Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 1, 37–46

# On Morozov's Method for Tikhonov Regularization as an Optimal Order Yielding Algorithm

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**Abstract.** It is shown that Tikhonov regularization for an ill-posed operator equation $Kx = y$ using a possibly unbounded regularizing operator $L$ yields an order-optimal algorithm with respect to certain stability set when the regularization parameter is chosen according to Morozov's discrepancy principle. A more realistic error estimate is derived when the operators $K$ and $L$ are related to a Hilbert scale in a suitable manner. The result includes known error estimates for ordininary Tikhonov regularization and also estimates available under the Hilbert scales approach.

**Keywords:** *Tikhonov regularization, ill-posed equations, order-optimal algorithms, interpolation inequalities, Hilbert scales*

**AMS subject classification:** 65 R 10, 65 J 10, 65 J 20, 65 R 20, 45 B 05, 45 L 10, 47 A 50

## 1. Introduction

Many problems in science and engineering have their mathematical formulation as an operator equation

$$Kx = y \tag{1.1}$$

where $K : X \to Y$ is a bounded linear operator between Hilbert spaces $X$ and $Y$ with its range $R(K)$ not closed in $Y$ (c.f. [1, 2]). It is well known that if $R(K)$ is not closed, then equation (1.1) or the problem of solving (1.1) is ill-posed (cf. [3]). A prototype of an ill-posed equation is the Fredholm integral equation of the first kind,

$$\int_a^b k(s,t)x(t)\,dt = y(s) \qquad (a \le s \le b)$$

with a non-degenerate kernel $k(\cdot,\cdot) \in L^2([a,b] \times [a,b])$ and $X = Y = L^2[a,b]$.

Regularization procedures are employed for obtaining stable approximate solutions of ill-posed equations of the type (1.1). These procedures are especially useful when the data available is inexact. That is, we may have an approximation $\tilde{y}$ of $y$ with a known error level $\delta > 0$, $\|y - \tilde{y}\| \le \delta$.

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Partially supported by a project under NBHM, DAE, Govt. of India

ISSN 0232-2064 / $ 2.50 

In this paper we consider the well known Tikhonov regularization method using a possibly unbounded regularizing operator $L$. In fact, we assume that

$$L : D(L) \subseteq X \to Z$$

is a closed densily defined linear operator between Hilbert spaces $X$ and $Z$. Then the Tikhonov regularization involves minimization of the map

$$x \mapsto \|Kx - \tilde{y}\|^2 + \alpha\|Lx\|^2 \qquad (x \in D(L)). \tag{1.2}$$

It is known that if $K$ and $L$ satisfy the relation

$$\|Kx\|^2 + \|Lx\|^2 \geq \gamma\|x\|^2 \qquad (x \in D(L)) \tag{1.3}$$

for some $\gamma > 0$, then the map in (1.2) attains its minimum at a unique element $x_\alpha(\tilde{y})$ in $D(L)$ (see, e.g., [5, 9, 10]). It is also known (cf. [5, 9, 10]) that if $y \in R(A) + R(A)^\perp$, $A = K_{|D(L)}$, then

- the set $S_y := \{x \in D(L) : \|Kx - y\| \leq \|Ku - y\| \ \forall\ u \in D(L)\}$ is non-empty
- there exists a unique $\hat{x}(y) \in S_y$ such that $\|L\hat{x}(y)\| \leq \|Lx\|$ for all $x \in S_y$
- $x_\alpha(y) \to \hat{x}(y)$ as $\alpha \to 0$.

What one would like to have is the convergence of $x_\alpha(\tilde{y})$ to some $\tilde{x}$ as $\alpha \to 0$, where $\tilde{x}$ is close to $\hat{x}(y)$ whenever $\delta$ is close to 0.

