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Oblique Derivative Problems for Elliptic Systems of Second Order Equations in Infinite Domains

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Dedicated to Prof. Dr. L. von Wolfersdorf on the occasion of his 65th birthday

Abstract. There are many problems in mechanics and physics, the mathematical models of which are some boundary value problems for nonlinear elliptic systems of first and second order equations in multiply connected domains including infinity. In this paper, we discuss oblique derivative problems for systems of second order equations.

Keywords: *Complex partial differential equations, oblique derivative problems*

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1. Formulation of the problems

Let D be an $(N+1)$-connected domain in $\mathbb{C}$ including infinity, with boundary $\Gamma = \cup_{j=0}^{N}\Gamma_j \in C_\alpha^2$ $(0 < \alpha < 1)$. Without loss of generality we may assume that D is a circular domain in $\{z \in \mathbb{C} : |z| > 1\}$, whose boundary consists of $N+1$ circles $\Gamma_0 = \Gamma_{N+1} = \{z \in \mathbb{C} : |z| = 1\}$ and $\Gamma_j = \{z \in \mathbb{C} : |z - z_j| = \gamma_j\}$ $(j = 1, \ldots, N)$, where $z_j \in \mathbb{C}$ are given points, $0 < \gamma_j \in \mathbb{R}$ are given constants (see, e.g., [2, 3]).

We consider the nonlinear elliptic system of second order equations in complex form

$$\left.\begin{aligned} w_{z\bar z} &= F(z, w, w_z, \overline{w}_z, w_{zz}, \overline{w}_{zz}) \\ F &= Q_1 w_{zz} + Q_2 \overline{w}_{zz} + A_1 w_z + A_2 \overline{w}_z + A_3 w + A_4 \\ Q_j &= Q_j(z, w, w_z, \overline{w}_z, w_{zz}, \overline{w}_{zz}) \quad (j = 1, 2) \\ A_j &= A_j(z, w, w_z, \overline{w}_z) \quad (j = 1, 2, 3, 4). \end{aligned}\right\} \tag{1.1}$$

Suppose that (1.1) satisfies the following conditions $(\mathrm{C})_1$ - $(\mathrm{C})_3$.

$(\mathrm{C})_1$ $Q_j(z, w, w_z, \overline{w}_z, U, V)$ and $A_j(z, w, w_z, \overline{w}_z)$ are continuous in $w, w_z, \overline{w}_z \in \mathbb{C}$ for almost every $z \in D$ and all $U, V \in \mathbb{C}$, and $Q_j = 0$ and $A_j = 0$ for $z \notin D$.

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(C)$_2$ $Q_j(z,w,w_z,\overline{w}_z,U,V)$ and $A_j(z,w,w_z,\overline{w}_z)$ are measurable in $z\in D$ for all continuously differentiable functions $w=w(z)$ on $\overline{D}$ and all measurable functions $U,V\in L_{p_0,2}(\overline{D})$, and satisfy

$$L_{p,2}\big[A_j(z,w,w_z,\overline{w}_z),\overline{D}\big]\le k_{j-1} \qquad (j=1,...,4) \tag{1.2}$$

in which p_0 and p with $2<p_0\le p$ and $k_j\ \ (j=0,1,2,3)$ are non-negative constants.

(C)$_3$ System (1.1) satisfies for any functions $w\in C^1(\overline{D})$ and constants $U^j,V^j\in\mathbb{C}\ \ (j=1,2)$ the uniform ellipticity condition

$$\begin{aligned}\big|F\big(z,w,w_z,\overline{w}_z,U^1,V^1\big)-F\big(z,w,w_z,\overline{w}_z,U^2,V^2\big)\big|\\ \le q_1|U^1-U^2|+q_2|V^1-V^2|\end{aligned} \tag{1.3}$$

for almost every point $z\in D$, where $q_1\ge 0$ and $q_2\ge 0$ are constants with $q_1+q_2<1$.

Now we formulate the oblique derivative problem, i.e. the Poincaré boundary value problem as follows (compare [5]).

Problem (P). In the domain D, find a solution $w=w(z)$ of system (1.1), which is continuously differentiable on $\overline{D}$ and satisfies the boundary condition

$$\left.\begin{aligned}\mathrm{Re}\,\big[\overline{\lambda_1(z)}w_z+a_{11}(z)w\big]=a_{12}(z)\\ \mathrm{Re}\,\big[\overline{\lambda_2(z)}w_{\bar z}+a_{21}(z)w\big]=a_{22}(z)\end{aligned}\right\}\qquad (z\in\Gamma) \tag{1.4}$$

where λ_j with $|\lambda_j(z)|=1$ and $a_{jk}\ \ (j,k=1,2)$ are known functions, which satisfy the conditions

$$C_\alpha[\lambda_j,\Gamma]\le k_0,\qquad C_\alpha[a_{j1},\Gamma]\le k_1,\qquad C_\alpha[a_{j2},\Gamma]\le k_4 \tag{1.5}$$

in which α with $\frac{1}{2}<\alpha<1$ and k_0,k_1,k_4 are non-negative constants.

