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Approximation of Lévy-Feller Diffusion by Random Walk

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Dedicated to Prof. L. von Wolfersdorf on occasion of his 65th birthday

Abstract. After an outline of W. Feller's inversion of the (later so called) Feller potential operators and the presentation of the semigroups thus generated, we interpret the two-level difference scheme resulting from the Grünwald-Letnikov discretization of fractional derivatives as a random walk model discrete in space and time. We show that by properly scaled transition to vanishing space and time steps this model converges to the continuous Markov process that we view as a generalized diffusion process. By re-interpretation of the proof we get a discrete probability distribution that lies in the domain of attraction of the corresponding stable Lévy distribution. By letting only the time-step tend to zero we get a random walk model discrete in space but continuous in time. Finally, we present a random walk model for the time-parametrized Cauchy probability density.

Keywords: *Stable probability distributions, Riesz-Feller potentials, pseudo-differential equations, Markov processes, random walks*

AMS subject classification: 26 A 33, 44 A 20, 45 K 05, 60 E 07, 60 J 15, 60 J 60

1. Introduction

Let

$$0 < \alpha \le 2 \qquad \text{and} \qquad |\theta| \le \begin{cases} \alpha & \text{if } 0 < \alpha \le 1 \\ 2-\alpha & \text{if } 1 < \alpha \le 2 \end{cases} \tag{1.1}$$

(θ real) and denote by $p_\alpha(x;\theta)$ for $x \in \mathbb{R}$ the stable probability density whose characteristic function (Fourier transform) is

$$\hat{p}_\alpha(\kappa;\theta) = \exp\left(-|\kappa|^\alpha e^{i(\operatorname{sign}\kappa)\frac{\theta\pi}{2}}\right) \qquad (\kappa \in \mathbb{R}) \tag{1.2}$$

(see, e.g., [4], [17], [19] for the general theory of stable probability distributions). In particular we recommend [4], Feller's parametrization being close to ours. For a generic function f on $\mathbb{R}$ we denote by $\hat{f}$ its Fourier transform

$$\hat{f}(\kappa) = \int_{-\infty}^{+\infty} e^{i\kappa x} f(x)\,dx \qquad (\kappa \in \mathbb{R}) \tag{1.3}$$

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and we then have, in the case of $\int\limits_{-\infty}^{+\infty} |\hat{f}(\kappa)|\, d\kappa < \infty$,

$$f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} e^{-i\kappa x} \hat{f}(\kappa)\, d\kappa \qquad (x \in \mathbb{R}).$$

For $t > 0$ we rescale p_α by the similarity variable $x\, t^{-\frac{1}{\alpha}}$ to obtain

$$g_\alpha(x, t; \theta) := t^{-\frac{1}{\alpha}} p_\alpha(x t^{-\frac{1}{\alpha}}; \theta) \qquad (x \in \mathbb{R},\ t > 0). \tag{1.4}$$

This function $g_\alpha(\cdot, t; \theta)$ again is a stable probability density, and by interpreting x as space and t as time variable we have in g_α a description of a Markov process that can be considered as a generalized diffusion process. In fact, we have in

$$g_2(x, t; 0) = \frac{1}{2\sqrt{\pi}} t^{-\frac{1}{2}} \exp\Big(-\frac{x^2}{4t} \Big)$$

the classical Gauss process and in

$$g_1(x, t; 0) = \frac{1}{\pi} \frac{t}{x^2 + t^2},$$

the Cauchy process. For a few other pairs (α, θ) leading to elementary or well-investigated special functions, see [19]. A general representation of all stable probability densities in terms of Fox H functions has been only recently achieved (see [18]). The Fourier transform of g_α being

$$\hat{g}_\alpha(\kappa, t; \theta) = \exp\big(-t |\kappa|^\alpha e^{i(\operatorname{sign}\kappa)\frac{\theta\pi}{2}} \big) \qquad (\kappa \in \mathbb{R}) \tag{1.5}$$

we recognize $g_\alpha(x, t; \theta)$ as the fundamental solution (Green function for the Cauchy problem) of the pseudo-differential equation

$$\frac{\partial u(x,t)}{\partial t} = D_\theta^\alpha u(x,t) \qquad (x \in \mathbb{R},\ t > 0) \tag{1.6}$$

where the pseudo-differential operator D_θ^α has the symbol $-|\kappa|^\alpha e^{i(\operatorname{sign}\kappa)\frac{\theta\pi}{2}}$. For initial values

$$u(x, 0) = f(x) \qquad (x \in \mathbb{R},\ f \in L_1(\mathbb{R})) \tag{1.7}$$

we then have as solution to (1.6)

$$u(x,t) = \int\limits_{-\infty}^{+\infty} g_\alpha(x - \xi, t; \theta) f(\xi)\, d\xi \tag{1.8}$$

and for all $t > 0$ then

$$u(\cdot, t) \in C^\infty \cap L_1(\mathbb{R}) \qquad \text{and} \qquad \int\limits_{-\infty}^{+\infty} u(x,t)\, dx = \int\limits_{-\infty}^{+\infty} f(x)\, dx.$$

William Feller in his pioneering paper [3] has shown that the pseudo-differential operator D_θ^α can be viewed as the operator inverse to the Feller potential operator (the name "Feller potential" is used in [16]) which is a linear combination of two Weyl integrals. Honouring both Lévy and Feller for their essential contributions [11], [12] and [3] we call the process described by (1.6) *Lévy-Feller diffusion.*