But examples can be easily constructed where this is no longer true. Therefore a strategy has to be adopted for choosing the regularization parameter $\alpha = \alpha(\delta, \tilde{y})$ so as to have the above situation. For this purpose we consider the simple procedure suggested by Morozov [8, 9], namely, to choose $\alpha = \alpha(\delta, \tilde{y})$ such that

$$\|K\tilde{x}_\alpha - \tilde{y}\| = \delta \tag{1.4}$$

where $\tilde{x}_\alpha = x_\alpha(\tilde{y})$. It is known that if

$$\|(I - P_L)y\| > 0 \qquad \text{and} \qquad \|(I - P_L)\tilde{y}\| > \delta, \tag{1.5}$$

where $P_L : Y \to Y$ is the orthogonal projection onto the closure of the set

$$\{Kx : x \in D(L) \text{ with } Lx = 0\},$$

then there exists a unique $\alpha$ depending on $\delta$ and $\tilde{y}$ satisfying (1.4) (cf. Morozov [9: Section 10]). Note that if $L$ is injective, then $P_L = 0$, and in that case (1.5) can be replaced by the assumption $\|y\| > 2\delta$.

We show that the Tikhonov regularization together with the parameter choice strategy (1.4) yield an order-optimal algorithm with respect to the stabilizing set

$$M_\rho = \{x \in D(L) : \|Lx\| \leq \rho\}.$$

That is, we show that

$$\|\hat{x} - \tilde{x}_\alpha\| = O(E(M_\rho, \delta))$$

where $\hat{x} = \hat{x}(y)$, $\tilde{x}_\alpha = x_\alpha(\tilde{y})$ and $E(M_\rho, \delta)$ is the *best possible maximal error* defined by

$$E(M_\rho, \delta) = \inf_R \sup \Big\{ \|x - Rv\| : x \in M_\rho \text{ and } v \in Y \text{ with } \|Kx - v\| \le \delta \Big\}.$$

In order to obtain more realistic error estimates, we relate the operators $K$ and $L$ with a Hilbert scale in a suitable manner. Better estimates are derived under additional assumptions on the *smoothness of the solution* $\hat{x}$. Particular cases include known estimates for ordinary Tikhonov regularization, i.e., for $L = I$, and also the well known estimates available under the Hilbert scales approach derived by Natterer [11].

In addition to all the above, our approach seems to be simpler and straight forward for the Hilbert scales setting.

## 2. Main results

Let $K : X \to Y$ and $L : D(L) \subseteq X \to Z$ be as in the earlier section satisfying the condition (1.3) and $y \in R(A)$ such that $\|(I - P_L)y\| > 0$, where $A = K_{|D(L)}$ and $P_L : Y \to Y$ is the orthogonal projection onto the closure of the set $\{Kx : x \in D(L) \text{ with } Lx = 0\}$. Let $\tilde{y} \in Y$ satisfy

$$\|y - \tilde{y}\| \le \delta < \|(I - P_L)\tilde{y}\|, \tag{2.1}$$

where $0 < \delta < 1$, and let $\alpha = \alpha(\delta, \tilde{y})$ be the unique positive real satisfying (1.4). We recall from [5] or [10] that the condition $y \in R(A)$ implies that $K\hat{x} = y$.

For $M \subseteq D(L)$, let

$$e(M, \delta) = \sup \big\{ \|x\| : x \in M \text{ with } \|Kx\| \le \delta \big\}.$$

If $M$ is a convex and balanced subset, then it is proved in [7] that

$$e(M, \delta) \le E(M, \delta) \le 2e(M, \delta)$$

where

$$E(M, \delta) = \inf_R \sup \Big\{ \|x - Rv\| : x \in M \text{ and } v \in M \text{ with } \|Kx - v\| \le \delta \Big\}.$$

**2.1 Order-optimal result.** For $\rho > 0$ let

$$M_\rho = \big\{ x \in D(L) : \|Lx\| \le \rho \big\}.$$

We note that $M_\rho$ is a convex and balanced subset of $X$. In the following, we use the notation $\hat{x}$ and $\tilde{x}_\alpha$ for $\hat{x}(y)$ and $x_\alpha(\tilde{y})$, respectively.

