cyclic Galois extension. Then the canonical automorphism α has order p^l for some $l \in \mathbb{N}$. By [Zh, Lemma 10] (which is true also for $\mathrm{char}(K) = p > 0$), $K \simeq K_0((t_1))((t_2))$ with

$$t_2at_2^{-1} = \alpha(a) + \delta_i(a)t_2^i + \delta_{i+p^l}(a)t_2^{i+p^l} + \delta_{i+p^l}(a)t_2^{i+2p^l} + \dots$$

where $i \in p^l\mathbb{N}$, $a \in K_1$. Suppose that $\alpha \neq 1$ and K_1 is not a cyclic extension of $\overline{Z(K)}$. Then there exists a field $F \subset K_1$, $F \not\subset Z(K)$ such that $\alpha|_F = 1$. If $a \in F$ then for some m the element a^{p^m} belongs to a cyclic extension of the field $\overline{Z(K)}$, hence $\delta_j(a^{p^m}) = 0$ for all j. But we can apply the same arguments as in the proof of the preceding theorem to show that if $\delta_i \neq 0$ then there exists a map δ_j such that $\delta_j(a^{p^m}) \neq 0$, a contradiction. We only need to apply [Zh, Prop.2] instead of [Zh, Cor.1] and note that $\alpha\delta = x\delta\alpha$ where δ is a derivation on K_1, $x \in K_1$, $x \equiv 1 \bmod \mathcal{M}_{K_1}$, because $\alpha(t_1)/t_1 \equiv 1 \bmod \mathcal{M}_{K_1}$.

Hence $t_2 a t_2^{-1} = \alpha(a)$ and $K_1/\overline{Z(K)}$ is a cyclic extension and K is a cyclic division algebra $(K_1(t_2^{p^k})/Z(K), \alpha, t_2^{p^k})$. □

Corollary 3. *Let $F = F_0((t_1))((t_2))$ be a two-dimensional local field, where F_0 is an algebraically closed field. Let A be a division algebra over F.*

Then $A \simeq B \otimes C$, where B is a cyclic division algebra of index prime to p and C is either cyclic (*as in Corollary* 2) *or C is a local skew field from the theorem of index p.*

Proof. Note that F is a C_2-field. Then A_1 is a field, A_1/F_1 is the compositum of a purely inseparable extension and a cyclic Galois extension, and $A_1 = F_0((u))$ for some $u \in A_1$. Hence A splits. So, A is a splitting two-dimensional local skew field.

It is easy to see that the index of A is $|\overline{A} : \overline{F}| = p^q m$, $(m,p) = 1$. Consider subalgebras $B = C_A(F_1)$, $C = C_A(F_2)$ where $F_1 = F(u^{p^q})$, $F_2 = F(u^m)$. Then by [M, Th.1] $A \simeq B \otimes C$.

The rest of the proof is clear. □

Now one can easily deduce that

Corollary 4. *The following conjecture: the exponent of A is equal to its index for any division algebra A over a C_2-field F* (*see for example* [PY, 3.4.5.])
has the positive answer for $F = F_0((t_1))((t_2))$.

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9. Local reciprocity cycles

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In this section we introduce a description of totally ramified Galois extensions of a local field with finite residue field (extensions have to satisfy certain arithmetical restrictions if they are infinite) in terms of subquotients of formal power series $\mathbb{F}_p^{\text{sep}}[[X]]^*$. This description can be viewed as a non-commutative local reciprocity map (which is not in general a homomorphism but a cocycle) which directly describes the Galois group in terms of certain objects related to the ground field. Abelian class field theory as well as metabelian theory of Koch and de Shalit [K], [KdS] (see subsection 9.4) are partial cases of this theory.

9.1. Group $U^{\diamond}_{N(\widehat{L/F})}$

Let F be a local field with finite residue field. Denote by $\varphi \in G_F$ a lifting of the Frobenius automorphism of F_{ur}/F.

Let F^{φ} be the fixed field of φ. The extension F^{φ}/F is totally ramified.

Lemma ([KdS, Lemma 0.2]). *There is a unique norm compatible sequence of prime elements π_E in finite subextensions E/F of F^{φ}/F.*

Proof. Uniqueness follows from abelian local class field theory, existence follows from the compactness of the group of units. □

In what follows we fix F^{φ} and consider Galois subextensions L/F of F^{φ}/F. Assume that L/F is *arithmetically profinite*, ie for every x the ramification group $\text{Gal}(L/F)^x$ is open in $\text{Gal}(L/F)$ (see also subsection 6.3 of Part II). For instance, a totally ramified p-adic Lie extension is arithmetically profinite.

