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A Multi-Dimensional Hausdorff Moment Problem: Regularization by Finite Moments

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Dedicated to Prof. Dr. G. Anger on occasion of his 70th birthday

Abstract. We consider the multi-dimensional Hausdorff moment problem over the unit cube: to reconstruct an unknown function from the (inaccurately) given values of the integrals of the unknown function multiplied by all power-products of the independent variables. We describe a regularization scheme using orthogonalization by the tensor product of (shifted) Legendre polynomials and "approximation" of the unknown function by a finite sum, the dimension of the space of approximation playing the role of the regularization parameter. For the case of square integrability of the unknown function we present an estimate of the regularization error that implies convergence if the data error tends to zero.

Keywords: *Ill-posed problems, Hausdorff moment problem, polynomial approximation*

AMS subject classification: Primary 65 R 30, secondary 41 A 10

1. Introduction

We propose to present a regularization of a Hausdorff moment problem, with error estimates given. The literature on the moment problem is impressive, we shall list only a few of them in the bibliography (cf. [2 - 8, 11, 13]). Of particular relevance to the present work are the papers by Ang, Vy and Gorenflo [3] and by Talenti [11] (see also Inglese [7]). In [11], the author considered the one-dimensional case and proved the stability of the finite moment approximation with stability estimates given. But the relationship of the finite moment approximation to the original moment problem was not exhibited as transparently as we feel desirable. In the present paper, we shall consider the multi-dimensional case, specifically the following problem.

Problem (MP). Find a function u in $L^2(I)$ satisfying the sequence of equations

$$\int_I u(x_1,\dots,x_d)x_1^{k_1}\cdots x_d^{k_d}\,dx_1\cdots dx_d = \mu_{k_1\cdots k_d} \qquad (k_i \in \mathbb{N}_0; i = 1,\dots,d) \tag{1}$$

where $d \in \mathbb{N}$, $I = [0,1]^d \subset \mathbb{R}^d$ and $\mu = (\mu_{k_1\cdots k_d})$ is a given bounded sequence of real numbers.

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It can be proved that (MP) is an ill-posed problem. Note also that (MP) does not always have a solution for an arbitrary bounded sequence $(\mu_{k_1 \ldots k_d})$ in the right-hand side of (1). Hence it is realistic to deal also with cases where problem (MP) does not have a solution. We refer to the paper [3] of which the present work can be seen as a continuation. However while [3] deals with a general moment problem, the Hausdorff moment problem has it own peculiarities that we have taken into account, in departure from the general case, especially in the crucial choice of the function $f(t)$, $[f^{-1}(\varepsilon^{\frac{1}{2}})]$ being the dimension of our finite approximation.

In our construction of finite-dimensional approximations, we obtain an orthonormalization of the basis functions $\{x_1^{k_1} \cdots x_d^{k_d}\}$ $(k_i \in \mathbb{N}_0; i = 1, \ldots, d)$ not through the Gram-Schmidt process but by using the polynomials

$$L_{k_1 \ldots k_d}(x_1, \ldots, x_d) = L_{k_1}(x_1) \cdots L_{k_d}(x_d), \tag{2}$$

where

$$L_m(x) = \sum_{j=0}^{m} C_{mj} x^j \qquad (m \in \mathbb{N}_0) \tag{3}$$

with

$$C_{mj} = (2m+1)^{\frac{1}{2}} (-1)^j \frac{(m+j)!}{(j!)^2 (m-j)!} \qquad (j = 0, 1, \ldots, m). \tag{4}$$

The sequence $(L_{k_1} \cdots L_{k_d})$ $(k_i \in \mathbb{N}_0; i = 1, \ldots, d)$ forms a complete orthonormal set in $L^2(I)$. In fact,

$$L_m(x) = (2m+1)^{\frac{1}{2}} P_m(1-2x), \tag{5}$$

where P_m is the Legendre polynomial:

$$P_m(t) = \sum_{j=0}^{m} \frac{(m+j)!}{(j!)^2 (m-j)! 2^j} (t-1)^j, \qquad \|P_m\|^2_{L^2(-1,1)} = \frac{2}{2m+1}$$

(cf. [10: page 227]). Thus (L_m) is a complete orthonormal basis in $L^2(0,1)$ and hence $(L_{k_1} \cdots L_{k_d})$ is a complete orthonormal set in $L^2(I)$. From (3), we have

$$L_{k_1 \cdots k_d}(x_1, \ldots, x_d) = \sum_{p_1=0}^{k_1} \cdots \sum_{p_d=0}^{k_d} C_{k_1 p_1} \cdots C_{k_d p_d} x_1^{p_1} \cdots x_d^{p_d}. \tag{6}$$