We now give, in our notation, a formal account of the essentials of Feller's theory (for more details see [8]). With the Weyl integrals

$$\left.\begin{aligned}(I_+^\alpha\varphi)(x) &= \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}\varphi(\xi)\,d\xi\\ (I_-^\alpha\varphi)(x) &= \frac{1}{\Gamma(\alpha)}\int_{x}^{+\infty}(\xi-x)^{\alpha-1}\varphi(\xi)\,d\xi\end{aligned}\right\}\qquad (x\in\mathbb{R}) \tag{1.9}$$

and (for $0<\alpha<2$ but $\alpha\neq 1$) the coefficients

$$c_+ = c_+(\alpha;\theta) = \frac{\sin(\frac{\pi}{2}(\alpha-\theta))}{\sin(\alpha\pi)},\qquad c_- = c_-(\alpha;\theta) = \frac{\sin(\frac{\pi}{2}(\alpha+\theta))}{\sin(\alpha\pi)} \tag{1.10}$$

and (by passing to the limit $\alpha = 2$)

$$c_+(2,0) = c_-(2,0) = -\frac{1}{2}, \tag{1.11}$$

the Feller potentials are given as

$$(I_\theta^\alpha\varphi)(x) = c_-(\alpha,\theta)(I_+^\alpha\varphi)(x) + c_+(\alpha,\theta)(I_-^\alpha\varphi)(x)\,. \tag{1.12}$$

Note that in accordance with [16] we omit the singular case $\alpha = 1$.

Feller [3] has shown the operator I_θ^α to possess the semigroup property

$$I_\theta^\alpha I_\theta^\beta = I_\theta^{\alpha+\beta}\qquad \text{for}\ \ 0<\alpha,\beta<1\ \ \text{with}\ \ \alpha+\beta<1,$$

and so analytic continuation to negative exponents can be justified to obtain the operator

$$D_\theta^\alpha := -I_\theta^{-\alpha} = -\left\{c_+(\alpha,\theta)I_+^{-\alpha} + c_-(\alpha,\theta)I_-^{-\alpha}\right\} \tag{1.13}$$

for $0<\alpha\leq 2$ but $\alpha\neq 1$, the parameter θ restricted as in (1.1), with (see [16])

$$I_\pm^{-\alpha} = \begin{cases}\pm\frac{d}{dx}I_\pm^{1-\alpha} & \text{if } 0<\alpha<1\\ \frac{d^2}{dx^2}I_\pm^{2-\alpha} & \text{if } 1<\alpha\leq 2.\end{cases} \tag{1.14}$$

From [3], equating $-\frac{\theta\pi}{2\alpha}$ to Feller's parameter δ, we take the symbol of the pseudo-differential operator D_θ^α as $\hat{D_\theta^\alpha} = -|\kappa|^\alpha e^{i(\operatorname{sign}\kappa)\theta\pi/2}$. In particular, we have $D_0^2 = \frac{d^2}{dx^2}$ but $D_0^1 \neq \frac{d}{dx}$.

For the rest of this paper we always keep in mind the distinction of the following two cases:

(a) $0 < \alpha < 1$ and $|\theta| \le \alpha$.

(b) $1 < \alpha \le 2$ and $|\theta| \le 2 - \alpha$.

Henceforth, for ease of notation, we shall omit the arguments of the coefficients $c_+ = c_+(\alpha, \theta)$ and $c_- = c_-(\alpha, \theta)$. We have

$$c_\pm \begin{cases} \ge 0 & \text{in the case (a)} \\ \le 0 & \text{in the case (b)} \end{cases} \tag{1.15}$$

and

$$c_+ + c_- = \frac{\cos \frac{\theta\pi}{2}}{\cos \frac{\alpha\pi}{2}} \begin{cases} > 0 & \text{in the case (a)} \\ < 0 & \text{in the case (b).} \end{cases} \tag{1.16}$$

The reader is asked not to worry about the foregoing purely formal description of Feller's considerations. It will merely serve us as a motivation for constructing a difference scheme via the Grünwald-Letnikov discretization of fractional derivatives, a difference scheme which by interpretation as a random walk model will be shown to converge (in a sense to be specified in Section 3).

2. Random walks, discrete in space and time

In this section we define a random variable Y assuming only integers as values, its probability distribution depending on three parameters α, θ and μ. By aid of this random variable we define a random walk on an equidistant grid $\{jh |\, j \in \mathbb{Z}\}$ with a space-step $h > 0$. We show that after introduction of a time-step $\tau > 0$ this random walk admits an interpretation as an explicit difference scheme for the Cauchy problem (1.6) - (1.7), namely for

$$\left.\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= D_\theta^\alpha u(x,t) \quad (x \in \mathbb{R},\ t > 0) \\ u(x,0) &= f(x). \end{aligned}\right\} \tag{2.1}$$

In the next section we shall show that the probability distribution of the discrete random variable Y belongs to the domain of attraction of the Lévy distribution with the parameters α and θ, proceeding in a way which simultaneously proves "convergence" of the random walk (if $\tau = \mu h^\alpha \to 0$) to the corresponding Lévy-Feller diffusion characterized by (1.4).

Let Y be a random variable assuming its values in $\mathbb{Z}$, $P(Y = k) = p_k$ for $k \in \mathbb{Z}$, with probabilities p_k defined as follows. With a parameter μ, restricted by

$$0 < \mu \le \begin{cases} \dfrac{\cos \frac{\alpha\pi}{2}}{\cos \frac{\theta\pi}{2}} & \text{in the case (a)} \\[2ex] \dfrac{1}{\alpha} \dfrac{|\cos \frac{\alpha\pi}{2}|}{\cos \frac{\theta\pi}{2}} & \text{in the case (b)} \end{cases} \tag{2.2}$$

put in the case (a)

$$\left.\begin{aligned} p_0 &= 1-\mu(c_+ + c_-) \\ p_k &= (-1)^{k+1}\mu c_+\binom{\alpha}{k} \\ p_{-k} &= (-1)^{k+1}\mu c_-\binom{\alpha}{k} \quad (k\in\mathbb{N}) \end{aligned}\right\} \tag{2.3}$$

and in the case (b)

$$\left.\begin{aligned} &p_0 = 1+\mu\alpha(c_+ + c_-) \\ &p_1 = -\mu\Big[c_+\binom{\alpha}{2}+c_-\Big], \quad p_{-1} = -\mu\Big[c_-\binom{\alpha}{2}+c_+\Big] \\ &p_k = (-1)^k\mu c_+\binom{\alpha}{k+1}, \quad p_{-k} = (-1)^k\mu c_-\binom{\alpha}{k+1} \quad (k\geq 2). \end{aligned}\right\} \tag{2.4}$$