For an arithmetically profinite extension L/F define its Hasse–Herbrand function $h_{L/F}\colon [0,\infty) \to [0,\infty)$ as $h_{L/F}(x) = \lim h_{M/F}(x)$ where M/F runs over finite subextensions of L/F (cf. [FV, Ch. III §5]).

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If L/F is infinite let $N(L/F)$ be the field of norms of L/F. It can be identified with $k_F((\Pi))$ where Π corresponds to the norm compatible sequence π_E (see subsection 6.3 of Part II, [W], [FV, Ch.III §5]).

Denote by φ the automorphism of $N(L/F)_{\mathrm{ur}}$ and of its completion $\widehat{N(L/F)}$ corresponding to the Frobenius automorphism of F_{ur}/F.

Definition. Denote by $U^{\diamond}_{\widehat{N(L/F)}}$ the subgroup of the group $U_{\widehat{N(L/F)}}$ of those elements whose $\widehat{F}$-component belongs to U_F. An element of $U^{\diamond}_{\widehat{N(L/F)}}$ such that its $\widehat{F}$-component is $\varepsilon \in U_F$ will be called a lifting of ε.

The group $U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}$ is a direct product of a quotient group of the group of multiplicative representatives of the residue field k_F of F, a cyclic group $\mathbb{Z}/p^a$ and a free topological $\mathbb{Z}_p$-module. The Galois group $\mathrm{Gal}(L/F)$ acts naturally on $U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}$.

9.2. Reciprocity map $\mathcal{N}_{L/F}$

To motivate the next definition we interpret the map $\Upsilon_{L/F}$ (defined in 10.1 and 16.1) for a finite Galois totally ramified extension L/F in the following way. Since in this case both π_Σ and π_L are prime elements of L_{ur}, there is $\varepsilon \in U_{L_{\mathrm{ur}}}$ such that $\pi_\Sigma = \pi_L\varepsilon$. We can take $\tilde\sigma = \sigma\varphi$. Then $\pi_L^{\sigma-1} = \varepsilon^{1-\sigma\varphi}$. Let $\eta \in U_{\widehat{L}}$ be such that $\eta^{\varphi-1} = \varepsilon$. Since $(\eta^{\sigma\varphi-1}\varepsilon^{-1})^{\varphi-1} = (\eta^{(\sigma-1)\varphi})^{\varphi-1}$, we deduce that $\varepsilon = \eta^{\sigma\varphi-1}\eta^{(1-\sigma)\varphi}\rho$ with $\rho \in U_L$. Thus, for $\xi = \eta^{\sigma\varphi-1}$

$$\Upsilon_{L/F}(\sigma) \equiv N_{\Sigma/F}\pi_\Sigma \equiv N_{\widehat{L}/\widehat{F}}\xi \bmod N_{L/F}L^*, \qquad \xi^{1-\varphi} = \pi_L^{\sigma-1}.$$

Definition. For a $\sigma \in \mathrm{Gal}(L/F)$ let $U_\sigma \in U_{\widehat{N(L/F)}}$ be a solution of the equation

$$\boxed{U^{1-\varphi} = \Pi^{\sigma-1}}$$

(recall that $\mathrm{id} - \varphi\colon U_{\widehat{N(L/F)}} \to U_{\widehat{N(L/F)}}$ is surjective). Put

$$\mathcal{N}_{L/F}\colon \mathrm{Gal}(L/F) \to U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}, \quad \mathcal{N}_{L/F}(\sigma) = U_\sigma \mod U_{N(L/F)}.$$

Remark. Compare the definition with Fontaine-Herr's complex defined in subsection 6.4 of Part II.

Properties.

(1) $\mathcal{N}_{L/F} \in Z^1(\mathrm{Gal}(L/F), U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)})$ is injective.

(2) For a finite extension L/F the $\widehat{F}$-component of $\mathcal{N}_{L/F}(\sigma)$ is equal to the value $\Upsilon_{L/F}(\sigma)$ of the abelian reciprocity map $\Upsilon_{L/F}$ (see the beginning of 9.2).