**Theorem 2.1.** *If $\hat{x} \in M_\rho$ for some $\rho > 0$, then*

$$\frac{1}{2}(\hat{x} - \tilde{x}_\alpha) \in M_\rho$$

*and*

$$\|\hat{x} - \tilde{x}_\alpha\| \leq 2e(M_\rho, \delta).$$

**Proof.** Since $\tilde{x}_\alpha$ minimizes the map (1.2), it follows from (1.4) and (2.1) that

$$\begin{aligned}\delta^2 + \alpha\|L\tilde{x}_\alpha\|^2 &= \|K\tilde{x}_\alpha - \tilde{y}\|^2 + \alpha\|L\tilde{x}_\alpha\|^2 \\ &\leq \|K\hat{x} - \tilde{y}\|^2 + \alpha\|L\hat{x}\|^2 \\ &\leq \delta^2 + \alpha\|L\hat{x}\|^2.\end{aligned}$$

Hence $\|L\tilde{x}_\alpha\| \leq \|L\hat{x}\|$. Using this, we obtain

$$\begin{aligned}\|L(\hat{x} - \tilde{x}_\alpha)\|^2 &= \langle L(\hat{x} - \tilde{x}_\alpha), L(\hat{x} - \tilde{x}_\alpha)\rangle \\ &= \langle L\hat{x}, L\hat{x}\rangle - 2Re\langle L\hat{x}, L\tilde{x}_\alpha\rangle + \langle L\tilde{x}_\alpha, L\tilde{x}_\alpha\rangle \\ &\leq 2(\langle L\hat{x}, L\hat{x}\rangle - \mathrm{Re}\langle L\hat{x}, L\tilde{x}_\alpha\rangle).\end{aligned}$$

Thus

$$\|L(\hat{x} - \tilde{x}_\alpha)\|^2 \leq 2|\langle L\hat{x}, L(\hat{x} - \tilde{x}_\alpha)\rangle|. \tag{2.2}$$

From this it follows that

$$\|L(\hat{x} - \tilde{x}_\alpha)\| \leq 2\rho.$$

Also, since $K\hat{x} = y$, $\|y - \tilde{y}\| \leq \delta$ and (1.4), we have

$$\|K(\hat{x} - \tilde{x}_\alpha)\| \leq 2\delta. \tag{2.3}$$

Thus,

$$\left\|L\left(\frac{\hat{x} - \tilde{x}_\alpha}{2}\right)\right\| \leq \rho \qquad \text{and} \qquad \left\|K\left(\frac{\hat{x} - \tilde{x}_\alpha}{2}\right)\right\| \leq \delta$$

so that $\frac{\hat{x} - \tilde{x}_\alpha}{2} \in M_\rho$ and $\|\hat{x} - \tilde{x}_\alpha\| \leq 2\,e(M_\rho, \delta)$ ∎

**2.2 Realistic estimates using Hilbert scales.** To obtain a more realistic estimate for the error $\|\hat{x} - \tilde{x}_\alpha\|$, we relate the operators $K$ and $L$ to a Hilbert scale $(X_s)_{s\in\mathbb{R}}$ (cf. [4]) with $X_0 = X$ in the following way :

(i) There exist $a > 0$ and $c > 0$ such that

$$\|Kx\| \geq c\|x\|_{-a} \qquad (x \in X). \tag{2.4}$$

(ii) There exist $b \geq 0$ and $d > 0$ such that $D(L) \subseteq X_b$ and

$$\|Lx\| \geq d\|x\|_b \qquad (x \in D(L)). \tag{2.5}$$

To obtain our results we shall make use of the *interpolation inequality* (cf. [4])

$$\|x\|_s \leq \|x\|_r^\theta \|x\|_t^{1-\theta} \qquad (x \in X_t)$$

where $r \leq s \leq t$ and $\theta = \frac{t-s}{t-r}$. Taking $r = -a$, $t = b$ and $s = 0$ in the above interpolation inequality it follows from (2.4) and (2.5) that

$$\|x\| \leq \left(\frac{\|Kx\|}{c}\right)^\theta \left(\frac{\|Lx\|}{d}\right)^{1-\theta} \qquad (\theta = \tfrac{b}{a+b}) \tag{2.6}$$

for every $x \in D(L)$.