One sees that all $p_k \geq 0$, and by rearrangement it turns out that

$$\sum_{k\in\mathbb{Z}} p_k = 1-\mu(c_+ + c_-)\sum_{j=0}^{+\infty}(-1)^j\binom{\alpha}{j} = 1-0.$$

Remark 2.1. It is worthwhile here to observe the fact which will also be useful in Section 3 that for all $\alpha > 0$ the series $\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}z^k$ for $(1-z)^\alpha$ converges absolutely and uniformly on the closed unit disk $|z|\leq 1$, due to the asymptotics $|\binom{\alpha}{k}| \sim \Gamma(\alpha+1)\frac{|\sin(\alpha\pi)|}{\pi}k^{-(\alpha+1)}$ for $k\to\infty$, valid for non-integer $\alpha > 0$. This asymptotics can be deduced by use of the reflection formula for the gamma function and Stirling's asymptotics.

We obtain a random walk on the grid $\{jh|\, j\in\mathbb{Z}\}$ starting at the point 0, by defining random variables

$$S_n = hY_1 + hY_2 + \ldots + hY_n \qquad (n\in\mathbb{N}) \tag{2.5}$$

with the Y_j as independent identically distributed random variables, all having the same probability distribution as the random variable Y.

Let us write our random walk in an alternative way. Discretizing the space variable x and the time variable t by grid points $x_j = jh$ and instants $t_n = n\tau$, with $h > 0, \tau > 0, j\in\mathbb{Z}, n\in\mathbb{N}_0$ and denoting by $y_j(t_n)$ the probability of sojourn of the random walker in point x_j at instant t_n, the recursion $S_{n+1} = S_n + hY_{n+1}$ (following from (2.5)) means

$$y_j(t_{n+1}) = \sum_{k\in\mathbb{Z}} p_k y_{j-k}(t_n) \qquad (j\in\mathbb{Z},\, n\in\mathbb{N}_0), \tag{2.6}$$

and the random walker starting at point $x_0 = 0$ means $y_0(0) = 1$ and $y_j(0) = 0$ for $j\neq 0$. However, in the recursion scheme (2.6) it is legitimate to use a more general initial sojourn probability distribution $\{y_j(0)|\, j\in\mathbb{Z}\}$. There is yet another possible interpretation of (2.6), namely as a redistribution scheme of an extensive quantity (e.g. mass, charge, or may be probability), $y_j(t_n)$ being imagined as a clump of this extensive quantity, sitting in point x_j at instant t_n. Then (2.6) is a conservative and non-negativity preserving redistribution scheme. In fact, from all $p_k \geq 0$ and $\sum_{k\in\mathbf{Z}} p_k = 1$ it follows immediately for all $n \in \mathbb{N}$ that

$$\sum_{j\in\mathbb{Z}} y_j(t_n) = \sum_{j\in\mathbb{Z}} y_j(0) \qquad \text{if} \quad \sum_{j\in\mathbb{Z}} |y_j(0)| < \infty$$
$$\text{all } y_j(t_n) \geq 0 \qquad \text{if all} \quad y_j(0) \geq 0.$$

Such redistribution schemes have been shown to be useful for discretization of diffusion processes modelled by second order linear parabolic differential equations (see, e.g., [6], [7], [9]) as they discretely imitate essential properties of the continuous process.

To come nearer to the Cauchy problem (2.1) we relate the time step τ to the space step h by the scaling relation

$$\tau = \mu h^\alpha \tag{2.7}$$

and remark that the $y_j(t_n)$ are then intended as approximations to

$$\int_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}} u(x, t_n)\, dx$$

which, if $u(\cdot, t_n)$ is continuous, is also $\approx hu(x_j, t_n)$. It is again a matter of rearrangement to show that (2.6) is equivalent to the explicit difference scheme

$$\frac{y_j(t_{n+1}) - y_j(t_n)}{\tau} = {}_hD_\theta^\alpha y_j(t_n) \tag{2.8}$$

where (in analogy to (1.13)) ${}_hD_\theta^\alpha = -\{c_+ \, {}_hI_+^{-\alpha} + c_- \, {}_hI_-^{-\alpha}\}$ with the Grünwald-Letnikov discretization (see [16]) of the fractional derivatives in the form

$$ {}_hI_\pm^{-\alpha} y_j = \begin{cases} h^{-\alpha} \sum\limits_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} y_{j\mp k} & \text{in the case (a)} \\ h^{-\alpha} \sum\limits_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} y_{j\pm 1\mp k} & \text{in the case (b).} \end{cases} \tag{2.9}$$

Notice the shift of index in the case (b) which among other things has the effect that in the special case $\alpha = 2$ (the classical diffusion equation) we obtain the standard symmetric three-point difference scheme. For more details and discussions see [8].

Instead of trying to work out a convergence proof for the difference scheme (2.8), thereby using the Lax-Richtmyer theory of consistency, stability and convergence (in effect the Lax equivalence theorem, see [5] or [14]) we prefer to present in the next section a proof in the true spirit of random walks. We leave the numerical analysis aspect to a forthcoming paper.