Denote

$$K_j=\frac{1}{2\pi}\Delta_\Gamma\arg\lambda_j(z)\qquad (j=1,2). \tag{1.6}$$

$[K_1,K_2]$ is called the *index* of Problem (P). When $K_1<0$ and $K_2<0$, then Problem (P) may not be solvable. Further, when $K_1\ge 0$ and $K_2\ge 0$, then the solution of Problem (P) is not necessarily unique. Hence we consider the well-posedness of Problem (P) with modified boundary conditions (see [1, 4]).

Problem (Q). Find a continuous solution $[w,U,V]$ of the complex system

$$\left.\begin{aligned}U_{\bar z}&=F(z,w,U,V,U_z,V_z)\\ F&=Q_1U_z+Q_2V_z+A_1U+A_2V+A_3w+A_4\\ V_{\bar z}&=\bar U_z\end{aligned}\right\} \tag{1.7}$$

satisfying the boundary condition

$$\mathrm{Re}\,\big[\overline{\lambda_j(z)}U_j(z)+a_{j1}(z)w(z)\big]=a_{j2}(z)+h_j(z)\qquad (j=1,2;\ z\in\Gamma) \tag{1.8}$$

and the relation

$$w(z) = -\int_1^z \left[\frac{U(\zeta)}{\zeta^2}\,d\zeta + \frac{\overline{V(\zeta)}}{\bar{\zeta}^2}\,d\bar{\zeta} - \sum_{m=1}^{N} \frac{d_m z_m}{\zeta(\zeta - z_m)}\,d\zeta\right] + c_0 \tag{1.9}$$

where $U_1 = U$, $U_2 = V$ and d_m are appropriate real constants such that the function determined by the integral in (1.9) is single-valued in D, and the undetermined functions h_j are of the form

$$h_j(z) = \begin{cases} 0 & \text{if } z \in \Gamma \quad (K_j \geq N) \\ h_{jk} & \text{if } z \in \Gamma_k \quad \begin{pmatrix} k = 1, \ldots, N - K_j \\ 0 \leq K_j < N \end{pmatrix} \\ 0 & \text{if } z \in \Gamma_k \quad \begin{pmatrix} k = N - K_j + 1, \ldots, N + 1 \\ 0 \leq K_j < N \end{pmatrix} \\ h_{jk} & \text{if } z \in \Gamma_k \quad \begin{pmatrix} k = 1, \ldots, N \\ K_j < 0 \end{pmatrix} \\ h_{j0} + \operatorname{Re} \sum_{m=1}^{-K_j - 1} (h_{jm}^{+} + i h_{jm}^{-})\, z^m & \text{if } z \in \Gamma_0 \quad (K_j < 0), \end{cases} \tag{1.10}$$

Here h_{jk} and $h_{jm}^{\pm}$ are unknown real constants to be determined appropriately. In addition, for $K_j \geq 0$ $(j = 1, 2)$ the solution w is assumed to satisfy the point conditions

$$\left.\begin{aligned} &\operatorname{Im}\big[\overline{\lambda_j(a_k)}\, U_j(a_k) + a_{j1} w(a_k)\big] = b_{jk} \quad (j = 1, 2;\ k \in J_j) \\ &J_j = \begin{cases} \{1, \ldots, 2K_j - N + 1\} & \text{if } K_j \geq N \\ \{N - K_j + 1, \ldots, N + 1\} & \text{if } 0 \leq K_j < N \end{cases} \end{aligned}\right\} \tag{1.11}$$

where $a_k \in \Gamma_k$ $(k = 1, \ldots, N)$ and $a_k \in \Gamma_0$ $(k = N + 1, \ldots, 2K_j - N + 1,$ with $K_j \geq N$ for $j = 1, 2)$ are distinct points and b_{jk} are all real constants satisfying the conditions

$$\sum_{j=1,2;\, k \in J_j} |b_{jk}| \leq k_5 \tag{1.12}$$

with a non-negative constant k_5 such that $|c_0| \leq k_5$.

Problem (Q)$_0$. This is a special case of Problem (Q), namely with $A_4 = 0$, $a_{j2} = 0$, $b_{jk} = 0$ $(j = 1, 2;\ k \in J_j)$ and $c_0 = 0$.

In order to prove the uniqueness of solutions for Problem (Q), we need to add the condition that for any functions $U^j, V^j, w^j \in \widetilde{C}(\overline{D})$ $(j = 1, 2)$ with $U_z^1, V_z^1 \in L_{p_0,2}(\overline{D})$ the equality

$$\begin{aligned} &F\big(z, w^1, U^1, V^1, U_z^1, V_z^1\big) - F\big(z, w^2, U^2, V^2, U_z^1, V_z^1\big) \\ &\qquad = \tilde{A}_1(U^1 - U^2) + \tilde{A}_2(V^1 - V^2) + \tilde{A}_3(w^1 - w^2) \end{aligned} \tag{1.13}$$

holds in almost every point $z \in D$, where $L_{p_0,2}[\tilde{A}_j, \overline{D}] < \infty$ $(j = 1, 2, 3)$.

2. A priori estimates for solutions of problem (Q)

In oder to prove the solvability of Problem (Q), we need to give some estimates of its solutions.

