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HÖLDER TYPE INEQUALITIES FOR ORTHOSYMMETRIC BILINEAR OPERATORS[1]

A. G. Kusraev

Dedicated to the memory of G. Ya. Lozanovskiĭ

An interplay between squares of vector lattice and homogeneous functional calculus is considered and Hölder type inequalities for orthosymmetric bilinear operators are obtained.

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1. Introduction

The homogeneous functional calculus on vector lattices is a useful tool in a variety of areas. One of the interesting application is the study of powers of Banach lattices initiated by G. Ya. Lozanovskiĭ [18]. Recently G. Buskes and A. van Rooij [8] introduced the concept of squares of Archimedean vector lattices which allows to represent orthoregular bilinear operators as linear regular operators. In particular, it is proved in [9] that the square of a relatively uniformly complete vector lattice can be constructed by well known p-convexification procedure (with $p = 1/2$) which is also based on the homogeneous functional calculus, see [17, 22].

The aim of this paper is to consider some interplay between squares of vector lattices and homogeneous functional calculus and obtain Hölder type inequalities for orthosymmetric bilinear operators. We also collect several useful facts concerning homogeneous functional calculus on relatively uniformly complete vector lattices some of which despite of their simplicity does not seem appeared in the literature.

There are different ways to introduce the homogeneous functional calculus on vector lattices, see [6, 13, 17, 19, 21, 22]. We follow the approach [6, 9] going back also to G. Ya. Lozanovskiĭ [19]. For the theory of vector lattices and positive operators we refer to the books [2] and [14]. All vector lattices in this paper are real and Archimedean.

1.1. We start by recalling some definitions and results from [7]. Let E and G be vector lattices. A bilinear operator $b : E \times E \to G$ is said to be *orthosymmetric* if $|x| \wedge |y| = 0$ implies $b(x, y) = 0$ for arbitrary $x, y \in E$, see [8]. If $b(x, y) \geqslant 0$ for all $0 \leqslant x \in E$ and $0 \leqslant y \in E$, then b is named *positive*. The difference of two positive orthosymmetric bilinear operators is called *orthoregular*. Denote by $BL_{or}(E; G)$ the space of all orthoregular bilinear operators from $E \times E$

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to G ordered by the cone of positive orthosymmetric operators. Aspects of orthosymmetric bilinear operators are presented in [7, 15, 16].

We say that a bilinear operator b is *symmetric* if $b(x,y) = b(y,x)$ for all $x,y \in E$. The following important property of orthosymmetric bilinear operators was established in [8, Corollary 2].

An orthosymmetric positive bilinear operator is symmetric.

A bilinear operator b is said to be *lattice bimorphism* if the mappings $y \mapsto b(e,y)$ $(y \in F)$ and $x \mapsto b(x,f)$ $(x \in E)$ are lattice homomorphisms for all $0 \leqslant e \in E$ and $0 \leqslant f \in F$. For a lattice bimorphism the converse is also true, see [7, Proposition 1.7].

A lattice bimorphism is orthosymmetric if and only if it is symmetric.

1.2. For an arbitrary vector lattice E there exists a vector lattice $E^{\odot}$ and an orthosymmetric lattice bimorphism $\odot : (x,y) \mapsto x \odot y$ from $E \times E$ to $E^{\odot}$ such that the following universal property holds: *whenever b is a symmetric lattice bimorphism from $E \times E$ to some vector lattice F, there is a unique lattice homomorphism $\Phi_b : E^{\odot} \to F$ with $b = \Phi_b \odot$.*

The pair $(E^{\odot}, \odot)$ is essentially unique, i.e. if for some vector lattice $E^{\circledcirc}$ and symmetric lattice bimorphism $\circledcirc : E \times E \to E^{\circledcirc}$ the pair $(E^{\circledcirc}, \circledcirc)$ obeys the said universal property, then there exists a lattice isomorphism ι from $E^{\odot}$ onto $E^{\circledcirc}$ such that $\iota\odot = \circledcirc$ (and, of course, $\iota^{-1}\circledcirc = \odot$).

The vector lattice $E^{\odot}$ and the lattice bimorphism $\odot$ are called the *square* of E and the *canonical bimorphism*, respectively. The following result claims that the mentioned universal property of squares can be essentially extended, see [7, Theorem 3.1] and [9, Theorem 9].

1.3. *Let E and G be vector lattices with G relatively uniformly complete. Then for every bilinear orthoregular operator $b : E \times E \to G$ there exists a unique linear regular operator $\Phi_b : E^{\odot} \to G$ such that*

$$b(x,y) = \Phi_b(x \odot y) \quad (x,y \in E).$$

The correspondence $b \mapsto \Phi_b$ is an isomorphism of the ordered vector spaces $BL_{or}(E,G)$ and $L_r(E^{\odot}, G)$.

2. Homogeneous functions on vector lattices

In this section we introduce homogeneous functional calculus on relatively uniformly complete vector lattices and state some useful facts.

