Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 1, 143–155

Global Solution of Optimal Shape Design Problems

A. Fakharzadeh J. and J. E. Rubio

Abstract. We consider optimal shape design problems defined by pairs of geometrical elements and control functions associated with linear or nonlinear elliptic equations. First, necessary conditions are illustrated in a variational form. Then by applying an embedding process, the problem is extended into a measure-theoretical one, which has some advantages. The theory suggests the development of a computational method consisting of the solution of a finite-dimensional linear programming problem. Nearly optimal shapes and related controls can thus be constructed. Two examples are also given.

Keywords: *Embedding methods, optimal shape design, Radon measures, linear programming, optimal shape, optimal control, elliptic equations*

AMS subject classification: 49

1. Introduction

In general, most known methods of solving an optimal shape design problem associated with the solution of a (partial) differential equation are concerned with the numerical solution of the differential equation. The exception is the mapping method which maps the solution spaces of the differential equations in an optimal shape design problem on a fixed domain (see [4], for example). Also, all optimal shape design problems considered have been based on not more than one geometrical element (which, indeed, has usually been a domain); thus efforts have been directed to obtain an optimal element as the optimal solution.

The paper [1] shows a new approach, an embedding method, for solving an optimal shape design problem defined in polar coordinates, in which no partial differential equation is involved. We introduce there a new approach to attack an optimal shape design problem by transferring the problem into a new one in which positive Radon measures are involved. Moreover, Rubio in [7] applied the embedding method for solving a control system governed by an elliptic equation to find the global control for the described system. The history of these ideas can be find, for instance, in [6]. Based on these papers, here we introduce a method to solve a new and larger set of optimal shape design problems associated with elliptic equations; those are defined in terms of a pair of geometrical elements (a domain and its boundary) and a pair of control functions.

A. Fakharzadeh J.: Shahid Chamran University of Ahwaz, Dept. Math., Ahwaz, Iran; e-mail: science@www.dci.co.ir
J. E. Rubio: University of Leeds, Dept. Appl. Math. Studies, Leeds, LS2 9JT, UK

ISSN 0232-2064 / $ 2.50

2. Problem

Here, we take $J = [0, 2\pi]$, $J^0 = (0, 2\pi)$, $A = [0, 1]$ and $W \subset \mathbb{R}$ bounded, and suppose

$$r : J \longrightarrow A \quad \text{to be an absolutely continuous}$$
$$w : J \longrightarrow W \quad \text{to be a Lebesgue-measurable}$$

function. This pair of functions satisfies the differential equation

$$\dot{r}(\theta) = w(\theta) \equiv g(\theta, r, w) \qquad (\theta \in J^0).$$

Let $\Omega = J \times A$, $\omega = J \times A \times W$ and ∂D a smooth simple closed curve (simple in the sense of the Green Theorem (see, for instance, [10: p. 1030]) that ∂D is not cut itself between its initial and final points). Here ∂D is the boundary of the Lebesgue-measurable set $D \subset \mathbb{R}^2$ in the polar plane; we know that in polar coordinates, when $r \geq 0$ and $0 \leq \theta \leq 2\pi$, the curve $r = r(\theta)$ is simple, and these conditions are considered when defining Ω and ω. We also assume that ∂D contains a fixed point (θ_a, r_a) and is defined by the equation $r = r(\theta)$, $r(\theta) \in A$ for all $\theta \in J$. It is also supposed that D has a fixed area.

Let $u : \Omega \longrightarrow \mathbb{R}$, a differentiable and bounded function in $C^2(D)$ with first derivatives bounded in D, be a bounded solution of the elliptic problem with Neumann condition

$$\operatorname{div}(k(\theta, r)\nabla u) - f(\theta, r, u) = 0 \tag{1}$$
$$\nabla u \cdot \mathbf{n}|_{\partial D} = v \tag{2}$$

that takes values in a bounded set $U \subset \mathbb{R}$. Here

k is a positive function in $C^1(D)$

$f : \Omega \times U \longrightarrow \mathbb{R}$ is a bounded function in $C(\Omega \times U)$

$\mathbf{n}$ is the outward normal vector on ∂D

$v : J \longrightarrow V \subset \mathbb{R}$ is a bounded Lebesgue-measurable function (V bounded).

