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Nonlinear Singular Integral Equations Involving the Hilbert Transform in Clifford Analysis

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Dedicated to Prof. L. von Wolfersdorf on the occasion of his 65th birthday

Abstract. We apply operator-theoretical methods for monotone and maximal monotone operators to prove the existence of solutions for nonlinear singular integral and integro-differential equations involving the Hilbert transform in the Clifford algebra $\mathcal{C}\ell_{n,0}$. Properties of the Hilbert transform are proved using Clifford analysis. We generalize well-known results concerning the complex Hilbert transform and the singular Cauchy integral operator to higher dimensions.

Keywords: *Clifford analysis, Cauchy-type integrals, nonlinear integro-differential equations*

AMS subject classification: Primary 30 G 35, secondary 45 E 05, 45 E 10, 47 H 30

1. Introduction

Clifford analysis is a generalization of complex function theory to higher dimensions. Therefore it is natural to look for similarities. Methods of monotone operator theory as given in [6, 7] have been applied to nonlinear singular integral equations on the unit circle and on the real line. The basic property used here is the monotonicity, i.e. positivity of the Hilbert transform. To get an overview of these results we recommend the papers [17, 18].

A generalization from the real line to the complex plane is done in [1]. Singular integral operators, especially the Cauchy transform, play an important role in quaternionic and Clifford analysis. We want to give some outline about Clifford analysis and singular integral operators.

A foundation of Clifford analysis was done in [5], quaternionic analysis is treated extensively in [9] and more recently in [10, 11]. These last books explain also some relations to physical problems. Connections between harmonic and monogenic functions are discussed in [8]. The Cauchy transform and some classes of singular integral operators and associated equations in a quaternionic context were investigated in [15] concerning

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Fredholm property. Cauchy transform and convolution singular integral operators on Lipschitz surfaces using Clifford analytical techniques were treated in [12, 13]. A good overview of monotonicity principles and their application to operator equations is given in [19]. Special nonlinear singular integral equations were considered in [3, 4].

Our paper is organized as follows. First, in Section 2, we review some basic properties of Clifford algebras, Clifford function theory and related function spaces. Then, in Section 3, we prove mapping and monotonicity properties of the Hilbert transform and related operators, especially the Nemyckii operator. In the final Section 4 we apply monotone operator theory to several kind of nonlinear singular integral equations involving the Hilbert transform.

2. Preliminaries

We shall briefly review some basic definitions and properties of the function theory corresponding to the Clifford algebra. For a more detailed investigation of this matters, we refer to [5, 8 - 11].

Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis in $\mathbb{R}^m$. Consider the 2^m-dimensional real Clifford algebra $\mathcal{C}\ell_{m,0}$ generated by $e_1, \ldots, e_m$ according to the multiplication rules

$$e_i e_j + e_j e_i = 2\delta_{ij} e_0$$

where e_0 is the identity of $\mathcal{C}\ell_{m,0}$. The elements $e_J \quad (J = \{h_1, \ldots, h_k\} \subseteq \{1, \ldots, m\})$ define a basis of $\mathcal{C}\ell_{m,0}$ where $e_J = e_{h_1} \cdots e_{h_k} \quad (1 \le h_1 < \ldots h_k \le m)$ and $e_\emptyset = e_0$. Thus, an arbitrary element $a \in \mathcal{C}\ell_{m,0}$ can be represented as

$$a = \sum_J a_J e_J \qquad (a_J \in \mathbb{R}).$$

Especially, the elements $\vec{x} \in \mathbb{R}^m$ will be identified with $\sum_{j=1}^m x_j e_j \in \mathcal{C}\ell_{m,0}$. We want to denote by $\operatorname{Sc} a = a_0 e_0 = a_0$ the *scalar part* of a and by $\operatorname{Vec} a = a - \operatorname{Sc} a$ the $(multi-)$ *vectorpart*.

We introduce an automorphism called *reversion*. The reversed element $\hat{a}$ of a is given by $\hat{a} = \sum_J a_J \hat{e}_J$ where $\hat{e}_0 = e_0$, $\hat{e}_j = e_j$ and $\hat{e}_J = \hat{e}_{h_k} \cdots \hat{e}_{h_2} \cdot \hat{e}_{h_1}$. Then by

$$[a, b] = \operatorname{Sc} \hat{a} b = \operatorname{Sc} a \hat{b} = \sum_J a_J b_J \qquad \text{and} \qquad |a|^2 = \operatorname{Sc} \hat{a} a = \operatorname{Sc} a \hat{a}$$

the scalar product and norm are given in $\mathcal{C}\ell_{m,0}$ and we have especially $x\hat{x} = \sum_{j=1}^m x_j^2 = |x|^2$.