(3) Let M/F be a Galois subextension of L/F and E/F be a finite subextension of L/F. Then the following diagrams of *maps* are commutative:

$$\begin{array}{ccccccc} \mathrm{Gal}(L/E) & \xrightarrow{\mathcal{N}_{L/E}} & U^{\diamond}_{\widehat{N(L/E)}}/U_{N(L/E)} & \mathrm{Gal}(L/F) & \xrightarrow{\mathcal{N}_{L/F}} & U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)} \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ \mathrm{Gal}(L/F) & \xrightarrow{\mathcal{N}_{L/F}} & U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)} & \mathrm{Gal}(M/F) & \xrightarrow{\mathcal{N}_{M/F}} & U^{\diamond}_{\widehat{N(M/F)}}/U_{N(M/F)}. \end{array}$$

(4) Let $U^{\diamond}_{n,\widehat{N(L/F)}}$ be the filtration induced from the filtration $U_{n,\widehat{N(L/F)}}$ on the field of norms. For an infinite arithmetically profinite extension L/F with the Hasse–Herbrand function $h_{L/F}$ put $\mathrm{Gal}(L/F)_n = \mathrm{Gal}(L/F)^{h^{-1}_{L/F}(n)}$. Then $\mathcal{N}_{L/F}$ maps $\mathrm{Gal}(L/F)_n \setminus \mathrm{Gal}(L/F)_{n+1}$ into $U^{\diamond}_{n,\widehat{N(L/F)}}U_{N(L/F)} \setminus U^{\diamond}_{n+1,\widehat{N(L/F)}}U_{N(L/F)}$.

(6) The set $\mathrm{im}(\mathcal{N}_{L/F})$ is not closed in general with respect to multiplication in the group
$U_{\widehat{N(L/F)}}/U_{N(L/F)}$. Endow $\mathrm{im}(\mathcal{N}_{L/F})$ with a new group structure given by $x \star y = x\mathcal{N}^{-1}_{L/F}(x)(y)$. Then clearly $\mathrm{im}(\mathcal{N}_{L/F})$ is a group isomorphic to $\mathrm{Gal}(L/F)$.

Problem. What is $\mathrm{im}(\mathcal{N}_{L/F})$?

One method to solve the problem is described below.

9.3. Reciprocity map $\mathcal{H}_{L/F}$

Definition. Fix a tower of subfields $F = E_0 - E_1 - E_2 - \dots$, such that $L = \cup E_i$, E_i/F is a Galois extension, and E_i/E_{i-1} is cyclic of prime degree. We can assume that $|E_{i+1} : E_i| = p$ for all $i \geqslant i_0$ and $|E_{i_0} : E_0|$ is relatively prime to p.

Let σ_i be a generator of $\mathrm{Gal}(E_i/E_{i-1})$. Denote

$$X_i = U_{\widehat{E}_i}^{\sigma_i - 1}.$$

The group X_i is a $\mathbb{Z}_p$-submodule of $U_{1,\widehat{E}_i}$. It is the direct sum of a cyclic torsion group of order p^{n_i}, $n_i \geqslant 0$, generated by, say, α_i ($\alpha_i = 1$ if $n_i = 0$) and a free topological $\mathbb{Z}_p$-module Y_i.

We shall need a sufficiently "nice" injective map from characteristic zero or p to characteristic p

$$f_i \colon U_{\widehat{E}_i}^{\sigma_i - 1} \to U_{\widehat{N(L/E_i)}} \to U_{N(L/F)}.$$

If F is a local field of characteristic zero containing a non-trivial pth root ζ and f_i is a homomorphism, then ζ is doomed to go to 1. Still, from certain injective maps (not homomorphisms) f_i specifically defined below we can obtain a subgroup $\prod f_i(U_{\widehat{E_i}}^{\sigma_i-1})$ of $U^{\diamond}_{\widehat{N(L/F)}}$.

Definition. If $n_i=0$, set $A^{(i)}\in U_{\widehat{N(L/E_i)}}$ to be equal to 1.
If $n_i>0$, let $A^{(i)}\in U_{\widehat{N(L/E_i)}}$ be a lifting of α_i with the following restriction: $A^{(i)}_{\widehat{E_{i+1}}}$ is not a root of unity of order a power of p (this condition can always be satisfied, since the kernel of the norm map is uncountable).