**Theorem 2.2.** *If $\hat{x} \in M_\rho$ for some $\rho > 0$, then*

$$\|\hat{x} - \tilde{x}_\alpha\| \leq 2\Big(\frac{\rho}{d}\Big)^{\frac{a}{a+b}}\Big(\frac{\delta}{c}\Big)^{\frac{b}{a+b}}.$$

**Proof.** From (2.6) it follows that, for every $x \in M_\rho$ with $\|Kx\| \leq \delta$,

$$\|x\| \leq \Big(\frac{\delta}{c}\Big)^{\theta}\Big(\frac{\rho}{d}\Big)^{1-\theta} \qquad \big(\theta = \tfrac{b}{a+b}\big)$$

so that

$$e(M_\rho, \delta) \leq \Big(\frac{\delta}{c}\Big)^{\frac{b}{a+b}}\Big(\frac{\rho}{d}\Big)^{\frac{a}{a+b}}.$$

Now the result follows from Theorem 2.1 ∎

Next we obtain an improved estimate under stronger assumptions on $\hat{x}$. For this we shall make use of the following lemma which is a particular case of a well-known moment inequality (cf. [13: Formula (2.49)]). For the sake of completion of the exposition, we include its proof as well.

**Lemma 2.3.** *If $B$ is a bounded self-adjoint operator on $X$ and $0 \leq \tau \leq 1$, then*

$$\|B^\tau x\| \leq \|Bx\|^\tau \|x\|^{1-\tau} \qquad (x \in X).$$

**Proof.** The result is obvious if either $\tau = 0$ or $\tau = 1$. Therefore assume that $0 < \tau < 1$. As a consequence of the spectral theorem we have

$$\|B^\tau x\|^2 = \int_J \lambda^{2\tau} d\langle E_\lambda x, x\rangle \qquad (x \in X)$$

where $J$ is an open interval containing the spectrum of $B$ and $\{E_\lambda\}_{\lambda \in J}$ is the spectral family for $B$. Now by Hölder's inequality we have

$$\|B^\tau x\|^2 \leq \Big(\int_J \lambda^2 d\langle E_\lambda x, x\rangle\Big)^{\tau}\Big(\int_J d\langle E_\lambda x, x\rangle\Big)^{1-\tau} = \|Bx\|^{2\tau}\|x\|^{2(1-\tau)}$$

for every $x \in X$ and $0 < \tau < 1$ ∎

**Theorem 2.4.** *Suppose $D(L^*L) \subset X_b$, $\hat{x} \in D(L^*L)$ and $L^*L\hat{x} = (K^*K)^\nu u$ for some $u \in X$ and $0 \leq \nu \leq \frac{1}{2}$. Then*

$$\|\hat{x} - \tilde{x}_\alpha\| \leq \kappa\delta^p$$

*where*

$$p = \frac{2(a\nu + b)}{2(a\nu + b) + a} \qquad \textit{and} \qquad \kappa = 2\Big(\frac{1}{c}\Big)^{\frac{2b}{2(a\nu+b)+a}}\Big(\frac{\sqrt{\|u\|}}{d}\Big)^{\frac{2a}{2(a\nu+b)+a}}.$$