3. Convergence and domain of attraction

We will show that for *fixed* $t = n\tau > 0$ the discrete distribution of the sojourn probabilities $y_j(t_n)$ $(j \in \mathbb{Z})$ with initial condition $y_j(0) = \delta_{j0}$ (Kronecker symbol) converges completely to the probability distribution with density

$$g_\alpha(x, t; \theta) = t^{-\frac{1}{\alpha}} p_\alpha(x t^{-\frac{1}{\alpha}}; \theta) \qquad (x \in \mathbb{R}) \tag{3.1}$$

as $n \to +\infty$. Let us remind that this probability distribution has the characteristic function

$$\hat{g}_\alpha(\kappa, t; \theta) = \int_{-\infty}^{+\infty} g_\alpha(x, t; \theta)\, e^{i\kappa x} dx = \exp\left(- t|\kappa|^\alpha e^{i(\operatorname{sign}\kappa)\frac{\theta\pi}{2}} \right). \tag{3.2}$$

To avoid confusion of language one meets in probability theory let us agree to use the terminology adopted in [10]. From this source we take Definitions 3.1 - 3.4, Remark 3.1 and Theorem 3.1.

Definition 3.1. Let (F_n) be a sequence of uniformly bounded, non-decreasing right-continuous functions defined on $\mathbb{R}$. We say that F_n *converges weakly* to a bounded non-decreasing right-continuous function F on $\mathbb{R}$ if $F_n(x) \to F(x)$ at all continuity points of F. In this case we write $F_n \overset{w}{\to} F$.

Definition 3.2. Let (F_n) be as in Definition 3.1. Then (F_n) is said to *converge completely* to F if

(i) $F_n \overset{w}{\to} F$ and

(ii) $F_n(\mp\infty) \to F(\mp\infty)$ as $n \to \infty$.

In this case we write $F_n \overset{c}{\to} F$.

Theorem 3.1 (Continuity theorem). *Let (F_n) be a sequence of probability distribution functions, and let (φ_n) be the sequence of the corresponding characteristic functions,*

$$\varphi_n(\kappa) = \int_{-\infty}^{+\infty} e^{i\kappa x}\, dF_n(x) \qquad (\kappa \in \mathbb{R}).$$

Then (F_n) converges completely to a probability distribution function F if and only if $\varphi_n(\kappa) \to \varphi(\kappa)$ for all $\kappa \in \mathbb{R}$ as $n \to \infty$, where $\varphi(\kappa)$ is continuous at $\kappa = 0$. In this case the limit function φ is the characteristic function of the limit distribution function F,

$$\varphi(\kappa) = \int_{-\infty}^{+\infty} e^{i\kappa x}\, dF(x) \qquad (\kappa \in \mathbb{R}).$$

Definition 3.3. In the cases where the functions F_n and F are probability distribution functions such that $F_n \overset{c}{\to} F$, let X_n and X be random variables corresponding to F_n and F, respectively. Then we say that X_n *converges in law* to X.

Definition 3.4. Let (X_n) be a sequence of independent identically distributed random variables with common probability distribution function F. Suppose there exist sequences (a_n) and (b_n) of constants, with $b_n > 0$, such that the sequence of sums $b_n^{-1} \sum_{k=1}^{n} X_k - a_n$ converges in law to some random variable with probability distribution function G. Then we say that F is *attracted* to G. The set of all probability distribution functions attracted to G is called the *domain of attraction* of the distribution function G.

Remark 3.1. A stable probability distribution is characterized by having a domain of attraction.

Let us now state, with the notations of Sections 1 and 2, our

Theorem 3.2. *Let the independent identically distributed random variables $Y_1, Y_2, Y_3, \ldots$ have the common probability distribution function F of the random variable Y with $P(Y = k) = p_k$, $k \in \mathbb{Z}$, and p_k given by (2.3), (2.4), respectively. Let $X(t)$ with $t > 0$ be the random variable with probability density $g_\alpha(x, t; \theta)$ and let $G_\alpha(\cdot, t; \theta)$ be the corresponding distribution function,*

$$G_\alpha(x, t; \theta) = \int_{-\infty}^{x} g_\alpha(\xi, t; \theta)\, d\xi .$$

Then F is attracted by $G_\alpha(\cdot, t; \theta)$, indeed: for $n \to \infty$ the distribution function of the random variable

$$X_n = \left(\frac{t}{\mu n}\right)^{\frac{1}{\alpha}} \{Y_1 + Y_2 + \ldots + Y_n\} \tag{3.3}$$

converges completely to $G_\alpha(\cdot, t; \theta)$, the distribution function of the random variable $X(t)$.

Proof. Using the scaling relation (2.7), namely $\tau = \mu h^\alpha$, and the substitution $t = n\tau$ of time, we get

$$h = (t/(\mu n))^{\frac{1}{\alpha}}, \tag{3.4}$$

and comparing (3.3) and (2.5) we see that $X_n = S_n$, the random variable taking values in the grid $\{jh | j \in \mathbb{Z}\}$ at the fixed instant $t_n = n\tau = t$. In view of Theorem 3.1 and the fact that $\hat{g}_\alpha(\kappa, t; \theta)$ is continuous at $\kappa = 0$, it only remains to prove that the characteristic function of the sojourn probabilities $y_j(t_n)$, namely the function

$$\hat{y}(\kappa, t; h) = \sum_{j \in \mathbb{Z}} y_j(t_n)\, e^{ij\kappa h}, \qquad \text{with } \ t_n = t, \tag{3.5}$$

tends for all $\kappa \in \mathbb{R}$ (as $h \to 0$) to $\hat{g}_\alpha(\kappa, t; \theta) = \exp\left(- t|\kappa|^\alpha e^{i(\operatorname{sign}\kappa)\frac{\theta\pi}{2}}\right)$. Let us calculate $\hat{y}(\kappa, t; h)$ for ease of notation via the generating functions

$$\tilde{p}(z) = \sum_{j \in \mathbb{Z}} p_j z^j \qquad \text{and} \qquad \tilde{y}(z, t_n) = \sum_{j \in \mathbb{Z}} y_j(t_n)\, z^j \tag{3.6}$$

of the transition probabilities and the sojourn probabilities. The series in (3.6) converge absolutely and uniformly on the periphery $|z| = 1$ of the unit circle, representing there a continuous function, and due to the fact that the random walk occurs on the grid $\{jh | j \in \mathbb{Z}\}$ change to characteristic functions $\hat{p}(\kappa)$ and $\hat{y}(\kappa, t_n)$ is accomplished via