In this paper, (v, w) is considered as a pair of control functions, and (r, u) is regarded as a pair of trajectory functions in a classical shape design (or optimal control) problem.

Definition 1. The quadruple $(D, \partial D, u, v)$ defined above is called *admissible* if the elliptic problem (1) - (2) has a bounded solution on D. The set of all admissible quadruples is denoted by $\mathcal{F}$.

The aim of this work is to find a minimizer for the following optimal shape design problem over the set $\mathcal{F}$ by applying a method similar to that introduced in [1]. Let $f_0 \in C(\Omega)$ and $h_0 \in C(\omega)$ be two given continuous functions; they can be regarded as any suitable objective function in a related physical system, like energy, heat distribution in surface and so on. We seek the solution of the following

Optimization problem. Minimize

$$I(D, \partial D, u, v) = \int_D f_0(\theta, r, u, \nabla u)\, dr d\theta + \int_{\partial D} h_0(\theta, r, w, v)\, ds \qquad (3)_1$$

subject to

$$\left.\begin{array}{r} (D, \partial D, u, v) \in \mathcal{F} \\ \operatorname{div}(k(\theta, r)\nabla u) - f(\theta, r, u) = 0 \\ \nabla u \cdot \mathbf{n}|_{\partial D} = v. \end{array}\right\} \qquad (3)_2$$

3. The elliptic equation in variational problem

In general, it is difficult to identify a classical solution for the general case of the elliptic Neumann problem. Thus usually it has been tried to find a *weak* (or *generalized*) solution for it. So we change the elliptic problem into the variational form in the following proposition; we follow Mikhailov in [2: Chapter IV] to prove it.

Proposition 1. *Let u be the classical solution of problem* $(1) - (2)$. *Then we have the integral equality*

$$\int_D (k\nabla u \nabla \varphi + f\varphi) r\, dr d\theta - \int_{\partial D} k\varphi v\, ds = 0 \qquad (\varphi \in H^1(D)) \qquad (4)$$

where $H^1(D)$ is the Sobolev space of order 1 *on D.*

Proof. Multiplying (1) by a function $\varphi \in H^1(D)$ and then integrating over D, we obtain

$$\int_D \varphi \operatorname{div}(k\nabla u) r\, dr d\theta - \int_D \varphi f r\, dr d\theta = 0.$$

Because $\operatorname{div}(k\nabla u) = k\Delta u + \nabla u \nabla k$ (see, for instance, [2]), thus

$$\int_D \varphi k \Delta u\, r\, dr d\theta + \int_D \varphi \nabla u \nabla k\, r\, dr d\theta - \int_D \varphi f\, r\, dr d\theta = 0. \qquad (5)$$

Green's formula (see [2]) gives rise to

$$\int_D \varphi k \Delta u\, r\, dr d\theta = \int_{\partial D} \varphi k \frac{\partial u}{\partial n}\, ds - \int_D \nabla u \nabla(\varphi k)\, r\, dr d\theta. \qquad (6)$$

But $\nabla(\varphi k) = \varphi \nabla k + k \nabla \varphi$. Hence by considering (2) and applying (6) in (5), equality (4) is obtained ∎

Definition 2. A bounded function $u \in H^1(D)$ is called a *bounded weak solution of problem* (1) - (2) if it satisfies equality (4) for all function $\varphi \in H^1(D)$.

4. Metamorphosis

Generally, the minimization of (3) over $\mathcal{F}$ is not easy, even if the function u satisfies (4) instead of (1). The infimum may not be attained at any admissible quadruple. If this is the case, it is not possible for instance to write necessary conditions for this problem. We proceed then to transform it into a measure-theoretical form. Because $u \in H^1(D)$ is bounded and the first order partial derivatives of u are also bounded, then ∇u is a bounded real-valued function. Let ∇u take values in the bounded set U'. Then we define $\Omega' = \Omega \times U \times U'$ and $\omega' = \omega \times V$ (that $\omega = J \times A \times W$).

