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LARGE TIME DECAY ESTIMATES OF THE SOLUTION TO THE CAUCHY PROBLEM OF DOUBLY DEGENERATE PARABOLIC EQUATIONS WITH DAMPING[#]

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Abstract. In this paper we study the large time behaviour of the solution and compactification of support to the Cauchy problem for doubly degenerate parabolic equations with strong gradient damping. Under the suitable assumptions on the structure of the equation and data of the problem we establish new sharp bound of solutions for a large time. Moreover, when the support of initial datum is compact we prove that the support of the solution contains in the ball with radius which is independent in time variable. In the critical case of the behaviour of the damping term the support of the solution depends on time variable logarithmically for a sufficiently large time. The main tool of the proof is based on nontrivial use of cylindrical Gagliardo–Nirenberg type embeddings and recursive inequalities. The sup-norm estimates of the solution is carried out by modified version of the classical method of De-Giorgi–Ladyzhenskaya–Uraltseva–DiBenedetto. The approach of the paper is flexible enough and can be used when studying the Cauchy–Dirichlet or Cauchy–Neumann problems in domains with non compact boundaries.

Key words: doubly degenerate parabolic equations, strong gradient damping, finite speed of propagation, large time behavior.

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1. Introduction

We look at the following Cauchy problem for the degenerate parabolic equation of the form:

$$\frac{\partial u^\beta}{\partial t} = \Delta_p(u) - |\nabla u|^q \quad \text{in} \ \ S_T = \mathbf{R}^N \times (0,T), \tag{1.1}$$

$$u^\beta(x,0) = u_0^\beta(x) \geqslant 0, \quad x \in \mathbf{R}^N, \ N \geqslant 1. \tag{1.2}$$

Here $x = (x_1, \ldots, x_N)$, $\nabla u = (u_{x_1}, \ldots, u_{x_N})$, $|\nabla u| = (u_{x_1}^2 + \ldots + u_{x_N}^2)^{1/2}$,

$$\Delta_p(u) := \operatorname{div}(|\nabla u|^{p-2}\,\nabla u).$$

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The main purpose of the paper is to establish the localization property and large time behavior of the solution to the problem (1.1), (1.2). It is well-known [1] that if

$$p-\beta-1>0, \quad p>1, \tag{1.3}$$

the equation $\partial u^\beta = \Delta_p(u)$ possesses the finite speed of propagation (FSP for short), which means that if $u(x,t_1) = 0$, $t_1 \geqslant 0$, then so does for any $t > t_1$. Clearly, that under assumptions (1.3) the FSP still holds for the nonnegative solutions to the equation (1.1) for any $q > 1$. Moreover, as we will show under the additional assumptions

$$0<\beta<q, \quad 1<q\leqslant p-1, \tag{1.4}$$

the radius of the support is independet of time (when $q < p-1$), or grows logarithmically (when $q = p-1$). Besides, the qualitative temporal decay estimate of sup norm of solution for a large time is done as well. We generalize some of the results of [2], where the case $\beta = 1$ was studied. Before to formulate the main results of the paper, we give the deffinition of the weak solution. Assume that u_0^β is a nonnegative locally integrable function.

Definition 1.1. We say that $u(x,t)$ is a weak solution of the problem (1.1), (1.2) in $S_T = \mathbf{R}^N \times (0,T)$, if $u \geqslant 0$, for any $|\nabla u|^p$, $|\nabla u|^q \in L_{1,\mathrm{loc}}(S_T)$, $u^\beta \in L_{\infty,\mathrm{loc}}(S_T) \cap C(0,T;L_{\beta+1,\mathrm{loc}})$, and for any $\eta \in C_0^1(S_T)$

$$-\int_0^T\int_{\mathbf{R}^N} u^\beta \eta_\tau \, dx\, d\tau + \int_0^T\int_{\mathbf{R}^N} \left(|\nabla u|^{p-2}\nabla u \nabla\eta + |\nabla u|^q \eta\right) dx\, d\tau = 0.$$

Moreover, for any $\zeta(x) \in C_0^\infty(\mathbf{R}^N)$

$$\lim_{t\to 0}\int_{\mathbf{R}^N} u^\beta(x,t)\zeta(x)\, dx = \int_{\mathbf{R}^N} u_0^\beta(x)\zeta(x)\, dx.$$

The existence of the weak solution can be done as in [3, 4]. The main results of the paper read as follows