We suppose $G \subset \mathbb{R}^m$ to be a domain with a smooth boundary Γ. We consider functions f defined on G with values in $\mathcal{C}\ell_{m,0}$. These functions can be written as

$$f(x) = \sum_J f_J(x) e_J \qquad (x \in G).$$

Properties such as continuity, differentiability, integrability, and so on, which are ascribed to f have to be possessed by all components $f_J(x)$. In this way the usual Banach spaces of these functions are denoted by C^α, L^2, H^1 and H_0^1. Further,

$$H^1(G) = \left\{ u \in L^2(G) : \frac{\partial u}{\partial x_k} \in L^2(G) \right\}$$

and $H_0^1(G)$ is the closure of $C_0^\infty(G)$ in H^1. We now define the Dirac operator D by

$$D = \sum_{k=1}^{m} e_k \frac{\partial}{\partial x_k}.$$

We consider in G the equation

$$(Du)(x) = 0$$

and look for its solutions which are called *left-monogenic* functions in G.

Now we define the Cauchy kernel in $\mathbb{R}^m$ by

$$e(x) = \frac{1}{\sigma_m} \frac{x}{|x|^m} \quad (x \neq 0) \qquad \text{with } \ \sigma_m = \frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)}.$$

It is well known that e (a fundamental solution of D) is monogenic in $\mathbb{R}^m \setminus \{0\}$. Using the function e we introduce the integral operators

$$\begin{aligned}
(T_G u)(x) &= \int_G e(x-y)u(y)\,dy && (x \in \mathbb{R}^m) \quad \text{(Teodorescu transform)}\\
(F_\Gamma u)(x) &= -\int_\Gamma e(x-y)n(y)u(y)\,d\Gamma_y && (x \notin \Gamma) \quad \text{(Cauchy type operator)}\\
(S_\Gamma u)(x) &= -\int_\Gamma 2e(x-y)n(y)u(y)\,d\Gamma_y && (x \in \Gamma) \quad \text{(singular integral operator)}
\end{aligned}$$

where $n(y) = \sum_{i=1}^m e_i n_i(y)$ is the exterior unit vector to Γ at the point y. The integral which defines the operator S_Γ has to be taken in the sense of the Cauchy principal value. We remark that the operators $F_\Gamma, S_\Gamma, P_\Gamma$ and Q_Γ are defined in spaces of Hölder continuous functions. It is possible to extend these operators to Sobolev spaces in the classical way by approximation (with Hölder continuous functions). We omit the detailed discussion here.

We introduce weighted L^2-spaces. Let G be a bounded or unbounded smooth domain in $\mathbb{R}^m$ and

$$L^{2,\alpha}(G, \mathcal{C}\ell_{m,0}) := \left\{ u : (1+|x|^2)^{\frac{\alpha}{2}} u \in L^2(G, \mathcal{C}\ell_{m,0}) \right\}.$$

These spaces are (real) Hilbert spaces with the scalar product

$$(u,v)_\alpha = \mathrm{Sc} \int_G (1+|x|^2)^\alpha \hat{u}(x) v(x)\,dx = \int_G (1+|x|^2)^\alpha [u(x), v(x)]\,dx$$

and the norm is $||u||_\alpha = \sqrt{(u,u)_\alpha}$. We set $(\cdot,\cdot) = (\cdot,\cdot)_0$ and $||\cdot|| = ||\cdot||_0$. Further, we will use the weighted Sobolev spaces

$$H^{1,\alpha}(G, \mathcal{C}\ell_{m,0}) = \\ \left\{u : (1+|x|^2)^{\frac{\alpha}{2}} u \in L^2(G, \mathcal{C}\ell_{m,0}) \text{ and } (1+|x|^2)^{\frac{\alpha+1}{2}} Du \in L^2(G, \mathcal{C}\ell_{m,0})\right\}.$$

It is easy to see that if G is a bounded domain, these weighted spaces coincide with $L^2(G, \mathcal{C}\ell_{m,0})$ and $H^1(G, \mathcal{C}\ell_{m,0})$, respectively.

From [2] we immediately get the following statements.

Lemma 1. *Let* $u \in H^{1,\alpha-1}(G, \mathcal{C}\ell_{m,0})$ $(-\frac{m}{2}+1 < \alpha < \frac{m}{2})$. *Then we have:*

(i) $F_\Gamma u + T_G Du = \begin{cases} u & \textit{for } x \in G \\ 0 & \textit{for } x \in \mathbb{R}^m \setminus \overline{G} \end{cases}$ (*Borel-Pompeiu formula*).

(ii) $DT_G u = \begin{cases} u & \textit{in } G \\ 0 & \textit{in } \mathbb{R}^m \setminus \overline{G} \end{cases}$.

(iii) $DF_\Gamma u = 0$ *in* $G \cup (\mathbb{R}^m \setminus \overline{G})$.