**Proof.** Since $\hat{x} - \tilde{x}_\alpha \in D(L^*L) \subset X_b$, from (2.6) we have

$$\|\hat{x} - \tilde{x}_\alpha\| \leq \Big(\frac{\|K(\hat{x} - \tilde{x}_\alpha)\|}{c}\Big)^{\theta}\Big(\frac{\|L(\hat{x} - \tilde{x}_\alpha\|}{d}\Big)^{1-\theta} \tag{2.7}$$

where $\theta = \frac{b}{a+b}$. Now using the fact that $L^*L\hat{x} = (K^*K)^\nu u \quad (0 \le \nu \le \frac{1}{2})$ the relation (2.2) implies

$$\begin{aligned}\|L(\hat{x}-\tilde{x}_\alpha)\|^2 &\le 2|\langle (K^*K)^\nu u, \hat{x}-\tilde{x}_\alpha\rangle| \\ &= 2|\langle u, (K^*K)^\nu(\hat{x}-\tilde{x}_\alpha)\rangle| \\ &\le 2\|u\|\|(K^*K)^\nu(\hat{x}-\tilde{x}_\alpha)\|.\end{aligned}$$

Taking $B = (K^*K)^{\frac{1}{2}}$ and $\tau = 2\nu$ in Lemma 2.3, and using (2.3), we obtain

$$\|(K^*K)^\nu(\hat{x}-\tilde{x}_\alpha)\| \le \|K(\hat{x}-\tilde{x}_\alpha)\|^{2\nu}\|\hat{x}-\tilde{x}_\alpha\|^{1-2\nu} \le (2\delta)^{2\nu}\|\hat{x}-\tilde{x}_\alpha\|^{1-2\nu}$$

where we used the relation $\|(K^*K)^{\frac{1}{2}}x\| = \|Kx\|$. Thus,

$$\|L(\hat{x}-\tilde{x}_\alpha)\| \le \sqrt{2\|u\|}(2\delta)^\nu\|\hat{x}-\tilde{x}_\alpha\|^{\frac{1-2\nu}{2}}.$$

Therefore, (2.7) gives

$$\|\hat{x}-\tilde{x}_\alpha\| \le \left(\frac{1}{c}\right)^\theta\left(\frac{\sqrt{2\|u\|}}{d}\right)^{1-\theta}(2\delta)^{\theta+\nu(1-\theta)}\|\hat{x}-\tilde{x}_\alpha\|^{(1-\theta)\frac{1-2\nu}{2}}$$

so that

$$\|\hat{x}-\tilde{x}_\alpha\|^{1-(1-\theta)\frac{1-2\nu}{2}} \le \left(\frac{1}{c}\right)^\theta\left(\frac{\sqrt{2\|u\|}}{d}\right)^{1-\theta}(2\delta)^{\theta+\nu(1-\theta)}.$$

From this the result follows by observing that

$$\theta+\nu(1-\theta) = \frac{a\nu+b}{a+b} \qquad \text{and} \qquad 1-\frac{(1-\theta)(1-2\nu)}{2} = \frac{2(a\nu+b)+a}{2(a+b)}.$$

Thus the statement is proved ∎

**Remark.** We note that

$$\frac{2(a\nu+b)}{2(a\nu+b)+a} - \frac{b}{a+b} = \frac{(2a\nu+b)a}{[2(a\nu+b)+a](a+b)}$$

so that if either $\nu \ne 0$ or $b \ne 0$, then the estimate in Theorem 2.4 is better than the estimate in Theorem 2.2.

**2.3 Particular cases.** The particular cases of Theorem 2.4 are worth noticing.

**Theorem 2.5.**

**(i)** *If $L = I$ and $\hat{x} = (K^*K)^\nu u$ for some $u \in X$ and $0 \le \nu \le \frac{1}{2}$, then*

$$\|\hat{x}-\tilde{x}_\alpha\| \le 2\|u\|^{\frac{1}{2\nu+1}}\delta^{\frac{2\nu}{2\nu+1}}.$$

**(ii)** *Suppose $\hat{x} \in D(L^*L)$ and $\hat{u} = L^*L\hat{x}$. Then*

$$\|\hat{x}-\tilde{x}_\alpha\| \le 2\left(\frac{1}{c}\right)^{\frac{2b}{2b+a}}\left(\frac{\sqrt{\|u\|}}{d}\right)^{\frac{2a}{2b+a}}\delta^{\frac{2b}{2b+a}}.$$