Theorem 1.1. *Let $u(x,t)$ be the solution of the problem* (1.1), (1.2) *in Q_T for any $T > 0$. Suppose that*

$$q\leqslant p-1, \quad q>1, \quad q>\beta>0. \tag{1.5}$$

Then for any $t > 0$, $\nu > 0$

$$\|u(t)\|_\infty \leqslant \gamma t^{-\frac{N}{N(q-\beta)+q\nu}} \left(\sup_{\frac{t}{4}<\tau<t}\int_{\mathbf{R}^N} u^\nu dx\right)^{\frac{q}{N(q-\beta)+q\nu}}. \tag{1.6}$$

Theorem 1.2. *Let $u(x,t)$ be a weak solution of the problem* (1.1), (1.2) *in Q_T for any $T > 0$. Suppose that support $u_0^\beta(x) \subset B_{R_0} = \{|x| < R_0\}$, $R_0 < \infty$. Then we have:*

i) *If $0 < \beta < q < p-1$, $q > 1$, then for any $t > 0$*

$$Z(t) := \operatorname{support} u(x,t) := \{\rho : u(x,t) \equiv 0,\ |x| > \rho\} \subset B_R, \tag{1.7}$$

where $R = R(\|u_0^\beta\|_1)$ is independent of t. Moreover, for any $t > 0$

$$\|u(t)\|_\infty \leqslant \gamma t^{-\frac{1}{q-\beta}}. \tag{1.8}$$

ii) *If $\beta < q = p - 1$, $q > 1$, then for any t large enough we have for some given $\theta \geqslant 0$ that*

$$Z(t) \subset B_{R(t)}, \quad R(t) = \Gamma\left(\|u_0\|_{\beta+\theta}, \delta\right) \log t. \tag{1.9}$$

Moreover, for t large enough

$$\|u(t)\|_\infty \leqslant \gamma t^{-\frac{1}{p-\beta-1}} \left[\log \Gamma t\right]^{\frac{p-1}{p-\beta-1}}. \tag{1.10}$$

Here and hereafter we denote: $\|u(t)\|_\infty := \|u(x,t)\|_{L_\infty, \mathbf{R}^N}$, $B_R := B_R(0)$. Besides, we denote the generic constant γ, which depends only on the parameter of the problem β, N, p, q and may vary from line to line.

Last decades to the investigation of qualitative behavior of solutions to (1.1) under the various interplay of parameters β, p, q were devoted many papers [5–19]. As can be seen from these articles, few works have been devoted to the case of equations with double non-linearity. In this paper we use energy approach as in [20–22] (seee also [17]) which allows us to extend our results for more general class of equations. To get (1.8) we need to prove the following integral estimate

$$\int\limits_{\mathbf{R}^N} u^{\beta+\theta} dx \leqslant \gamma t^{-\frac{\beta+\theta}{q-\beta}} \quad \text{for some} \quad \theta > 0. \tag{1.11}$$

Note that (1.11) holds when support of initial datum is finite (see also [2, 17]) for $\beta = 1$. Then (1.8) is a consequence of (1.11) and (1.6) with $\nu = \beta + \theta$. If support of initial datum is unbounded, then the possible way to get (1.11) is as follows. Multiplying the both sides of (1.1) by u^θ, $\theta > 0$, and integrating by parts, and applying the Hardy and the Hölder inequalities, we have

$$\begin{aligned} \frac{\beta}{\beta+\theta} \frac{d}{dt} \int\limits_{\mathbf{R}^N} u^{\beta+\theta}\, dx &\leqslant -\int\limits_{\mathbf{R}^N} u^\theta \left|\nabla u\right|^q\, dx \leqslant -\gamma \int\limits_{\mathbf{R}^N} \frac{u^{q+\theta}}{|x|^q}\, dx \\ &\leqslant -\gamma \left(\int\limits_{\mathbf{R}^N} u^{\beta+\theta}\, dx\right)^{\frac{q-\beta+\theta}{\theta}} \left(\int\limits_{\mathbf{R}^N} u^\beta\, |x|^{\frac{q\theta}{q-\beta}}\, dx\right)^{-\frac{q-\beta}{\theta}}. \end{aligned} \tag{1.12}$$

Integrating (1.12), we get

$$\int\limits_{\mathbf{R}^N} u^{\beta+\theta}\, dx \leqslant \gamma t^{-\frac{\theta}{q-\beta}} \sup_{0<t<\infty} \int\limits_{\mathbf{R}^N} u^\beta\, |x|^{\frac{q\theta}{q-\beta}}\, dx. \tag{1.13}$$

From (1.13) it follows that asymptotically $\theta \to \infty$ we arrive at the sharp bound. Therefore, the weighted estimates of the solution are needed. We hope to devote a separate paper to this problem.