Lemma 2 (Plemelj-Sokhotzkij formulas). *Let* $u \in C^{0,\alpha}(G, \mathcal{C}\ell_{m,0})$ $(0 < \alpha < 1)$. *Then we have*

(i) $\lim_{G \ni x \to \xi \in \Gamma} (F_\Gamma u)(x) = P_\Gamma u(\xi)$

(ii) $\lim_{\mathbb{R}^m \setminus \overline{G} \ni x \to \xi \in \Gamma} (F_\Gamma u)(x) = -Q_\Gamma u(\xi)$

for any $\xi \in \Gamma$.

The operator $P_\Gamma := \frac{1}{2}(I + S_\Gamma)$ denotes the projection onto the space of all $\mathcal{C}\ell_{m,0}$-valued functions which have a left monogenic extension into the domain G. Further, $Q_\Gamma := \frac{1}{2}(I - S_\Gamma)$ denotes the projection onto the space of all $\mathcal{C}\ell_{m,0}$-valued functions which have a left monogenic extension into the domain $\mathbb{R}^m \setminus \overline{G}$ and vanish at infinity.

Corollary 1. *Let* $u \in L^2(\Gamma, \mathcal{C}\ell_{m,0})$. *Then the equations*

(i) $S_\Gamma^2 u = u$

(ii) $F_\Gamma P_\Gamma u = F_\Gamma u$

(iii) $P_\Gamma^2 u = P_\Gamma u$

(iv) $Q_\Gamma^2 u = Q_\Gamma u$

are valid on Γ.

Corollary 2. *Let* $u \in H^{1,\alpha-1}(G, \mathcal{C}\ell_{m,0})$ $(-\frac{m}{2}+1 < \alpha < \frac{m}{2})$. *Then*

(i) $T_G Du = u$ *in* $G \iff \operatorname{tr} u \in \operatorname{im} Q_\Gamma$

(ii) $TDu = u$ *in* $\mathbb{R}^m$.

3. Monogenicity and Hilbert transform

The main properties of the usual one-dimensional Hilbert transform are the relation to boundary values of holomorphic functions and the connection of conjugated harmonic functions. Both properties are also fulfilled by the Hilbert transform in Clifford analysis which has been studied be many authors, e.g., in the books [8] and [11]. We want to show that the Hilbert transform used by us connects conjugated harmonic functions and we will further use this property.

Lemma 3. *Let $W \in H^{1,\alpha}(\mathbb{R}^{n+1}_+, \mathcal{C}\ell_{n+1,0})$ be a monogenic function. Then there exist $U, V \in H^{1,\alpha}(\mathbb{R}^{n+1}_+, \mathcal{C}\ell_{n,0})$ such that $W = U + e_{n+1}V$ and $\Delta U = \Delta V = 0$.*

Proof. Because of $\mathcal{C}\ell_{n+1,0} = \mathcal{C}\ell_{n,0} + e_{n+1}\mathcal{C}\ell_{n,0}$ there exist $U, V \in \mathcal{C}\ell_{n,0}$ such that $W = U + e_{n+1}V$. We want to denote by D the Dirac operator in $\mathbb{R}^n$, i.e. $D = \sum_{k=1}^n e_k \frac{\partial}{\partial x_k}$. Then

$$\begin{aligned}
\left(D + e_{n+1}\frac{\partial}{\partial x_{n+1}}\right) W &= \left(D + e_{n+1}\frac{\partial}{\partial x_{n+1}}\right)(U + e_{n+1}V) = 0 \\
\iff \; & DU - e_{n+1}DV + e_{n+1}\frac{\partial}{\partial x_{n+1}}U + \frac{\partial}{\partial x_{n+1}}V = 0 \\
\iff \; & \begin{cases} DU + \frac{\partial}{\partial x_{n+1}}V = 0 \\ \frac{\partial}{\partial x_{n+1}}U - DV = 0. \end{cases}
\end{aligned} \tag{1}$$

Thus $\Delta U = \Delta V = 0$ in $\mathbb{R}^{n+1}_+$ where Δ is the Laplacian in $\mathbb{R}^{n+1}$ ∎

Let G be a bounded or unbounded smooth domain in $\mathbb{R}^n$. Then we define the *Hilbert transform* by

$$(H_G u)(x) = \int_G 2e(x-y)u(y)\,dy.$$

If G is the hole space $\mathbb{R}^n$, we denote the Hilbert transform by H. If we interpret $G \subset \mathbb{R}^n$ as a subset of the boundary $\mathbb{R}^n$ of $\mathbb{R}^{n+1}_+$ with outer normal $-e_{n+1}$, we get $H_G(-e_{n+1})u = S_G u$.

3.1 Properties of H_G, H and HD. Here, we want to summarize properties of the Hilbert transform H as a singular integral operator and the integro-differential operator HD.