Theorem 2.1. *Suppose that Conditions* (C)$_1$ - (C)$_3$ *hold and the constants* q_2 *and* k_1, k_2 *in* $(1.2), (1.3)$ *and* (1.5) *are small enough. Then any solution* $[w, U, V]$ *of Problem* (Q) *satisfies the estimates*

$$\left.\begin{aligned} &L_1 = L(U) = C_\beta[U, \overline{D}] + L_{p_0,2}[|U_{\bar z}| + |U_z|, \overline{D}] \le M_1 \\ &L_2 = L(V) \le M_1 \end{aligned}\right\} \tag{2.1}$$

and

$$S = S(w) = C^1_\beta[w, \overline{D}] + L_{p_0,2}[|w_{z\bar z}| + |w_{zz}| + |\overline{w}_{zz}|, \overline{D}] \le M_2 \tag{2.2}$$

where $\beta = \min(\alpha, 1 - \frac{2}{p_0})$, p_0 *with* $2 < p_0 \le p$, $M_j = M_j(q_0, p_0, k, \alpha, K, D)$ $(j = 1, 2;\ k = (k_0, \ldots, k_5))$ *are non-negative constants and* $K = (K_1, K_2)$.

Proof. Let the solution $[w, U, V]$ of Problem (Q) be substituted into system (1.7), the boundary conditions (1.8) and (1.11), and relation (1.9). It is clear that (1.7) and (1.8) can be rewritten in the form

$$\left.\begin{aligned} &U_{\bar z} - Q_1 U_z - A_1 U = A \\ &A = Q_2 V_z + A_2 V + A_3 w + A_4 \\ &U_{\bar z} = \overline{V}_z \end{aligned}\right\} \quad \text{in } D \tag{2.3}$$

and

$$\left.\begin{aligned} &\mathrm{Re}[\overline{\lambda_j(z)} U_j(z)] = r_j(z) + h_j(z) \\ &\qquad \text{with } r_j(z) = a_{j2}(z) - \mathrm{Re}[a_{j1}(z) w(z)] \\ &\mathrm{Im}[\overline{\lambda_j(a_k)} U_j(a_k)] = s_{jk} \\ &\qquad \text{with } s_{jk} = b_{jk} - \mathrm{Im}[\overline{\lambda_j(a_k)}(a_k)] \end{aligned}\right\} \quad (j = 1, 2;\ k \in J_j;\ z \in \Gamma) \tag{2.4}$$

where A and r_j, s_{jk} satisfy the inequalities

$$\begin{aligned} L_{p_0,2}[A, \overline{D}] &\le q_2 L_{p_0,2}[V_z, \overline{D}] + L_{p_0,2}[A_2, \overline{D}] C[V, \overline{D}] \\ &\qquad + L_{p_0,2}[A_3, \overline{D}] C[w, \overline{D}] + L_{p_0,2}[A_4, \overline{D}] \\ &\le q_2 L_2 + k_1 L_2 + k_2 S_1 + k_3 \end{aligned} \tag{2.5}$$

and

$$\left.\begin{aligned} &C_\alpha[r_j, \Gamma] \le C_\alpha[a_{j1}, \Gamma] C[w, \Gamma] + C_\alpha[a_{j2}, \Gamma] \le k_1 S_1 + k_4 \\ &\qquad |s_{jk}| \le k_1 S_1 + k_5 \end{aligned}\right\} \quad (j = 1, 2;\ k \in J_j) \tag{2.6}$$

in which $S_1 = C[w, \overline{D}]$. In accordance with the estimates on Problem B for (2.3) in [4], we obtain

$$\begin{aligned} L_1 &\le M_3\Big[(q_2+k_1)L_2 + k_2S_1 + k_3 + 2k_1S_1 + k_4 + k_5\Big] \\ &= M_3\Big[(q_2+k_1)L_2 + (k_2+2k_1)S_1 + k_3 + k_4 + k_5\Big] \end{aligned} \tag{2.7}$$

where $M_3 = M_3(q_0, p_0, k, \alpha, K, D)$. Moreover, noting that V is a solution of the modified Riemann-Hilbert problem for $U_{\bar z} = \overline{V}_z$, we have

$$L_2 \le M_3\big[L_1 + 2k_1S_1 + k_4 + k_5\big]. \tag{2.8}$$

In addition, from (1.9) it can be derived that

$$S_1 = C[w, \overline{D}] \le k_5 + M_4\big[C(U, \overline{D}) + C(V, \overline{D})\big] \le k_5 + M_4(L_1 + L_2) \tag{2.9}$$

where $M_4 = M_4(D)$. Combining (2.7) - (2.9), it is derived that

$$\begin{aligned} L_2 &\le M_3\Big\{M_3\Big[(q_2+k_1)L_2 + (k_2+2k_1)\big(k_5 + M_4(L_1+L_2)\big) + k_3 + k_4 + k_5\Big] \\ &\quad + 2k_1\big(k_5 + M_4(L_1+L_2)\big) + k_4 + k_5\Big\} \\ &\le M_3\Big\{(q_2+k_1)M_3L_2 + (k_2+2k_1)(1+M_3)M_4(L_1+L_2) \\ &\quad + k_5(k_2+2k_1)(1+M_3) + (k_3+k_4+k_5)(1+M_3)\Big\}. \end{aligned} \tag{2.10}$$

Provided that the constants q_2 and k_1, k_2 are sufficiently small, for instance, when

$$M_3\Big[(q_2+k_1)M_3 + (k_2+2k_1)(1+M_3)M_4\Big] < \frac{1}{2},$$

we thus have

$$\begin{aligned} L_2 &\le 2M_3\Big[(k_2+2k_1)(1+M_3)M_4L_1 \\ &\qquad + k_5(k_2+2k_1)(1+M_3) + (k_3+k_4+k_5)(1+M_3)\Big] \\ &= M_5L_1 + M_6. \end{aligned} \tag{2.11}$$