The rest of the paper is organized as follows. The Chapter 2 is devoted to auxiliary statements. In Chapters 3 and 4 we prove Theorems 1.1 and 1.2 correspondingly. For the sake of simplicity, in the proofs of the main results, the solution will be understood almost everywhere.

2. Auxiliary Results

We start with classical iterative Lemma 5.6 of [23, Chapter 2].

Lemma 2.1. *Let $\{Y_n\}$ be a sequence of positive numbers satisfying the recursive inequalities*

$$Y_{n+1} \leqslant C b^n Y_n^{1+\alpha},$$

where $C, b > 1$ and $\alpha > 0$ given numbers. Then, if

$$Y_0 \leqslant C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}},$$

then $\{Y_n\}$ converges to zero as $n \to \infty$.

Next, the classical Sobolev inequality reads: for any $f \in W_0^{1,q}(\Omega)$, where Ω is bounded or unbounded domain in $\mathbf{R}^N$, we have

$$\left(\int\limits_{\Omega} |f|^{\frac{Nq}{N-q}} \, dx \right)^{\frac{N-q}{Nq}} \leqslant \gamma(N,q) \left(\int\limits_{\Omega} |\nabla f|^q \, dx \right)^{\frac{1}{q}}, \tag{2.1}$$

provided

$$1 \leqslant q < N. \tag{2.2}$$

Let Ω is bounded: $|\Omega| := \text{meas}_N \Omega < \infty$. Then, applying the Hölder inequality, from (2.1) we have the Poincare–Fridrich inequality:

$$\int\limits_{\Omega} |f|^q dx \leqslant \gamma |\Omega|^{\frac{q}{N}} \left(\int\limits_{\Omega} |f|^{\frac{Nq}{N-q}} \, dx \right)^{\frac{N-q}{N}} \leqslant \gamma(N,q) \, |\Omega|^{\frac{q}{N}} \int\limits_{\Omega} |\nabla f|^q \, dx. \tag{2.3}$$

Another application of the Sobolev inequality is the Gagliardo–Nirenberg inequality, which one can obtain applying the Hölder inequality

$$\|f\|_{b,\Omega} \leqslant \gamma \, \|\nabla f\|_{q,\Omega}^{a} \, \|f\|_{r,\Omega}^{1-a}, \tag{2.4}$$

where $0 < r < b \leqslant \frac{Nq}{N-q}$, and $0 < a \leqslant 1$ is defined as follows

$$\frac{N}{b} = \frac{N-q}{q} a + \frac{N}{r}(1-a).$$

3. Proof of Theorem 1.1

Let $0 < a_2 < a_1$, $0 < \tau_2 < \tau_1$. Then we have (see [22] for the proof) the following version of the Caccioppoli inequality:

$$\begin{aligned} \sup_{\tau_1 < \tau < t} \int\limits_{\mathbf{R}^N} (u(\tau) - a_1)_+^{\beta+\theta} \, dx + \int\limits_{\tau_1}^{t} \int\limits_{\mathbf{R}^N} \left| \nabla (u - a_1)_+^{\frac{p+\theta-1}{p}} \right|^p \, dx \, d\tau \\ - \int\limits_{\tau_1}^{t} \int\limits_{\mathbf{R}^N} \left| \nabla (u - a_1)_+^{\frac{q+\theta}{q}} \right|^q \, dx \, d\tau \\ \leqslant \gamma \left(\frac{a_1}{a_1 - a_2} \right)^{(1-\beta)_+} (\tau_1 - \tau_2)^{-1} \int\limits_{\tau_2}^{t} \int\limits_{\mathbf{R}^N} (u - a_2)_+^{\beta+\theta} \, dx \, d\tau. \end{aligned} \tag{3.1}$$

Define for $h_0 > h_\infty > 0$, $\tau_0 > \tau_\infty > 0$, and $i = 0, 1, 2, \dots,$

$$k_i = h_\infty + (h_0 - h_\infty)2^{-i}, \quad t_i = \tau_\infty + (\tau_0 - \tau_\infty)2^{-i}, \quad v_i = (u - k_i)_+^{\frac{q+\theta}{q}}.$$