$z = e^{i\kappa h}$ $(\kappa \in \mathbb{R})$. Using the binomial series for $(1-z)^\alpha$, absolutely convergent on $|z| = 1$ if $\alpha > 0$ (see Remark 2.1), we readily verify the identities

$$\tilde{p}(z) = \begin{cases} 1 - \mu\{c_+(1-z)^\alpha + c_-(1-z^{-1})^\alpha\} & \text{in the case (a)} \\ 1 - \mu\{c_+ z^{-1}(1-z)^\alpha + c_- z(1-z^{-1})^\alpha\} & \text{in the case (b).} \end{cases} \tag{3.7}$$

From the discrete convolution (2.6) we deduce $\tilde{y}(z, t_n) = \tilde{y}(z, 0)\,(\tilde{p}(z))^n$, and the special initial condition $y_j(0) = \delta_{j0}$ for $j \in \mathbb{Z}$ gives $\tilde{y}(z, 0) \equiv 1$, hence

$$\tilde{y}(z, t_n) = (\tilde{p}(z))^n. \tag{3.8}$$

In view of (3.4), (3.8) and the fixation $t = t_n = n\tau$ we have to show that, with $z = e^{i\kappa h}$,

$$(\tilde{p}(z))^n \to \exp\left(-t|\kappa|^\alpha e^{i(\operatorname{sign}\kappa)\frac{\theta\pi}{2}}\right) = \hat{g}_\alpha(\kappa, t; \theta)$$

as $n \to \infty$. More clearly, using (2.7) and $t = t_n = n\tau$, we have to show that the function

$$\hat{y}(\kappa, t; h) = (\tilde{p}(e^{i\kappa h}))^{\frac{t}{\mu h^\alpha}} \qquad (\kappa \in \mathbb{R}) \tag{3.9}$$

has the property

$$\lim_{h\to 0} \hat{y}(\kappa, t; h) = \hat{g}_\alpha(\kappa, t; \theta). \tag{3.10}$$

Let us first treat the case (a): $0 < \alpha < 1$ and $|\theta| \le \alpha$. Then $\kappa = |\kappa| \operatorname{sign}\kappa$ and

$$\tilde{p}(e^{i\kappa h}) = 1 - \mu\Big\{c_+\big(1 - e^{i|\kappa|h\operatorname{sign}\kappa}\big)^\alpha + c_-\big(1 - e^{-i|\kappa|h\operatorname{sign}\kappa}\big)^\alpha\Big\}.$$

We see that $\tilde{p}(e^{i0h}) = 1$, whereas we can get the result for $\kappa < 0$ by complex conjugation of that for $\kappa > 0$. So, for notational ease, we treat in detail the case $\kappa > 0$. In this case

$$\tilde{p}(e^{i\kappa h}) = 1 - \mu\{c_+(1-e^{i\kappa h})^\alpha + c_-(1-e^{-i\kappa h})^\alpha\} \tag{3.11}$$

and for small h by Taylor

$$\begin{aligned}(1-e^{i\kappa h})^\alpha &= (-i\kappa h + O(h^2))^\alpha \\ &= (-i)^\alpha(\kappa h)^\alpha(1+O(h))^\alpha \\ &= e^{-i\frac{\pi\alpha}{2}}(\kappa h)^\alpha + O(h^{\alpha+1})\end{aligned}$$

and

$$(1-e^{-i\kappa h})^\alpha = e^{i\frac{\pi\alpha}{2}}(\kappa h)^\alpha + O(h^{\alpha+1}).$$

Inserting this into (3.7) we find

$$\tilde{p}(e^{i\kappa h}) = 1 - \mu\kappa^\alpha h^\alpha\{c_+ e^{-i\frac{\pi\alpha}{2}} + c_- e^{i\frac{\pi\alpha}{2}}\} + O(h^{\alpha+1}).$$

By use of (1.10) for c_+ and c_- and the complex representation of $\sin\frac{\pi(\alpha\mp\theta)}{2}$ a straightforward calculation yields for (fixed) $\kappa > 0$

$$\tilde{p}(e^{i\kappa h}) = 1 - \mu\kappa^\alpha h^\alpha e^{i\frac{\pi\theta}{2}} + O(h^{\alpha+1}) = 1 - \mu|\kappa|^\alpha h^\alpha e^{i\frac{\pi\theta}{2}} + O(h^{\alpha+1}),$$

for $\kappa < 0$ by complex conjugation

$$\tilde{p}(e^{i\kappa h}) = 1 - \mu|\kappa|^\alpha h^\alpha e^{-i\frac{\pi\theta}{2}} + O(h^{\alpha+1}).$$

So, finally, for all $\kappa \in \mathbb{R}$,

$$\tilde{p}(e^{i\kappa h}) = 1 - \mu|\kappa|^\alpha h^\alpha e^{i(\operatorname{sign}\kappa)\frac{\pi\theta}{2}} + O(h^{\alpha+1}) \tag{3.12}$$

and by (3.9)

$$\begin{aligned}\log \hat{y}(\kappa, t; h) &= \frac{t}{\mu h^\alpha}\left\{ - \mu|\kappa|^\alpha h^\alpha e^{i(\operatorname{sign}\kappa)\frac{\theta\pi}{2}} + O(h^{\alpha+1})\right\}\\ &= -t|\kappa|^\alpha e^{i(\operatorname{sign}\kappa)\frac{\theta\pi}{2}} + O(h),\end{aligned}$$

hence, as desired, (3.10).

