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# Hyperbolic Functional-Differential Equations with Unbounded Delay

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**Abstract.** The phase space for quasilinear equations with unbounded delay is constructed. Carathéodory solutions of initial problems are investigated. A theorem on the existence, uniqueness and continuous dependence upon initial data is given. The method of bicharacteristics and integral inequalities are used.

**Keywords:** *Unbounded delay, local existence, Carathéodory solutions*

**AMS subject classification:** 35 L 50, 35 D 05

## 1. Introduction

For any metric spaces $U$ and $V$ we denote by $C(U, V)$ the class of all continuous functions defined on $U$ and taking values in $V$. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Let

$$E = [-r_0, 0] \times [-r, +r] \subset \mathbb{R}^{1+n}$$

where $r_0 \in \mathbb{R}_+ := [0, +\infty)$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$. Assume that $a > 0$, $(t, x) = (t, x_1, \ldots, x_n) \in [0, a] \times \mathbb{R}^n$ and $z : [-r_0, a] \times \mathbb{R}^n \to \mathbb{R}$. We define a function $z_{(t,x)} : E \to \mathbb{R}^n$ by

$$z_{(t,x)}(\tau, s) = z(t + \tau, x + s) \qquad ((\tau, s) \in E).$$

For each $(t, x) \in [0, a] \times \mathbb{R}^n$ the function $z_{(t,x)}$ is the restriction of $z$ to the set $[t - r_0, t] \times [x - r, x + r]$ and this restriction is shifted to the set $E$. Suppose that

$$F : [0, a] \times \mathbb{R}^n \times C(E, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$$

is a given function. In this time numerous papers were published concerning various problems for the equation

$$D_t z(t, x) = F\big(t, x, z_{(t,x)}, D_x z(t, x)\big)$$

where $D_x z = (D_{x_1} z, \ldots, D_{x_n} z)$ and for adequate weakly coupled hyperbolic systems. The following questions were considered: functional-differential inequalities, uniqueness

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of initial or initial-boundary value problems, difference-functional inequalities, approximate solutions of initial or initial-boundary value problems, existence of classical or generalized solutions (see [1 - 3, 5 - 8, 10 - 15]). All these problems have the property that the set $E$ is bounded.

In the paper we start the investigation of first order partial functional-differential equations with unbounded delay. We give sufficient conditions for the existence and uniqueness of Carathéodory solutions of initial problems for quasilinear equations with unbounded delay. We consider functional-differential equations in a Banach space. The theory of ordinary functional-differential equations with unbounded delay is given in monographs [4, 9].

We formulate the problem. Let $B$ be a Banach space with norm $\|\cdot\|$ and $D = (-\infty, 0] \times [-r, +r] \subset \mathbb{R}^{1+n}$ $(r \in \mathbb{R}^n_+)$. The norm in $\mathbb{R}^n$ will also be denoted by $\|\cdot\|$. For a function $z : (-\infty, b] \times \mathbb{R}^n \to B$ $(b \geq 0)$ and for a point $(t, x) \in [0, b] \times \mathbb{R}^n$ we define a function $z_{(t,x)} : D \to B$ by

$$z_{(t,x)}(\tau, s) = z(t + \tau, x + s) \qquad ((\tau, s) \in D).$$

The phase space $X$ for equations with unbounded delay is a linear space, with norm $\|\cdot\|_X$ and consisting of functions mapping the set $D$ into $B$. Let $a > 0$ be fixed and suppose that

$$\varrho = (\varrho_1, \ldots, \varrho_n) : [0, a] \times \mathbb{R}^n \times X \to \mathbb{R}^n$$
$$f : [0, a] \times \mathbb{R}^n \times X \to B$$
$$\varphi : (-\infty, 0] \times \mathbb{R}^n \to B$$

are given functions. We consider the quasilinear equation

$$D_t z(t, x) + \sum_{i=1}^{n} \varrho_i(t, x, z_{(t,x)})\, D_{x_i} z(t, x) = f(t, x, z_{(t,x)}) \tag{1}$$

with the intial condition

$$z(t, x) = \varphi(t, x) \qquad \text{on } (-\infty, 0] \times \mathbb{R}^n. \tag{2}$$

We will deal with Carathéodory solutions of problem (1) - (1.2). A function $\bar{u} : (-\infty, b] \times \mathbb{R}^n \to B$ where $0 < b \leq a$ is a *solution* of the above problem provided:

(i) $\bar{u}$ is continuous on $[0, b] \times \mathbb{R}^n$ and the derivatives $D_t \bar{u}(t, x)$ and $D_x \bar{u}(t, x)$ exist for almost all $(t, x) \in [0, b] \times \mathbb{R}^n$.

(ii) $\bar{u}$ satisfies equation (1.1) almost everywhere on $[0, b] \times \mathbb{R}^n$ and condition (1.2) holds.