If $\mu = (\mu_{k_1 \cdots k_d})$ is a real sequence, we define the sequence $\lambda = \lambda(\mu) = (\lambda_{k_1 \cdots k_d})$ as follows:

$$\lambda_{k_1 \cdots k_d} = \lambda_{k_1 \cdots k_d}(\mu) = \sum_{p_1=0}^{k_1} \cdots \sum_{p_d=0}^{k_d} C_{k_1 p_1} \cdots C_{k_d p_d} \mu_{p_1 \cdots p_d}.$$

Now put

$$p^n = p^n(\mu) = \sum_{k_1, \ldots, k_d = 0}^{n} \lambda_{k_1 \cdots k_d}(\mu) L_{k_1 \cdots k_d}. \tag{7}$$

2. Results

We now give our results in form of three theorems that we shall prove in Section 3.

Theorem 1. *Let $\mu = (\mu_{k_1\cdots k_d})$ $(k_i \in \mathbb{N}_0; i = 1, \ldots, d)$ be a given sequence of real numbers. If u is the (unique) solution of (1) in $L^2(I)$, then*

$$\sum_{k_1,\ldots k_d=0}^{\infty} \left(\sum_{p_1,\ldots,p_d=0}^{\infty} C_{k_1p_1} \cdots C_{k_dp_d} \mu_{p_1\cdots p_d} \right)^2 < \infty \tag{8}$$

and

$$p^n(\mu) \to u \qquad \text{in } L^2(I) \text{ as } n \to \infty, \tag{9}$$

where C_{ij} is defined by (4) if $j \le i$ and $C_{ij} = 0$ if $j > i$. Moreover, if the solution u is in $H^1(I)$, then

$$\|p^n(\mu) - u\| \le \frac{1}{n+1}(F(u))^{\frac{1}{2}} \qquad (n \in \mathbb{N}) \tag{10}$$

where $\|\cdot\|$ is the $L^2(I)$-norm and

$$F(u) = \sum_{i=1}^{d} \int_I x_i(1-x_i) \left| \frac{\partial u}{\partial x_i} \right|^2 dx_1 \cdots dx_d.$$

In Theorem 1 above, it is assumed that a solution exists. In the general case that a solution may not exist, Theorem 2 below will be useful.

Theorem 2. *Let $u_0 \in L^2(I)$ be the unique solution of (1) corresponding to $\mu^0 = (\mu^0_{k_1\cdots k_d})$ $(k_i \in \mathbb{N}_0; i = 1, \ldots, d)$ in the right-hand side of (1). Let*

$$f(t) = \frac{1}{\sqrt{(2\pi)^d}}(3 + 2\sqrt{2})^{dt+d} \tag{11}$$

and for $0 < \varepsilon < 1$, put

$$n(\varepsilon) = [f^{-1}(\varepsilon^{-\frac{1}{2}})] \tag{12}$$

where $[x]$ is the largest integer $n \le x$. Then there exists a function $\eta(\varepsilon)$ $(0 < \varepsilon < 1)$ such that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and that for all sequences μ satisfying

$$\|\mu - \mu^0\|_\infty := \sup_{k_1,\ldots,k_d \in \mathbb{N}_0} \left| \mu_{k_1\cdots k_d} - \mu^0_{k_1\cdots k_d} \right| \le \varepsilon$$

we have $\|p^{n(\varepsilon)}(\mu) - u_0\| \le \eta(\varepsilon)$. Moreover, if $u_0 \in H^1(I)$ and $0 < \varepsilon < \left(\frac{\sqrt{2\pi}}{3+2\sqrt{2}}\right)^{2d}$, then

$$\left\|p^{n(\varepsilon)}(\mu) - u_0\right\| \le \varepsilon^{\frac{1}{2}} + \frac{|F(u_0)|^{\frac{1}{2}}}{C_1 + C_2 \ln\left(\frac{1}{\varepsilon}\right)}, \tag{13}$$

where

$$C_1 = \frac{\ln\left(\sqrt{2\pi}(3 - 2\sqrt{2})\right)}{\ln(3 + 2\sqrt{2})} \qquad \text{and} \qquad C_2 = \frac{1}{2d \ln(3 + 2\sqrt{2})}. \tag{14}$$

Remark. In the first version of this paper (see also [2]), we had derived (13) with C_1 and C_2 replaced by

$$\widetilde{C}_1 = \frac{2\ln\left(\frac{8\sqrt{5}}{27\sqrt{2}}\right)}{2\ln 3 + \frac{1}{2}\ln 2} \qquad \text{and} \qquad \widetilde{C}_2 = \frac{1}{d\left(2\ln 3 + \frac{1}{2}\ln 2\right)},$$

respectively.