Lemma ([F]). *If $A^{(i)}\neq 1$, then $\beta_{i+1}={A^{(i)}_{\widehat{E_{i+1}}}}^{p^{n_i}}$ belongs to X_{i+1}.*

Note that every β_{i+1} when it is defined doesn't belong to X_{i+1}^p. Indeed, otherwise we would have ${A^{(i)}_{\widehat{E_{i+1}}}}^{p^{n_i}}=\gamma^p$ for some $\gamma\in X_{i+1}$ and then ${A^{(i)}_{\widehat{E_{i+1}}}}^{p^{n_i-1}}=\gamma\zeta$ for a root ζ of order p or 1. Taking the norm down to $\widehat{E_i}$ we get $\alpha_i^{p^{n_i-1}}=N_{\widehat{E_{i+1}}/\widehat{E_i}}\gamma=1$, which contradicts the definition of α_i.

Definition. Let $\beta_{i,j}$, $j\geqslant 1$ be free topological generators of Y_i which include β_i whenever β_i is defined. Let $B^{(i,j)}\in U_{\widehat{N(L/E_i)}}$ be a lifting of $\beta_{i,j}$ (i.e. ${B^{(i,j)}}_{\widehat{E_i}}=\beta_{i,j}$), such that if $\beta_{i,j}=\beta_i$, then $B^{(i,j)}_{\widehat{E_k}}=B^{(i)}_{\widehat{E_k}}={A^{(i-1)}_{\widehat{E_k}}}^{p^{n_i-1}}$ for $k\geqslant i$.

Define a map $X_i\to U_{\widehat{N(L/E_i)}}$ by sending a convergent product $\alpha_i^c\prod_j\beta_{i,j}^{c_j}$, where $0\leqslant c\leqslant n_i-1$, $c_j\in\mathbb{Z}_p$, to ${A^{(i)}}^c\prod_j {B^{(i,j)}}^{c_j}$ (the latter converges). Hence we get a map

$$f_i\colon U_{\widehat{E_i}}^{\sigma_i-1}\to U_{\widehat{N(L/E_i)}}\to U_{\widehat{N(L/F)}}$$

which depends on the choice of lifting. Note that $f_i(\alpha)_{\widehat{E_i}}=\alpha$.

Denote by Z_i the image of f_i. Let

$$Z_{L/F}=Z_{L/F}(\{E_i,f_i\})=\left\{\prod_i z^{(i)}: z^{(i)}\in Z_i\right\},$$

$$Y_{L/F}=\{y\in U_{\widehat{N(L/F)}}: y^{1-\varphi}\in Z_{L/F}\}.$$

Lemma. *The product of $z^{(i)}$ in the definition of $Z_{L/F}$ converges. $Z_{L/F}$ is a subgroup of $U^{\diamond}_{\widehat{N(L/F)}}$. The subgroup $Y_{L/F}$ contains $U_{N(L/F)}$.*

Theorem ([F]). *For every* $(u_{\widehat{E_i}}) \in U^{\diamond}_{\widehat{N(L/F)}}$ *there is a unique automorphism* τ *in the group* $\mathrm{Gal}(L/F)$ *satisfying*

$$(u_{\widehat{E_i}})^{1-\varphi} \equiv \Pi^{\tau-1} \mod Z_{L/F}.$$

If $(u_{\widehat{E_i}}) \in Y_{L/F}$*, then* $\tau = 1$.

Hint. Step by step, passing from $\widehat{E_i}$ to $\widehat{E_{i+1}}$. □

Remark. This theorem can be viewed as a non-commutative generalization for finite k of exact sequence $(*)$ of 16.2.

Corollary. *Thus, there is map*

$$\mathcal{H}_{L/F}: U^{\diamond}_{\widehat{N(L/F)}} \to \mathrm{Gal}(L/F), \quad \mathcal{H}_{L/F}((u_{\widehat{E_i}})) = \tau.$$

The composite of $\mathcal{N}_{L/F}$ *and* $\mathcal{H}_{L/F}$ *is the identity map of* $\mathrm{Gal}(L/F)$.