We adopt the following notations. If $z : (-\infty, b] \times \mathbb{R}^n \to B$ $(0 < b \leq a)$ is a function such that $z$ is continuous on $[0, b] \times \mathbb{R}^n$, then we put for $(t, x) \in [0, b] \times \mathbb{R}^n$

$$\|z\|_{[0,t;x]} = \max\Big\{ \|z(\tau, s)\| : (\tau, s) \in [0, t] \times [x - r, x + r] \Big\}$$
$$\|z\|_{[0,t;\mathbb{R}^n]} = \sup\Big\{ \|z(\tau, s)\| : (\tau, s) \in [0, t] \times \mathbb{R}^n \Big\}$$

and

$$\operatorname{Lip} z\Big|_{[0,t;x]} = \sup\left\{ \frac{\|z(\tau, s) - z(\tau, \bar{s})\|}{\|s - \bar{s}\|} : (\tau, s), (\tau, \bar{s}) \in [0, t] \times [x - r, x + r] \right\}.$$

The fundamental axioms assumed on $X$ are the followings.

**Assumption H[X].** Suppose the following:

**1)** $(X, \|\cdot\|_X)$ is a Banach space.

**2)** If $z : (-\infty, b] \times \mathbb{R}^n \to B \quad (0 < b \leq a)$ is a function such that $z_{(0,x)} \in X$ for $x \in \mathbb{R}^n$ and $z$ is continuous on $[0, b] \times \mathbb{R}^n$, then $z_{(t,x)} \in X$ for $(t, x) \in (0, b] \times \mathbb{R}^n$ and

(i) for $(t, x) \in [0, b] \times \mathbb{R}^n$ we have $\|z_{(t,x)}\|_X \leq K \|z\|_{[0,t;x]} + L \|z_{(0,x)}\|_X$ where $K, L \in R_+$ are constant independent on $z$

(ii) the function $(t, x) \to z_{(t,x)}$ is continuous on $[0, b] \times \mathbb{R}^n$.

**3)** The linear subspace $X_L \subset X$ is such that

(i) $X_L$ endowed with the norm $\|\cdot\|_{X_L}$ is a Banach space

(ii) if $z : (-\infty, b] \times \mathbb{R}^n \to B \quad (0 < b \leq a)$ is a function such that $z_{(0,x)} \in X_L$ for $x \in \mathbb{R}^n$, $z$ is continuous on $[0, b] \times \mathbb{R}^n$ and $z(t, \cdot) : \mathbb{R}^n \to B$ satisfies the Lipschitz condition with a constant independent on $t \quad (t \in [0, b])$, then

$(\alpha)$ $z_{(t,x)} \in X_L$ for $(t, x) \in (0, b] \times \mathbb{R}^n$

$(\beta)$ for $(t, x) \in [0, b] \times \mathbb{R}^n$ we have

$$\|z_{(t,x)}\|_{X_L} \leq K_0\big(\|z\|_{[0,t;x]} + \mathrm{Lip}\, z|_{[0,t;x]}\big) + L_0\|z_{(0,x)}\|_{X_L}$$

where $K_0, L_0 \in \mathbb{R}_+$ are constants independent on $z$.

Examples of phase spaces are given in Section 4.

Let us denote by $L([\alpha, \beta], \mathbb{R}) \quad ([\alpha, \beta] \subset \mathbb{R})$ the class of functions

$$L([\alpha, \beta], \mathbb{R}) = \Big\{\mu : [\alpha, \beta] \to \mathbb{R} :\ \mu \text{ integrable on } [\alpha, \beta]\Big\}.$$

Further, we will use the symbol $\Theta$ to denote the set of functions

$$\Theta = \left\{ \gamma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+ \ \middle|\ \begin{array}{l} \gamma(t, \cdot) \text{ is non-decreasing for a.a. } t \in [0, a] \\ \gamma(\cdot, \tau) \in L([0, a], \mathbb{R}_+) \ \text{ for all } \tau \in \mathbb{R}_+ \end{array} \right\}.$$

Further, write

$$X[\kappa] = \big\{w \in X :\ \|w\|_X \leq \kappa\big\}$$
$$X_L[\kappa] = \big\{w \in X_L :\ \|w\|_{X_L} \leq \kappa\big\}$$

where $\kappa \in \mathbb{R}_+$.

## 2. Bicharacteristics of functional-differential equations

We start with assumptions on the initial function $\varphi$.

**Assumption H$_0$.** Suppose that $\varphi : (-\infty, 0] \times \mathbb{R}^n \to B$ and

**(i)** $\varphi_{(0,x)} \in X_L$ for $x \in \mathbb{R}^n$

**(ii)** there are $\tilde{L}, L \in \mathbb{R}_+$ such that $\|\varphi(0,x)\| \le \tilde{L}$ for $x \in \mathbb{R}^n$ and $\|\varphi_{(0,x)} - \varphi_{(0,\bar{x})}\|_X \le L\|x - \bar{x}\|$ for $x, \bar{x} \in \mathbb{R}^n$.

**Assumption H$[\varrho]$.** Suppose the following:

**1)** The function $\varrho(\cdot, x, w) : [0,a] \to \mathbb{R}^n$ is measurable for $(x,w) \in \mathbb{R}^n \times X$ and $\varrho(t,\cdot) : \mathbb{R}^n \times X \to \mathbb{R}^n$ is continuous for almost all $t \in [0,a]$.