Estimate (13) with C_1 and C_2 given by (14) was derived using formula (31) as suggested by one of the referees. Clearly, the constants C_1 and C_2 give a better approximation than do $\widetilde{C}_1$ and $\widetilde{C}_2$, respectively. While it is not known to us whether C_1 and C_2 are the best constants, we shall show below that the estimate of the order of magnitude of the error for $\varepsilon \to 0$ as given by (13) is optimal in the sense of the following theorem.

Theorem 3. *Assume that there exist two functions* $m : (0,1] \to \mathbb{N}$ *and* $\alpha : (0,1] \to (0,\infty)$ *such that* $\lim_{\varepsilon\downarrow 0} m(\varepsilon) = +\infty$ *and* $\lim_{\varepsilon\downarrow 0} \alpha(\varepsilon) = 0$, *and*

$$\left\| p^{m(\varepsilon)}(\mu) - u_0 \right\| \le \left(1 + (F(u_0))^{\frac{1}{2}}\right) \alpha(\varepsilon) \tag{15}$$

for all $u_0 \in H^1(I)$, $I = [0,1]^d$, *and all* $\mu = (\mu_{k_1\cdots k_d})$ *satisfying*

$$\|\mu - \mu_0\|_\infty = \sup_{k_1,\ldots,k_d \ge 0} \left|\mu_{k_1\cdots k_d} - \mu^0_{k_1\cdots k_d}\right| \le \varepsilon,$$

where

$$\mu^0_{k_1\ldots k_d} = \int_I u_0(x_1,\ldots,x_d) x_1^{k_1}\cdots x_d^{k_d}\, dx_1\cdots dx_d.$$

Then we have

$$\liminf_{\varepsilon\downarrow 0} \alpha(\varepsilon)\ln\left(\frac{1}{\varepsilon}\right) \ge \frac{d}{2}\ln(3 + 2\sqrt{2}). \tag{16}$$

3. Proofs

For definiteness, we shall give the proofs of Theorems 1 - 3 in the case $d = 3$ only. The proofs carry almost *verbatim* to the general case. For $d = 3$, we put $(x_1,x_2,x_3) = (x,y,z)$, $(k_1,k_2,k_3) = (i,j,k)$, $\mu_{k_1k_2k_3} = \mu_{ijk}$, $\lambda_{k_1k_2k_3} = \lambda_{ijk}$ and $L_{k_1k_2k_3}(x,y,z) = L_i(x)L_j(y)L_k(z) = L_{ijk}(x,y,z)$.

Proof of Theorem 1. We easily get the first result of Theorem 1 using the orthonormality property of the $L_{ijk}{}'$s. In fact, one has in view of (3)

$$\begin{aligned}
\int_I uL_{ijk}\, dxdydz &= \int_I u(x,y,z) \sum_{p=0}^{i}\sum_{q=0}^{j}\sum_{r=0}^{k} C_{ip}C_{jq}C_{kr}\, x^p y^q z^r\, dxdydz \\
&= \sum_{p=0}^{i}\sum_{q=0}^{j}\sum_{r=0}^{k} C_{ip}C_{jq}C_{kr} \left(\int_I u(x,y,z)\, x^p y^q z^r\, dxdydz\right) \\
&= \sum_{p=0}^{i}\sum_{q=0}^{j}\sum_{r=0}^{k} C_{ip}C_{jq}C_{kr}\mu_{ijk} \\
&= \lambda_{ijk}.
\end{aligned}$$

Hence, by completeness and the orthonormality property of the $L_{ijk}{}'$s in $L^2(I)$ one has

$$u = \sum_{i,j,k=0}^{\infty} \lambda_{ijk}(\mu) L_{ijk} \qquad \text{and} \qquad \|u\|^2 = \sum_{i,j,k=0}^{\infty} |\lambda_{ijk}(\mu)|^2 < \infty, \tag{17}$$

i.e. (8) holds. To prove (9), we subtract the first equation in (17) from (7) to get

$$p^n(\mu) - u = - \sum_{max(i,j,k)>n} \lambda_{ijk}(\mu) L_{ijk}.$$

Hence

$$\|p^n(\mu) - u\|^2 = \sum_{max(i,j,k)>n} |\lambda_{ijk}(\mu)|^2. \tag{18}$$

Combining (8) and (18) gives (9).