Substituting (2.11) and (2.9) into (2.7), it can be obtained that

$$\begin{aligned} L_1 &\le M_3\Big[(q_2+k_1)(M_5L_1+M_6) + (k_2+2k_1)M_4(L_1+L_2) \\ &\quad + k_5(k_2+2k_1) + k_3 + k_4 + k_5\Big] \\ &\le M_3\Big\{\Big[(q_2+k_1)M_5 + (k_2+2k_1)M_4(1+M_5)\Big]L_1 \\ &\quad + (q_2+k_1)M_6 + (k_2+2k_1)M_4M_6 + k_5(k_2+2k_1) + k_3 + k_4 + k_5\Big\}. \end{aligned} \tag{2.12}$$

Moreover, choose q_2 and k_1, k_2 small enough such that

$$M_3\Big[(q_2+k_1)M_5+(k_2+2k_1)(1+M_5)M_4\Big] < \frac{1}{2}.$$

Then it can be concluded that

$$\begin{aligned} L_1 &\le 2M_3\Big[(q_2+k_1)M_6+(k_2+2k_1)M_4M_6+k_5(k_2+2k_1)+k_3+k_4+k_5\Big] \\ &= M_7 \end{aligned} \tag{2.13}$$

and

$$L_2 \le M_5M_7+M_6 \le M_1 = \max(M_7, M_5M_7+M_6). \tag{2.14}$$

Furthermore, from (1.9) it follows that (2.2) holds ∎

From Theorem 2.1 we can derive the following result.

Theorem 2.2. *Under the conditions of Theorem* 2.1, *any solution* $[w,\ U,\ V]$ *of Problem* (Q) *satisfies the estimates*

$$\left.\begin{aligned} L_1 = L(U) \le M_8k^* \\ L_2 = L(V) \le M_8k^* \end{aligned}\right\} \tag{2.15}$$

and

$$S = S(w) \le M_9k^* \tag{2.16}$$

where $k^* = k_3+k_4+k_5$ *and* $M_j = M_j(q_0, p_0, k_0, \alpha, K, D)\quad (j=8,9)$.

Proof. If $k^*=0$, i.e. $k_3=k_4=k_5=0$, the estimates in (2.15) can be derived by Theorem 3.1 below. If $k^*>0$, it is clear that the system of functions $[w^*, U^*, V^*] = [\frac{w^*}{k^*}, \frac{U^*}{k^*}, \frac{V^*}{k^*}]$ is a solution of the boundary value problem

$$\left.\begin{aligned} &U^*_{\bar z} = Q_1U^*_z+Q_2V^*_z+A_1U^*+A_2V^*+A_3w^*+\frac{A_4}{k^*} \\ &V^*_{\bar z} = \overline{U^*}_z \end{aligned}\right\} \tag{2.17}$$

$$\operatorname{Re}\big[\overline{\lambda_j(z)}U^*(z)+a_{j1}(z)w^*(z)\big] = \frac{a_{j2}(z)+h_j(z)}{k^*} \qquad (z\in\Gamma) \tag{2.18}$$

$$\operatorname{Im}\big[\overline{\lambda_j(z)}U^*(z)+a_{j1}(z)w^*(z)\big]\Big|_{z=a_k} = \frac{b_{jk}}{k^*} \tag{2.19}$$

where $j=1,2$ and $k\in J_j$, and

$$w^*(z) = -\int_0^z\left[\frac{U^*(\zeta)}{\zeta^2}\,d\zeta - \sum_{m=1}^N\frac{d_mz_m}{k^*\zeta(\zeta-z_m)}\,d\zeta+\frac{\overline{V^*(\zeta)}}{\bar\zeta^2}\,d\bar\zeta\right]+\frac{c_0}{k^*}. \tag{2.20}$$

From (1.2), (1.5) and (1.12) we see that

$$L_{p,2}\left[\frac{A_4}{k^*},\overline{D}\right]\le 1,\quad C_\alpha\left[\frac{a_{j2}}{k_*},\Gamma\right]\le 1,\qquad \sum_{j=1,2;\,k\in J_j}\frac{|b_{jk}|}{k_*}\le 1,\quad \frac{|c_0|}{k_*}\le 1.$$

On the basis of the estimates in Theorem 2.1, we obtain for the solution $[w^*, U^*, V^*]$ of the boundary value problem (2.17) - (2.20) the estimate

$$L(U^*)\le M_8,\qquad L(V^*)\le M_8,\qquad S(w^*)\le M_9. \tag{2.21}$$

From the above estimates it immediately follows that estimates (2.15) and (2.16) hold ∎