Then plugging in (3.1) $a_1 = k_i$, $a_2 = k_{i+1}$, $\tau_1 = t_i$, $\tau_2 = t_{i+1}$ and dropping the second summand in the left-hand side of (3.1), we get

$$\sup_{t_i<\tau<t} \int_{\mathbf{R}^N} v_i^\mu \, dx + \int_{t_i}^{t} \int_{\mathbf{R}^N} |\nabla v_i|^q \, dx \, d\tau$$
$$\leqslant \gamma \left(\frac{h_0}{h_0 - h_\infty} \right)^{(1-\beta)_+} \frac{2^i}{\tau_0 - \tau_\infty} \int_{t_{i+1}}^{t} \int_{\mathbf{R}^N} v_{i+1}^\mu \, dx \, d\tau, \quad \mu = \frac{q(\beta+\theta)}{q+\theta}. \tag{3.2}$$

By the Gagliardo–Nirenberg inequality (2.4), we have

$$\int_{\mathbf{R}^N} v_{i+1}^\mu \, dx \leqslant \gamma \left(\int_{\mathbf{R}^N} |\nabla v_{i+1}|^q \, dx \right)^{\frac{\mu A}{q}} \left(\int_{\mathbf{R}^N} v_{i+1}^{\nu_1} \, dx \right)^{\mu \frac{1-A}{\nu_1}},$$

where

$$\nu_1 = \frac{q\nu}{q+\theta}, \quad 0 < \nu < \beta + \theta,$$

and A is defined by the dimensional analysis; so that

$$C_1 = \frac{\mu}{q} A = \frac{N(\beta+\theta-\nu)}{N(q+\theta-\nu)+\nu q}, \quad C_2 = \mu \frac{1-A}{\nu_1} = \frac{N(q-\beta)+q(\beta+\theta)}{N(q+\theta-\nu)+\nu q}.$$

The Young inequality yields:

$$\int_{\mathbf{R}^N} v_{i+1}^\mu \, dx \leqslant \varepsilon^{\frac{1}{C_1}} C_1 \int_{\mathbf{R}^N} |\nabla v_{i+1}|^q \, dx + (1 - C_1) \, \varepsilon^{-\frac{1}{1-C_1}} \left(\int_{\mathbf{R}^N} v_{i+1}^\mu dx \right)^{\frac{C_2}{1-C_1}}.$$

Next, integrating in time, we have

$$\gamma \left(\frac{h_0}{h_0 - h_\infty} \right)^{(1-\beta)_+} \frac{2^i}{\tau_0 - \tau_\infty} \int_{t_{i+1}}^{t} \int_{\mathbf{R}^N} v_{i+1}^\mu \, dx \, d\tau$$
$$\leqslant \gamma \left(\frac{h_0}{h_0 - h_\infty} \right)^{(1-\beta)_+} \frac{2^i}{\tau_0 - \tau_\infty} \varepsilon^{\frac{1}{C_1}} C_1 \int_{t_{i+1}}^{t} \int_{\mathbf{R}^N} |\nabla v_{i+1}|^q \, dx \, d\tau \tag{3.3}$$
$$+ \gamma \left(\frac{h_0}{h_0 - h_\infty} \right)^{(1-\beta)_+} \frac{(t - \tau_\infty)2^i}{\tau_0 - \tau_\infty} (1 - C_1) \, \varepsilon^{-\frac{1}{1-C_1}} \sup_{\tau_\infty<\tau<t} \left(\int_{\mathbf{R}^N} v_{i+1}^{\nu_1} \, dx \right)^{\frac{C_2}{1-C_1}}.$$

Choose the free parameter ε as follows

$$\gamma \left(\frac{h_0}{h_0 - h_\infty} \right)^{(1-\beta)_+} \frac{2^i}{\tau_0 - \tau_\infty} \, \varepsilon^{\frac{1}{C_1}} C_1 = \varepsilon_1,$$

where ε_1 will be chosen later.

Therefore, (3.2) and (3.3) yield

$$Y_i := \sup_{t_i<\tau<t} \int\limits_{\mathbf{R}^N} v_i^{\mu}\, dx + \int\limits_{t_i}^{t} \int\limits_{\mathbf{R}^N} |\nabla v_i|^q \, dx\, d\tau \leqslant \varepsilon_1 Y_{i+1} + \gamma \left(\frac{h_0}{h_0 - h_\infty} \right)^{\frac{C_1(1-\beta)_+}{1-C_1}}$$
$$\times \frac{\varepsilon_1^{-\frac{C_1}{1-C_1}} 2^{\frac{i}{1-C_1}} (t-\tau_\infty)}{(\tau_0 - \tau_\infty)^{1+\frac{C_1}{1-C_1}}} \sup_{\tau_\infty<\tau<t} \left(\int\limits_{\mathbf{R}^N} v_\infty^{\nu_1}\, dx \right)^{\frac{C_2}{1-C_1}}. \quad (3.4)$$