9.4. Main Theorem

Theorem ([F]). *Put*

$$\mathcal{H}_{L/F}: U^{\diamond}_{\widehat{N(L/F)}}/Y_{L/F} \to \mathrm{Gal}(L/F), \quad \mathcal{H}_{L/F}((u_{\widehat{E}})) = \tau$$

where τ *is the unique automorphism satisfying* $(u_{\widehat{E}})^{1-\varphi} \equiv \Pi^{\tau-1} \mod Z_{L/F}$. *The injective map* $\mathcal{H}_{L/F}$ *is a bijection. The bijection*

$$\mathcal{N}_{L/F}: \mathrm{Gal}(L/F) \to U^{\diamond}_{\widehat{N(L/F)}}/Y_{L/F}$$

induced by $\mathcal{N}_{L/F}$ *defined in 9.2 is a 1-cocycle.*

Corollary. Denote by q the cardinality of the residue field of F. Koch and de Shalit [K], [KdS] constructed a sort of metabelian local class field theory which in particular describes totally ramified metabelian extensions of F (the commutator group of the commutator group is trivial) in terms of the group

$$\mathfrak{n}(F) = \left\{(u \in U_F, \xi(X) \in \mathbb{F}_p^{\mathrm{sep}}[[X]]^*) : \xi(X)^{\varphi-1} = \{u\}(X)/X\right\}$$

with a certain group structure. Here $\{u\}(X)$ is the residue series in $\mathbb{F}_p^{\mathrm{sep}}[[X]]^*$ of the endomorphism $[u](X) \in O_F[[X]]$ of the formal Lubin–Tate group corresponding to π_F, q, u.

Let M/F be the maximal totally ramified metabelian subextension of F_φ, then M/F is arithmetically profinite. Let R/F be the maximal abelian subextension of M/F. Every coset of $U^{\diamond}_{\widehat{N(M/F)}}$ modulo $Y_{M/F}$ has a unique representative in

$\mathrm{im}(\mathcal{N}_{M/F})$. Send a coset with a representative $(u_{\widehat{Q}}) \in U^{\diamond}_{\widehat{N(M/F)}}$ ($F \subset Q \subset M$, $|Q:F| < \infty$) satisfying $(u_{\widehat{Q}})^{1-\varphi} = (\pi_Q)^{\tau-1}$ with $\tau \in \mathrm{Gal}(M/F)$ to

$$\left(u_{\widehat{F}}^{-1}, (u_{\widehat{E}}) \in U^{\diamond}_{\widehat{N(R/F)}}\right) \quad (F \subset E \subset R, |E:F| < \infty).$$

It belongs to $\mathfrak{n}(F)$, so we get a map

$$g\colon U^{\diamond}_{\widehat{N(M/F)}}/Y_{M/F} \to \mathfrak{n}(F).$$

This map is a bijection [F] which makes Koch–de Shalit's theory a corollary of the main results of this section.

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10. Galois modules and class field theory

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In this section we shall try to present the reader with a sample of several significant instances where, on the way to proving results in Galois module theory, one is lead to use class field theory. Conversely, some contributions of Galois module theory to class fields theory are hinted at. We shall also single out some problems that in our opinion deserve further attention.

10.1. Normal basis theorem

The Normal Basis Theorem is one of the basic results in the Galois theory of fields. In fact one can use it to obtain a proof of the fundamental theorem of the theory, which sets up a correspondence between subgroups of the Galois group and subfields. Let us recall its statement and give a version of its proof following E. Noether and M. Deuring (a very modern proof!).

Theorem (Noether, Deuring)**.** *Let K be a finite extension of $\mathbb{Q}$. Let L/K be a finite Galois extension with Galois group $G = \mathrm{Gal}(L/K)$. Then L is isomorphic to $K[G]$ as a $K[G]$-module. That is: there is an $a \in L$ such that $\{\sigma(a)\}_{\sigma\in G}$ is a K-basis of L. Such an a is called a normal basis generator of L over K.*

Proof. Use the isomorphism

$$\varphi\colon L \otimes_K L \to L[G], \quad \varphi(x \otimes y) = \sum_{\sigma\in G} \sigma(x) y \sigma^{-1} ,$$

then apply the Krull–Schmidt theorem to deduce that this isomorphism descends to K. Note that an element a in L generates a normal basis of L over K if and only if $\varphi(a) \in L[G]^*$.

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10.1.1. Normal integral bases and ramification.

Let us now move from dimension 0 (fields) to dimension 1, and consider rings of algebraic integers.