Theorem 1. *We have the following mapping properties:*

(i) $\begin{cases} H_G : L^{2,\alpha}(G, \mathcal{C}\ell_{n,0}) \to L^{2,\alpha}(G, \mathcal{C}\ell_{n,0}) \\ H : L^{2,\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \to L^{2,\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \end{cases} \quad (-\frac{n}{2} < \alpha < \frac{n}{2}).$

(ii) $\begin{cases} (H_G u, v) = -(u, H_G v) \\ (Hu, v) = -(u, Hv) \end{cases}$ *for all* $\begin{cases} u \in L^{2,\alpha}(G, \mathcal{C}\ell_{n,0}) \\ v \in L^{2,-\alpha}(G, \mathcal{C}\ell_{n,0}) \end{cases} \quad (-\frac{n}{2} < \alpha < \frac{n}{2}).$

(iii) $(H_G u, u) = 0$ *and* $(Hu, u) = 0$ *for all* $u \in L^{2,\alpha}(G, \mathcal{C}\ell_{n,0})$ $(0 \le \alpha < \frac{n}{2})$.

(iv) $H^2 = -I$ *on* $L^{2,\alpha}(G, \mathcal{C}\ell_{n,0})$ $(-\frac{n}{2} < \alpha < \frac{n}{2})$.

Proof. Property (i) follows from the fact that H is a singular integral operator, see for example [14:p. 284] or [16:p. 205]. To prove property (ii) let $x \in G$ be fixed and $y \in G$ variable and put for $N \geq 1$

$$u_N(y) = \begin{cases} u(y) & \text{if } |y-x| \geq \frac{1}{N} \\ 0 & \text{if } |y-x| < \frac{1}{N}. \end{cases}$$

Then $||u_N - u|| \to 0$ as $N \to \infty$ and so $||w_N - w|| \to 0$ as $N \to \infty$ where $w_N(x) = \int_G \hat{u}_N(y) e\widehat{(x-y)}\, dy$ and $w(x) = \int_G \hat{u}(y) e\widehat{(x-y)}\, dy$. This and Hölders inequality lead to

$$\int_G w(x)v(x)\, dx = \lim_{N\to\infty} \int_G w_N(x)v(x)\, dx = \lim_{N\to\infty} \int_G \int_G \hat{u}_N(y) e\widehat{(x-y)}\, dy\, v(x)\, dx.$$

In the last integral the order of integration can be reversed and because of

$$\hat{u}_N(y) e\widehat{(x-y)} v(x) = \hat{u}(y) e\widehat{(x-y)} v_N(x)$$

we get

$$\begin{aligned} \int_G w(x)v(x)\, dx &= \lim_{N\to\infty} \int_G \int_G \hat{u}(y) e\widehat{(x-y)}\, dy\, v_N(x)\, dx \\ &= \int_G \hat{u}(y) \int_G -e(y-x)\, v(x)\, dx\, dy. \end{aligned}$$

Property (iii) follows from

$$(H_G u, u) = -(u, H_G u) = -(H_G u, u).$$

For the second relation we use the embedding

$$L^{2,\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \subset L^2(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \subset L^{2,-\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}).$$

Thus H maps $L^{2,\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ into $L^{2,-\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ $(\frac{n}{2} > \alpha \geq 0)$. To prove property (iv) we remember that $S^2 = I$ and thus $H^2 u = -H(-e_{n+1})H(-e_{n+1})u = -S^2 u = -u$ ∎

Lemma 4. *Let $u \in L^{2,\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ $(-\frac{n}{2} < \alpha < \frac{n}{2})$. Then there exists a $v \in L^{2,\alpha}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ such that $w = u + e_{n+1}v \in \operatorname{im} P_{\mathbb{R}^n}$.*

Proof. Assume $w = u + e_{n+1}v \in \operatorname{im} P_{\mathbb{R}^n}$. Then

$$\begin{aligned} Sw = w \text{ on } \mathbb{R}^n \quad &\Longleftrightarrow \quad H(-e_{n+1})w = H(-e_{n+1})(u + e_{n+1}v) = w = u + e_{n+1}v \\ &\Longleftrightarrow \quad H(-v - e_{n+1}u) = e_{n+1}Hu - Hv = u + e_{n+1}v \\ &\Longleftrightarrow \quad \begin{cases} -Hv = u \\ Hu = v. \end{cases} \end{aligned} \tag{2}$$

Now, set $v = Hu$. Then $Hv = H^2 u = -u$ and going backwards inside the relations given before we obtain the desired relation ∎

Theorem 2. *Let* $u, v \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$. *Then*

$$(HDu, v) = (u, DHv) \qquad \textit{and} \qquad (-HDu, u) \geq 0.$$

Proof. First of all we have

$$\begin{aligned}(Du, v) &= \mathrm{Sc} \sum_{k=1}^{n} \int_{\mathbb{R}^n} \widehat{e_k \frac{\partial}{\partial x_k} u(x)} v(x)\, dx \\ &= \mathrm{Sc} \sum_{k=1}^{n} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_k} \hat{u}(x) \hat{e}_k v(x)\, dx \\ &= -\mathrm{Sc} \sum_{k=1}^{n} \int_{\mathbb{R}^n} \hat{u}(x) e_k \frac{\partial}{\partial x_k} v(x)\, dx \\ &= -(u, Dv).\end{aligned}$$

Putting this together with Theorem 1/(i) we get $(HDu, v) = -(Du, Hv) = (u, DHv)$.