2.1. Denote by $\mathscr{H}(\mathbb{R}^N)$ the vector lattice of all continuous positively homogeneous functions $f : \mathbb{R}^N \to \mathbb{R}$. In accordance with [6] we say that $f(x_1,\dots,x_N)$ exists in E and write $y = f(x_1,\dots,x_N)$ if there is an element $y \in E$ such that $\omega(y) = f(\omega(x_1),\dots,\omega(x_N))$ for every $\mathbb{R}$-valued lattice homomorphism ω on the sublattice of E generated by $\{x_1,\dots,x_N,y\}$. The definition is correct in the sense that if L is any vector sublattice of E containing $\{x_1,\dots,x_N,y\}$ and $\omega(y) = f(\omega(x_1),\dots,\omega(x_N))$ $(\omega \in \Omega)$ for some point separating set Ω of $\mathbb{R}$-valued lattice homomorphisms on L, then $y = f(x_1,\dots,x_N)$. It is immediate from the definition that $f(x,\dots,x) = xf(1,\dots,1)$ whenever $0 \leqslant x \in E$. Define $dt_j \in \mathscr{H}(\mathbb{R}^N)$ by $dt_j(t_1,\dots,t_N) = t_j$ $(j := 1,\dots,N)$.

2.2. Theorem. *Let E be a relatively uniformly complete vector lattice and $x_1,\dots,x_N \in E$. Then $f(x_1,\dots,x_N)$ exists for any $f \in \mathscr{H}(\mathbb{R}^N)$ and the mapping*

$$f \mapsto f(x_1,\dots,x_N) \quad \big(f \in \mathscr{H}(\mathbb{R}^N)\big)$$

is a unique lattice homomorphism from $\mathscr{H}(\mathbb{R}^N)$ into E with $dt_j(x_1,\dots,x_N) = x_j$ $(j := 1,\dots,N)$.

In particular, if $f, g \in \mathscr{H}(\mathbb{R}^N)$ and $f \leqslant g$, then $f(x_1,\dots,x_N) \leqslant g(x_1,\dots,x_N)$ for all $(x_1,\dots,x_N) \in E^N$. Moreover, the inequality holds:

$$|f(x_1,\dots,x_N)| \leqslant \|f\| \bigvee_{j=1}^{N} |x_j|,$$

where $\|f\| := \sup\{f(t_1,\dots,t_N) : (t_1,\dots,t_N) \in \mathbb{R}^N, \max_j |t_j| = 1\}$.

2.3. Let $K, M, N \in \mathbb{N}$ and consider positively homogeneous functions $f_1,\dots,f_M \in \mathscr{H}(\mathbb{R}^N)$ and $g_1,\dots,g_K \in \mathscr{H}(\mathbb{R}^M)$. Denote $f := (f_1,\dots,f_M)$ and $g := (g_1,\dots,g_K)$. Then $g_1 \circ f,\dots,g_K \circ f \in \mathscr{H}(\mathbb{R}^N)$ and, for any $x = (x_1,\dots,x_N) \in E^N$ and $y = (y_1,\dots,y_M) \in E^M$, the elements $f(x) := (f_1(x),\dots,f_M(x)) \in E^M$ and $g(y) := (g_1(y),\dots,g_K(y)) \in E^K$ are well defined. Moreover,

$$(g \circ f)(x) = g(f(x)) \quad \big(x = (x_1,\dots,x_N) \in E^N\big).$$

In particular, if $N = M = K$ and $g = f^{-1}$, then

$$f^{-1}(f(x)) = x, \quad f(f^{-1}(y)) = y \quad \big(x, y \in E^N\big).$$

We define also $f_1 \times g_1 : E^{N+M} \to E^2$ by $(f_1 \times g_1)(x, y) := (f_1(x), g_1(y))$.

2.4. Let E and F be relatively uniformly complete vector lattices, $h : E \to F$ be a lattice homomorphism, $x_1,\dots,x_N \in E$, and $f \in \mathscr{H}(\mathbb{R}^N)$. Then

$$h(f(x_1,\dots,x_N)) = f(h(x_1),\dots,h(x_N))).$$

If E is a relatively uniformly complete vector sublattice of F containing $x_1,\dots,x_N \in F$ and h is the inclusion map $E \to F$, then $f(x_1,\dots,x_N)$ relative to F is contained in E and its meaning relative to E is the same.

2.5. Assume that $f \in \mathscr{H}(\mathbb{R}^N)$ possesses the following property:

$$(\forall t_1,\dots,t_N \in \mathbb{R})\, t_1 t_2 \cdot \ldots \cdot t_N = 0 \;\Rightarrow\; f(t_1, t_2,\dots,t_N) = 0.$$

Then for any $u, x_1,\dots,x_N \in E$ and fixed integer $1 \leqslant k \leqslant N$ we have

$$x_k \perp u \;\Rightarrow\; f(x_1,\dots,x_{k-1}, x_k, x_{k+1},\dots,x_N) \perp u.$$

Moreover, for any band $L \subset E$ there holds $f(x_1,\dots,x_{k-1}, x_k, x_{k+1},\dots,x_N) \in L$ whenever $x_k \in L$. If L admits a band projection π, then

$$\pi f(x_1,\dots,x_{k-1}, x_k, x_{k+1},\dots,x_N) = f(x_1,\dots,x_{k-1}, \pi x_k, x_{k+1}\dots,x_N).$$

Now, we consider concrete examples of homogeneous functions.