Let p be a prime number congruent to 1 modulo an (odd) prime l. Let $L_1 = \mathbb{Q}(\mu_p)$, and let K be the unique subfield of L_1 of degree l over $\mathbb{Q}$. Then $G = \mathrm{Gal}(K/\mathbb{Q})$ is cyclic of order l and K is tamely ramified over $\mathbb{Q}$. One can construct a normal basis for the ring $\mathcal{O}_K$ of integers in K over $\mathbb{Z}$: indeed if ζ denotes a primitive p-th root of unity, then ζ is a normal basis generator for $L_1/\mathbb{Q}$ and the trace of ζ to K gives the desired normal integral basis generator. Let now $L_2 = \mathbb{Q}(\mu_{l^2})$. It is easy to see that there is no integral normal basis for L_2 over $\mathbb{Q}$. As noticed by Noether, this is related to the fact that L_2 is a wildly ramified extension of the rationals. However there is the following structure result, which gives a complete and explicit description of the Galois module structure of rings of algebraic integers in absolute abelian extensions.

Theorem (Leopoldt 1959)**.** *Let K be an abelian extension of $\mathbb{Q}$. Let $G = \mathrm{Gal}(K/\mathbb{Q})$. Define*

$$\Lambda = \{\lambda \in \mathbb{Q}[G] : \lambda\mathcal{O}_K \subset \mathcal{O}_K\}$$

where $\mathcal{O}_K$ is the ring of integers of K. Then $\mathcal{O}_K$ is isomorphic to Λ as a Λ-module.

Note that the statement is not true for an arbitrary global field, nor for general relative extensions of number fields. The way to prove this theorem is by first dealing with the case of cyclotomic fields, for which one constructs explicit normal basis generators in terms of roots of unity. In this step one uses the criterion involving the resolvent map φ which we mentioned in the previous theorem. Then, for a general absolute abelian field K, one embeds K into the cyclotomic field $\mathbb{Q}(f_K)$ with smallest possible conductor by using the Kronecker–Weber theorem, and one "traces the result down" to K. Here it is essential that the extension $\mathbb{Q}(f_K)/K$ is essentially tame. Explicit class field theory is an important ingredient of the proof of this theorem; and, of course, this approach has been generalized to other settings: abelian extensions of imaginary quadratic fields (complex multiplication), extensions of Lubin–Tate type, etc.

10.1.2. Factorizability.

While Leopoldt's result is very satisfactory, one would still like to know a way to express the relation there as a relation between the Galois structure of rings of integers in general Galois extensions and the most natural integral representation of the Galois group, namely that given by the group algebra. There is a very neat description of this which uses the notion of factorizability, introduced by A. Fröhlich and A. Nelson. This leads to an equivalence relation on modules which is weaker than local equivalence (genus), but which is non-trivial.

Let G be a finite group, and let $S = \{H : H \leqslant G\}$. Let T be an abelian group.

Definition. A map $f: S \to T$ is called *factorizable* if every relation of the form

$$\sum_{H \in S} a_H \operatorname{ind}_H^G 1 = 0$$

with integral coefficients a_H, implies the relation

$$\prod_{H \in S} f(H)^{a_H} = 1 .$$

Example. Let $G = \mathrm{Gal}(L/K)$, then the discriminant of L/K defines a factorizable function (conductor-discriminant formula).

Definition. Let $i: M \to N$ be a morphism of $\mathcal{O}_K[G]$-lattices. The lattices M and N are said to be *factor-equivalent* if the map $H \to |L^H : i(M)^H|$ is factorizable.

Theorem (Fröhlich, de Smit)**.** *If $G = \mathrm{Gal}(L/K)$ and K is a global field, then $\mathcal{O}_L$ is factor-equivalent to $\mathcal{O}_K[G]$.*

Again this result is based on the isomorphism induced by the resolvent map φ and the fact that the discriminant defines a factorizable function.

10.1.3. Admissible structures.

Ideas related to factorizability have very recently been used to describe the Galois module structure of ideals in local field extensions. Here is a sample of the results.

Theorem (Vostokov, Bondarko)**.** *Let K be a local field of mixed characteristic with finite residue field. Let L be a finite Galois extension of K with Galois group G.*

(1) *Let I_1 and I_2 be indecomposable $\mathcal{O}_K[G]$-submodules of $\mathcal{O}_L$. Then I_1 is isomorphic to I_2 as $\mathcal{O}_K[G]$-modules if and only if there is an a in K^* such that $I_1 = aI_2$.*
(2) *$\mathcal{O}_L$ contains decomposable ideals if and only if there is a subextension E/L of L/K such that $|L : E|\mathcal{O}_L$ contains the different $\mathcal{D}_{L/E}$.*
(3) *If L is a totally ramified Galois p-extension of K and $\mathcal{O}_L$ contains decomposable ideals, then L/K is cyclic and $|L : K|\mathcal{O}_L$ contains the different $\mathcal{D}_{L/K}$.*

In fact what is remarkable with these results is that they do *not* involve class field theory.