Now, let be $u \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$. Then using Lemma 3 and Lemma 4 we have with $v := Hu \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ that $w = u + e_{n+1}v = \mathrm{tr}\, W = \mathrm{tr}\,(U + e_{n+1}V)$ where W is a monogenic function in $\mathbb{R}^{n+1}_+$. Therefore if $U, V \in C^2(\overline{\mathbb{R}^{n+1}_+}, \mathcal{C}\ell_{n,0}(\mathbb{R}))$, we conclude from (1) and (2)

$$\frac{\partial}{\partial x_{n+1}} U\Big|_{x_{n+1}=0} = \mathrm{tr}\, \frac{\partial}{\partial x_{n+1}} U = \mathrm{tr}\, DV = DV\big|_{x_{n+1}=0} = D\,\mathrm{tr}\, V = Dv = DHu$$

and thus

$$\begin{aligned}(-HDu, u) &= (u, -DHu) \\ &= -\Big(U\big|_{x_{n+1}=0}, \frac{\partial}{\partial x_{n+1}} U\Big|_{x_{n+1}=0}\Big) \\ &= -\sum_J \int_{\mathbb{R}^n} U_J\big|_{x_{n+1}=0} \frac{\partial}{\partial x_{n+1}} U_J\Big|_{x_{n+1}=0} \\ &= \sum_J \int_{\mathbb{R}^{n+1}_+} U_J \Delta U_J\, dx + \sum_{k=1}^{n+1} \sum_J \int_{\mathbb{R}^{n+1}_+} \Big(\frac{\partial}{\partial x_k} U_J\Big)^2 dx \\ &\geq 0\end{aligned}$$

because the first integral is zero due to $\Delta U = 0$. The space $C^2(\overline{\mathbb{R}^n}, \mathcal{C}\ell_{n,0})$ is dense in $H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ and we get the desired relation ∎

3.2 The Nemyckii operator. We want to study two types of non-singular integral equations. First, we require the properties of the so-called Nemyckii operator F in a Clifford-analysis context. This operator is defined as

$$(\Phi u)(x) = \varphi\big(x, u_0(x), u_1(x), \ldots, u_N(x)\big) = \varphi(x, u(x)) \qquad (N = 2^n)$$

with $u = \sum_J u_J(x)e_J$. We make the following assumptions:

(AC) Carathéodory condition: $\varphi : G \times \mathcal{C}\ell_{n,0} \to \mathcal{C}\ell_{n,0}$ is a given function, where G is a non-empty, measurable set in $\mathbb{R}^n$ $(n \geq 1)$. Moreover,

$$\begin{aligned} x \to \varphi(x,u) &\quad \text{is measurable on } G \text{ for all } u \in \mathcal{C}\ell_{n,0} \\ u \to \varphi(x,u) &\quad \text{is continuous on } \mathcal{C}\ell_{n,0} \text{ for almost all } x \in G. \end{aligned}$$

We call φ a Carathéodory function if φ fulfills (AC).

(Aα) Growth condition: For all $(x,u) \in G \times \mathcal{C}\ell_{n,0}$ and $\alpha \in \mathbb{R}$

$$(1+|x|^2)^{-\frac{\alpha}{2}}|\varphi(x,u)| \leq a(x) + b|u|(1+|x|^2)^{\frac{\alpha}{2}}$$

where b is a fixed positive number and $a \in L^{2,\alpha}(G)$ is a real-valued non-negative function.

Proposition 1. *Under the two assumptions* (AC) *and* (Aα) *the following statements are valid:*

(i) (Continuity and boundedness of Φ). *The Nemyckii operator* $\Phi : L^{2,-\alpha}(G,\mathcal{C}\ell_{n,0}) \to L^{2,\alpha}(G,\mathcal{C}\ell_{n,0})$ *is continuous and bounded with*

$$||\Phi u||_{L^{2,\alpha}} \leq C\left(||a||_{L^{2,-\alpha}} + ||u||_{L^{2,-\alpha}}\right)$$

and

$$(\Phi u, u) = \mathrm{Sc}\int_G \varphi(\widehat{x,u(x)})u(x)\,dx \qquad \textit{for all} \;\; u \in L^{2,-\alpha}(G,\mathcal{C}\ell_{n,0}).$$

(ii) (Monotonicity of Φ). *The function φ is monotone with respect to u, i.e.*

$$[\varphi(x,u) - \varphi(x,v), u - v] \geq 0$$

for all $u, v \in L^{2,-\alpha}(G,\mathcal{C}\ell_{n,0})$ implies Φ is monotone.