In the case (b): $1 < \alpha \leq 2$ and $|\theta| \leq 2 - \alpha$, we have by (3.7)

$$\tilde{p}(e^{i\kappa h}) = 1 - \mu\left\{c_+ e^{-i\kappa h}(1 - e^{i\kappa h})^\alpha + c_- e^{i\kappa h}(1 - e^{-i\kappa h})^\alpha\right\}$$

and in comparison to the case (a) we have because of $e^{\mp i\kappa h} = 1 + O(h)$ within $\{\ldots\}$ the additional asymptotic term

$$O(h)(1 - e^{i\kappa h})^\alpha + O(h)(1 - e^{-i\kappa h})^\alpha = O(h^{\alpha+1}),$$

hence again (3.12) for all $\kappa \in \mathbb{R}$, and again we arrive at (3.10) ∎

It is instructive to take a look at the very special case $\alpha = 2$ and $\theta = 0$ (the classical diffusion equation $u_t = u_{xx}$). In this case

$$\begin{aligned}\tilde{p}(z) &= 1 + \mu\left\{\frac{1}{z} - 2 + z\right\}\\ \tilde{p}(e^{i\kappa h}) &= 1 + \mu\left\{e^{i\frac{\kappa h}{2}} - e^{-i\frac{\kappa h}{2}}\right\}^2 = 1 - 4\mu \sin^2\frac{\kappa h}{2}\end{aligned}$$

and one finds $(\tilde{p}(e^{i\kappa h}))^{\frac{t}{\mu h^2}} \to \exp(-t\kappa^2)$ as $h \to 0$.

4. A random walk model, discrete in space, continuous in time

Consider the difference scheme (2.8) which is equivalent to the redistribution scheme (or random walk model) (2.6) with the coefficients given by (2.3) or (2.4), respectively. By sending the parameter $\mu \to 0$ (letting the time step τ tend to 0) we obtain an infinite system of ordinary differential equations

$$\left.\begin{aligned} y_j'(t) &= {}_hD_\theta^\alpha y_j(t)\\ y_j(0) &\quad \text{given}\end{aligned}\right\}\qquad (j \in \mathbb{Z}) \tag{4.1}$$

describing a time-continuous redistribution scheme over the grid $\{jh|\, j \in \mathbb{Z}\}$ in time $t \geq 0$ of the form

$$y_j'(t) = \sum_{k\in\mathbb{Z}} q_k y_{j-k}(t) \qquad (j \in \mathbb{Z}). \tag{4.2}$$

Interpreting $y_j(t)$ as a clump of an extensive quantity sitting in point $x_j = jh$ at instant t we have, for (4.2) to describe such a redistribution scheme, the balancing conditions

$$\left.\begin{array}{l} q_0 < 0 \\ q_k \geq 0 \quad (0 \neq k \in \mathbb{Z}) \end{array}\right\} \tag{4.3}$$

$$\sum_{k\in\mathbb{Z}} q_k = 0. \tag{4.4}$$

In analogy to our redistribution scheme (2.6) of Section 2 system (4.2) also is conservative and non-negativity preserving. In fact, it can be shown (we leave this as an exercise to the reader) that system (4.2) under conditions (4.3) - (4.4) is uniquely solvable if

$$\sum_{j\in\mathbb{Z}} |y_j(0)| < \infty \tag{4.5}$$

and that then $\sum_{j\in\mathbb{Z}} |y_j(t)| < \infty$ for all $t > 0$. It can further be shown that then

$$\sum_{j\in\mathbb{Z}} y_j'(t) = 0, \qquad \text{hence} \quad \sum_{j\in\mathbb{Z}} y_j(t) \equiv \sum_{j\in\mathbb{Z}} y_j(0)$$

for all $t > 0$. If furthermore $y_j(0) \geq 0 \;\; (j \in \mathbb{Z})$, then $y_j(t) \geq 0 \;\; (j \in \mathbb{Z})$ for all $t > 0$.

The interpretation of (4.2) with (4.3) and (4.4) as a redistribution scheme means: $|q_0| y_j(t)$ is the rate of outflow from the point $x_j = jh$ being transferred to other points, and this must equal the sum of the rates $q_k y_j(t)$, received by the points x_{j+k} $(k \neq 0)$.

Using in (4.1) again the Grünwald-Letnikov discretization (2.9) we find the following.

In the case (a) $0 < \alpha < 1$ and $|\theta| \leq \alpha$:

$$\left.\begin{aligned} q_0 &= -h^{-\alpha}(c_+ + c_-) = -h^{-\alpha}\,\frac{\cos\frac{\theta\pi}{2}}{\cos\frac{\alpha\pi}{2}} \\ q_k &= h^{-\alpha}(-1)^{k+1} c_+ \binom{\alpha}{k} \quad (k \in \mathbb{N}) \\ q_{-k} &= h^{-\alpha}(-1)^{k+1} c_- \binom{\alpha}{k} \quad (k \in \mathbb{N}). \end{aligned}\right\} \tag{4.6}$$

In the case (b) $1 < \alpha \leq 2$ and $|\theta| \leq 2 - \alpha$:

$$\left.\begin{aligned} q_0 &= h^{-\alpha}(c_+ + c_-)\alpha = -\alpha h^{-\alpha}\,\frac{\cos\frac{\theta\pi}{2}}{|\cos\frac{\alpha\pi}{2}|} \\ q_1 &= -h^{-\alpha}\left[c_+\binom{\alpha}{2} + c_-\right], \quad q_{-1} = -h^{-\alpha}\left[c_-\binom{\alpha}{2} + c_+\right] \\ q_k &= (-1)^k h^{-\alpha}\, c_+ \binom{\alpha}{k+1} \quad (k \geq 2) \\ q_{-k} &= (-1)^k h^{-\alpha}\, c_- \binom{\alpha}{k+1} \quad (k \geq 2). \end{aligned}\right\} \tag{4.7}$$

By playing again with infinite sums of binomial coefficients it is readily verified that (4.3) and (4.4) are fulfilled.