An admissible quadruple $(D, \partial D, u, v) \in \mathcal{F}$ introduces the two following functionals:

Any bounded weak solution u of problem (1) - (2) determines a linear, bounded and positive functional

$$u_D : F \longrightarrow \int_D F(\theta, r, u, \nabla u)\, dr d\theta \tag{7}$$

on the space $C(\Omega')$. Also, a control function v defined on ∂D and satisfying (2) introduces a linear, bounded and positive functional

$$v_{\partial D} : G \longrightarrow \int_J G(\theta, r, w, v)\, d\theta \qquad \left(\equiv \int_{\partial D} \frac{1}{\sqrt{r^2 + w^2}}\, G\, ds \right) \tag{8}$$

on the space $C(\omega')$. On the base of the Riesz representation theorem (see [9: Theorem 2.14]) the above functionals represent two positive Radon measures λ_u and σ_v, respectively (see, for instance, [1: Proposition 3.1] and [5: Chapter 1]), so that

$$\begin{aligned} \lambda_u(F) &= \int_D F(\theta, r, u, \nabla u)\, dr d\theta \equiv u_D(F) \qquad (F \in C(\Omega')) \\ \sigma_v(G) &= \int_J G(\theta, r, w, v)\, d\theta \equiv v_{\partial D}(G) \qquad (G \in C(\omega')). \end{aligned} \tag{9}$$

Thus each admissible quadruple $(D, \partial D, u, v) \in \mathcal{F}$ can be considered as a pair of measures (λ_u, σ_v) in the appropriate subset of $\mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega')$, say $\mathcal{F}$ again. Hence there exists the transformation

$$(D, \partial D, u, v) \in \mathcal{F} \;\longrightarrow\; (\lambda_u, \sigma_v) \in \mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega').$$

As we showed in [1: Proposition 3.2], this transformation is injective. Hence someone may think that nothing is changed and the same difficulties as before (existence of the optimal pair in $\mathcal{F}$ and so on) still remain. So, we will extend the underlying space of the problem: Instead of seeking the optimal pair of $(\lambda_u^*, \sigma_v^*)$ in the set $\mathcal{F}$, we look for the minimizer of the functional

$$i : (\lambda, \sigma) \in \mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega') \longrightarrow \lambda\big(f_0\big) + \sigma\big(h_0 \sqrt{r^2 + w^2}\big)$$

in a subset of $\mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega')$ defined by some linear equalities in terms of the properties of admissible quadruples, which will be explained later.

5. Conditions

According to the new formulation and considering the properties of the weak solution, an admissible pair of measures (λ, σ) must satisfy

$$\lambda(F_\varphi) + \sigma(G_\varphi) = 0 \qquad (\varphi \in H^1(D)) \tag{10}$$

where

$$F_\varphi = rk\nabla u \nabla\varphi + rf\varphi \qquad \text{and} \qquad G_\varphi = -k\varphi v\sqrt{r^2 + w^2}.$$

Below we shall apply the measures λ and σ to functions of the variables (θ, r) and $(\theta, r, \dot{r})$, respectively, in relationships embodying the geometrical features of the problem. Thus, the measures here will actually be the projection of the measures λ and σ in the appropriate spaces; we shall use the same names for these projections, λ and σ.

The admissibility of the curve ∂D (and hence the set D) has been characterized in [1: Section 3]. We assumed there that $\dot{r}(\theta) = w(\theta) \equiv g(\theta, r, w)$ $(\theta \in J^0)$ and $\omega = J \times A \times W$ where $w \in W$, $W \subset \mathbb{R}$ bounded. Let B be an open ball in $\mathbb{R}^2$ containing $J \times A$, and denote by $C'(B)$ the space of real-valued continuously differentiable functions on B such that they and their first derivatives are bounded on B. For $\phi \in C'(B)$ we define

$$\phi^g(\theta, r, w) = \phi_r(\theta, r)w + \phi_\theta(\theta, r) \qquad ((\theta, r, w) \in \omega). \tag{11}$$