Remark. Through the mapping $z = z(\zeta) = \frac{1}{\zeta}$ the complex equation (1.1) can be reduced to the form

$$\left.\begin{array}{l} w_{\zeta\bar{\zeta}} = G(z, w, w_\zeta, \overline{w}_\zeta, w_{\zeta\zeta}, \bar{w}_{\zeta\zeta}) \\ G = \tilde{Q}_1 w_{\zeta\zeta} + \tilde{Q}_2 \overline{w}_{\zeta\zeta} + \tilde{A}_1 w_\zeta + \tilde{A}_2 \overline{w}_\zeta + \tilde{A}_3 w + \tilde{A}_4 \end{array}\right\} \quad \big(z \in \widetilde{D} = \zeta(D)\big) \tag{2.22}$$

in which

$$\widetilde{Q}_j = \frac{Q_j \zeta^2}{\bar{\zeta}^2},\ \widetilde{A}_j = -\frac{A_j}{\bar{\zeta}^2}\ \ (j = 1, 2) \quad \text{and} \quad \widetilde{A}_j = \frac{A_j}{|\zeta|^4}\ \ (j = 3, 4) \qquad (\zeta \in \widetilde{D})$$

and $\zeta = \zeta(z) = \frac{1}{z}$. By Condition (C), the above coefficients satisfy the conditions

$$|\widetilde{Q}_1| + |\widetilde{Q}_2| \le q_0 \ \ (\zeta \in \tilde{D}) \quad \text{and} \quad L_{p,2}[\widetilde{A}_j, \overline{\widetilde{D}}] \le k_{j-1} \ \ (j = 1, 2, 3, 4). \tag{2.23}$$

If the function w is a solution of the complex equation (1.1) with Condition (C) in D, then $w(z) = w[z(\zeta)] = w[\frac{1}{\zeta}]$ is a solution of the complex equation (2.22) in $\widetilde{D}$. Noting that $w_{z\bar{z}} = |\zeta|^4 w_{\zeta\bar{\zeta}}$ and $w_{zz} = \zeta^4 w_{\zeta\zeta}$, we see that if $w(z) \in W^2_{p_0,4}(D)$ $(2 < p_0 \le p)$, then $w[z(\zeta)] \in W^2_{p_0}(\widetilde{D})$. The inverse result is also true.

Moreover, denoting $U(z) = U[z(\zeta)] = U(\frac{1}{\zeta})$, we have $U_{\bar{z}} = -\bar{\zeta}^2 U_{\bar{\zeta}}$ and $U_z = -\zeta^2 U_\zeta$, and we see that if $U(z) \in W^1_{p_0,2}(D)$ $(2 < p_0 \le p)$, then $U[z(\zeta)] \in W^1_{p_0}(\widetilde{D})$. The inverse result is also true.

If $f(z) \in L_{p_0,2}(\overline{D})$, then

$$\left.\begin{array}{l} Tf = -\dfrac{1}{\pi}\displaystyle\iint_D \frac{f(\zeta)}{\zeta - z}\, d\sigma_\zeta = -\frac{1}{\pi}\iint_{\tilde{D}} \frac{f(\frac{1}{\zeta})}{\bar{\zeta}^2 \zeta(1 - \zeta z)}\, d\sigma_\zeta = S(0) - S\Big(\frac{1}{z}\Big) \\ S(z) = -\dfrac{1}{\pi}\displaystyle\iint_{\widetilde{D}} \frac{\tilde{f}(\zeta)}{\zeta - z}\, d\sigma_\zeta, \qquad \tilde{f}(\zeta) = \frac{f(\frac{1}{\zeta})}{\bar{\zeta}^2}. \end{array}\right\} \tag{2.24}$$

This shows that $\tilde{f}(z) \in L_{p_0}(\overline{\widetilde{D}})$. Hence

$$\left.\begin{array}{r} \widetilde{C}_\alpha[S(z), \widetilde{D}] \le M L_{p_0}[\tilde{f}(z), \overline{\widetilde{D}}] \\ \widetilde{C}_\alpha\Big[S(0) - S\Big(\dfrac{1}{z}\Big), \tilde{D}\Big] \le M L_{p_0}[\tilde{f}(z), \overline{\widetilde{D}}] \end{array}\right. \tag{2.25}$$

in which $\alpha = 1 - \frac{2}{p_0}$ and $M = M(p_0)$. Thus by using the method of continuity and the contracting mapping principle, we can prove that there exist the solutions $\psi = Tf$ and $\phi = Tg \in W^1_{p_0,2}(\overline{D})$ of

$$\psi_{\bar{z}} = Q_1 \psi_z + A_1 \psi + A, \quad A = Q_2 V_z + A_2 V + A_3 w + A_4 \tag{2.26}$$

$$\phi_{\bar{z}} = Q_1 \phi_z + A_1 \tag{2.27}$$

in D. Moreover, we can also find the solution $\chi(z) = \frac{1}{z} + Th$ of the equation

$$W_{\bar{z}} = Q W_z \quad \text{or} \quad h(z) = Q(z)\Big[-\frac{1}{z^2} + \Pi h\Big]. \tag{2.28}$$

It is clear that $-\frac{Q(z)}{z^2} \in L_{p,2}(\overline{D})$, and then $h(z) \in L_{p_0,2}(\overline{\widetilde{D}})$. Due to the fact that the function $\chi(\frac{1}{z}) = z + S(0) - S(z) = z + S(0) - T[\tilde{h}]$ is a solution of

$$\tilde{h}(z) = \frac{\widetilde{Q}(z)z^2}{\bar{z}^2[1+\Pi\tilde{h}]} \qquad \text{in } \widetilde{D}$$

where $\tilde{h}(\zeta) = \frac{h(\frac{1}{\zeta})}{\bar{\zeta}^2}$, the above function $\chi(\frac{1}{z})$ is a homeomorphism in $\widetilde{D}$. Obviously, $\chi(z)$ is also a homeomorphism in D.