For solving system (4.2) with initial values $y_j(0)$ $(j \in \mathbb{Z})$ with $\sum_j |y_j(0)| < \infty$ and q_k given by (4.6) - (4.7), we apply the method of generating functions. The series

$$\tilde{q}(z) = \sum_{k\in\mathbb{Z}} q_k z^k \qquad \text{and} \qquad \tilde{y}(z,t) = \sum_{k\in\mathbb{Z}} y_k(t) z^k \tag{4.8}$$

converge absolutely and uniformly on the periphery $|z| = 1$ of the unit circle, and system (4.2) is then (with $|z| = 1$) equivalent to

$$\frac{\partial \tilde{y}(z,t)}{\partial t} = \sum_{j\in\mathbb{Z}} y_j'(t)\, z^j = \sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} q_k\, y_{j-k}(t) \right) z^j ,$$

hence we have the z-parameterized ordinary differential equation problem

$$\left. \begin{aligned} \frac{\partial \tilde{y}(z,t)}{\partial t} &= \tilde{q}(z)\tilde{y}(z,t) \quad (t \geq 0,\ |z| = 1) \\ \tilde{y}(z,0) &= \sum_{k\in\mathbb{Z}} y_k(0)\, z^k. \end{aligned} \right\}$$

The solution is $\tilde{y}(z,t) = \tilde{y}(z,0)\, e^{t\tilde{q}(z)}$, or simply

$$\tilde{y}(z,t) = e^{t\tilde{q}(z)} \tag{4.9}$$

in the special case $y_j(0) = \delta_{j0}$ for $j \in \mathbb{Z}$ which means $\tilde{y}(z,0) \equiv 1$.

By inspection (using the binomial series) we see that

$$\tilde{q}(z) = \begin{cases} -h^{-\alpha}\big\{c_+(1-z)^\alpha + c_-(1-z^{-1})^\alpha\big\} & \text{in the case (a)} \\ -h^{-\alpha}\big\{c_+z^{-1}(1-z)^\alpha + c_-z(1-z^{-1})^\alpha\big\} & \text{in the case (b).} \end{cases}$$

Changing to characteristic functions via $z = e^{i\kappa h}$ $(\kappa \in \mathbb{R})$, we take from our calculations of Section 3 for small h

$$\tilde{q}(e^{i\kappa h}) = -|\kappa|^\alpha e^{i(\operatorname{sign}\kappa)\frac{\pi\theta}{2}} + O(h).$$

Then with (4.9) we get for $\hat{y}(\kappa,t;h) := \tilde{y}(e^{i\kappa h},t)$ in analogy to (3.10) the limit relation

$$\lim_{h\to 0} \hat{y}(\kappa,t;h) = \exp\big(-t|\kappa|^\alpha e^{i(\operatorname{sign}\kappa)\frac{\theta\pi}{2}}\big) = \hat{g}_\alpha(\kappa,t;\theta) \qquad (\kappa \in \mathbb{R}).$$

We have interpreted (4.2) as a time-continuous redistribution scheme. We can interpret it probabilistically as a random walk model discrete in space (over the grid $\{jh|\, j \in \mathbb{Z}\}$), but continuous in time. At any instant t of time the random walker can jump to another grid point. After arriving at a point x_m he will remain sitting there for a random time interval whose length is exponentially distributed. More precisely:

when we know that at instant t^* he is sitting at point x_m, then the conditional sojourn probabilities for sitting at points x_j are $\eta_j(t^*) = \delta_{mj} \quad (j \in \mathbb{Z})$ and (4.2) gives by re-conditioning the equation

$$\eta'_m(t) = -|q_0|\,\eta_m(t),$$

for the time interval $[t^*,\ t^*+\tilde{t}\,]$ of sojourn at x_m. Its solution is $\eta_m(t) = 1 - e^{-|q_0|(t-t^*)}$ $(t \geq t^*)$, from which we deduce that the time $\tilde{t}$ the wanderer remains sitting at any point x_m is exponentially distributed with parameter $|q_0|$. Hence, the random walker, after arriving at point x_m sits there for a random time interval of length $\tilde{t}$ and then jumps to another point x_j in instant $t = t^* + \tilde{t}$. The conditional probability of jumping to the point x_j (with $j \neq m$) is then given as $\frac{q_{j-m}}{|q_0|}$. For general information on time-continuous discrete Markov processes we refer the reader to [15]. It should finally be remarked that the conditional density $\eta_m(t) \quad (t \geq t^*)$ can also be obtained in the limit of $\tau \to 0$ from the conditional geometric probability distribution relevant in the random walk model (2.6) with the transition probabilities of (2.3) - (2.4) and the scaling condition (2.7).

We can now state a theorem analogous to Theorem 3.2, namely

Theorem 4.1. *Let a random walker start in point* 0 *at instant* $t = 0$ *and jump over the grid points* $jh \quad (j \in \mathbb{Z})$ *with* $h > 0$*, the probabilities* $y_j(t)$ *of sojourn in point* jh *at instant* $t \geq 0$ *evolving according to* (4.2) *with* $y_j(0) = \delta_{j0}$*. Then for fixed* $t > 0$ *the distribution function* $G_\alpha(\cdot, t; \theta; h)$ *given as* $G_\alpha(x, t; \theta; h) = \sum_{kh \leq x} y_k(t)$ *converges completely to* $G_\alpha(\cdot, t; \theta)$ *as* $h \to 0$.