For the proof of (10) we rely, with $\alpha_k = \int_0^1 v(x) L_k(x)\, dx$, on the identity (cf. [11])

$$\int_0^1 x(1-x)|v'(x)|^2\, dx = \sum_{k=0}^{\infty} k(k+1)\alpha_k^2 \qquad \forall\ v \in H^1(0,1). \tag{19}$$

In view of (17) and (19), we have

$$\sum_{i,j,k=1}^{\infty} i^2 \lambda_{ijk}^2 \le \int_I x(1-x)\Big|\frac{\partial u}{\partial x}(x,y,z)\Big|^2 dxdydz \tag{20}$$

$$\sum_{i,j,k=1}^{\infty} j^2 \lambda_{ijk}^2 \le \int_I y(1-y)\Big|\frac{\partial u}{\partial y}(x,y,z)\Big|^2 dxdydz \tag{21}$$

$$\sum_{i,j,k=1}^{\infty} k^2 \lambda_{ijk}^2 \le \int_I z(1-z)\Big|\frac{\partial u}{\partial z}(x,y,z)\Big|^2 dxdydz. \tag{22}$$

Adding (20) - (22) together, we get

$$\sum_{i,j,k=1}^{\infty} \big(i^2 + j^2 + k^2\big)\lambda_{ijk}^2 \le F(u), \tag{23}$$

where, we recall,

$$F(u) = \int_I \Big(x(1-x)\Big|\frac{\partial u}{\partial x}(x,y,z)\Big|^2 + y(1-y)\Big|\frac{\partial u}{\partial y}(x,y,z)\Big|^2 + z(1-z)\Big|\frac{\partial u}{\partial z}(x,y,z)\Big|^2\Big) dxdydz.$$

Note that

$$i^2 + j^2 + k^2 \ge (n+1)^2 \qquad \text{for} \quad \max(i,j,k) > n. \tag{24}$$

We get in view of (18), (23) and (24) that

$$(n+1)^2 \big\|p^n(\mu) - u\big\|^2 \le \sum_{max(i,j,k)>n} \big(i^2 + j^2 + k^2\big)\lambda_{ijk}^2(\mu) \le F(u),$$

which gives the desired inequality. The proof of Theorem 1 is completed ∎

Proof of Theorem 2. We have

$$\left\|p^n(\mu) - u_0\right\| \leq \left\|p^n(\mu) - p^n(\mu^0)\right\| + \left\|p^n(\mu^0) - u_0\right\| \qquad (n \in \mathbb{N}). \tag{25}$$

We shall estimate the two terms in the right-hand side of (25). Using (18) gives

$$\left\|p^n(\mu^0) - u_0\right\|^2 = \sum_{\max(i,j,k)>n} \left(\sum_{p,q,r=0}^{\infty} C_{ip} C_{jq} C_{kr} \mu^0_{pqr} \right)^2. \tag{26}$$

For the first term, we get in view of (7)

$$p^n(\mu) - p^n(\mu^0) = \sum_{i,j,k=0}^{n} \sum_{p,q,r=0}^{\infty} C_{ip} C_{jq} C_{kr} \left(\mu_{pqr} - \mu^0_{pqr}\right) L_{ijk}. \tag{27}$$

Hence

$$\begin{aligned} \left\|p^n(\mu) - p^n(\mu^0)\right\|^2 &= \sum_{i,j,k=0}^{n} \left(\sum_{p,q,r=0}^{\infty} C_{ip} C_{jq} C_{kr} \left(\mu_{pqr} - \mu^0_{pqr}\right) \right)^2 \\ &\leq \|\mu - \mu^0\|_\infty^2 \sum_{i,j,k=0}^{n} \left(\sum_{p,q,r=0}^{\infty} |C_{ip} C_{jq} C_{kr}| \right)^2. \end{aligned} \tag{28}$$

Noting that $C_{mj} = 0$ for $j > m$, we have

$$\sum_{p,q,r=0}^{\infty} |C_{ip} C_{jq} C_{kr}| = \sum_{p=0}^{i} |C_{ip}| \sum_{q=0}^{j} |C_{jq}| \sum_{r=0}^{k} |C_{kr}|. \tag{29}$$

Substituting (29) into (28) gives

$$\left\|p^n(\mu) - p^n(\mu^0)\right\|^2 \leq \|\mu - \mu^0\|_\infty^2 \left(\sum_{i=0}^{n} \left(\sum_{p=0}^{i} |C_{ip}| \right)^2 \right)^3. \tag{30}$$

From (3) - (5), one gets

$$\sum_{p=0}^{i} |C_{ip}| = L_i(-1) = (2i+1)^{\frac{1}{2}} P_i(3). \tag{31}$$

On the other hand, one has (cf. [9: page 48], or [1: Equation (22.10.10)])

$$P_i(3) = \frac{1}{\pi} \int_0^\pi \left(3 + 2\sqrt{2}\, \cos\phi\right)^i d\phi \qquad (i \in \mathbb{N}_0).$$