2.6. Homogeneous functional calculus is used to introduce the so called p-convexification and p-concavification procedures for a Banach lattice, see [17, 22]. Consider three functions $\sigma_{\alpha,N}, \sigma'_{\alpha,N} : \mathbb{R}^N \to \mathbb{R}$, and $J : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\sigma_{\alpha,N}(t_1,\dots,t_N) := \theta_\alpha^{-1}(\theta_\alpha(t_1) + \ldots + \theta_\alpha(t_N)),$$
$$\sigma'_{\alpha,N}(t_1,\dots,t_N) := \theta_\alpha(\theta_\alpha^{-1}(t_1) + \ldots + \theta_\alpha^{-1}(t_N)),$$
$$J(r, s) := \theta_2^{-1}(rs) \quad (r, s, t_1,\dots,t_N \in \mathbb{R}),$$

where $0 < \alpha \in \mathbb{R}$ and $\theta_\alpha : t \mapsto \operatorname{sgn}(t)t^\alpha$ is an order preserving bijection of $\mathbb{R}$. Obviously, $\sigma_{\alpha,N}, \sigma'_{\alpha,N}$ belong to $\mathscr{H}(\mathbb{R}^N)$ and J belongs to $\mathscr{H}(\mathbb{R}^2)$, so that $\sigma_{\alpha,N}(x_1,\dots,x_N)$, $\sigma'_{\alpha,N}(x_1,\dots,x_N)$, and $J(x,y)$ are well defined for all $x, y, x_1, \dots, x_N$ in a relatively uniformly complete vector lattice E. From the above definitions the following implication is easily deduced

$$(\forall\, x, y \in E) \quad |x| \wedge |y| = 0 \;\Rightarrow\; \sigma_{\alpha,2}(x,y) = \sigma'_{\alpha,2}(x,y) = x + y,$$

since it is true in the real context. Denote for brevity $\theta := \theta_2$, $\sigma = \sigma_{2,2}$, and $\sigma' = \sigma'_{2,2}$.

Given a relatively uniformly complete vector lattice $E := (E, +, \cdot, \leqslant)$ and $0 < \alpha \in \mathbb{R}$, the *$\alpha$-convexification* $E^{(\alpha)}$ of E is defined as the same underlying set equipped with the same order and new vector operations

$$x \tilde{+} y := \sigma'_{\alpha,2}(x,y) := (x^{1/\alpha} + y^{1/\alpha})^\alpha \quad (x, y \in E),$$
$$\lambda * x := \theta_\alpha(\lambda) \cdot x := \lambda^\alpha \cdot x \quad (\lambda \in \mathbb{R},\, x \in E),$$

so that $E^{(\alpha)} := (E, \tilde{+}, *, \leqslant)$. Then $E^{(\alpha)}$ is also a relatively uniformly complete vector lattice. Moreover, $E^{(1)} = E$, $(E^{(\alpha)})^{(\beta)} = E^{(\alpha\beta)}$, and $(E^{(\alpha)})^{(1/\alpha)} = E$, where identities meant in the sense of vector and lattice isomorphism, see [22, Proposition 4.8].

In particular, the square $(E^{\odot}, \odot)$ can be defined as $E^{\odot} := E^{(1/2)} := (E, \tilde{+}, *, \leqslant)$ and $\odot := J$, where $x \tilde{+} y := \sigma(x,y)$, $\lambda * x := \theta^{-1}(\lambda)x$, and $\leqslant$ is the given ordering in E, see [9, Theorem 9].

2.7. We say that a function $f \in \mathscr{H}(\mathbb{R}^N)$ is *multiplicative* or *modulus preserving* if respectively $f(s_1t_1, \dots, s_Nt_N) = f(s_1,\dots,s_N)f(t_1,\dots,t_N)$ and $f(|t_1|,\dots,|t_N|) = |f(t_1,\dots,t_N)|$ for all $s_1, t_1, \dots, s_N, t_N \in \mathbb{R}$. The general form of a nonzero positively homogeneous multiplicative and modulus preserving function is given by

$$t_1 t_2 \cdot \ldots \cdot t_N = 0 \;\Rightarrow\; f(t_1, t_2, \dots, t_N) = 0,$$
$$f(t_1,\dots,t_N) = f(|t_1|,\dots,|t_N|)\operatorname{sgn} f(t_1,\dots,t_N),$$
$$f(|t_1|,\dots,|t_N|) = \exp(g_1(\ln|t_1|)) \cdot \ldots \cdot \exp(g_N(\ln|t_N|)),$$

where $g_1, \dots, g_N$ are some additive functions in $\mathbb{R}$ (i. e. solutions to Cauchy functional equation, see [3]) with $\sum_{i=1}^N g_i = I_{\mathbb{R}}$. In the case of continuous $g_1, \dots, g_N$ we get a Kobb–Duglas type function f and if, in addition, f is nonnegative, then $f(t_1,\dots,t_N) = c|t_1|^{p_1} \cdot \ldots \cdot |t_N|^{p_N}$ with $0 \leqslant c \in \mathbb{R}$, $p_1, \dots, p_N \in \mathbb{R}$ and $\sum_{i=1}^N p_i = 1$. Therefore, the expression $|x_1|^{p_1} \cdot \ldots \cdot |x_N|^{p_N}$ is well defined in E for $p_k \geqslant 0$, $p_1 + \ldots + p_N = 1$. Moreover,

$$|x_1|^{p_1} \cdot \ldots \cdot |x_N|^{p_N} \leqslant p_1|x_1| + \ldots + p_N|x_N|$$

by the inequality between the weighted arithmetic and geometric means.