From Theorem 2.1 we see that the solution $w = w(z)$ satisfies the estimate

$$U(z), V(z) = O(|z|^{\frac{2}{p_0}-1}) \quad \text{as } z \to \infty \qquad \text{and} \qquad \int_{\widetilde{\Gamma}} [U(z)\,dz + V d\bar{z}] = 0$$

where $\widetilde{\Gamma} = \{z \in \mathbb{C} : |z| = R\}$. Herein R is a sufficiently large number. Hence w is in $\overline{D}$ continuously differentiable.

3. Solvability of boundary value problems

On the basis of proper a priori estimates nonlinear problems are often solved by the Leray-Schauder technique. This method is extensively used in [2, 3] for different problems. In this way here the solvability of problems (P) and (Q) are discussed.

Theorem 3.1. *If Conditions* $(\mathrm{C})_1$ - $(\mathrm{C})_3$ *and* (1.13) *hold, and the constants* q_2 *and* k_1, k_2 *in* (1.2), (1.3) *and* (1.5) *are small enough, then the solution* $[w, U, V]$ *of Problem* (Q) *is unique.*

Proof. Denote by $[w^j, U^j, V^j]$ $(j = 1, 2)$ two solutions of Problem (Q) and substitute them into (1.7) - (1,9) and (1.11). Then $[w, U, V] = [w^1 - w^2, U^1 - U^2, V^1 - V^2]$ is a solution of the homogeneous boundary value problem

$$\left.\begin{aligned} U_{\bar{z}} &= \tilde{Q}_1 U_z + \tilde{Q}_2 V_z + \tilde{A}_1 U + \tilde{A}_2 V + \tilde{A} w \\ V_{\bar{z}} &= \bar{U}_z \end{aligned}\right\} \tag{3.1}$$

$$\operatorname{Re}\big[\overline{\lambda_j}(z) U_j(z) + a_{j1}(z) w(z)\big] = h_j(z) \qquad (z \in \Gamma) \tag{3.2}$$

$$\operatorname{Im}\big[\overline{\lambda_j(z)} U(z) + a_{j1}(z) w(z)\big]\Big|_{z=a_k} = 0 \qquad (j = 1, 2;\ k \in J_j) \tag{3.3}$$

$$w(z) = -\int_1^z \left[\frac{U(\zeta)}{\zeta^2}\,d\zeta - \sum_{m=1}^{N} \frac{d_m z_m}{\zeta(\zeta - z_m)}\,d\zeta + \frac{\overline{V(\zeta)}}{\bar{\zeta}^2}\,d\bar{\zeta}\right] \tag{3.4}$$

the coefficients of which satisfy conditions (1.7) - (1.9) and (1.11), but $k_3 = k_4 = k_5 = 0$. On the basis of Theorem 2.2, provided q_2 and k_1, k_2 are sufficiently small, we can derive that $U = V = w = 0$ on $\bar{D}$, i.e. $w^1 = w^2, U^1 = U^2$ and $V^1 = V^2$ on $\bar{D}$ ∎

In the following, we use the foregoing estimates of solutions and the Leray-Schauder theorem to prove the solvability of Problem (Q) for the nonlinear elliptic system.

Theorem 3.2. *Suppose that the conditions of Theorem* 2.1 *are satisfyed. Then Problem* (Q) *is solvable.*

Proof. First of all, we assume that $F(z,w,U,V,U_z,V_z) = 0$ from (1.7) in the neighbourhood D^* of the boundary Γ, namely

$$\left.\begin{aligned} &U^*_{\bar z} = t\,F^*(z,w,U,V,U^*_z,V^*_z)\\ &V^*_{\bar z} = t\,\overline{U^*}_z \end{aligned}\right\} \qquad (0 \le t \le 1). \tag{3.5}$$

We introduce the Banach space

$$B = W^1_{p_0,2}(D) \times W^1_{p_0,2}(D) \times C^1(\overline{D}) \qquad (2 < p_0 \le p).$$

Denote by B_M the set of triples of continuous functions $\omega = [w,U,V]$ satisfying the inequalities

$$\left.\begin{aligned} L(U) = C_\beta[U,\overline{D}] + L_{p_0,2}[|U_{\bar z}| + |U_z|,\overline{D}] &< M_{10}\\ L(V) &< M_{10}\\ C^1[w(z),\overline{D}] &< M_{11} \end{aligned}\right\} \tag{3.6}$$

where $M_{10} = M_1 + 1$ and $M_{11} = M_2 + 1$, with β and M_1, M_2 being non-negative constants as stated in (2.1) and (2.2). It is evident that B_M is a bounded open set in B.