The function ϕ^g is in the space $C(\omega)$ and

$$\int_J \phi^g(\theta, r(\theta), w(\theta))\, d\theta = \int_0^{2\pi} \dot{\phi}(\theta, r(\theta))\, d\theta \equiv d_\phi \qquad (\phi \in C'(B)). \tag{12}$$

We now consider a special case of (11). Let $\mathcal{D}(J^0)$ be the space of infinitely differentiable real-valued functions with compact support in J^0 and define

$$\psi^g(\theta, r, w) = r(\theta)\psi'(\theta) + w(\theta)\psi(\theta). \tag{13}$$

Then

$$\int_J \psi^g(\theta, r(\theta), w(\theta))\, d\theta = r(2\pi)\psi(2\pi) - r(0)\psi(0) \qquad (\psi \in \mathcal{D}(J^0)). \tag{14}$$

Since the function ψ has compact support in J^0, $\psi(0) = \psi(2\pi) = 0$. It is important to single out this special case of (12), because later on, when we want to consider the approximation problem, it will be necessary to include some of the functions in $\mathcal{D}(J^0)$.

The same situation arises for another special choice of functions in $C'(B)$. Put

$$\phi(\theta, r, w) = f(\theta) \qquad ((\theta, r, w) \in \omega). \tag{15}$$

Then $\phi^g(\theta, r, w) = \dot{f}(\theta)$ $((\theta, r, w) \in \omega)$ also is a function of θ only. We are led thus to consider a subset of $C(\omega)$, to be denoted by $C_1(\omega)$, of those functions in this space which depend only on the variable θ. Thus

$$\int_J f(\theta, r(\theta), w(\theta))\, d\theta = a_f \qquad (f \in C_1(\omega)) \tag{16}$$

where a_f is the integral of $f(\cdot, r, u)$ over $[0, 2\pi]$, independent of r and u.

Thus, the properties of an admissible pair have been shown by sets of equalities in (12), (14) and (16), in the classical formulation of the optimal shape design problem. So we have

$$\left.\begin{aligned} \sigma(\phi^g) &= d_\phi \qquad && (\phi \in C'(B)) \\ \sigma(\psi^g) &= 0 && (\psi \in \mathcal{D}(J^0)) \\ \sigma(f) &= a_f && (f \in C_1(\omega)). \end{aligned}\right\} \tag{17}$$

The simple and closed curve ∂D is the boundary of D. This fact introduces a relation between the measures λ and σ. In [1: Section 6], this relationship has been considered by computing the inside measure μ in terms of the boundary measure ν by use of a special function. Here we are going to do this in another way. Let $\rho, \tau \in C^1(\Omega)$. Then by the Stokes theorem we have

$$\int_D \Big(\frac{\partial}{\partial r}(r\rho) - \frac{\partial \tau}{\partial \theta}\Big) dr d\theta = \int_J (\tau w + \rho r)\, d\theta.$$

Therefore

$$\lambda\Big(\rho + r\,\frac{\partial \rho}{\partial r} - \frac{\partial \tau}{\partial \theta}\Big) - \sigma(\tau w + \rho r) = 0 \qquad (\rho, \tau \in C^1(\Omega)). \tag{18}$$

Moreover, the definition of the functional u_D shows that, for any $(\theta, r, u, t) \in \Omega'$, there is a relation between the variables $u \in U$ and $t \in U'$. These variables are not independent from each other and this dependency should be regarded in the determination of the measures λ and σ. It is also very important to regard this fact in numerical examples when we identify the variables u and t just by some (finite) values (see Example). By use of the Green formula (see [2: p. 104]) this relation can be come into account as

$$\int_D (u\Delta\varphi + \nabla u \nabla \varphi)\, r\, dr d\theta = \int_{\partial D} \varphi v\, ds \qquad (\varphi \in H^1(D)).$$

Thus we have the relation

$$\lambda(ru\Delta\varphi + r\nabla u \nabla\varphi) = \sigma(\varphi v\sqrt{r^2 + w^2}) \qquad (\varphi \in H^1(D)). \tag{19}$$

As a result, to find the minimizer of I over $\mathcal{F}$ in (3), one can search for the minimizer of the functional

$$(\lambda, \sigma) \longrightarrow \lambda(f_0) + \sigma(h_0\sqrt{r^2 + w^2})$$

over a subset Q of $\mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega')$ defined by all pairs (λ, σ) which satisfied conditions (10) and (17) - (19). Thus, instead of solving problem (3), we look for the minimizer of the following new problem over Q.