We claim that

$$P_i(3) \leq 2\sqrt{\frac{3 + 2\sqrt{2}}{\pi\sqrt{2}}} \cdot \frac{1}{\sqrt{2i+1}} \, (3 + 2\sqrt{2})^i \qquad (i \in \mathbb{N}_0) \tag{32}$$

the proof of which will be given later. Using (31) and (32) one gets after some rearrangements

$$\sum_{p=0}^{i} |C_{ip}| \leq 2\sqrt{\frac{3+2\sqrt{2}}{\pi\sqrt{2}}}\,(3+2\sqrt{2})^i \qquad (i \in \mathbb{N}_0)$$

and

$$\sum_{i=0}^{n}\left(\sum_{p=0}^{i}|C_{ip}|\right)^2 \leq \frac{4(3+2\sqrt{2})}{\pi\sqrt{2}}\sum_{i=0}^{n}\left(3+2\sqrt{2}\right)^{2i} \leq \frac{1}{2\pi}\left(3+2\sqrt{2}\right)^{2n+2}.$$

Substituting the latter inequality into (30) gives

$$\left\|p^n(\mu) - p^n(\mu^0)\right\| \leq \|\mu - \mu^0\|_\infty \frac{1}{\sqrt{8\pi^3}}\left(3+2\sqrt{2}\right)^{3n+3}. \tag{33}$$

Put

$$f(t) = \frac{1}{\sqrt{8\pi^3}}\left(3+2\sqrt{2}\right)^{3t+3} \qquad \text{and} \qquad n(\varepsilon) = \left[f^{-1}(\varepsilon^{-\frac{1}{2}})\right].$$

Then we have in view of (33)

$$\left\|p^n(\mu) - p^n(\mu^0)\right\| \leq \varepsilon^{\frac{1}{2}}. \tag{34}$$

If we let

$$\eta(\varepsilon) = \varepsilon^{\frac{1}{2}} + \sum_{max(i,j,k)>n(\varepsilon)}\left(\sum_{p,q,r=0}^{\infty} C_{ip}C_{jq}C_{kr}\mu^0_{pqr}\right)^2,$$

then, in view of (25), (26) and (34), we have $\|p^{n(\varepsilon)}(\mu) - u_0\| \leq \eta(\varepsilon)$. As $\varepsilon \to 0$, we have $f^{-1}(\varepsilon^{-\frac{1}{2}}) \to \infty$ and $n(\varepsilon) \to \infty$. By (8)

$$\sum_{max(i,j,k)>n(\varepsilon)}\left(\sum_{p,q,r=0}^{\infty} C_{ip}C_{jq}C_{kr}\mu^0_{pqr}\right)^2 \to 0 \qquad \text{as } n(\varepsilon) \to 0.$$

Hence $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now, let $u_0 \in H^1(I)$. Then by Theorem 1 one has

$$\left\|p^{n(\varepsilon)}(\mu^0) - u_0\right\| \leq \frac{(F(u_0))^{\frac{1}{2}}}{n(\varepsilon)+1}. \tag{35}$$

The definition of $n(\varepsilon)$ implies $n(\varepsilon)+1 > f^{-1}(\varepsilon^{-\frac{1}{2}})$, and f being a monotone increasing function, $f(n(\varepsilon)+1) > \varepsilon^{-\frac{1}{2}}$ or, equivalently,

$$\frac{1}{\sqrt{8\pi^3}}\left(3+2\sqrt{2}\right)^{3(n(\varepsilon)+1)+3} > \varepsilon^{-\frac{1}{2}}.$$

It follows that

$$\left(3+2\sqrt{2}\right)^{n(\varepsilon)+2} > \sqrt{2\pi}\varepsilon^{-\frac{1}{6}},$$

i.e.,

$$n(\varepsilon)+2>\frac{1}{\ln\left(3+2\sqrt{2}\right)}\ln\left(\frac{\sqrt{2\pi}}{\varepsilon^{1/6}}\right).$$

Thus we get after some calculations

$$n(\varepsilon)+1>\frac{\ln\left(\sqrt{2\pi}(3-2\sqrt{2})\right)}{\ln(3+2\sqrt{2})}+\frac{1}{6\ln(3+2\sqrt{2})}\ln\left(\frac{1}{\varepsilon}\right).$$