Iterating the recursive inequality (3.4), we can easely deduce that if $\varepsilon_1 2^{\frac{1}{1-C_1}} < 1$, then

$$\sup_{\tau_0<\tau<t} \int\limits_{\mathbf{R}^N} (u-h_0)_+^{\beta+\theta} dx \leqslant \gamma \left(\frac{h_0}{h_0 - h_\infty} \right)^{\frac{C_1(1-\beta)_+}{1-C_1}}$$
$$\times \frac{(t-\tau_\infty)}{(\tau_0 - \tau_\infty)^{1+\frac{C_1}{1-C_1}}} \sup_{\tau_\infty<\tau<t} \left(\int\limits_{\mathbf{R}^N} (u-h_\infty)_+^{\nu} dx \right)^{\frac{C_2}{1-C_1}}. \quad (3.5)$$

Let $K_n = k(1-2^{-n-1})$, $\overline{K}_n = \frac{K_n+K_{n+1}}{2}$, $t'_n = t(1-2^{-n-1})$. Choose in (3.5) $\tau_0 = t'_{n+1}$, $\tau_\infty = t'_n$, $h_0 = \overline{K}_n$, $h_\infty = K_n$. Then we have

$$I_{n+1} := \sup_{t'_{n+1}<\tau<t} \int\limits_{\mathbf{R}^N} (u-K_{n+1})^{\nu}\, dx \leqslant \gamma b^n k^{-(\beta+\theta-\nu)} t^{-\frac{N(\beta+\theta-\nu)}{H_q}} I_n^{1+\frac{q(\beta+\theta-\nu)}{H_q}}, \quad (3.6)$$

where $b = b(\beta, q, \theta) > 1$, $H_q = N(q-\beta) + q\nu$. Hence, by iterative Lemma 2.1 it follows from (3.6) that $I_n \to 0$ as $n \to \infty$, $u \leqslant k$, provided

$$I_0^{\frac{q}{H_q}} k^{-1} t^{-\frac{N}{H_q}} \leqslant \delta_1,$$

where δ_1 is sufficiently small constant depending only on the data of the problem. Since

$$I_0 \leqslant \sup_{\frac{t}{4}<\tau<t} \int\limits_{\mathbf{R}^N} u^{\nu}\, dx,$$

we may choose

$$k = \frac{1}{2}\delta_1 \left(\sup_{\frac{t}{4}<\tau<t} \int\limits_{\mathbf{R}^N} u^{\nu}\, dx \right)^{\frac{q}{H_q}} t^{-\frac{N}{H_q}}$$

to complete the proof Theorem 1.1.

4. Proof of Theorem 1.2

Let for $\rho > R_0$ and $\theta > 0$ small enough

$$A_{n+1} = \rho'_n < |x| < \rho''_n, \quad \rho'_n = \frac{\rho}{2} - \sigma\frac{\rho}{2^n}, \quad \rho''_n = \rho + \sigma 2^{-n}\rho, \quad 0 < \sigma < \frac{1}{4},\ \theta > 0,$$

$$v_n = u^{\frac{p+\theta-1}{p}} \eta_n^s, \quad s > p,$$

η_n — the cutoff function of A_n.

Then proceeding exactly as in the proof of Theorem 1.1, owing in mind that support of solution is bounded, we have

$$Y_{n+1} := \sup_{0<\tau<t} \int\limits_{A_{n+1}} v_{n+1}^{a}\, dx + \frac{1}{\rho^q} \int\limits_0^t \int\limits_{A_{n+1}} v_n^{q_1}\, dx\, d\tau + \int\limits_0^t \int\limits_{A_{n+1}} |\nabla v_{n+1}|^p\, dx\, d\tau \leqslant \gamma \frac{2^{np}}{\sigma^p \rho^p} \int\limits_0^t \int\limits_{\mathbf{R}^N} v_n^p\, dx\, d\tau, \tag{4.1}$$

where

$$a = \frac{(\beta+\theta)p}{p+\theta-1}, \quad q_1 = \frac{(q+\theta)p}{p+\theta-1}.$$

Let $b : a < b < q_1$, will be chosen later, then applying the Gagliardo–Nirenberg inequality we have

$$\int\limits_{\mathbf{R}^N} v_n^p dx \leqslant \gamma \left(\int\limits_{\mathbf{R}^N} |\nabla v_n|^p\, dx \right)^B \left(\int\limits_{\mathbf{R}^N} v_n^b\, dx \right)^{\frac{(1-B)p}{b}}, \tag{4.2}$$

where B is defined as

$$\frac{N}{p} = \frac{(N-p)B}{p} + \frac{N(1-B)}{b}.$$

Next, by the Hölder inequality we have

$$\int\limits_{\mathbf{R}^N} v_n^b\, dx \leqslant \left(\int\limits_{\mathbf{R}^N} v_n^{q_1}\, dx \right)^{\frac{q_1-b}{q_1-a}} \left(\int\limits_{\mathbf{R}^N} v_n^{a}\, dx \right)^{\frac{b-a}{q_1-a}}. \tag{4.3}$$