3. Gauges and Hölder type inequalities

Now we consider some interplay between the square of a vector lattice and homogeneous functional calculus and deduce some Hölder type inequalities. In the sequel E denotes a relatively uniformly complete vector lattice.

3.1. A *gauge* is a nonnegative sublinear function defined on a convex cone contained in $\mathbb{R}^N$. The *polar* k° of a gauge k defined by

$$k^\circ(t) := \inf\{\lambda > 0 : \; (\forall s \in \mathbb{R}^N)\langle s, t\rangle \leqslant \lambda k(s)\} \quad (y \in \mathbb{R}^N)$$

is also a gauge. Moreover, $k^{\circ\circ} := (k^\circ)^\circ = k$ if and only if k is lower semicontinuous (for more details see [20]).

A gauge $k : \mathbb{R}^N \to \mathbb{R}$ is *strictly positive* provided that $k(s) > 0$ for every $s \neq 0$. Here we consider only strictly positive gauges defined everywhere on $\mathbb{R}^N$. The totality of such gauges on $\mathbb{R}^N$ will be denoted by $\mathscr{G}(\mathbb{R}^N)$. Every gauge from $\mathscr{G}(\mathbb{R}^N)$ is continuous. The polar of a gauge $k \in \mathscr{G}(\mathbb{R}^N)$ is also contained in $\mathscr{G}(\mathbb{R}^N)$ and can be calculate by

$$k^\circ(t) = \sup_{s\neq 0} \frac{\langle s, t\rangle}{k(s)} = \sup\{\langle s, t\rangle : \ k(s) \leqslant 1\} \quad (t \in \mathbb{R}^N).$$

Since $\mathscr{G}(\mathbb{R}^N) \subset \mathscr{H}(\mathbb{R}^N)$, there exist $k(x_1, \dots, x_N) \in E$ and $k^\circ(x_1, \dots, x_N) \in E$ for any $x_1, \dots, x_N \in E$. Moreover, the mapping $(x_1, \dots, x_N) \mapsto k(x_1, \dots, x_N)$ is a sublinear operator from E^N to E and

$$|k(x_1, \dots, x_N) - k(y_1, \dots, y_N)| \leqslant \|k\| \bigvee_{i=1}^{N} |x_i - y_i|.$$

3.2. *If $k \in \mathscr{G}(\mathbb{R}^N)$ and $x_1, \dots, x_N \in E$, then the representation holds*

$$k^\circ(x_1, \dots, x_N) = \sup\left\{ \sum_{i=1}^{N} \lambda_i x_i : \ (\lambda_1, \dots \lambda_N) \in \mathbb{R}^N, \ k(\lambda_1, \dots \lambda_N) \leqslant 1 \right\}.$$

Moreover, $k^\circ(x_1, \dots, x_N)$ is a relatively uniform limit of an increasing sequence which is comprised of the finite suprema of linear combinations of the form $\sum_{i=1}^N \lambda_i x_i$ with $k(\lambda_1, \dots \lambda_N) \leqslant 1$.

◁ Observe that the set $U := \left\{ \sum_{i=1}^N \lambda_i x_i \ : \ k(\lambda_1, \dots \lambda_N) \leqslant 1 \right\}$ is norm totally bounded in the AM-space E_u, $u := |x_1| \vee \dots \vee |x_N|$, since it is the image of the compact set $\{\lambda \in \mathbb{R}^N \ : \ k(\lambda) \leqslant 1\}$ under the map $\lambda = (\lambda_1, \dots, \lambda_N) \mapsto \sum_{k=1}^N \lambda_i x_i$. Denote by $U^\vee$ the subset of E consisting of the suprema of the finite subsets of U. Then by Krengel's Lemma (see [1, Lemma 3.13]) $y := \sup U$ exists in E_u and belongs to the norm closure $\overline{U^\vee}$ of $U^\vee$. Since $U^\vee$ is upward directed, $U^\vee$ is norm convergent to y. Therefore, for any $\mathbb{R}$-valued homomorphism ω on E_u we have

$$\omega(y) = \lim_{u\in U^\vee} \omega(u) = \sup\{\omega(u) : \ u \in U^\vee\} \sup\{\omega(u) : \ u \in U\} = k^\circ(\omega(x_1), \dots, \omega(x_N)).$$

Thus, $y = k^\circ(x_1, \dots, x_N)$ by [6, Corollary 3.4]. ▷

3.3. Take a gauge $k_{p,N} \ : \ (t_1, \dots, t_N) \ \mapsto \ \left(\sum_{i=1}^N |t_i|^p\right)^{\frac{1}{p}}$ with $1 \ \leqslant \ p \ \leqslant \ \infty$. For the corresponding mapping from E^N into E an expressive notation is used, see [17, 21, 22]:

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} := k_{p,N}(x_1, \dots, x_N) \quad (x_1, \dots, x_N \in E).$$

For $p = \infty$, we define $k_{p,N}(t_1, \dots, t_N) = \max\left\{|t_i| \ : \ i := 1, \dots, N\right\}$ and, obviously, $k_{p,N}(x_1, \dots, x_N) = x_1 \vee \dots \vee x_N$. Of course, $k_{p,N} \in \mathscr{H}(\mathbb{R}^N)$ and the mapping $(x_1, \dots, x_N) \mapsto k_{p,N}(x_1, \dots, x_N) \in E$ is well defined even if $0 < p < 1$, but in this case $k_{p,N} \notin \mathscr{G}(\mathbb{R}^N)$ and the corresponding mapping is not sublinear.