It can be checked directly that the right-hand side of the latter inequality is positive if and only if

$$0<\varepsilon<\left(\frac{\sqrt{2\pi}}{3+2\sqrt{2}}\right)^{6}.$$

Let ε satisfy this inequality. By the preceding arguments, (35) gives

$$\left\|p^{n(\varepsilon)}(\mu^0)-u_0\right\|\le\frac{(F(u_0))^{\frac{1}{2}}}{C_1+C_2\ln\left(\frac{1}{\varepsilon}\right)}\tag{36}$$

with

$$C_1=\frac{\ln\left(\sqrt{2\pi}(3-2\sqrt{2})\right)}{\ln(3+2\sqrt{2})}\qquad\text{and}\qquad C_2=\frac{1}{6\ln(3+2\sqrt{2})}.$$

Combining (34) and (36) gives

$$\left\|p^{n(\varepsilon)}(\mu)-u_0\right\|\le\varepsilon^{\frac{1}{2}}+\frac{(F(u_0))^{\frac{1}{2}}}{C_1+C_2\ln\left(\frac{1}{\varepsilon}\right)}.$$

Now the proof of Theorem 2 will be completed once (32) is proved. In fact, one has

$$\begin{aligned}P_i(3)&=\frac{1}{\pi}\int_0^{\pi}\left(3+2\sqrt{2}\,\cos\phi\right)^i d\phi\\&=\frac{1}{\pi}\left(\int_0^{\pi/2}+\int_{\pi/2}^{\pi}\right)\left(3+2\sqrt{2}\,\cos\phi\right)^i d\phi\\&=\frac{1}{\pi}\int_0^{\pi/2}\left(\left(3+2\sqrt{2}\,\cos\phi\right)^i+\left(3-2\sqrt{2}\,\cos\phi\right)^i\right)d\phi.\end{aligned}\tag{37}$$

From (37), with $t=\cos\phi$, it follows that

$$P_i(3)\le\frac{2}{\pi}\int_0^{\pi/2}\left(3+2\sqrt{2}\,\cos\phi\right)^i d\phi\le\frac{2}{\pi}\int_0^1\frac{(3+2\sqrt{2}t)^i}{\sqrt{1-t^2}}\,dt\le\frac{2}{\pi}\int_0^1\frac{(3+2\sqrt{2}t)^i}{\sqrt{1-t}}\,dt.$$

Putting $s=\sqrt{1-t}$ and $\zeta=s\sqrt{\frac{2\sqrt{2}}{3+2\sqrt{2}}}$ we get, after some computations,

$$P_i(3)\le\frac{4}{\pi}(3+2\sqrt{2})^i\sqrt{\frac{3+2\sqrt{2}}{2\sqrt{2}}}\int_0^1(1-\zeta^2)^i d\zeta.$$

Letting $\zeta = \cos\theta$, we have

$$P_i(3) \le \frac{4}{\pi}(3+2\sqrt{2})^i \sqrt{\frac{3+2\sqrt{2}}{2\sqrt{2}}} \int_0^{\pi/2} \sin^{2i+1}\theta \, d\theta. \tag{38}$$

Putting $I_n = \int_0^{\pi/2} \sin^n \theta \, d\theta$ we have (cf. [12: p. 681])

$$I_{2k} = \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots(2k)}\cdot\frac{\pi}{2} \qquad \text{and} \qquad I_{2k-1} = \frac{2\cdot 4\cdots(2k-2)}{1\cdot 3\cdots(2k-1)}.$$

It follows that

$$I_m I_{m-1} = \frac{\pi}{2m} \qquad (m \in \mathbb{N}). \tag{39}$$

But

$$I_{m-1} = \int_0^{\pi/2} \sin^{m-1}\theta \, d\theta > \int_0^{\pi/2} \sin^m \theta \, d\theta = I_m.$$

Hence (39) gives

$$I_m^2 \le I_m I_{m-1} = \frac{\pi}{2m}. \tag{40}$$

Substituting (40) into (38), we get

$$P_i(3) \le \frac{4}{\pi}(3+2\sqrt{2})^i \sqrt{\frac{3+2\sqrt{2}}{2\sqrt{2}}} \cdot \sqrt{\frac{\pi}{2(2i+1)}}.$$