Problem. Minimize

$$i(\lambda, \sigma) = \lambda(f_0) + \sigma(h_0\sqrt{r^2 + w^2}) \tag*{$(20)_1$}$$

subject to

$$\left.\begin{aligned}
\sigma(\phi^g) &= d_\phi && (\phi \in C'(B)) \\
\sigma(\psi^g) &= 0 && (\psi \in \mathcal{D}(J^0)) \\
\sigma(f) &= a_f && (f \in C_1(\omega)) \\
\lambda(F_\varphi) + \sigma(G_\varphi) &= 0 && (\varphi \in H^1(D)) \\
\lambda\Big(\rho + r\,\frac{\partial\rho}{\partial r} - \frac{\partial\tau}{\partial\theta}\Big) - \sigma(\tau w + \rho r) &= 0 && (\rho, \tau \in C^1(\Omega)) \\
\lambda(ru\Delta\varphi + r\nabla u\nabla\varphi) &= \sigma(\varphi v\sqrt{r^2 + w^2}) && (\varphi \in H^1(D)).
\end{aligned}\right\} \qquad (20)_2$$

6. Existence and approximation

The following theorem states that the above problem has a minimizer. To prove the theorem the reader can follow Rubio [5: p. 16/Theorem II.1] or [1: Theorem 4.1].

Theorem 1. *There exists an optimal pair of measures* (λ^*, σ^*) *in the set* $Q \subset \mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega')$ *for which*

$$i(\lambda^*, \sigma^*) \leq i(\lambda, \sigma) \qquad ((\lambda, \sigma) \in Q)$$

holds.

We remind the reader that, since the set $\mathcal{F}$ of admissible quadruples can be considered, by means of the mentioned injective transformation, as a subset of Q, then $\inf_{\mathcal{F}} I(D, \partial D, u, v) \geq \inf_Q i(\lambda, \sigma)$. Thus, in (20) the minimization is global. So in the non-classical form of the optimal shape design problem (problem (20)), the global minimizer will be illustrated.

All the equations in problem (20) are linear in their arguments λ and σ. It is an infinite linear program; the number of equations and the dimension of the underlying space are infinite.

We are going now to approximate the solution of the problem by the solution of an appropriate finite linear programming problem so that not only the number of equations is finite, but the underlying space on which minimization takes place on it, will be finite-dimensional. This important can be happened by use of a total set in each space $H^1(D), C^1(\Omega), C_1(\omega), \mathcal{D}(J^0)$ and $C'(B)$.

The total sets in the spaces $C_1(\omega), \mathcal{D}(J^0)$ and $C'(B)$ were introduced in [1: Section 5] and [5: pp. 52 - 53]. Here we identify those for the other spaces. Let P be the $\mathbb{C}$-vector space with the basis $\{Z^n, \bar{Z}^n : Z \in \Omega\}$. Under multiplication, P is an algebra and satisfies the conditions of the Stone-Weierstrass Theorem (see [8: Theorem 7.33]). Hence it is dense in $C(\Omega)$. Moreover, since each $Z \in \Omega$ can be rewritten as $Z = r(\cos\theta + i\sin\theta)$,

$$\left.\begin{aligned}
Z^n &= r^n(\cos n\theta + i\sin n\theta) \\
\bar{Z}^n &= r^n(\cos n\theta - i\sin n\theta)
\end{aligned}\right\}$$

So the set of functions $\{r^n \cos n\theta, r^n \sin n\theta\}_{n\geq 1}$ is a base for P. Hence the set of functions $\{\varphi_n\}_{n\geq 1}$ with $\varphi_n = r^n \cos n\theta$ or $\varphi_n = r^n \sin n\theta$ is dense in $C^1(\Omega) \subset C(\Omega)$. By [2: Chapter III/Theorem 3] $C^1(\Omega)$ is dense in $H^1(D)$. Therefore the set of functions $\{\varphi_n\}_{n\geq 1}$ is total in $H^1(D)$.