5. A random walk model for the Cauchy process

For completeness we present a random walk model, discrete in space and time, for the omitted case $\alpha = 1$, namely for the Cauchy process ($\alpha = 1$ and $\theta = 0$). This model cannot be obtained via the Grünwald-Letnikov approach, neither directly nor by a passage to the limit $\alpha \to 1$. We have, with the notations of Sections 1 - 3 the process

$$g_1(x, t; 0) = \frac{1}{\pi}\frac{t}{x^2 + t^2} = t^{-1} p_1\Big(\frac{x}{t}; 0\Big) \qquad (x \in \mathbb{R},\ t > 0)$$

with the Cauchy probability density

$$p_1(x, 0) = \frac{1}{\pi(x^2 + 1)} \qquad (x \in \mathbb{R}).$$

The corresponding characteristic functions are

$$\hat{p}_1(\kappa; 0) = e^{-|\kappa|} \qquad \text{and} \qquad \hat{g}_1(\kappa, t; 0) = e^{-t|\kappa|}.$$

Let Y be a random variable assuming its values in $\mathbb{Z}$ with $P(Y = k) = p_k$ for $k \in \mathbb{Z}$ defined as follows. With a parameter μ restricted to $0 < \mu \leq \frac{\pi}{2}$ put

$$\left.\begin{aligned} p_0 &= 1 - \frac{2\mu}{\pi} \\ p_k &= \frac{\mu}{\pi|k|(|k|+1)} \quad (0 \neq k \in \mathbb{Z}). \end{aligned}\right\} \tag{5.1}$$

Then all $p_k \geq 0$ and $\sum_{k\in\mathbb{Z}} p_k = 1$. Indeed,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \Big(\frac{1}{k} - \frac{1}{k+1}\Big) = 1.$$

Proceeding as in Sections 2 and 3 we produce a random walk on the grid $\{jh|\, j \in \mathbb{Z}\}$, with $h > 0$, letting the walker start in point 0 at instant 0. We define

$$S_n = hY_1 + hY_2 + \ldots + hY_n \qquad (n \in \mathbf{N})$$

where the Y_k are independent identically distributed random variables all having their distribution common with the random variable Y. By allowing the walker to jump in instants $t_n = n\tau$ $(\tau > 0,\, n \in \mathbf{N})$ and using S_n as the random variable assuming values in the grid $\{jh|\, j \in \mathbb{Z}\}$ we have a random walk. We relate the steps h and τ of space and time by $\tau = \mu h$ and again define generating functions

$$\tilde{p}(z) = \sum_{j\in\mathbb{Z}} p_j z^j \qquad \text{and} \qquad \tilde{y}(z, t_n) = \sum_{j\in\mathbb{Z}} y_j(t_n)\, z^j \tag{5.2}$$

with $y_j(t_n)$ as probability of sojourn in point $x_j = jh$ at instant $t_n = n\tau$. Then $\tilde{y}(z, t_n) = (\tilde{p}(z))^n$, and in view of $\frac{t}{\tau} = \frac{t}{\mu h}$ our aim is to show that for all $\kappa \in \mathbb{R}$ and $t > 0$ the limit relation

$$\tilde{p}(e^{i\kappa h})^{\frac{t}{\mu h}} \to e^{-t|\kappa|} \qquad \text{as} \ \ h \to 0 \tag{5.3}$$

holds. This limit relation then implies that *the distribution of the random variable Y lies in the domain of attraction of the Cauchy distribution, more precisely that the distribution of the random variable*

$$X_n = \frac{t}{\mu n}\{Y_1 + Y_2 + \ldots + Y_n\}$$

converges completely to the Cauchy distribution $G_1(\cdot, t; 0)$ with

$$G_1(x, t; 0) = \int_{-\infty}^{x} \frac{t}{\pi(t^2 + \xi^2)}\, d\xi = \frac{1}{2} + \frac{1}{\pi} \arctan \frac{x}{t}.$$

Let us prove (5.3). We observe that the series $\sum_{j\in\mathbb{Z}} p_j\, z^j$ converges absolutely and uniformly on the periphery $|z| = 1$ of the unit circle, hence represents there a continuous function so that the following calculations are legitimate. In fact,

$$\sum_{k=1}^{\infty} \frac{z^k}{k(k+1)} = 1 - (1 - z^{-1}) \log(1 - z) \qquad \text{for} \ \ |z| \leq 1.$$

We tacitly take the limit 1 for $z = 1$. Then

$$\tilde{p}(z) = 1 - \frac{\mu}{\pi}\{(1 - z^{-1}) \log(1 - z) + (1 - z) \log(1 - z^{-1})\}. \tag{5.4}$$

Passing to the characteristic function via $z = e^{i\kappa h}$ $(\kappa \in \mathbb{R})$ we get

$$\tilde{p}(e^{i\kappa h}) = 1 - \frac{2\mu}{\pi} \left\{ (1 - \cos(\kappa h)) \log|1 - e^{i\kappa h}| - \sin(\kappa h) \ \arctan \frac{\sin(\kappa h)}{1 - \cos(\kappa h)} \right\}.$$

Using $\lim_{u\to\pm\infty} \arctan u = \pm\frac{\pi}{2}$ we obtain for $h \to 0$ the asymptotics

$$\tilde{p}(e^{i\kappa h}) = 1 - \frac{2\mu}{\pi} \left\{ |\kappa| h \frac{\pi}{2} + o(|\kappa| h) \right\} = 1 - \mu|\kappa|h + o(h)$$

which implies (5.3).

6. Concluding remarks

It is instructive to observe that in view of Theorem 3.1 our proof of Theorem 3.2 can be re-interpreted as a proof of existence of Lévy's stable distributions (for $\alpha \neq 1$). In fact, assuming to be ignorant of these we can find them as limiting distributions by sending $n \to \infty$ in (3.3). And gratis (the discrete probabilities being all non-negative cannot become negative in the limit) we get that the limiting densities are everywhere non-negative, for all values of the parameter α between 0 and 2, with omission of the value 1. For this we actually need neither the theory of the inversion of the Feller potentials nor the method of positive-definite functions. Thus we have an alternative way of solving a problem that surmounted Cauchy's capabilities [2] who had considered the functions $\exp(-|\kappa|^\alpha)$ as candidates of cosine transforms of probability densities but could only prove them to have this property in the special cases $\alpha = 1$ and $\alpha = 2$. Lévy in [11], [12] introduced the whole scale of stable densities, Bochner in [1] has given an elegant proof for the full range $0 < \alpha \leq 2$ that the inverse Fourier transforms of the functions $\exp(-|\kappa|^\alpha)$ are non-negative, hence probability densities. He used the theory of positive-definite functions that we can avoid. A well readable account of Bochner's method can be looked up in [13].

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