**2)** There exist $\alpha, \beta \in \Theta$ such that $\|\varrho(t,x,w)\| \le \alpha(t,\kappa)$ for $(x,w) \in \mathbb{R}^n \times X[\kappa]$ almost everywhere on $[0,a]$ and

$$\big\|\varrho(t,x,w) - \varrho(t,\bar{x},\bar{w})\big\| \le \beta(t,\kappa)\big[\|x - \bar{x}\| + \|w - \bar{w}\|_X\big] \tag{3}$$

for $(x,w), (\bar{x},\bar{w}) \in \mathbb{R}^n \times X_L[\kappa]$ almost everywhere on $[0,a]$.

Suppose that $\varphi : (-\infty, 0] \times \mathbb{R}^n \to B$ and $\varphi_{(0,x)} \in X_L$ for $x \in \mathbb{R}^n$. Let $c \in [0,a]$, $d = (d_0, d_1) \in \mathbb{R}_+^2$ and $\omega \in L([0,a], \mathbb{R}_+)$. The symbol $Y_{c.\varphi}[\omega, d]$ denotes the function class

$$Y_{c,\varphi}[\omega, d] =$$

$$\left\{ z : (-\infty, c] \times \mathbb{R}^n \to B \;\middle|\; \begin{array}{l} z(t,x) = \varphi(t,x) \quad \text{on} \quad (-\infty, 0] \times \mathbb{R}^n \\[2ex] \|z(t,x)\| \le d_0 \quad \text{on} \quad [0,c] \times \mathbb{R}^n \\[2ex] \|z(t,x) - z(\bar{t},\bar{x})\| \le \left| \displaystyle\int_t^{\bar{t}} \omega(\tau)\, d\tau \right| + d_1 \|x - \bar{x}\| \\[2ex] \text{for } (t,x), (\bar{t},\bar{x}) \in [0,c] \times \mathbb{R}^n. \end{array} \right\}$$

For the above $\varphi$ and for $z \in Y_{c.\varphi}[\omega, d]$ consider the Cauchy problem

$$\left.\begin{array}{l} \eta'(\tau) = \varrho\big(\tau, \eta(\tau), z_{(\tau,\eta(\tau))}\big) \\ \eta(t) = x \end{array}\right\} \tag{4}$$

where $(t,x) \in [0,c] \times \mathbb{R}^n$. We consider Carathéodory solutions of problem (4). Denote by $g[z](\cdot, t, x)$ the solution of the above problem. The function $g[z]$ is the bicharacteristic of equation (1) corresponding to $z \in Y_{c.\varphi}[\omega, d]$.

Let $\Delta_c = [0,c] \times [0,c] \times [0,c]$. For $\varphi$ satisfying Assumption H$_0$ define

$$\|\varphi\|_{(X,\infty)} = \sup\big\{\|\varphi_{(0,x)}\|_X : x \in \mathbb{R}^n\big\}.$$

**Lemma 2.1.** *Suppose that Assumptions* H$[X]$ *and* H$[\varrho]$ *are satisfied and*

**1)** *the functions* $\varphi, \bar{\varphi} : (-\infty, 0] \times \mathbb{R}^n \to B$ *satisfy Assumption* $\mathrm{H}_0$

**2)** $c \in [0, a]$, $z \in Y_{c.\varphi}[\omega, d]$ *and* $\bar{z} \in Y_{c,\bar{\varphi}}[\omega, d]$.

*Then the solutions* $g[z](\cdot, t, x)$ *and* $g[\bar{z}](\cdot, t, x)$ *are defined on* $[0, c]$ *and they are unique. Moreover, we have the estimates on* $\Delta_c$

$$\begin{aligned} &\big\| g[z](\tau, t, x) - g[z](\tau, \bar{t}, \bar{x}) \big\| \leq \\ &\quad \left[ \|x - \bar{x}\| + \left| \int_t^{\bar{t}} \alpha(\xi, \bar{\kappa})\, d\xi \right| \right] \exp \left[ \bar{d} \left| \int_t^{\tau} \beta(\xi, \kappa_0)\, d\xi \right| \right] \end{aligned} \tag{5}$$

*and*

$$\begin{aligned} &\big\| g[z](\tau, t, x) - g[\bar{z}](\tau, t, x) \big\| \leq \\ &\quad \left| \int_t^{\tau} \beta(\xi, \kappa_0) \Big[ K \|z - \bar{z}\|_{[0,\xi;\mathbb{R}^n]} + M \|\varphi - \bar{\varphi}\|_{(X,\infty)} \Big]\, d\xi \, \exp \left[ \bar{d} \int_t^{\tau} \beta(\xi, \kappa_0)\, d\xi \right] \right| \end{aligned} \tag{6}$$

*where*

$$\left. \begin{aligned} \bar{d} &= 1 + K d_0 + ML \\ \bar{\kappa} &= K d_0 + M \, \|\varphi\|_{(X,\infty)} \\ \kappa_0 &= K_0(d_0 + d_1) + L_0 \sup \big\{ \|\varphi_{(0,x)}\|_{X_L} : x \in \mathbb{R}^n \big\}. \end{aligned} \right\}$$