Now consider the following problem which results from (20) just by choosing a finite number of functions in the mentioned total sets.

Problem. Minimize

$$i(\lambda,\sigma) = \lambda(f_0) + \sigma\big(h_0\sqrt{r^2+w^2}\big) \tag*{$(21)_1$}$$

subject to

$$\left.\begin{aligned}
\sigma(\phi_k^g) &= d_{\phi_k} && (k=1,2,\ldots,M_1)\\
\sigma(\chi_l) &= 0 && (l=1,2,\ldots,M_2)\\
\sigma(f_s) &= a_s && (s=1,2,\ldots,M_3)\\
\lambda(F_i)+\sigma(G_i) &= 0 && (i=1,2,\ldots,M_4)\\
\lambda(D_j)+\sigma(E_j) &= 0 && (j=1,2,\ldots,M_5)\\
\lambda(H_r)+\sigma(I_r) &= 0 && (r=1,2,\ldots,M_6).
\end{aligned}\right\} \tag*{$(21)_2$}$$

Here

$$\left.\begin{aligned}
D_j &= ru\Delta\varphi_j + r\nabla u\nabla\varphi_j\\
E_j &= -(\varphi_j v\sqrt{r^2+w^2})\\
F_i &= F_{\varphi_i}\\
G_i &= G_{\varphi_i}\\
H_r = H_{ij} &= \varphi_i + r\frac{\partial\varphi_i}{\partial r} - \frac{\partial\varphi_j}{\partial\theta}\\
I_r = I_{ij} &= -(\varphi_j w + \varphi_i r).
\end{aligned}\right\} \tag{22}$$

Then we have the following proposition which shows that the solution of problem (20) can be approximated by the solution of problem (21). For proof one can follow Rubio in [5: p. 25/Proposition III.1].

Proposition 2. *For positive integers M_i $(i=1,\ldots,6)$ let $Q_{M'}$ be the set of pairs $(\lambda,\sigma) \in \mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega')$ which satisfy the constraints of problem* (21)*. If $M_i \to \infty$, then*

$$\inf_{Q_{M'}} i(\lambda,\sigma) \longrightarrow \inf_{Q} i(\lambda,\sigma).$$

In other words, the solution of problem (21) *tends to the solution of problem* (20).

We have already limited the number of constraints of problem (20) in the first stage of approximation. But the underlying space $Q_{M'}$ is still infinite-dimensional. Let (λ^*,σ^*) be the optimal solution of problem (21) (the existence of (λ^*,σ^*) can be resulted from Theorem 1). By applying [5: Theorem A.5] one can obtain

$$\lambda^* = \sum_{n=1}^{N} \alpha_n^*\delta(Z_n^*) \qquad \text{and} \qquad \sigma^* = \sum_{m=1}^{M} \beta_m^*\delta(z_m^*)$$

that for each $n = 1, 2, \ldots, N$ and $m = 1, 2, \ldots, M$ we have $\alpha_n^* \geq 0$ and $\beta_m^* \geq 0$, and also that Z_n^* and z_m^* belong to a dense subsets of Ω' and ω', respectively. Here M and N are positive integers and $\delta(z)$ is a unitary atomic measure with support the singleton point set $\{z\}$ ∎

In the next stage of approximation, let $E_{\Omega'}$ and $E_{\omega'}$ be two countable dense subsets of Ω' and ω', respectively. Then (as a result of [5: Proposition III.3]) the measures λ^* and σ^* can be approximated by

$$\lambda = \sum_{n=1}^{N} \alpha_n \delta(Z_n) \qquad \text{and} \qquad \sigma = \sum_{m=1}^{M} \beta_m \delta(z_m)$$

where $Z_m \in E_{\Omega'}$ and $z_n \in E_{\omega'}$. This result suggests that problem (21) can be approximated by the following linear programming one which the points Z_n and z_m are chosen from a finite subset of a countable dense subsets in the appropriate space by putting discretization on Ω' and ω'. So the only unknowns are the coefficients α_n and β_m.