Next, we arbitrarily select a system of functions $\omega = [w,U,V] \in B_M$ and substitute it into the appropriate positions of (1.7) - (1.9) and (1.11), and then consider the boundary value problem (Q)$'$ with parameter $t \in [0,1]$

$$\left.\begin{aligned} &U^*_{\bar z} = t\,F^*(z,w,U,V,U^*_z,V^*_z)\\ &V^*_{\bar z} = t\,\overline{U^*}_z \end{aligned}\right\} \qquad \text{in } D \tag{3.7}$$

$$\operatorname{Re}\left[\overline{\lambda_j(z)}U^*(z) + ta_{j1}(z)w(z)\right] = a_{j2}(z) + h_j(z) \qquad (z \in \Gamma) \tag{3.8}$$

$$\operatorname{Im}\left[\overline{\lambda_j(z)}U^*(z) + ta_{j1}(z)w(z)\right]\Big|_{z=a_k} = b_{jk} \qquad (j = 1,2;\ k \in J_j) \tag{3.9}$$

$$w^*(z) = -\int_0^z \left[\frac{U^*(\zeta)}{\zeta^2} - \sum_{m=1}^N \frac{d_m z_m}{\zeta(\zeta - z_m)}\right] d\zeta + \frac{\overline{V^*(\zeta)}}{\bar\zeta^2}\, d\bar\zeta \tag{3.10}$$

where w,U,V are known functions as stated before. Noting that Problem (Q)$'$ consists of two modified Riemann-Hilbert boundary value problems for elliptic complex equations of first order and applying [4: Theorem 3.2], there exist the solution $U^*, V^* \in W^1_{p_0,2}(D)$ $(2 < p_0 \le p)$. From (3.10), the single-valued function w^* on $\overline{D}$ is determined. Denote by $\omega^* = [w^*,U^*,V^*] = T(\omega,t)$ $(0 \le t \le 1)$ this mapping from ω onto ω^*. According to Theorem 2.1, if $\omega = [w,U,V] = T(\omega,t)$, then $\omega = [w,U,V]$ satisfies estimates

(2.1) and (2.2), consequently $\omega \in B_M$. Setting $B_0 = B_M \times [0,1]$, we shall verify that the mapping $\omega^* = T(\omega, t)$ satisfies the three conditions of the Leray-Schauder theorem:

(1) When $t = 0$, by Theorem 2.1, it is evident that $\omega^* = T(\omega, t) \in B_M$.

(2) As stated before, the solution $\omega = [w, U, V]$ of the functional equation $\omega = T(\omega, t)$ satisfies estimates (2.1) and (2.2) which shows that $\omega = T(\omega, t)$ does not have a solution $\omega = [w, U, V]$ on the boundary $\partial B_M = \overline{B_M} \setminus B_M$.

(3) $\omega^* = T(\omega, t)$ continuously maps the Banach space B into itself, and is completely continuous on B_M. Besides, for $\omega \in \overline{B_M}$, $T(\omega, t)$ is uniformly continuous with respect to t.

In fact, let us choose any sequence $\{\omega_n\}_{n\in\mathbb{N}} = \{[w_n, U_n, V_n]\}_{n\in\mathbb{N}} \subset \overline{B_M}$. By Theorem 2.1, it is not difficult to see that $\omega_n^* = [w_n^*, U_n^*, V_n^*] = T(\omega_n, t)$ $(0 \le t \le 1)$ satisfies the estimates

$$L(U_n^*) \le M_{12}, \qquad L(V_n^*) \le M_{12}, \qquad S(w_n^*) \le M_{13} \tag{3.11}$$

where $M_j = M_j(q_0, p_0, k, \alpha, K, D, M)$ $(j = 12, 13)$. Hence there can be selected subsequences of $\{w_n^*\}, \{U_n^*\}$ and $\{V_n^*\}$, which uniformly converge to w_0^*, U_0^* and V_0^* on $\overline{D}$, and $\{U_{nz}^*\}, \{U_{n\bar z}^*\}$ and $\{V_{nz}^*\}, \{V_{n\bar z}^*\}$ in D weakly converge to U_{0z}^*, $U_{0\bar z}^*$ and $V_{0z}^*, V_{0\bar z}^*$, respectively. For convenience, denote by the same symbols as before these subsequences. From $\omega_n^* = T(\omega_n, t)$ and $\omega_0^* = T(\omega_0, t)$ $(0 \le t \le 1)$ we obtain

$$\left.\begin{aligned} U_{n\bar z}^* - U_{0\bar z}^* &= t\Big[F\big(z, w_n, U_n, V_n, U_{nz}^*, V_{nz}^*\big) \\ &\qquad - F\big(z, w_n, U_n, V_n, U_{0z}^*, V_{0z}^*\big) + c_n\Big] \\ c_n &= F\big(z, w_n, U_n, V_n, U_{0z}^*, V_{0z}^*\big) - F\big(z, w_0, U_0, V_0, U_{0z}^*, V_{0z}^*\big) \\ V_{n\bar z}^* - V_{0\bar z}^* &= t\,\big[\overline{U}_{nz}^* - \overline{U}_{0z}^*\big] \end{aligned}\right\} \quad (z \in D) \tag{3.12}$$

$$\operatorname{Re}\big[\overline{\lambda_j(z)}(U_n^* - U_0^*) + t a_{j1}(z)(w_n - w_0)\big] = h_j(z) \qquad (z \in \Gamma) \tag{3.13}$$

$$\operatorname{Im}\big[\overline{\lambda_j}(U_n^* - U_0^*) + t a_{j1}(w_n - w_0)\big]\Big|_{z=a_k} = b_{jk} \qquad (j = 1, 2;\ k \in J_j) \tag{3.14}$$

$$\begin{aligned} w_n^*(z) - w_0^*(z) = -\int_1^z &\left[\frac{U_n^*(\zeta) - U_0^*(\zeta)}{\zeta^2} - \sum_{m=1}^N \frac{d_m z_m}{\zeta(\zeta - z_m)}\right] d\zeta \\ &+ \left[\frac{\overline{V_n^*(\zeta)} - \overline{V_0^*(\zeta)}}{\bar\zeta^2}\right] d\bar\zeta. \end{aligned} \tag{3.15}$$