**Proof.** Suppose that $(\xi, \eta), (\xi, \bar{\eta}) \in [0, c] \times \mathbb{R}^n$ and a function $\tilde{z} : (-\infty, c] \times \mathbb{R}^n \to B$ is defined by

$$\tilde{z}(\tau, s) = z(\tau, s + \bar{\eta} - \eta) \qquad \big( (\tau, s) \in (-\infty, 0] \times \mathbb{R}^n \big).$$

Then $\tilde{z}_{(\xi,\eta)} = z_{(\xi,\bar{\eta})}$. It follows from Assumptions $\mathrm{H}[X]$ and $\mathrm{H}_0$ that

$$\|z_{(\xi,\eta)} - z_{(\xi,\bar{\eta})}\|_X = \|(z - \tilde{z})_{(\xi,\eta)}\|_X \leq (K d_0 + ML) \, \|\eta - \bar{\eta}\|.$$

The existence and uniqueness of the solutions of problem (4) follows from classical theorems. On this purpose, note that the right-hand side of the differential system satisfies the Carathéodory assumptions, and the Lipschitz condition

$$\big\| \varrho(\tau, \eta, z_{(\tau,\eta)}) - \varrho(\tau, \bar{\eta}, z_{(\tau,\bar{\eta})}) \big\| \leq \bar{d} \, \beta(\tau, \kappa_0) \, \|\eta - \bar{\eta}\|$$

holds on $[0, c] \times \mathbb{R}^n$. The function $g[z](\cdot, t, x)$ satisfies the integral equation

$$g[z](\tau, t, x) = x + \int_t^{\tau} \varrho\big(\xi, g[z](\xi, t, x), z_{(\xi, g[z](\xi,t,x))}\big)\, d\xi.$$

For $(\tau, t, x), (\tau, \bar{t}, \bar{x}) \in \Delta_c$ we have

$$\left. \begin{aligned} \big\| z_{(\tau, g[z](\tau,t,x))} \big\|_X &\leq \bar{\kappa} \\ \big\| z_{(\tau, g[z](\tau,t,x))} \big\|_{X_L} &\leq \kappa_0 \end{aligned} \right\} \tag{7}$$

and

$$\left\|z_{(\tau,g[z](\tau,t,x))} - z_{(\tau,g[z](\tau,\bar{t},\bar{x}))}\right\|_X \le (Kd_1 + ML)\left\|g[z](\tau,t,x) - g[z](\tau,\bar{t},\bar{x})\right\|. \tag{8}$$

It follows from Assumption H$[X]$ and from (7) - (8) that the integral inequality

$$\begin{aligned}&\left\|g[z](\tau,t,x) - g[z](\tau,\bar{t},\bar{x})\right\| \\ &\quad \le \|x-\bar{x}\| + \left|\int_t^{\bar{t}} \alpha(\xi,\bar{\kappa})\,d\xi\right| + \bar{d}\left|\int_t^{\tau} \beta(\xi,\kappa_0)\,\left\|g[z](\xi,t,x) - g[z](\xi,\bar{t},\bar{x})\right\|d\xi\right|\end{aligned}$$

is sastisfied. Now we obtain (5) by the Gronwall inequality.

For $z \in Y_{c.\varphi}[\omega,d]$ and $\bar{z} \in Y_{c.\bar{\varphi}}[\omega,d]$ we have the estimate

$$\begin{aligned}&\left\|z_{(\xi,g[z](\xi,t,x))} - \bar{z}_{(\xi,g[\bar{z}](\xi,t,x))}\right\|_X \\ &\quad \le (Kd_1 + M\tilde{L})\,\left\|g[z](\xi,t,x) - g[\bar{z}](\xi,t,x)\right\| \\ &\quad\quad + K\|z-\bar{z}\|_{[0,\xi;\mathbb{R}^n]} + M\,\|\varphi-\bar{\varphi}\|_{(X,\infty)}.\end{aligned} \tag{9}$$

It follows from Assumption H$[X]$ and from (9) that the integral inequality

$$\begin{aligned}&\left\|g[z](\tau,t,x) - g[\bar{z}](\tau,t,x)\right\| \\ &\quad \le \left|\int_t^{\tau} \beta(\xi,\kappa_0)\left[\|z-\bar{z}\|_{[0,\xi;\mathbb{R}^n]} + M\|\varphi-\bar{\varphi}\|_{(X,\infty)}\right]d\xi\right| \\ &\quad\quad + \bar{d}\left|\int_t^{\tau} \beta(\xi,\kappa_0)\left\|g[z](\xi,t,x) - g[\bar{z}](\xi,t,x)\right\|d\xi\right|\end{aligned}$$

is satisfied. Now we obtain (6) by the Gronwall inequality. This completes the proof of the lemma ∎

## 3. Existence and uniqueness of solutions

Now we construct an integral operator corresponding to problem (1) - (2). Suppose that the function $\varphi$ satisfies Assumption H$_0$, $c \in (0,a]$, $z \in Y_{c.\varphi}[\omega,d]$ and $g[z](\cdot,t,x)$ is the bicharacteristic corresponding to $z$. Let us define the operator $U_\varphi$ for all $z \in Y_{c.\varphi}[\omega,d]$ be the formulas

$$U_\varphi z(t,x) = \varphi\big(0, g[z](0,t,x)\big) + \int_0^t f\left(\tau, g[z](\tau,t,x), z_{(\tau,g[z](\tau,t,x))}\right)d\tau \tag{10}$$

where $(t,x) \in [0,c] \times \mathbb{R}^n$ and

$$U_\varphi z(t,x) = \varphi(t,x) \qquad \text{on } \ (-\infty,0] \times \mathbb{R}^n. \tag{11}$$

**Remark 3.1.** The operator $U_\varphi$ is obtained by integration of equation (1) along bicharacteristics.

Now we give sufficient conditions for the solvability of the equation $z = U_\varphi z$ on $Y_{c.\varphi}[\omega,d]$.