(iii) (Strictly monotonicity of Φ). *The function φ is strictly monotone with respect to u, i.e.*

$$[\varphi(x,u) - \varphi(x,v), u - v] > 0$$

for all $u, v \in L^{2,-\alpha}(G,\mathcal{C}\ell_{n,0})$ implies Φ is strictly monotone.

(iv) (Coerciveness of Φ). *The inequality*

$$[\varphi(x,u), u] \geq d(1+|x|^2)^{-\alpha}|u|^2 + g(x)$$

where $g \in L^1(G)$ implies Φ is coercive and $(\Phi u, u) \geq d||u||_{-\alpha}^2 + \int_G g(x)\,dx$ for all $u \in L^{2,-\alpha}(G,\mathcal{C}\ell_{n,0})$.

4. Monotonicity principles for integral operators

Monotonicity principles went back to Brèzis, Browder and Minty. A comprehensive description of these principles is done in [19]. In our considerations we will use the following theorem on maximal monotone operators by Browder [7].

Theorem 3. *Let X be a real (separable) reflexive Banach space and $A = A_1 + A_2$ where $0 \in \mathrm{Dom}(A_1)$ and $A_1 : \mathrm{Dom}(A_1) \subset X \to X^*$ is maximal monotone, and $A_2 : X \to X^*$ is bounded, monotone, coercive and (hemi-) continuous. Then A is surjective. If A is strictly monotone, then A is injective.*

This theorem also holds if $A = A_2$. In our setting $X = X^* = L^2(G, \mathcal{C}\ell_{n,0})$ or $X = L^{2,-\frac{\alpha}{2}}(G, \mathcal{C}\ell_{n,0})$ and $X^* = L^{2,\frac{\alpha}{2}}(G, \mathcal{C}\ell_{n,0})$.

4.1 Integral equations. To apply monotonicity principles to this integral equations we only use that the Hilbert transform is a linear, bounded and positive operator, which also implies monotonicity.

Theorem 4 (Hammerstein-type equations). *Let G be a bounded or unbounded smooth domain and φ be a monotone, coercive Carathéodory function on $G \times \mathcal{C}\ell_{n,0}$ satisfying assumption (Aα) with $\alpha = 0$ and let K be a linear bounded, positive operator from $L^2(G, \mathcal{C}\ell_{n,0}) \to L^2(G, \mathcal{C}\ell_{n,0})$. Then*

$$u + (\lambda H_G + K)\Phi u = f$$

has a solution $u \in L^2(G, \mathcal{C}\ell_{n,0})$ for any $f \in L^2(G, \mathcal{C}\ell_{n,0})$ and each $\lambda \in \mathbb{R}$. If Φ or K are strictly monotone, this solution is unique.

Proof. The operator $\lambda H_G + K$ is linear bounded and monotone, the Nemyckii operator Φ is monotone, bounded and coercive. Both operators map $L^2(G, \mathcal{C}\ell_{n,0}) \to L^2(G, \mathcal{C}\ell_{n,0})$. Now, an application of [19: Theorem 32.B] gives the desired result. ∎

Theorem 5. *Let G be a bounded or unbounded smooth domain and φ be a monotone, coercive Carathéodory function on $G \times \mathcal{C}\ell_{n,0}$ and K a linear bounded, positive operator from $L^{2,\alpha}(G, \mathcal{C}\ell_{n,0}) \to L^{2,-\alpha}(G, \mathcal{C}\ell_{n,0})$ $(\frac{n}{2} > \alpha \geq 0)$. Then*

$$\Phi u + \lambda H_G u + K u = g$$

has a solution $u \in L^{2,\alpha}(G, \mathcal{C}\ell_{n,0})$ $(\frac{n}{2} > \alpha \geq 0)$ for any $g \in L^{2,-\alpha}(G, \mathcal{C}\ell_{n,0})$ $(\frac{n}{2} > \alpha \geq 0)$ and each fixed $\lambda \in \mathbb{R}^n$. This solution is unique if $\Phi + \lambda H + K$ is strictly monotone.

Proof. The operator $\lambda H_G + K : L^{2,\alpha}(G, \mathcal{C}\ell_{n,0}) \to L^{2,-\alpha}(G, \mathcal{C}\ell_{n,0})$ $(\frac{n}{2} > \alpha \geq 0)$ is linear bounded and monotone, the properties of the Carathéodory function φ imply that the Nemyckii operator $\Phi : L^{2,\alpha}(G, \mathcal{C}\ell_{n,0}) \to L^{2,-\alpha}(G, \mathcal{C}\ell_{n,0})$ $(\frac{n}{2} > \alpha \geq 0)$ is monotone, bounded, coercive and continuous. Thus $\Phi + \lambda H_G + K$ fulfills the assumptions of Theorem 3. ∎

4.2 Maximal monotone operators and integro-differential equations. For the consideration of integro-differential equations we will use the property of maximal monotonicity.