3.3. If $k \in \mathscr{G}(\mathbb{R}^N)$, $x_1, \dots, x_N \in E$, and T is a positive operator from E to a F, then

$$k(Tx_1, \dots, Tx_N) \leqslant T(k(x_1, \dots, x_N)).$$

and equality holds if T is a lattice homomorphism. The inequality can be derived from the representation 3.2 using the same arguments as in 3.6 below. In particular, specializing $k := k_{p,N}$ with $1 \leqslant p \leqslant \infty$ yields

$$\left(\sum_{k=1}^{n} |Tx_k|^p\right)^{\frac{1}{p}} \leqslant T\left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$$

and equality holds if T is a lattice homomorphism [17, 21]. On the contrary, if $0 < p < 1$, then

$$T\left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{k=1}^{n} |Tx_k|^p\right)^{\frac{1}{p}}.$$

A more general fact will be proved below in 4.3.

3.4. *For any $k \in \mathscr{G}(\mathbb{R}^N)$ and $x_1, \dots, x_N, y_1, \dots, y_N \in E$ the inequality holds*

$$\sum_{i=1}^{N} x_i \odot y_i \leqslant k(x_1, \dots, x_N) \odot k^\circ(y_1, \dots, y_N).$$

$\lhd$ It is an easy exercise to check that the inequality (see 2.6 for definitions of $\sigma_{2,N}$ and J)

$$\sigma_{2,N}\big(J(s_1, t_1), \dots, J(s_N, t_N)\big) \leqslant J\big(k(s_1, \dots, s_N), k^\circ(t_1, \dots, t_N)\big).$$

is equivalent to the well known property of gauges [20]:

$$\langle s, t\rangle \leqslant k(s)k^\circ(t) \quad (s = (s_1, \dots, s_N),\ t = (t_1, \dots, t_N) \in \mathbb{R}^N).$$

Combining this with 2.3 and 3.6 we obtain the desired inequality. $\rhd$

In the special case of $k := k_{p,N}$ and $k^\circ = k_{q,N}$, $1 \leqslant p$, $q \leqslant \infty$, $1/p + 1/q = 1$, we have

$$\sum_{i=1}^{N} |x_i \odot y_i| \leqslant \left(\sum_{i=1}^{N} |x_i|^p\right)^{\frac{1}{p}} \odot \left(\sum_{i=1}^{N} |y_i|^q\right)^{\frac{1}{q}}.$$

3.5. *If $b : E \times E \to G$ is a positive orthosymmetric bilinear operator and $x_i, y_i \in E$, $i := 1, \dots, N$, then*

$$\sum_{k=1}^{N} |b(x_k, y_k)| \leqslant b\big(k(x_1, \dots, x_N), k^\circ(y_1, \dots, y_N)\big).$$

$\lhd$ Apply Φ_b to 3.4 and use 1.3. $\rhd$

Again, if $1 \leqslant p$, $q \leqslant \infty$ and $1/p + 1/q = 1$, then

$$\sum_{k=1}^{N} |b(x_k, y_k)| \leqslant b\left(\left(\sum_{k=1}^{N} |x_k|^p\right)^{\frac{1}{p}}, \left(\sum_{k=1}^{N} |y_k|^q\right)^{\frac{1}{q}}\right).$$

3.6. *Let $b : E \times E \to G$ be a positive orthosymmetric bilinear operator and $b = \Phi_b \odot$ for a positive linear operator Φ_b from $E^{\odot}$ to G. Then for $x_1, y_1, \dots, x_N, y_N \in E$ and $k \in \mathscr{G}(\mathbb{R}^N)$ we have*

$$k(b(x_1, y_1), \dots, b(x_N, y_N)) \leqslant \Phi_b\big(k(x_1 \odot y_1, \dots, x_N \odot y_N)\big).$$

In particular, if $1 \leqslant p,\ q \leqslant \infty$ and $1/p + 1/q = 1$, then

$$\left(\sum_{i=1}^{N} |b(x_i, y_i|)^p\right)^{\frac{1}{p}} \leqslant \Phi_b\left(\left(\sum_{i=1}^{N} |x_i \odot y_i|^p\right)^{\frac{1}{p}}\right).$$

$\lhd$ Taking into consideration 1.3, 3.2, and positivity of Φ_b we deduce

$$\sum_{i=1}^{N} \lambda_i b(x_i, y_i) = \Phi_b\left(\sum_{i=1}^{N} \lambda_i x_i \odot y_i\right) \leqslant \Phi_b\big(k(x_1 \odot y_1, \dots, x_N \odot y_N)\big)$$

for any finite collection $\lambda_1, \dots \lambda_N \in \mathbb{R}$ with $k^{\circ}(\lambda_1, \dots \lambda_N) \leqslant 1$. It remains to apply 3.2 again. $\rhd$

3.7. *If in* 3.6 *b is a symmetric lattice bimorphism, then*

$$k(b(x_1, y_1), \dots, b(x_N, y_N)) = \Phi_b\big(k(x_1 \odot y_1, \dots, x_N \odot y_N)\big).$$

In particular, if $1 \leqslant p, q \leqslant \infty$ and $1/p + 1/q = 1$, then we have

$$\left(\sum_{i=1}^{N} |b(x_i, y_i|^p\right)^{\frac{1}{p}} = \Phi_b\left(\left(\sum_{i=1}^{N} |x_i \odot y_i|^p\right)^{\frac{1}{p}}\right).$$

$\lhd$ Since Φ_b is a lattice homomorphism by 1.2, we only need to apply 2.4 and 1.3. $\rhd$

4. Inequalities with monomials

In this section we prove several inequalities containing homogeneous expressions of the form $|x_1|^{p_1} \cdot \ldots \cdot |x_N|^{p_N}$ with $p_1 + \ldots + p_N = 1$, see 2.7.