**Assumption H$[f]$.** Suppose the following:

**1)** The function $f(\cdot,x,w):[0,a]\to B$ is measurable for $(x,w)\in\mathbb{R}^n\times X$ and $f(t,\cdot):\mathbb{R}^n\times X\to B$ is continuous for almost all $t\in[0,a]$.

**2)** For $(x,w)\in\mathbb{R}^n\times X[\kappa]$ and for almost all $t\in[0,a]$ we have

$$\|f(t,x,w)\|\le\alpha(t,\kappa). \tag{12}$$

**3)** For $(x,w),(\bar{x},\bar{w})\in\mathbb{R}^n\times X_L[\kappa]$ and for almost all $t\in[0,a]$ we have

$$\big\|f(t,x,w)-f(t,\bar{x},\bar{w})\big\|\le\beta(t,\kappa)\,\big[\|x-\bar{x}\|+\|w-\bar{w}\|_X\big].$$

**Remark 3.2.** We prove a theorem on the existence and uniqueness of solutions of problem (1) - (2). For simplicity of notations, we have assumed the same estimation for $\varrho$ and for $f$. We have assumed also the Lipschitz condition for these functions with the same coefficient.

**Lemma 3.3.** *Suppose that Assumptions* H$[X]$, H$_0$, H$[\varrho]$ *and* H$[f]$ *are satisfied. Then there are* $(d_0,d_1)=d\in\mathbb{R}^2_+$, $c\in(0,a]$ *and* $\omega\in L([0,c],\mathbb{R}_+)$ *such that* $U_\varphi:Y_{c.\varphi}[\omega,d]\to Y_{c.\varphi}[\omega,d]$.

**Proof.** Suppose that the constants $(d_0,d_1)=d$ and $c\in(0,a]$ and the function $\omega\in L([0,c],\mathbb{R}_+)$ satisfy the conditions

$$\left.\begin{aligned} d_0&\ge\tilde{L}+\int_0^c\alpha(\tau,\bar{\kappa})\,d\tau\\ d_1&\ge\Gamma_c\\ \omega(t)&\ge(1+\Gamma_c)\,\alpha(t,\bar{\kappa})\end{aligned}\right\}$$

where

$$\Gamma_c=\left[L+\bar{d}\int_0^c\beta(\tau,\kappa_0)\,d\tau\right]\exp\left[\bar{d}\int_0^c\beta(\tau,\kappa_0)\,d\tau\right]. \tag{13}$$

Suppose that $z\in Y_{c.\varphi}[\omega,d]$. Then we have

$$\|U_\varphi z(t,x)\|\le\tilde{L}+\int_0^c\alpha(\tau,\bar{\kappa})\,d\tau\le d_0\qquad\text{on }\;[0,c]\times\mathbb{R}^n. \tag{14}$$

If $(t,x),(\bar{t},\bar{x})\in[0,c]\times\mathbb{R}^n$, then using Lemma 2.1 and (10) we obtain

$$\begin{aligned}&\big\|U_\varphi z(t,x)-U_\varphi z(\bar{t},\bar{x})\big\|\\ &\qquad\le\big\|\varphi\big(0,g[z](0,t,x)\big)-\varphi\big(0,g[z](0,\bar{t},\bar{x})\big)\big\|+\left|\int_t^{\bar{t}}\alpha(\tau,\bar{\kappa})\,d\tau\right|\end{aligned}$$

$$+\int_0^t \beta(\tau,\kappa_0)\Big[\big\|g[z](\tau,t,x)-g[z](\tau,\bar t,\bar x)\big\| + \big\|z_{(\tau,g[z](\tau,t,x))}-z_{(\tau,g[z](\tau,\bar t,\bar x))}\big\|_X\Big]\,d\tau$$

$$\le \Gamma_c\left[\|x-\bar x\| + \left|\int_t^{\bar t}\alpha(\tau,\bar\kappa)\,d\tau\right|\right] + \left|\int_t^{\bar t}\alpha(\tau,\bar\kappa)\,d\tau\right|.$$

Thus we see that

$$\big\|U_\varphi z(t,x)-U_\varphi z(\bar t,\bar x)\big\| \le d_1\|x-\bar x\| + \left|\int_t^{\bar t}\omega(\tau)\,d\tau\right|. \tag{15}$$

It follows from (14) and (15) that $U_\varphi z \in Y_{c.\varphi}[\omega,d]$ which completes the proof of the lemma ∎

Next we will show that there exists exactly one solution of problem (1) - (2). The solution is local with respect to $t$.