10.2. Galois module theory in geometry

Let X be a smooth projective curve over an algebraically closed field k. Let a finite group G act on X. Put $Y = X/G$.

Theorem (Nakajima 1975)**.** *The covering X/Y is tame if and only if for every line bundle $\mathcal{L}$ of sufficiently large degree which is stable under the G-action $H^0(X, \mathcal{L})$ is a projective $k[G]$-module.*

This is the precise analogue of Ullom's version of Noether's Criterion for the existence of a normal integral basis for ideals in a Galois extension of discrete valuation rings. In fact if (X, G) is a *tame* action of a finite group G on any reasonable proper scheme over a ring A like $\mathbb{Z}$ or $\mathbb{F}_p$, then for any coherent G-sheaf $\mathcal{F}$ on X one can define an equivariant Euler–Poincaré characteristic $\chi(\mathcal{F}, G)$ in the Grothendieck group $K_0(A[G])$ of finitely generated *projective* $A[G]$-modules. It is an outstanding problem to compute these equivariant Euler characteristics. One of the most important results in this area is the following. Interestingly it relies heavily on results from class field theory.

Theorem (Pappas 1998)**.** *Let G be an abelian group and let $\mathcal{X}$ be an arithmetic surface over $\mathbb{Z}$ with a free G-action. Then $2\chi(\mathcal{O}_{\mathcal{X}}, G) = 0$ in $K_0(\mathbb{Z}[G])/\langle \mathbb{Z}[G] \rangle$.*

10.3. Galois modules and L-functions

Let a finite group G act on a projective, regular scheme X of dimension n defined over the finite field $\mathbb{F}_q$ and let $Y = X/G$. Let $\zeta(X, t)$ be the zeta-function of X. Let e_X be the l-adic Euler characteristic of X. Recall that

$$\zeta(X,t) = \pm (q^n t^2)^{-e_X/2} \zeta(X, q^{-n} t^{-1}), \quad e_X \cdot n = 2 \sum_{0 \leqslant i \leqslant n} (-1)^i (n-i) \chi(\Omega^i_{X/\mathbb{F}_q})$$

the latter being a consequence of the Hirzebruch–Riemann–Roch theorem and Serre duality. It is well known that the zeta-function of X decomposes into product of L-functions, which also satisfy functional equations. One can describe the constants in these functional equations by "taking isotypic components" in the analogue of the above expression for $e_X \cdot n/2$ in terms of equivariant Euler-Poincaré characteristics. The results that have been obtained so far do *not* use class field theory in any important way. So we are lead to formulate the following problem:

Problem. Using Parshin's adelic approach (sections 1 and 2 of Part II) find another proof of these results.

Let us note that one of the main ingredients in the work on these matters is a formula on ε-factors of T. Saito, which generalizes one by S. Saito inspired by Parshin's results.

10.4. Galois structure of class formations

Let K be a number field and let L be a finite Galois extension of K, with Galois group $G = \mathrm{Gal}(L/K)$. Let S be a finite set of primes including those which ramify in L/K and the archimedean primes. Assume that S is stable under the G-action. Put $\Delta S = \ker(\mathbb{Z}S \to \mathbb{Z})$. Let U_S be the group of S-units of L. Recall that $U_S \otimes \mathbb{Q}$ is isomorphic to $\Delta S \otimes \mathbb{Q}$ as $\mathbb{Q}[G]$-modules. There is a well known exact sequence

$$0 \to U_S \to A \to B \to \Delta S \to 0$$

with finitely generated A, B such that A has finite projective dimension and B is projective. The latter sequence is closely related to the fundamental class in global class field theory and the class $\Omega = (A) - (B)$ in the projective class group $\mathrm{Cl}\,(\mathbb{Z}[G]$ is clearly related to the Galois structure of S-units. There are local analogues of the above sequence, and there are analogous sequences relating (bits) of higher K-theory groups (the idea is to replace the pair (U_S, ΔS) by a pair ($K_i'(\mathcal{O})$, $K_{i-1}'(\mathcal{O})$)).

Problem. Using complexes of G-modules (as in section 11 of part I) can one generalize the local sequences to higher dimensional fields?

For more details see [E].

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