Definition 1 (see [7]). A subset $M \subset X \times X^*$ is said to be a *monotone set* if for each pairs $\{u_1, w_1\}, \{u_2, w_2\} \in M$ we have $(w_2 - w_1, u_2 - u_1) \geq 0$. Such a set M is said to be *maximal monotone* if it is maximal among monotone sets in the sense of inclusion, and a mapping A is said to be *maximal monotone* if its graph is a maximal monotone set.

Theorem 6. *Let G be a smooth domain in $\mathbb{R}^n$ and γ a real-valued, continuous, positive function and let $C, c > 0$ be constants such that*

$$c \leq \inf_{x\in G} (1 + |x|^2)^{\frac{1}{2}} \gamma(x) \leq \sup_{x\in G} (1 + |x|^2)^{\frac{1}{2}} \gamma(x) \leq C.$$

Then the operator

$$A = D + \gamma(x)I, \qquad \mathrm{Dom}(A) = \big\{u \in H^{1,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0}) \big| \operatorname{tr} u \in \operatorname{im} Q_\Gamma\big\}$$

is a maximal monotone mapping $\mathrm{Dom}(A) \to L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$. *If* $G = \mathbb{R}^n$, *then* $\mathrm{Dom}(A) = H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$.

Proof. The operator $\gamma(x)I$ maps $L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$ uniquely onto $L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$ because of

$$\begin{aligned} ||\gamma(x)u||^2_{\frac{1}{2}} &= \int_G (1 + |x|^2)^{\frac{1}{2}} |\gamma(x)|^2 |u(x)|^2 dx \\ &\leq \sup_{x\in G} \big\{(1 + |x|^2)|\gamma(x)|^2\big\} \int_G (1 + |x|^2)^{-\frac{1}{2}} |u(x)|^2 dx \\ &\leq C\, ||u||^2_{-\frac{1}{2}}. \end{aligned}$$

From Theorem 2 we get that $(Du, u) = 0$. Hence

$$\big(Du + \gamma(x)u, u\big) = (Du, u) + (\gamma(x)u, u) = (\gamma(x)u, u) = \int_G \gamma(x)|u(x)|^2 dx \geq 0.$$

To prove maximal monotonicity we show the existence of a uniquely determined inverse operator with domain $L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$. From [2:p.74] we know that T_G maps $L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$ into $L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$. It is easily seen that $(T_G u, u) = 0$. Therefore, the operator

$$\gamma^{-1}(x)I + T_G : L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0}) \to L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$$

is linear, bounded, strictly monotone and coercive due to

$$\begin{aligned} \big(\gamma^{-1}(x)v + T_G v, v\big) &= (\gamma^{-1}(x)v, v) \\ &= \int_G \gamma^{-1}(x)(1 + |x|^2)^{-\frac{1}{2}} (1 + |x|^2)^{\frac{1}{2}} |v(x)|^2 dx \\ &\geq k\, ||v||^2_{\frac{1}{2}}. \end{aligned}$$

Now, because of $\operatorname{tr} u \in \operatorname{im} Q_\Gamma$ we have

$$Du + \gamma(x)u = f \iff u + T_G\gamma(x)u = T_G f \iff \big(\gamma^{-1}(x)I + T\big)\gamma(x)u = T_G f.$$

There exists a unique $v \in L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$ such that $(\gamma^{-1}(x)I + T_G)v = T_G f$, and there exists a unique $u \in L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$ with $u = \gamma^{-1}(x)v$. Thus

$$u + T_G\gamma(x)u = T_G f$$

has a unique solution u. Moreover, $u = -T_G(\gamma(x)u - f)$ and hence $\operatorname{tr} u \in \operatorname{im} Q_\Gamma$ and

$$Du = -\gamma(x)u + f \in L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$$

is well-defined ∎

Theorem 7. *Let G be a bounded domain and φ be a monotone, coercive Carathéodory function on $G \times \mathcal{C}\ell_{n,0}$ satisfying assumption* (Aα) *with $\alpha = 0$ and K be a linear bounded, positive operator from $L^2(G, \mathcal{C}\ell_{n,0}) \to L^2(G, \mathcal{C}\ell_{n,0})$. Then*

$$\Phi u + (\lambda H_G + K)u + Du + \gamma(x)u = g$$

has a unique solution $u \in L^2(G, \mathcal{C}\ell_{n,0})$ for any $g \in L^2(G, \mathcal{C}\ell_{n,0})$ and each $\lambda \in \mathbb{R}^n$.