**Theorem 3.4.** *Suppose that Assumptions* H$[X]$, H$_0$, H$[\varrho]$ *and* H$[f]$ *are satisfied. Then there are $(d_0,d_1)=d\in\mathbb{R}^2_+$, $c\in(0,a]$ and $\omega\in L([0,c],\mathbb{R}_+)$ such that problem (1) – (2) has exactly one solution $u\in Y_{c.\varphi}[\omega,d]$.*

*If $\bar\varphi:(-\infty,0]\times\mathbb{R}^n\to B$ satisfies Assumption* H$_0$ *and $\bar u\in Y_{c.\varphi}[\omega,d]$ is a solution of equation (1) with the initial condition $z=\bar\varphi$ on $(-\infty,0]\times\mathbb{R}^n$, then there is $\Lambda_c\in\mathbb{R}_+$ such that*

$$\|u-\bar u\|_{[0,t;\mathbb{R}^n]} \le \Lambda_c\Big[\|\varphi-\bar\varphi\|_{(X,\infty)} + \sup_{y\in\mathbb{R}^n}\big\|\varphi(0,y)-\bar\varphi(0,y)\big\|\Big] \tag{16}$$

*where $t\in[0,c]$.*

**Proof.** Lemma 3.3 shows that there are $(d_0,d_1)=d$, $c\in(0,a]$ and $\omega\in L([0,c],\mathbb{R}_+)$ such that $U_\varphi : Y_{c.\varphi}[\omega,d]\to Y_{c.\varphi}[\omega,d]$. Write

$$\lambda_c = K(1+\Gamma_c)\int_0^c \beta(\tau,\kappa_0)\,d\tau$$

where $\Gamma_c$ is given by (13). Let $c\in(0,a]$ be such a constant that $\lambda_c<1$. Now we prove that $U_\varphi$ is a contraction on $Y_{c.\varphi}[\omega,d]$. If $z,\tilde z\in Y_{c.\varphi}[\omega,d]$, then

$$\begin{aligned}\big\|U_\varphi z(t,x)-U_\varphi\tilde z(t,x)\big\| \le{}& L\,\big\|g[z](0,t,x)-g[\tilde z](0,t,x)\big\|\\ &+\int_0^t \beta(\tau,\kappa_0)\Big[\big\|g[z](\tau,t,x)-g[\tilde z](\tau,t,x)\big\|\\ &+\big\|z_{(\tau,g[z](\tau,t,x))}-\tilde z_{(\tau,g[z](\tau,t,x))}\big\|_X\Big]\,d\tau.\end{aligned}$$

The estimate

$$\begin{aligned}&\big\|z_{(\tau,g[z](\tau,t,x))}-\tilde z_{(\tau,g[z](\tau,t,x))}\big\|_X\\ &\qquad\le (Kd_1+M\tilde L)\,\big\|g[z](\tau,t,x)-g[\tilde z](\tau,t,x)\big\| + K\|z-\tilde z\|_{[0,\tau;\mathbb{R}^n]}\end{aligned}$$

and Lemma 2.1 imply

$$\big\|U_\varphi z(t,x) - U_\varphi \tilde{z}(t,x)\big\| \le K(1+\Gamma_c)\int\limits_0^t \beta(\tau,\kappa_0)\|z-\tilde{z}\|_{[0,\tau;\mathbb{R}^n]}d\tau$$

for all $(t,x)\in[0,c]\times\mathbb{R}^n$, and consequently

$$\|U_\varphi z - U_\varphi \tilde{z}\|_{[0,c;\mathbb{R}^n]} \le \lambda_c\|z-\tilde{z}\|_{[0,c;\mathbb{R}^n]}.$$

By the Banach fixed point theorem there exists a unique solution $u\in Y_{c.\varphi}[\omega,d]$ of the equation $z=U_\varphi z$.

Now we prove that $u$ is a solution of (1). We have proved that

$$u(t,x) = \varphi\big(0,g[u](0,t,x)\big) + \int\limits_0^t f\big(\tau, g[u](\tau,t,x), u_{(\tau,g[u](\tau,t,x))}\big)d\tau \tag{17}$$

on $[0,c]\times\mathbb{R}^n$. For given $x\in\mathbb{R}^n$ let us put $\eta = g[u](0,t,x)$. It follows that $g[u](\tau,t,x) = g[u](\tau,0,\eta)$ for $\tau\in[0,c]$ and that $x = g[u](t,0,\eta)$. The relations $\eta = g[u](0,t,x)$ and $x=g[u](t,0,\eta)$ are equivalent for $x,\eta\in\mathbb{R}^n$. It follows from (17) that

$$u\big(t,g[u](t,0,\eta)\big) = \varphi(0,\eta) + \int\limits_0^t f\big(\tau, g[u](\tau,0,\eta), u_{(\tau,g[u](\tau,0,\eta))}\big)d\tau \tag{18}$$

where $(t,\eta)\in[0,c]\times\mathbb{R}^n$. By differentiating (18) with respect to $t$ and by using the transformation $\eta = g[u](0,t,x)$ which preserves sets of measure zero, we obtain that $u$ satisfies equation (1) for almost all $(t,x)\in[0,c]\times\mathbb{R}^n$. It follows from (11) that $u$ satisfies also condition (2).