4.1. *Assume that a homogeneous function $f \in \mathscr{H}(\mathbb{R}^N)$ is multiplicative and modulus preserving. Then for all $x_1, y_1, \dots, x_N, y_N \in E$ we have*

$$f(x_1 \odot y_1, \dots, x_N \odot y_N) = f(x_1, \dots, x_N) \odot f(y_1, \dots, y_N).$$

In particular, if $0 \leqslant p_1, \dots, p_N \in \mathbb{R}$, $p_1 + \ldots + p_N = 1$, then

$$\prod_{i=1}^{N} |x_i \odot y_i|^{p_i} = \left(\prod_{i=1}^{N} |x_i|^{p_i}\right) \odot \left(\prod_{i=1}^{N} |y_i|^{p_i}\right).$$

$\lhd$ If f is multiplicative and modulus preserving, then $\theta(f(s_1, \dots, s_N)) = f(\theta(s_1), \dots, \theta(s_N))$ and the equality $f \circ (J \times \ldots \times J) = J \circ (f \times f)$ holds, see 2.6. Applying 2.3 and 2.6 we come to the desired inequalities. $\rhd$

4.2. Theorem (The generalized Hölder inequality). *Assume that E and G be relatively uniformly complete vector lattices. If a mapping $f : E_+ \to G$ is increasing and sublinear $\big(f(x + y) \leqslant f(x) + f(y),\ f(\lambda x) = \lambda f(x);\ x, y \in E,\ 0 < \lambda \in \mathbb{R}\big)$, then*

$$f\left(\prod_{i=1}^{N} |x_i|^{p_i}\right) \leqslant \prod_{i=1}^{N} f(|x_i|)^{p_i}$$

for $x_1, \dots, x_N \in E$ *and* $0 \leqslant p_1, \dots, p_N \in \mathbb{R}$ *with* $p_1 + \ldots + p_N = 1$. *Equality holds if* $f : E \to G$ *is lattice homomorphism.*

$\lhd$ Without loss of generality we may assume that $0 \leqslant x_i$ and $0 < p_i < 1$ for all $i := 1, \dots, N$. Indeed, if $\{i_1, \dots, i_k\} = \{j \leqslant N : \ p_j \neq 0\}$, then $|x_1|^{p_1} \cdot \ldots \cdot |x_N|^{p_N} = |x_{i_1}|^{p_{i_1}} \cdot \ldots \cdot |x_{i_k}|^{p_{i_k}}$. Now we observe that, for $0 \leqslant x, y \in E$ and $0 < p < 1$, the equality holds

$$x^p y^{1-p} = \inf\{p\lambda^{1/p}x + (1-p)\lambda^{-1/(1-p)}y : \ 0 < \lambda \in \mathbb{Q}\}.$$

Indeed, by 2.6 for an arbitrary $0 < \lambda \in \mathbb{R}$ the inequality is valid:

$$x^p y^{1-p} = \big(\lambda^{1/p}x\big)^p \big(\lambda^{-1/(1-p)}y\big)^{1-p} \leqslant p\lambda^{1/p}x + (1-p)\lambda^{-1/(1-p)}y.$$

Assume that $v \leqslant \varphi_\lambda := p\lambda^{1/p}x + (1-p)\lambda^{-1/(1-p)}y$ for all $0 < \lambda \in \mathbb{Q}$. Actually this inequality is true for all $0 < \lambda \in \mathbb{R}$, since $|\varphi_\lambda - \varphi_\mu| \leqslant C(\varepsilon)|\lambda - \mu|(x+y)$ whenever $0 < \varepsilon < \lambda, \mu < 1/\varepsilon$. By the Kreĭns–Kakutani Representation Theorem we can view the principal ideal E_u generated by $u = x + y + |v|$ as $C(S)$ for some compact space S. Then v, x, y, φ_λ, and $x^p y^{1-p}$ lie in $C(S)$ and for $0 < \lambda \in \mathbb{R}$ the pointwise inequality $v(s) \leqslant \varphi_\lambda(s)$ is true. If $x(s) = 0$, then trivially

$$v(s) \leqslant \inf\Big\{(1-p)\lambda^{-1/(1-p)}y(s) : \ 0 < \lambda \in \mathbb{Q}\Big\} = 0 = x(s)^p y(s)^{1-p}.$$

If $x(s) \neq 0$, then for $\lambda := (y(s)/x(s))^{p(1-p)}$ we have $\varphi_\lambda(s) = x(s)^p y(s)^{1-p} \geqslant v(s)$. Thus, $v \leqslant x^p y^{1-p}$ and the desired representation for $x^p y^{1-p}$ follows.