**(iii)** *Suppose $\hat{x} \in D(L^*L)$ and $L^*L\hat{x} = K^*u$ for some $u \in X$. Then*

$$\|\hat{x}-\tilde{x}_\alpha\| \le 2\left(\frac{1}{c}\right)^{\frac{b}{a+b}}\left(\frac{\sqrt{\|u\|}}{d}\right)^{\frac{a}{a+b}}\delta^{\frac{a+2b}{2a+2b}}.$$

**Proof.** The estimates in (i) - (iii) are obtained from Theorem 2.4 by taking $b = 0$, $\nu = 0$ and $\nu = \frac{1}{2}$, respectively ∎

**2.4 Concluding remarks. (a)** Recently Mair [6] obtained results similar to the ones in Theorems 2.1 and 2.2 with $\sqrt{2}$ in place of 2, but under an a priori choice of the parameter $\alpha$, namely, $\alpha = \frac{\delta^2}{\rho^2}$.

It should be observed that the error bound given in Theorem 2.2 need not be order optimal for the set $M_\rho$, unless the inequalities (2.4) and (2.5) are replaced by

$$c_1\|x\|_{-a} \geq \|Kx\| \geq c\|x\|_{-a} \qquad (x \in X)$$

and

$$d_1\|x\|_b \geq \|Lx\| \geq d\|x\|_b \qquad (x \in D(L)),$$

respectively. In order to see this we note that

$$e(M_\rho, \delta) \leq e(\widetilde{M}_{\frac{\rho}{d}}, \delta),$$

where

$$\widetilde{M}_r = \{x \in X_b : \|x\|_b \leq r\},$$

and recall (cf. [11]) that the error bound in Theorem 2.2 is order optimal for the set $\widetilde{M}_{\frac{\rho}{d}}$. But $e(M_\rho, \delta)$ can be of better order. For example, consider $L$ such that $D(L) \subseteq X_\beta$ and $\|Lx\| = \|x\|_\beta$ for all $x \in D(L)$ with $\beta > b$. Then we have

$$e(M_\rho, \delta) = O\left(\delta^{\frac{\beta}{\beta+a}}\right).$$

Note that

$$\frac{\beta}{\beta + a} > \frac{b}{b+a}.$$

**(b)** The error bound in Theorem 2.4 may be compared with the results obtained by Neubauer [14: Theorem 2.6] (also refer [13: Theorem 8.5]) and Schröter and Tautenhahn [12: Theorem 2] in the setting of Hilbert scales. In Hilbert scales setting the Tikhonov functional (1.2) is replaced by

$$x \mapsto \|Kx - \tilde{y}\|^2 + \alpha\|x\|_b^2 \qquad (x \in X_b).$$

The above map is a special case of (1.2) with $(L^*L)^{\frac{1}{2}} = T^b$, where $T$ is the operator which generates the Hilbert scale.

We recall that the bound obtained in [12: Theorem 2] is for the error in the Hilbert scale norm $\|\cdot\|_r$. Such error bound is also possible under the assumptions in Theorem 2.4. In fact, using the error bound in Theorem 2.4 and the interpolation inequality, it can be proved that

$$\|\hat{x} - \tilde{x}_\alpha\|_r \leq \kappa_1 \delta^\mu$$

with

$$\mu = \frac{2(a\nu + b) - r}{2(a\nu + b) + a}$$

and

$$\kappa_1 = 2\Big(\frac{1}{c}\Big)^{\frac{2b-r(2\nu+1)}{2(a\nu+b)+a}}\Big(\frac{\sqrt{\|u\|}}{d}\Big)^{\frac{2a+2r}{2(a\nu+b)+a}}.$$

Note that $r = 0$ corresponds to the result in Theorem 2.4.