It is assumed in (23) that

$$Z_n = (\theta_n, r_n, u_n, t_n) \in \Omega' \qquad \text{and} \qquad z_m = (\theta_m, r_m, w_m, v_m) \in \omega'.$$

Moreover, the last equation in (23) represents the area condition as explained in [1].

Problem. Minimize

$$\sum_{n=1}^{N} \alpha_n f_0(Z_n) + \sum_{m=1}^{M} \beta_m h_0(z_m) \sqrt{r_m^2 + w_m^2} \tag{23_1}$$

subject to

$$\left.\begin{aligned}
\alpha_n &\geq 0 & &(n = 1, 2, \ldots, N) \\
\beta_m &\geq 0 & &(m = 1, 2, \ldots, M) \\
\sum_{m=1}^{M} \beta_m \phi_k^g(z_m) &= d_{\phi_k} & &(k = 1, 2, \ldots, M_1) \\
\sum_{m=1}^{M} \beta_m \chi_l(z_m) &= 0 & &(l = 1, 2, \ldots, M_2) \\
\sum_{m=1}^{M} \beta_m f_s(z_m) &= a_s & &(s = 1, 2, \ldots, M_3) \\
\sum_{n=1}^{N} \alpha_n F_i(Z_n) + \sum_{m=1}^{M} \beta_m G_i(z_m) &= 0 & &(i = 1, 2, \ldots, M_4) \\
\sum_{n=1}^{N} \alpha_n D_j(Z_n) + \sum_{m=1}^{M} \beta_m E_j(z_m) &= 0 & &(j = 1, 2, \ldots, M_5) \\
\sum_{n=1}^{N} \alpha_n H_r(Z_n) + \sum_{m=1}^{M} \beta_m I_r(z_m) &= 0 & &(r = 1, 2, \ldots, M_6) \\
\sum_{m=1}^{M} \beta_m \left(\frac{1}{2} r_m^2\right) &= \text{given area.}
\end{aligned}\right\} \tag{23_2}$$

7. Numerical examples

For the following two examples, we choose $f_0 = 0$, $h_0 = v^2$, $f = u(u - 0.5)$ and $k(\theta, r) = 1$. We remind the reader that in polar coordinates

$$\nabla\varphi = \frac{\partial\varphi}{\partial r}u_r + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}u_\theta \qquad (\varphi \in H^1(D))$$

(see [3]). Also, it is supposed that $\nabla u = u_1 u_r + u_2 u_\theta$ where $u_1 \in U_1$ and $u_2 \in U_2$.

Example 1. For this example we choose

$$W = [-0.3, 0.3], \qquad V = [-10, 10], \qquad U = [-5, 5], \qquad U' = U_1 \times U_2 = [-15, 15]^2$$

and discretized ω' with $M = 10^4$ nodes $z = (\theta, r, w, v)$ by selecting:

10 angles for θ as $\frac{\pi}{10}, \frac{3\pi}{10}, \ldots, \frac{19\pi}{10}$ on $J = [0, 2\pi]$

10 values for r as $0, \frac{1}{9}, \frac{2}{9}, \ldots, 1$ in A

10 values for w as $-0.3, \frac{-21}{90}, \frac{-15}{90}, \ldots, 0.3$ in W

10 values for v as $-10, \frac{-70}{9}, \frac{-50}{9}, \ldots, 10$ in V.

Figure 1: The suboptimal control function w

And also discretized Ω' with $M = 10^5$ nodes $Z = (\theta, r, u, u_1, u_2)$ as:

10 values in each sets J and A for θ and r as above

10 values for u as $-5, \frac{-35}{9}, \frac{-25}{9}, \ldots, 5$ in U

10 values for u_1 and u_2 as $-15, \frac{-105}{9}, \frac{-75}{9}, \ldots, 15$ in each sets U_1 and U_2.