Now, taking into consideration that f is sublinear and increasing, we deduce

$$\begin{aligned} f(x^p y^{1-p}) &\leqslant \inf\Big\{f(p\lambda^{1/p}x + (1-p)\lambda^{-1/(1-p)}y) : \ 0 < \lambda \in \mathbb{Q}\Big\} \\ &\leqslant \inf\Big\{p\lambda^{1/p}f(x) + (1-p)\lambda^{-1/(1-p)}f(y) : \ 0 < \lambda \in \mathbb{Q}\Big\} = f(x)^p f(y)^{1-p}. \end{aligned}$$

The general case is handled by induction. Suppose $f\left(x_1^{q_1} \cdot \ldots \cdot x_{N-1}^{q_{N-1}}\right) \leqslant f(x_1)^{q_1} \cdot \ldots \cdot f(x_{N-1})^{q_{N-1}}$, whenever $q_1 + \ldots + q_{N-1} = 1$. Put $p := p_1 + \ldots + p_{N-1}$, $q_i := p_i/p$ $(i := 1, \dots, N-1)$, and $u := \big(x_1^{p_1} \cdot \ldots \cdot x_{N-1}^{p_{N-1}}\big)^{1/p} = x_1^{q_1} \cdot \ldots \cdot x_{N-1}^{q_{N-1}}$. Then

$$\begin{aligned} f\big(x_1^{p_1} \cdot \ldots \cdot x_N^{p_N}\big) &= f(u^p x_N^{p_N}) \leqslant f(u)^p f(x_N)^{p_N} \\ &= f\big(x_1^{q_1} \cdot \ldots \cdot x_{N-1}^{q_{N-1}}\big)^p f(x_N)^{p_N} \leqslant f(x_1)^{p_1} \cdot \ldots \cdot f(x_N)^{p_N}, \end{aligned}$$

and the required inequality follows. The remaining part is obvious. $\rhd$

4.3. Theorem (The generalized Minkowski inequality). *Assume that* E, G, f, *and* $x_1, \dots, x_N$ *are the same as in 4.2 and* $0 < p < 1$. *Then the inequality holds:*

$$f\bigg(\Big(\sum_{i=1}^N |x_i|^p\Big)^{1/p}\bigg) \leqslant \Big(\sum_{i=1}^N (f(|x_i|))^p\Big)^{1/p}.$$

Equality holds if $f : E \to G$ *is a lattice homomorphism.*

$\lhd$ The same line of reasoning as in 4.4 works. We may assume without loss of generality that $x_i \geqslant 0$ $i := 1, \dots, N$. First we prove that, for $0 \leqslant x, y \in E$ and $0 < p < 1$, the representation holds

$$(x^p + y^p)^{1/p} = \inf\{\lambda^{-1/p}x + (1-\lambda)^{-1/p}y : \ 0 < \lambda < 1, \, \lambda \in \mathbb{Q}\}.$$

Next, applying f to this equality and taking into account that f is increasing and sublinear we deduce

$$f\big((x^p + y^p)^{1/p}\big) \leqslant \big(f(x)^p + f(y)^p\big)^{1/p}.$$

The general case is again settled by induction. Put $u := \big(x_1^p + \ldots + x_{N-1}^p\big)^{1/p}$ and observe that

$$\big(u^p + x_N^p\big)^{1/p} = \big(x_1^p + \ldots + x_N^p\big)^{1/p}.$$

Moreover, by induction $f(u) \leqslant \big(f(x_1)^p + \ldots + f(x_{N-1})^p\big)^{1/p}$. Thus

$$f\big((x_1^p + \ldots + x_N^p)^{1/p}\big) = f\big(u^p + x_N^p\big)^{1/p}$$
$$\leqslant \big(f(u)^p + f(x_N)^p\big)^{1/p} \leqslant \big(f(x_1)^p + \ldots + f(x_N)^p\big)^{1/p}$$

and we are done. $\rhd$.

4.4. We can take in 4.2 an arbitrary increasing gauge $k \in \mathscr{G}(\mathbb{R}^M)$ instead of f and consider the corresponding sublinear operator from E^M to E. Suppose that $M \in \mathbb{N}$ and for every $j := 1, \ldots, M$ a finite collection of elements $(x_{1j}, \ldots, x_{Nj}) \in E^M$ is given. Replacing f, for example, by $k_{p,M}$ $(1 \leqslant p \leqslant \infty)$ we arrive at the following version of Hölder inequality:

$$\left(\sum_{j=1}^{M}\Big(|x_{1j}|^{p_1}\cdot\ldots|x_{Nj}^{p_N}|\Big)^p\right)^{1/p} \leqslant \left(\sum_{j=1}^{M}|x_{1j}|^p\right)^{p_1/p}\cdot\ldots\cdot\left(\sum_{j=1}^{M}|x_{Nj}|^p\right)^{p_N/p}.$$