**(c)** We observe that Theorem 2.4 holds if $\hat{x}$ belongs to the set

$$M_{\rho,\nu} = \Big\{x \in D(L^*L) : L^*Lx = (K^*K)^\nu u \text{ and } \|u\| \le \rho\Big\} \qquad \big(0 \le \nu \le \tfrac{1}{2}\big)$$

and the error bound obtained is order optimal for

$$\widetilde{M}_q = \big\{x \in X_q : \|x\|_q \le \rho\big\} \qquad \big(q = 2(a\nu + b)\big).$$

One may ask whether this rate is order optimal for $M_{\rho,\nu}$. The answer, in general, is not affirmative. In fact, under the assumptions (2.4) and (2.5), it can be proved that

$$e(M_{\rho,\nu}, \delta) \le \kappa\delta^p$$

where $\kappa$ and $p$ are as in Theorem 2.4. The rate for $M_{\rho,\nu}$ can be better than $O(\delta^p)$. To see this consider the case where $K^*K$ is injective and $T = (K^*K)^{-\frac{1}{2}}$ is the operator which generates the Hilbert scale $(X_s)_{s\in\mathbb{R}}$. Let $L$ be such that $L^*L = (K^*K)^{-t}$ for some $t > b$. Then it can be seen that $a = 1$ and

$$M_{\rho,\nu} = \big\{x \in D(L^*L) : \|x\|_{2t+2\nu} \le \rho\big\}.$$

Hence $O\big(\delta^{\frac{2t+2\nu}{2t+2\nu+1}}\big)$ is the order optimal rate for $e(M_{\rho,\nu}, \delta)$ whereas the rate in Theorem 2.4 is $O\big(\delta^{\frac{2b+2\nu}{2b+2\nu+1}}\big)$. Note that

$$\frac{2t+2\nu}{2t+2\nu+1} > \frac{2b+2\nu}{2b+2\nu+1}.$$

**(d)** We note that in Theorem 2.5, the result (i) is the well known optimal order result for ordinary Tikhonov regularization, and (iii) is the best rate obtained by Natterer [11] under an a priori choice of the parameter in the frame work of Hilbert scales, and later by Neubauer [14] under an a posteriori choice. In fact, the rates in (ii) and (iii) in Theorem 2.5 are the order optimal rates for the sets $\widetilde{M}_{2b}$ and $\widetilde{M}_{2b+a}$, respectively. Also, as expected since additional smoothness conditions are imposed, the estimates in (ii) and (iii) are of better order than the classical result in Theorem 2.2.

We observe that the error bound in Theorem 2.5/(ii) holds if $\hat{x}$ belongs to

$$\big\{x \in D(L^*L) : \|L^*Lx\| \le \rho\big\}$$

and the error bound in Theorems 2.1 and 2.2 are for the set $M_\rho$ which can also be written as

$$\big\{x \in D(L) : \|(L^*L)^{\frac{1}{2}}x\| \le \rho\big\}.$$

It is yet to investigate the question whether an order optimal result of the type Theorem 2.1 or a result of the form Theorem 2.2 can be proved for the set

$$\widehat{M}_{\rho,\mu} = \{x : \|(L^*L)^\mu x\| \leq \rho\}$$

for an arbitrary $\mu \geq 0$. Such a result is desirable since, in applications, one may not know precisely the smoothness properties of $\hat{x}$.

Of course, in the special case $L^*L = (K^*K)^{-t}$ $(t \geq b > 0)$ Theorem 2.4 does provide an error bound corresponding to the set $\widehat{M}_{\rho,\mu}$ with $1 \leq \mu \leq \frac{2b+1}{2b}$, for in this case

$$L^*L\hat{x} = (K^*K)^\nu u \qquad \text{if and only if} \qquad (L^*L)^\mu x = u$$

where

$$\mu = 1 + \frac{\nu}{t} \qquad \text{with } 0 \leq \nu \leq \frac{1}{2} \text{ and } t \geq b > 0.$$

**Acknowledgements.** The first version of this paper was written during my visit to Fachbereich Mathematik, Universität Kaiserslautern, Germany. I thank Professor Dr. Eberhard Schock for the invitation and also for the useful and lively discussions I had with him. Also, I thank the referee for many useful suggestions.

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Received 24.04.1998; in revised form 24.11.1998