Proof. Because G is a bounded domain we have $L^{2,-\alpha}(G, \mathcal{C}\ell_{n,0}) \equiv L^{2,\alpha}(G, \mathcal{C}\ell_{n,0}) \equiv L^2(G, \mathcal{C}\ell_{n,0})$. The operator $D + \gamma(x)I$ is maximal monotone $\mathrm{Dom}(D + \gamma(x)I) \subset L^2(G, \mathcal{C}\ell_{n,0}) \to L^2(G, \mathcal{C}\ell_{n,0})$, the operator $\lambda H_G + K$ is a linear bounded, positive operator and the Nemyckii operator Φ is monotone, bounded, coercive and continuous. Now, apply Theorem 3. ∎

Theorem 8. *Let G be a bounded or unbounded smooth domain and φ be a monotone, coercive Carathéodory function on $G \times \mathcal{C}\ell_{n,0}$ satisfying assumption* (Aα) *with $\alpha = -\frac{1}{2}$ and K a linear bounded, positive operator from $L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0}) \to L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$. Then*

$$\Phi u + Ku + Du + \gamma(x)u = f$$

has a unique solution $u \in \left\{u \in H^{1,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0}) \,\middle|\, \operatorname{tr} u \in \operatorname{im} Q_\Gamma\right\}$ for any $f \in L^{2,\frac{1}{2}}(G, \mathcal{C}\ell_{n,0})$.

Proof. We recall that the operator $D + \gamma(x)I$ is maximal monotone, the operator K is supposed to be linear bounded and positive, the Nemyckii operator is, due to the properties of the Carathéodory function φ, bounded, continuous and coercive. Using Theorem 3 gives the desired result. ∎

Theorem 9. *The operator*

$$-HD : H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \subset L^{2,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \to L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$$

is a maximal monotone mapping.

Proof. From Theorem 2 we get the monotonicity of $-HD$ and from Theorem 1/(iv) we know that $-H$ is invertible and its inverse is given by H. Using Corollary 2 we get

$$-HDu = f \iff Du = Hf \iff u = TDu = THf.$$

Thus for arbitrary $f \in L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ there exists a unique $u \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ such that $-HDu = f$ ∎

Theorem 10. *Let Φ be a monotone, coercive Carathéodory function on $\mathbb{R}^n \times \mathcal{C}\ell_{n,0}$ satisfying assumption* $(\mathrm{A}\alpha)$ *with* $\alpha = -\frac{1}{2}$ *and K a linear bounded, positive operator from* $L^{2,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \to L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$. *Then*

$$\Phi u + Ku - HDu = f$$

has a solution $u \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ *for any* $f \in L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$. *This solution is unique if Φ or K are strictly monotone.*

Proof. The operator $-HD : \mathrm{Dom}(-HD) \subset L^{2,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \to L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ is maximal monotone. The operator $\Phi + K : L^{2,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \to L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ is bounded, monotone, coercive and continuous. Now apply Theorem 3. ∎

Theorem 11. *For $\mu \in \mathbb{R}$ the operator*

$$\mu D - HD : H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \subset L^{2,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \to L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$$

is a maximal monotone mapping.

Proof. We have already seen that $\mu D - HD$ is monotone. Now we want to show that there exist an inverse operator. We have

$$\mu Du - HDu = (\mu I - H)Du = f \iff (\mu I - H)v = f \text{ in } L^{2,-\frac{1}{2}}(G, \mathcal{C}\ell_{n,0}).$$

The operator $\mu I - H$ is invertible and its inverse is given by $\frac{1}{1+\mu^2}(\mu I + H)$. Thus

$$Du = v = \frac{1}{1+\mu^2}(\mu I + H)f \iff u = TDu = \frac{1}{1+\mu^2}(\mu T + TH)f \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$$

for any $f \in L^{2,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ ∎

Theorem 12. *Let Φ be a monotone, coercive Carathéodory function on $\mathbb{R}^n \times \mathcal{C}\ell_{n,0}$ satisfying assumption* $(\mathrm{A}\alpha)$ *with* $\alpha = -\frac{1}{2}$ *and K a linear bounded, positive operator from* $L^{2,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \to L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$. *Then*

$$\Phi u + Ku + \mu Du - HDu = f$$

has a solution $u \in H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ *for any* $f \in L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$ *and each* $\mu \in \mathbb{R}$. *This solution is unique if Φ or K are strictly monotone.*

Proof. The operator $\mu D - HD$ is a maximal monotone operator $H^{1,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \subset L^{2,-\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0}) \to L^{2,\frac{1}{2}}(\mathbb{R}^n, \mathcal{C}\ell_{n,0})$. The remaining operator $\Phi + K$ is bounded, monotone, coercive and continuous. Thus an application of Theorem 3 completes the proof. ∎

Remark. We dealt with the Clifford algebra $\mathcal{C}\ell_{n,0}$, i.e. $e_j^2 = +1$. This seems to be unusual. But the operators H_G and H are not monotone if we use $\mathcal{C}\ell_{0,n}$. Nevertheless the operators iH_G and iH are monotone in spaces over the complexified Clifford algebras $\mathcal{C}\ell_{0,n}(\mathbb{C})$.

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