4.5. Let (Ω, Σ, μ) be a measure space with a σ-finite positive measure μ and F be a Banach lattice. Let $\mathscr{L}^1(\Omega, \Sigma, \mu, F)$ be the space of all Bochner integrable functions on Ω with values in F and $E := L^1(\mu, F) := \mathscr{L}(\Omega, \Sigma, \mu, F)/\sim$ denotes the space of all equivalence classes (of almost everywhere equal) functions from $\mathscr{L}^1(\Omega, \Sigma, \mu, F)$. Then $E = L^1(\mu, F)$ is also a Banach lattice and hence $f(x_1, \ldots, x_N)$ is well defined in E for $f \in \mathscr{H}(\mathbb{R}^M)$ and $x_1, \ldots, x_N \in E$. Denote by $\tilde{x}$ the equivalence class of $x \in \mathscr{L}^1(\Omega, \Sigma, \mu, F)$. Making use of the continuity of functional calculus (see [9, Theorem 7]) one can deduce that the equality $f(\tilde{x}_1, \ldots, \tilde{x}_N)(\omega) = f(x_1(\omega), \ldots, x_N(\omega))$ is true for almost all $\omega \in \Omega$ $\big($or more precisely $f(\tilde{x}_1, \ldots, \tilde{x}_N)(\omega)$ is the equivalence class of the measurable function $\omega \mapsto f(x_1(\omega), \ldots, x_N(\omega))\big)$ for any finite collection $x_1, \ldots, x_N \in \mathscr{L}^1(\Omega, \Sigma, \mu, F)$. Since the Bochner integral defines a linear and increasing operator from E to F, we can replace f in 4.2 and 4.3 by the Bochner integral. Thus, we get the following inequalities $(0 \leqslant p_1, \ldots, p_N \in \mathbb{R}$, $p_1 + \ldots + p_N = 1$, $0 < p < 1)$:

$$\int_\Omega \left(\prod_{i=1}^{N}|x_i(\omega)|^{p_i}\right) d\mu(\omega) \leqslant \prod_{i=1}^{N}\left(\int_\Omega |x_i(\omega)|\, d\mu(\omega)\right)^{p_i},$$

$$\int_\Omega \left(\sum_{i=1}^{N}|x_i(\omega)|^{p}\right)^{1/p} d\mu(\omega) \leqslant \left(\sum_{i=1}^{N}\left(\int_\Omega |x_i(\omega)|\, d\mu(\omega)\right)^{p}\right)^{1/p}$$

for $x_1(\cdot), \ldots, x_N(\cdot) \in \mathscr{L}^1(\Omega, \Sigma, \mu, F)$.

4.6. *Let E, F, and G be relatively uniformly complete vector lattices, $f, g : E_+ \to F$ be ingreasing sublinear operators, and $b : F \times F \to G$ be a positive orthosymmetric bilinear operator. Then*

$$b\left(f\left(\prod_{i=1}^{N}|x_i|^{p_i}\right), g\left(\prod_{i=1}^{N}|y_i|^{p_i}\right)\right) \leqslant \prod_{i=1}^{N} b\big(f(|x_i|), g(|y_i|)\big)^{p_i}.$$

for all $x_1, y_1, \dots, x_N, y_N \in E$ and $0 \leqslant p_1, \dots, p_N \in \mathbb{R}$ with $p_1 + \ldots + p_N = 1$. Equality holds if $f, g : E \to F$ are lattice homomorphisms and b is a symmetric lattice bimorphism.

$\lhd$ By applying 4.2 to f and g and using 4.1 we obtain

$$f\bigg(\prod_{i=1}^{N} |x_i|^{p_i}\bigg) \odot g\bigg(\prod_{i=1}^{N} |y_i|^{p_i}\bigg) \leqslant \prod_{i=1}^{N} \big(f(|x_i|) \odot g(|y_i|)\big)^{p_i}.$$

Now, apply Φ_b to the last inequality, use again 4.2 with $f := \Phi_b$, and take 1.3 into account. If f and g are lattice homomorphisms and b is a lattice bimorphism, then we apply 2.4 and 4.1 and observe that, according to 1.2, Φ_b is a lattice homomorphism if and only if b is a lattice bimorphism. $\rhd$

4.7. Remarks 1. The generalized Hölder inequalities 4.2 for increasing sublinear operator between relatively uniformly complete vector lattices was obtained in [16, Theorem 5.2]. Both, the generalized Hölder and Minkowski inequalities 4.2 and 4.3 for operators between spaces of measurable functions were established in [12, Remark 1.2 (5)] and [12, Remark 1.2 (6)].

2. In [10] and [11] some interesting estimates for the Hadamard weighted geometric means of positive kernel operators on Banach function spaces were obtained. Hölder type inequalities for operators can be useful in such studies. For example, the inequalities (1) of [11, Theorem 2.1] and (4) of [11, Theorem 2.2] are the easy consequences of 4.2 ($f(k) := \|K\|$) and 4.6 ($b(H, K) := H \cdot K$, $f(h) := H$, $g(k) := K$), respectively. More applications see in [12] and [16].

3. In [12, Proposition 1.1] a Jensen type convexity inequality for sublinear operator on spaces of measurable functions was proved. (The scalar case see in [5, § I.1, Proposition 1].) Certainly, this result can be generalized to operators on vector lattice so that 4.2 and 4.3 will become its special cases. The corresponding improved version of homogeneous functional calculus on vector lattice will be developed in a forthcoming paper.

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Anatoly G. Kusraev
Institute of Applied Mathematics and Informatics
Vladikavkaz Science Center of the RAS
Vladikavkaz, 362040, RUSSIA
E-mail: kusraev@alanianet.ru