This establishes (32) and completes the proof of Theorem 2 ∎

Proof of Theorem 3. We choose

$$u_0(x,y,z) = \frac{1}{m(\varepsilon)} L_{m(\varepsilon)+1}(x) = \frac{1}{m(\varepsilon)} L_{m(\varepsilon)+1,0,0}(x,y,z),$$

$$\mu_{pqr}^0 = \int_I u_0(x,y,z) x^p y^q z^r \, dxdydz.$$

Since $\int_0^1 L_i(x) x^j \, dx = 0 \;\; (0 \le j < i)$ one has

$$\mu_{pqr}^0 = 0 \qquad \text{for } \; p \le m(\varepsilon), \tag{41}$$

i.e., u_0 is orthogonal for all the L_{ijk}'s, $0 \le i \le m(\varepsilon)$. Put $\mu^0 = (\mu_{pqr}^0)$ and $\mu = (\mu_{pqr})$ with

$$\mu_{pqr} = \varepsilon(-1)^{p+q+r} + \mu_{pqr}^0. \tag{42}$$

It is clear that $\|\mu - \mu^0\|_\infty = \sup_{p,q,r} |\mu_{pqr} - \mu_{pqr}^0| = \varepsilon$. Hence, from assumption (15) of Theorem 3, one has

$$\left\| p^{m(\varepsilon)}(\mu) - u_0 \right\| \le \alpha(\varepsilon)\big(1 + (F(u_0))^{\frac{1}{2}}\big). \tag{43}$$

We estimate the left-hand side of (43). The orthonormality of the sequence (L_{ijk}) implies

$$\|p^{m(\varepsilon)}(\mu)-u_0\|^2=\|p^{m(\varepsilon)}(\mu)\|^2+\|u_0\|^2=\|p^{m(\varepsilon)}(\mu)\|^2+\frac{1}{m^2(\varepsilon)}. \tag{44}$$

On the other hand, one has in view of (42), (29) and (31)

$$\|p^{m(\varepsilon)}(\mu)\|^2=\varepsilon^2\left(\sum_{i=0}^{m(\varepsilon)}\left(\sum_{p=0}^{i}|C_{ip}|\right)^2\right)^3=\varepsilon^2\left(\sum_{i=0}^{m(\varepsilon)}(2i+1)|P_i(3)|^2\right)^3. \tag{45}$$

From (37) we conclude

$$P_i(3)\geq\frac{1}{\pi}\int_0^{\pi/2}(3+2\sqrt{2})^i\,d\phi. \tag{46}$$

We shall make use of the inequality

$$(2i+1)|P_i(3)|^2\geq\frac{(3+2\sqrt{2})^{2i}}{2\pi^2}, \tag{47}$$

the proof of which will be given later. Combining (47) with (45) gives

$$\begin{aligned}\|p^{m(\varepsilon)}(\mu)\|^2&\geq\varepsilon^2\left(\sum_{i=0}^{m(\varepsilon)}\frac{(3+2\sqrt{2})^{2i}}{2\pi^2}\right)^3\\&=\frac{\varepsilon^2}{8\pi^6}\left(\frac{(3+2\sqrt{2})^{2(m(\varepsilon)+1)}-1}{(3+2\sqrt{2})^2-1}\right)^3\\&\geq\frac{\varepsilon^2}{64\pi^6}(3+2\sqrt{2})^{6m(\varepsilon)}.\end{aligned} \tag{48}$$

From (43), (45) and (48) we get

$$\left(1+(F(u_0))^{\frac{1}{2}}\right)^2\alpha^2(\varepsilon)\geq\frac{\varepsilon^2}{64\pi^6}(3+2\sqrt{2})^{6m(\varepsilon)}+\frac{1}{m^2(\varepsilon)}\geq\min_{t>0}g(t), \tag{49}$$

where

$$g(t)=\frac{\varepsilon^2}{64\pi^6}(3+2\sqrt{2})^{6t}+\frac{1}{t^2}.$$

It is clear that g attains its minimum at $t_\varepsilon>0$ satisfying

$$g'(t_\varepsilon)=\frac{3\varepsilon^2}{32\pi^6}(3+2\sqrt{2})^{6t_\varepsilon}\ln(3+2\sqrt{2})-\frac{2}{t_\varepsilon^3}=0. \tag{50}$$

From (49) one has

$$\left(1+(F(u_0))^{\frac{1}{2}}\right)^2\alpha^2(\varepsilon)\geq g(t_\varepsilon)\geq t_\varepsilon^{-2},$$

which implies

$$\alpha(\varepsilon)\ln\left(\frac{1}{\varepsilon}\right) \geq \frac{1}{1+(F(u_0))^{\frac{1}{2}}}\cdot\frac{\ln\left(\frac{1}{\varepsilon}\right)}{t_\varepsilon}. \tag{51}$$

From (50) one has

$$t_\varepsilon^3(3+2\sqrt{2})^{6t_\varepsilon} = \frac{64\pi^6}{3\varepsilon^2\ln(3+2\sqrt{2})}.$$

Hence

$$3\ln t_\varepsilon + 6t_\varepsilon\ln(3+2\sqrt{2}) = \ln\left(\frac{64\pi^6}{3\ln(3+2\sqrt{2})}\right) + 2\ln\left(\frac{1}{\varepsilon}\right).$$