It is not difficult to see that $c_n \to 0$ for almost every point $z \in D$ as $n \to \infty$. Hence we can prove that $L_{p_0}[c_n, \overline{D}] \to 0$ for $n \to \infty$ as follows: Choosing two arbitrary sufficiently small positive constants ε_1 and ε_2, there exist a subset $D_* \subset D$ and a sufficiently large positive integer N such that $\operatorname{meas} D_* < \varepsilon_1$ and $|c_n| < \varepsilon_2$ on $\overline{D} \setminus D_*$ for $n > N$. By the

Hölder and Minkowski inequalities we have

$$\begin{aligned} L_{p_0,2}[c_n,\overline{D}] &\le L_{p_0,2}[c_n,D_*]+L_{p_0,2}[c_n,\overline{D}\setminus D_*] \\ &\le L_{p_1,2}[c_n,D_*]L_{p_2,2}[1,D_*]+\varepsilon_2 L_{p_0,2}[1,\overline{D}\setminus D_*] \\ &\le M_{14}\varepsilon_1^{1/p_2}+\varepsilon_2\pi^{1/p_0} \\ &=\varepsilon \end{aligned} \qquad (n>N)$$

where $p_2=\frac{p_0p_1}{p_1-p_0}$, $2<p_0<p_1<p_2<\infty$ and M_{14} is a non-negative constant. On the basis of Theorem 2.2, it can be derived that

$$\left.\begin{array}{r} L(U_n-U_0) \\ L(V_n-V_0) \\ S(w_n-w_0) \end{array}\right\} \to 0 \qquad (n\to\infty).$$

Because of the completeness of the Banach space B, there exists a system of functions $\omega_0=[w_0,U_0,V_0]\in B$ such that

$$\left.\begin{array}{r} L(U_n-U_0) \\ L(V_n-V_0) \\ S(w_n-w_0) \end{array}\right\} \to 0 \qquad (n\to\infty).$$

This shows the complete continuity of $\omega^*=T(\omega,t)$ $(0\le t\le 1)$ on $\overline{B_M}$. By a similar method we can also prove that $\omega^*=T(\omega,t)$ continuously maps $\overline{B_M}$ into B and $T(\omega,t)$ is uniformly continuous with respect to t for $\omega\in\overline{B_M}$.

Hence by the Leray-Schauder theorem, we see that the functional equation $\omega=T(\omega,t)$ $(0\le t\le 1)$ with $t=1$, i.e. Problem (Q) has a solution.

Finally, we can eliminate the assumption of $F(z,w,U,V,U_z,V_z)=0$ in D^* and prove the solvability of Problem (Q) for the general nonlinear elliptic system (1.7) in D. This completes the proof ∎

Theorem 3.3. *Under the same conditions as in Theorem* 3.2, *the result of solvability of Problem* (P) *for the complex equation is as follows:*

(1) *If* $K_n=\frac{1}{2\pi}\Delta_\Gamma\arg\lambda_n(z)\ge N$ $(n\in\mathbb{N})$, *then Problem* (P) *has* $2N$ *solvability conditions, and the general solution depends on* $2(K_1+K_2-2N+2)$ *arbitrary real constants.*

(2) *If* $0\le K_j<N$ $(j=1,2)$, *the total number of solvability conditions of Problem* (P) *is not greater than* $4N-K_1-K_2$, *and the general solution depends on* K_1+K_2+4 *arbitrary real constants.*

(3) *If* $K_j<0$ $(j=1,2)$, *then Problem* (P) *has* $4N-2K_1-2K_2-2$ *solvability conditions, and the general solution depends on two real constants.*

We can also write solvability conditions of Problem (P) *in other cases.*

Proof. We only discuss the case $0\le K_j<N$ $(j=1,2)$. Let the solution $[w,U,V]$ of Problem (Q) be substituted into (1.7) - (1.9) and (1.11). The functions h_j $(j=1,2)$

and the complex constants d_m $(m = 1, \ldots, N)$ are then determined. If the functions and the constants are equal to zero, namely

$$h_j(z) = h_{jk} \quad (j = 1, \ldots, N - K_j) \qquad \text{when} \quad 0 \le K_j < N \quad (j = 1, 2) \tag{3.16}$$

and

$$d_m = 0 \qquad (m = 1, \ldots, N), \tag{3.17}$$

then $w_z = U(z)$ and $\overline{w}_z = V(z)$, and w is a solution of Problem (P). Hence when $0 \le K_j < N$ $(j = 1, 2)$, Problem (P) has $4N - K_1 - K_2$ solvability conditions. In addition, the real constants b_{jk} $(k = N - K_j + 1, \ldots, N + 1;\ j = 1, 2)$ in (1.11) and the complex constant c_0 in (1.9) may be arbitrary. This shows that the general solution of Problem (P) $(0 \le K_j < N;\ j = 1, 2)$ depends on $K_1 + K_2 + 4$ arbitrary real constants ∎

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