Therefore, (4.2) and (2.3) imply

$$\int\limits_{\mathbf{R}^N} v_n^p\, dx \leqslant \gamma \rho^{qC_2} \left(\int\limits_{\mathbf{R}^N} |\nabla v_n|^p\, dx \right)^{C_1} \left(\rho^{-q} \int\limits_{\mathbf{R}^N} v_n^{q_1}\, dx \right)^{C_2} \left(\int\limits_{\mathbf{R}^N} v_n^{a}\, dx \right)^{C_3}, \tag{4.4}$$

where

$$C_1 = B = \frac{N(p-b)}{N(p-b)+bp}, \quad C_2 = \frac{b-a}{q_1-a}\frac{(1-B)p}{b}, \quad C_3 = \frac{q_1-b}{q_1-a}\frac{(1-B)p}{b}.$$

Choose now b as follows

$$C_1 + C_2 = 1,$$

that is

$$b = \frac{pa}{p+a-q_1} = \frac{(\beta+\theta)p}{p+\beta-1-q+\theta}, \quad C_2 = \frac{p(\beta+\theta)}{N(p-q-1)+p(\beta+\theta)}.$$

Note that $a < b < q_1$ under the assumptions i) for θ small enough. Therefore, integrating in time (4.4), we have

$$\int\limits_0^t \int\limits_{\mathbf{R}^N} v_n^p\, dx\, d\tau \leqslant \gamma \rho^{qC_2} \left(\sup_{0<\tau<t} \int\limits_{\mathbf{R}^N} v_n^a\, dx \right)^{C_3} \times \int\limits_0^t \int\limits_{\mathbf{R}^N} \left(|\nabla v_n|^p + \frac{1}{\rho^q} v_n^{q_1} \right) dx\, d\tau \leqslant \gamma \rho^{qC_2} Y_n^{1+C_3}. \tag{4.5}$$

Combinig now (4.1) and (4.5), we arrive at

$$Y_{n+1} \leqslant \gamma \frac{2^{np}}{\sigma^p \rho^{p-qC_2}} Y_n^{1+C_3}, \tag{4.6}$$

where

$$C_3 = \frac{q(p-1-q)-\theta(q-\beta)}{(q-\beta)(N(p-1-q)+p(\beta+\theta))} > 0 \quad \text{for} \quad \theta < \frac{q(p-1-q)}{q-\beta},$$

$$p - qC_2 = \frac{p(N(p-q-1)+(\beta+\theta)(p-q))}{N(p-q-1)+p(\beta+\theta)} > 0.$$

Thus, by the Lemma 2.1 we conclude that $Y_n \to 0$, provided

$$\rho^{-(p-qC_2)} Y_0^{C_3} \leqslant \delta, \quad \text{where } \delta = \delta(N, \beta, p, q, \theta) \text{ is small enough.} \tag{4.7}$$

Next, we will show that

$$Y_0 \leqslant \int\limits_{\mathbf{R}^N} u_0^{\beta+\theta}\, dx, \quad \theta > 0. \tag{4.8}$$

Indeed, multiplying both sides of (1.1) by u^θ and integrating over R^N, we have

$$\frac{\beta}{\beta+\theta} \frac{d}{dt} \int\limits_{\mathbf{R}^N} u^{\beta+\theta}\, dx = -\theta\, |\nabla u|^p\, u^{\theta-1}\, dx - \int\limits_{\mathbf{R}^N} |\nabla u|^q\, u^\theta\, dx \leqslant 0. \tag{4.9}$$

Thus, integrating (4.9) in time between 0 and t, we arrive at (4.8). Finally, choosing in (4.7)

$$\rho = \rho_1 = \left(\frac{1}{2} \delta^{-1} \left(\int\limits_{\mathbf{R}^N} u_0^{\beta+\theta}\, dx \right)^{C_3} \right)^{\frac{1}{p-qC_2}},$$

we deduce that support $u \subset B_{R_1}(0)$, $R_1 = 4R_0 + \rho_1$.