Multiplying this equality by t_ε^{-1}, letting $\varepsilon \to 0$ and noting that $t_\varepsilon \to +\infty$ as $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\downarrow 0}\frac{\ln\left(\frac{1}{\varepsilon}\right)}{t_\varepsilon} = 3\ln(3+2\sqrt{2}). \tag{52}$$

On the other hand, in view of (19) and the definitions of $F(u_0)$ and u_0 we have

$$\begin{aligned} F(u_0) &= \int_I x(1-x)\left|\frac{\partial u_0}{\partial x}(x,y,z)\right|^2 dxdydz \\ &\quad + \int_I y(1-y)\left|\frac{\partial u_0}{\partial y}(x,y,z)\right|^2 dxdydz \\ &\quad + \int_I z(1-z)\left|\frac{\partial u_0}{\partial z}(x,y,z)\right|^2 dxdydz \\ &= \frac{1}{m^2(\varepsilon)}(m(\varepsilon)+1)(m(\varepsilon)+2). \end{aligned} \tag{53}$$

Hence, in view of (52) and (53), inequality (51) implies the desired inequality (16).

To complete the proof of Theorem 3, we shall prove inequality (47). Putting $t = \cos\phi$ and $s = \sqrt{1-t}$, and then letting $\zeta = s\sqrt{\frac{2\sqrt{2}}{3+2\sqrt{2}}}$, we get the inequality

$$P_i(3) \geq \frac{1}{\pi}(3+2\sqrt{2})^i\sqrt{\frac{3+2\sqrt{2}}{\sqrt{2}}}J, \tag{54}$$

where

$$J = \int_0^{\sqrt{\frac{2\sqrt{2}}{3+2\sqrt{2}}}}(1-\zeta^2)^i d\zeta.$$

We shall estimate J. One has

$$J = I_{2i+1} - K_i,$$

where

$$I_{2i+1} = \int_0^{\pi/2}\sin^{2i+1}\theta\, d\theta \qquad \text{and} \qquad K_i = \int_{\sqrt{\frac{2\sqrt{2}}{3+2\sqrt{2}}}}^1 (1-\zeta^2)^i d\zeta.$$

Using (39) one has $I_{2i+1}I_{2i+2} = \frac{\pi}{2(2i+2)}$. Hence

$$I_{2i+1}^2 \geq I_{2i+1}I_{2i+2} = \frac{\pi}{2(2i+2)}. \tag{55}$$

On the other hand,

$$\begin{aligned} K_i &= \int_{\frac{2\sqrt{2}}{3+2\sqrt{2}}}^{1} (1-\zeta)^i \frac{d\zeta}{2\sqrt{\zeta}} \\ &\leq \frac{1}{2}\sqrt{\frac{3+2\sqrt{2}}{2\sqrt{2}}} \int_{\frac{2\sqrt{2}}{3+2\sqrt{2}}}^{1} (1-\zeta)^i d\zeta \\ &= \frac{1}{2(i+1)}\sqrt{\frac{3+2\sqrt{2}}{2\sqrt{2}}} \left(\frac{3}{3+2\sqrt{2}}\right)^{i+1}. \end{aligned} \tag{56}$$

From (55) and (56) we deduce

$$\begin{aligned} J &\geq \sqrt{\frac{\pi}{2(2i+2)}} - \frac{1}{2(i+1)}\sqrt{\frac{3+2\sqrt{2}}{2\sqrt{2}}} \left(\frac{3}{3+2\sqrt{2}}\right)^{i+1} \\ &= \frac{1}{\sqrt{i+1}}\left(\sqrt{\frac{\pi}{4}} - \frac{1}{2\sqrt{i+1}}\sqrt{\frac{3+2\sqrt{2}}{2\sqrt{2}}} \left(\frac{3}{3+2\sqrt{2}}\right)^{i+1}\right) \\ &\geq \frac{1}{\sqrt{i+1}}\left(\sqrt{\frac{\pi}{4}} - \frac{1}{2}\sqrt{\frac{3+2\sqrt{2}}{2\sqrt{2}}} \cdot \frac{3}{3+2\sqrt{2}}\right) \\ &> \frac{1}{2\sqrt{i+1}}. \end{aligned}$$

In view of the latter inequality, (54) gives

$$(2i+1)|P_i(3)|^2 \geq \frac{2i+1}{4(i+1)\pi^2\sqrt{2}}(3+2\sqrt{2})^{2i+1} \geq \frac{(3+2\sqrt{2})^{2i}}{2\pi^2}.$$

This establishes (47) and completes the proof of Theorem 3 ∎

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