We also choose the area as 0.6, $(0, 0.5) = (2\pi, 0.5)$ as the fixed point, $M_1 = 2, M_2 = 8, M_3 = 10, M_4 = 5, M_5 = 2$ and $M_6 = 2$. Then the linear program (23) was run with 30 equations and 110000 variables. We applied the $E04MBF$ NAG-Routine to solve the problem. The optimal value of performance criterion was 274.23683327352.

Based on the equation $w(\theta) = \frac{dr}{d\theta}$ (as Rubio did in [5]), the suboptimal control function w and 20 different points of the boundary of the (approximate) optimal shape were obtained. We remind the reader that by increasing the number of equations, the number of points will be increased.

Figure 2: Boundary points of the nearly optimal shape

Regardless the fault of changing the coordinates, the suboptimal control w and the boundary points of the nearly optimal shape are plotted in Figures 1 and 2 in Cartesian coordinates. To show the simplity of the nearly optimal shape, the points are linked to gather with segments. Indeed, Figure 2 can be regarded as an approximation of the nearly optimal shape by broken lines (so the area is less than it should be). Moreover, one may use the curve fitting methods, increase the number of equations, or use the other methods (like we use in [1], for instance) to get the shape.

Example 2. For the second example, to decrease the optimal valve of the performance criteria, we choose everything the same as Example 1, just $V = U = U_1 = U_2 = [-1, 1]$. So for discretization 10 values as $-1, \frac{-7}{9}, \frac{-5}{9}, \ldots, 1$ in each set were selected. As a result, the optimal value of the objective function was decreased to 2.8021342962670.

Figure 3: The suboptimal control function w

By the same limitation as previous example, the suboptimal control w and the boundary points of the nearly optimal shape are plotted in Figures 3 and 4 in Cartesian coordinates.

Figure 4: Boundary points of the nearly optimal shape

8. Conclusions

In this paper, we have shown that the embedding method is applicable to solve a large set of optimal shape design problems which are defined in terms of a pair of geometrical elements and control functions and moreover they are involved with a solution of linear or nonlinear elliptic partial differential equations with a boundary condition. The method changes the classical problem (in polar coordinates) into a nonclassical measure-theoretical one in a bigger space, which is defined in term of a pair of positive Radon measures. These measures satisfy some necessary linear conditions and hence the computational method for the new problem is much easier. Also the method allow us to approximate the optimal solution of the new problem in term of the solution of an appropriate finite-dimensional linear program. In this manner the optimal shape for the original problem can be identified from the results of the finite linear program.

References

[1] Fakharzadeh, A. J. and J. E. Rubio: *Shapes and Measures.* IMA J. Math. Control and Inf. (to appear).

[2] Mikhailov, V. P.: *Partial Differential Equations.* Moscow: MIR Publ. 1978.

[3] Philips, H. B.: *Vector Analysis.* New York – London: John Wiley & Sons 1960.

[4] Pironneau, O.: *Optimal Shape Design for Elliptic System.* New York et a.: Springer-Verlag 1983.

[5] Rubio, J. E.: *Control and Optimization: The Linear Treatment of Nonlinear Problems.* Manchester: Univ. Press 1086.

[6] Rubio, J. E.: *Modern trends in the Calculus of Variations and Optimal Control Theory.* In: Vortragsauszüge, Mathematiker-Kongress 1990 (ed.: R. Geise). Berlin: Math. Ges. 1990, pp. 155 – 163.

[7] Rubio, J. E.: *The global control of nonlinear elliptic equation.* J. Franklin Inst. 330 (1993), 29 – 35.

[8] Rudin, W.: *Principles of Mathematical Analysis* (2nd ed.). New York: McGraw-Hill 1976.

[9] Rudin, W.: *Real and Complex Analysis* (2nd ed.). New Delhi: Tata McGraw-Hill 1983.

[10] Thomas, G. B. and R. L. Finny: *Calculus and Analytic Geometry* (7th ed.). New York: Addison-Wesley 1988.

Received 04.04.1997; in revised form 21.09.1998