Now we prove relation (16). If $u=U_\varphi u$ and $\bar{u} = U_{\bar\varphi}\bar{u}$, then

$$\begin{aligned}
&\big\|u(t,x)-\bar{u}(t,x)\big\| \\
&\quad\le \sup_{y\in\mathbb{R}^n}\big\|\varphi(0,y)-\bar\varphi(0,y)\big\| + L\,\big\|g[u](0,t,x)-g[\bar u](0,t,x)\big\| \\
&\qquad + \int_0^t \beta(\tau,\kappa_0)\Big[\bar d\,\big\|g[u](\tau,t,x)-g[\bar u](\tau,t,x)\big\| \\
&\qquad + K\,\|u-\bar u\|_{[0,\tau;\mathbb{R}^n]} + M\|\varphi-\bar\varphi\|_{(X,\infty)}\Big]d\tau
\end{aligned}$$

where $(t,x)\in[0,c]\times\mathbb{R}^n$. Put

$$A_c = (1+\Gamma_c)M\int\limits_0^t \beta(\tau,\kappa_0)\,d\tau \qquad \textit{and} \qquad \gamma(t) = K(1+\Gamma_c)\,\beta(t,\kappa_0).$$

Then we get the integral inequality

$$\begin{aligned}
&\|u-\bar u\|_{[0,t;\mathbb{R}^n]} \\
&\quad\le \sup_{y\in\mathbb{R}^n}\big\|\varphi(0,y)-\bar\varphi(0,y)\big\| + A_c\|\varphi-\bar\varphi\|_{(X,\infty)} + \int\limits_0^t \gamma(\tau)\|u-\bar u\|_{[0,\tau;\mathbb{R}^n]}d\tau
\end{aligned}$$

for all $t\in[0,c]$. It follows from the Gronwall inequality that we have estimate (16) for $\Lambda_c = \exp\big[\int_0^c \gamma(\tau)\,d\tau\big]$. This completes the proof of the theorem ∎

## 4. Phase spaces

We give examples of spaces $X$ satisfying Assumption H[$X$].

**Example 4.1.** Let $X$ be the class of all function $w : (-\infty, 0] \times [-r, +r] \to B$ which are uniformly continuous and bounded on $(-\infty, 0] \times [-r, +r]$. For $w \in X$ we write

$$\|w\|_X = \sup \Big\{ \|w(\tau, s)\| : \, (\tau, s) \in (-\infty, 0] \times [-r, +r] \Big\}.$$

Let $X_L \subset X$ denote the set of all $w \in X$ such that

$$|w|_L = \sup \left\{ \frac{\|w(\tau, s) - w(\tau, \bar{s})\|}{\|s - \bar{s}\|} : \, (\tau, s), (\tau, \bar{s}) \in (-\infty, 0] \times [-r, +r] \right\} < +\infty. \tag{19}$$

Write $\|w\|_{X_L} = \|w\|_X + |w|_L$ where $w \in X_L$. Then Assumption H[$X$] is satisfied.

**Example 4.2.** Let $X$ be the class of all functions $w : (-\infty, 0] \times [-r, +r] \to B$ such that

**(i)** $w$ is continuous and bounded on $(-\infty, 0] \times [-r, +r]$

**(ii)** the limit $\lim_{t \to -\infty} w(t, x)$ exists uniformly with respect to $x \in [-r, +r]$.

Let

$$\|w\|_X = \sup \Big\{ \|w(\tau, s)\| : \, (\tau, s) \in (-\infty, 0] \times [-r, +r] \Big\}.$$

Let $X_L \subset X$ denote the class of all $w \in X$ such that the Lipschitz condition (19) is satisfied. Write $\|w\|_{X_L} = \|w\|_X + |w|_L$ where $w \in X_L$. Then Assumption H[$X$] is satisfied.

**Example 4.3.** Let $\gamma : (-\infty, 0] \to (0, +\infty)$ be a continuous function. Assume also that $\gamma$ is non-increasing on $(-\infty, 0]$. Let $X$ be the space of continuous functions $w : (-\infty, 0] \times [-r, +r] \to B$ for which

$$\lim_{\tau \to \infty} \frac{\|w(\tau, x)\|}{\gamma(\tau)} = 0 \qquad (x \in [-r, +r]).$$

Put

$$\|w\|_X = \sup \left\{ \frac{\|w(\tau, s)\|}{\gamma(\tau)} : \, (\tau, s) \in (-\infty, 0] \times [-r, +r] \right\}.$$

Denote by $X_L \subset X$ the set of all $w \in X$ such that

$$|w|_{\gamma.L} = \sup \left\{ \frac{\|w(\tau, s) - w(\tau, \bar{s})\|}{\gamma(\tau)\|s - \bar{s}\|} : \, (\tau, s), (\tau, \bar{s}) \in (-\infty, 0] \times [-r, +r] \right\} < +\infty.$$

For $w \in X_L$ put $\|w\|_{X_L} = \|w\|_X + |w|_{\gamma.L}$. Then Assumption H[$X$] is satisfied.