In order to prove (1.8), we apply the Poincare–Fridrich inequlity (2.3) with $\Omega = B_{R_1}$:

$$\int\limits_{B_{R_1}} u^{q+\theta}\, dx \leqslant C R_1^q \int\limits_{B_{R_1}} |\nabla u|^q\, u^\theta\, dx.$$

Thus, from (4.9) we have

$$\frac{\beta}{\beta+\theta} \frac{d}{dt} \int\limits_{B_{R_1}} u^{\beta+\theta}\, dx \leqslant -C^{-1} R_1^{-q} \int\limits_{B_{R_1}} u^{q+\theta}\, dx \leqslant -C^{-1} R_1^{-q}\, |B_{R_1}|^{-\frac{q-\beta}{\beta+\theta}} \left(\int\limits_{B_{R_1}} u^{\beta+\theta}\, dx \right)^{\frac{q+\theta}{\beta+\theta}}.$$

Integrating this inequality, we arrive at

$$\int\limits_{B_{R_1}} u^{\beta+\theta}\, dx \leqslant C(R_1) t^{-\frac{\beta+\theta}{q-\beta}}, \quad t > 0. \tag{4.10}$$

Let $K_q = N(q-\beta) + (\beta+\theta)q$, then by Theorem 1.1 with $\nu = \beta + \theta$, $\theta \geqslant 0$, and (4.10) we have

$$\|u(t)\|_{\infty, B_{R_1}} \leqslant \gamma t^{-\frac{N}{K_q}} \left(\sup_{\frac{t}{4} < \tau < t} \int\limits_{B_{R_1}} u^{\beta+\theta}\, dx \right)^{\frac{q}{K_q}} \leqslant \gamma t^{-\frac{1}{q-\beta}}. \tag{4.11}$$

As required.

The case $\beta < q = p - 1$. Proceeding exactly as in the previous case, we have

$$Y_{n+1} := \sup_{0<\tau<t} \int\limits_{A_{n+1}} v_{n+1}^a \, dx + \frac{1}{\rho^{p-1}} \int\limits_0^t \int\limits_{A_{n+1}} v_n^p \, dx \, d\tau + \int\limits_0^t \int\limits_{A_{n+1}} |\nabla v_{n+1}|^p \, dx \, d\tau \leqslant \gamma \frac{2^{np}}{\sigma^p \rho^p} \int\limits_0^t \int\limits_{\mathbf{R}^N} v_n^p \, dx \, d\tau, \tag{4.12}$$

where $a = \frac{p(\beta+\theta)}{p+\theta-1}$, $\theta > 0$. By the Gagliardo–Nirenberg inequality we obtain

$$\int\limits_{R^N} v_n^p \, dx \leqslant \gamma \left(\int\limits_{R^N} |\nabla v_n|^p \, dx \right)^A \left(\int\limits_{R^N} v_n^a \, dx \right)^{\frac{p}{a}(1-A)},$$

where A is defined as follows

$$\frac{N}{p} = \frac{(N-p)\,A}{p} + \frac{N(1-A)}{a}.$$

Integrating this inequality in time, and applying the Hölder inequality, we have

$$\int\limits_0^t \int\limits_{\mathbf{R}^N} v_n^p \, dx \, d\tau \leqslant \gamma \left(\int\limits_0^t \int\limits_{\mathbf{R}^N} |\nabla v_n|^p \, dx \, d\tau \right)^A,$$

$$\left(\sup_{0<\tau<t} \int\limits_{\mathbf{R}^N} v_n^a \, dx \right)^{\frac{p}{a}(1-A)} \leqslant \gamma t^{1-A} Y_n^{1+(1-A)(\frac{p}{a}-1)}. \tag{4.13}$$

Therefore, (4.12) and (4.13) yield

$$Y_{n+1} \leqslant \gamma \frac{2^{np} t^{1-A}}{\sigma^p \rho^p} Y_n^{1+(1-A)\left(\frac{p}{a}-1\right)}.$$

Thus, by the Lemma 2.1 we have $Y_n \to 0$, provided

$$\frac{t^{1-A}}{\rho^p} Y_0^{(1-A)\left(\frac{p}{a}-1\right)} \leqslant \delta = \delta(p, \beta, \sigma) \quad \text{is small enough.} \tag{4.14}$$

Since support $u \subset B_{R_0}$, we have (see also [1, 20, 21])

$$\operatorname{supp} u \subset B_{R_2(t)} \ \text{ with } \ R_2(t) = 4R_0 + \gamma t^{\frac{1+\theta}{H_\theta}} \left(\int\limits_{B_{R_0}} u_0^{\beta+\theta} dx \right)^{\frac{p-\beta-1}{H_\theta}},$$

where $H_\theta = N(p - \beta - 1) + (1 + \theta)p$, $\theta \geqslant 0$. In order to improve the bound of the support estimate, we want to show the exponential decay estimate of Y_0. To this end we need the following classical Stampachia lemma:

Lemma 4.1. *Let $\varphi(s)$ be nonincreasing non negative function defined on $[k_0, \infty)$, such that for all $l > k \geqslant k_0$*

$$\varphi(l) \leqslant \frac{C}{(l-k)^{\tau}}\varphi(k),$$

where C and τ are positive constants. Then for any $k > k_0$ he following estimate holds true

$$\varphi(l) \leqslant \varphi(k_0) \exp\left[1 - (Ce)^{-\frac{1}{\tau}}(k-k_0)\right].$$

Let

$$\varphi(\rho) := \frac{1}{\rho^{p-1}} \int\limits_0^t \int\limits_{|x|>\frac{\rho}{2}} u^{p+\theta-1}\, dx\, d\tau.$$

Multiplying both sides of (1.1) by $u^{\theta}\zeta^{s}(x)$, where $s \geqslant p$, ζ is the standard cutoff function of the ball $B_\rho(0)$, $\rho \leqslant R_2$, integrating by parts, we can easily found that $\varphi(\rho) \leqslant \gamma\rho^{-1}\varphi\left(\frac{\rho}{2}\right)$. Therefore, by Lemma 4.1 we have

$$Y_0 \leqslant \varphi\left(\frac{3\rho}{4}\right) \leqslant \gamma \exp(-\gamma\rho) \quad \text{for } \rho \text{ large enough.}$$

Now the condition (4.14) reads

$$\gamma_1 \frac{t^{1-A}}{\rho^p} \exp\left(-\gamma_2(1-A)\left(\frac{p}{a}-1\right)\rho\right) \leqslant \delta.$$

Hence, after elementary calculations one can choose ρ for t large enough as follows

$$\rho = R(t) = \Gamma\big(\|u_0\|_{\beta+\theta}, \delta\big) \log t.$$

Next, proceeding exactly as in the proof of (4.10), we have that for t large enough

$$E_{\beta+\theta}(t) := \int\limits_{B_R} u^{\beta+\theta}\, dx \leqslant \gamma R(t)^{\frac{N(p-\beta-1)+(p-1)(\beta+\theta)}{p-\beta-1}} t^{-\frac{\beta+\theta}{p-\beta-1}}.$$

Combining this inequality with (4.10), where we set $q = p-1$, we have

$$\|u(t)\|_\infty \leqslant \gamma t^{-\frac{N}{N(p-\beta-1)+(p-1)(\beta+\theta)}} \left(\sup_{\frac{t}{4}<\tau<t} \int\limits_{B_R} u^{\beta+\theta} dx\right)^{\frac{p-1}{N(p-\beta-1)+(p-1)(\beta+\theta)}} \leqslant \gamma t^{-\frac{1}{p-\beta-1}} \left[\log \Gamma t\right]^{\frac{p-1}{p-\beta-1}}.$$

As required. ▷

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Владикавказский математический журнал
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УБЫВАНИЕ РЕШЕНИЯ ЗАДАЧИ КОШИ ПРИ НЕОГРАНИЧЕННОМ ВОЗРАСТАНИИ ВРЕМЕНИ ДВАЖДЫ ВЫРОЖДЕННЫХ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ С ДЕМПФИРОВАНИЕМ

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Аннотация. В этой статье мы изучаем поведение решения при неограниченном возрастании времени и компактификацию носителя задачи Коши для дважды вырождающихся параболических уравнений с сильным градиентным демпфированием. При соответствующих предположениях на структуру уравнения и данные задачи устанавливается новая точная оценка решений при неограниченном возрастании времени. Более того, когда носитель начальных данных компактен, мы доказываем, что носитель решения содержится в шаре с радиусом, не зависящим от времени. При критическом поведении члена с демпфированием носитель решения зависят от времени логарифмически при достаточно больших значениях времени. Основной инструмент доказательства основан на нетривиальных цилиндрических вложениях типа Гальярдо — Ниренберга и итерационных неравенствах. Равномерные оценки решения доказываются модифицированным вариантом классического метода Де-Джорджи — Ладыженской — Уральцевой — ДиБенедетто. Подход статьи достаточно гибкий и может быть использован при дальнейшем изучении задач Коши-Дирихле и Коши — Неймана в областях с некомпактными границами.

Ключевые слова: дважды вырождающиеся параболические уравнения, сильный градиент демпфирование, конечная скорость распространения, поведение на большом времени.

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