**Example 4.4.** Let $\delta \in \mathbb{R}_+$ and $p \geq 1$ be fixed. Denote by $X$ the class of all functions $w : (-\infty, 0] \times [-r, +r] \to B$ such that

**(i)** $w$ is continuous on $[-\delta, 0] \times [-r, +r]$

**(ii)** for $x \in [-r, +r]$ we have $\int_{-\infty}^{-\delta} \|w(\tau, x\|^p \, d\tau < +\infty$

**(iii)** $w(t,\cdot) : [-r,+r] \to B$ is continuous for $t \in (-\infty,-\delta]$.

Write

$$\|w\|_X = \sup\Big\{\|z(\tau,s)\| : (\tau,s) \in [-\delta,0]\times[-r,+r]\Big\}$$
$$+\sup\Big\{\Big(\int_{-\infty}^{-\delta}\|w(\tau,x\|^p d\tau\Big)^{1/p} : x \in [-r,+r]\Big\}.$$

Let $X_L \subset X$ be the set of functions $w \in X$ such that the Lipschitz condition (19) is satisfied. Write $\|w\|_{X_L} = \|w\|_X + |w|_L$ where $w \in X_L$. Then Assumption H$[X]$ is satisfied.

**Remark 4.5.** Differential equations with a deviated argument and differential-integral equations can be obtained from equation (1) by specializing operators $\varrho$ and $f$.

**Remark 4.6.** It is important in our considerations that we have assumed the Lipschitz condition for given functions on some special function spaces. More precisely, we have assumed that the functions $\varrho(t,\cdot)$ and $f(t,\cdot)$ satisfy the Lipschitz condition on the space $\mathbb{R}^n \times X_L$ for almost all $t \in [0,a]$, and the condition is local with respect to the functional variable.

Let us consider simplest assumption on $\varrho$ and $f$. Suppose that there is $P \in \mathbb{R}_+$ such that for almost all $t \in [0,a]$ we have

$$\big\|\varrho(t,x,w) - \varrho(t,\bar{x},\bar{w})\big\| \le P\big[\|x-\bar{x}\| + \|w-\bar{w}\|_X\big] \tag{20}$$
$$\big\|f(t,x,w) - f(t,\bar{x},\bar{w})\big\| \le P\big[\|x-\bar{x}\| + \|w-\bar{w}\|_X\big] \tag{21}$$

where $(x,w),(\bar{x},\bar{w}) \in \mathbb{R}^n \times X$. Of course, our results are true if we assume (20), (21) instead of (3), (12).

Now we show that formulations (3), (12) are important. We show that there is a class of quasilinear equations satisfying (3), (12) but not satisfying (20), (21). Let $X$ and $X_L$ be the spaces given in Example 4.1. Consider the equation with a deviated argument

$$D_t z(t,x) + \sum_{i=1}^{n} \tilde{\varrho}_i\big(t,x,z(\psi_0(t),\psi(t,x))\big)D_{x_i}z(t,x) = \tilde{f}\big(t,x,z(\psi_0(t),\psi(t,x))\big) \tag{22}$$

where

$$\left.\begin{aligned} \tilde{\varrho} = (\tilde{\varrho}_1,\ldots,\tilde{\varrho}_n) : [0,a]\times\mathbb{R}^n\times B \to \mathbb{R}^n \\ f : [0,a]\times\mathbb{R}^n\times B \to B \\ \psi_0 : [0,a] \to (-\infty,a] \\ \psi : [0,a]\times\mathbb{R}^n \to \mathbb{R}^n. \end{aligned}\right\}$$

We assume that $\psi(t) \le t$ and $-r \le \psi(t,x) - x \le +r$ for $(t,x) \in [0,a]\times\mathbb{R}^n$. We get (22) by putting in (1)

$$\varrho(t,x,w) = \tilde{\varrho}\big(t,x,w\big(\psi_0(t)-t,\psi(t,x)-x\big)\big)$$
$$f(t,x,w) = \tilde{f}\big(t,x,w\big(\psi_0(t)-t,\psi(t,x)-x\big)\big).$$

From now we consider the function $\varrho$ only. Suppose that there are $\bar{C}, \tilde{C} \in \mathbb{R}_+$ such that

$$\begin{aligned}\big\|\tilde{\varrho}(t,x,\zeta) - \tilde{\varrho}(t,\bar{x},\bar{\zeta})\big\| &\le \bar{C}\big[\|x-\bar{x}\| + \|\zeta - \bar{\zeta}\|\big] \\ \big\|\psi(t,x) - \psi(t,\bar{x})\big\| &\le \tilde{C}\|x-\bar{x}\|.\end{aligned}$$

It is evident that for $(x,w),(\bar{x},\bar{w}) \in \mathbb{R}^n \times X_L[\kappa]$ and for almost all $t \in [0,a]$ we have

$$\big\|\varrho(t,x,w) - \varrho(t,\bar{x},\bar{w})\big\| \le \bar{C}\big[1 + \kappa(1+\tilde{C})\big]\,\|x-\bar{x}\| + \bar{C}\,\|w-\bar{w}\|_X.$$

Then condition (3) is satisfied.

We see at once the the function $\varrho(t,\cdot)$ does not satisfy the global Lipschitz condition (20) for $(x,w),(\bar{x},\bar{w}) \in \mathbb{R}^n \times X$. Similar consideration apply to $f$.

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