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Determining the Relaxation Kernel in Nonlinear One-Dimensional Viscoelasticity

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Dedicated to L. von Wolfersdorf on the occasion of his 65th birthday

Abstract. We consider a viscoelastic string whose mechanical behavior is governed by a nonlinear stress-strain relationship. This constitutive law is characterized by a time-dependent relaxation kernel k which is assumed to be unknown. The resulting motion equation is then associated with initial and Dirichlet boundary conditions. We show that the traction measurement at one end allows to identify k. More precisely, we prove an existence and uniqueness result on a small time interval. Also, we show how the solution continuously depends on the data.

Keywords: *Inverse problems, viscoelasticity of integral type, hyperbolic integro-differential equations*

AMS subject classification: 35 R 30, 45 K 05, 73 F 05, 73 F 15

1. Introduction

Consider a viscoelastic string of length $L > 0$ and indicate by $u(x,t)$ its transversal displacement at point $x \in [0,L]$ and at time $t \in [0,T]$, $T > 0$ being a fixed final time. Denote $Q_T = (0,L) \times (0,T)$. A quite general stress-strain relationship which describes the mechanical behavior of the string has the form (see, e.g., [9, 17] and references therein)

$$\sigma(u_x)(x,t) = \varphi(x, u_x(x,t)) + \int_0^t k(\tau)\psi\big(x, u_x(x,t-\tau)\big)\,d\tau \tag{1.1}$$

for all $(x,t) \in Q_T$ where φ and ψ are suitable given functions, k is the so-called *relaxation kernel*, and the string is supposed to be at rest for $t < 0$. Denoting by ϱ the string mass density, the evolution of u is then ruled by the Volterra integro-differential equation

$$\varrho u_{tt} - (\sigma(u_x))_x = f \qquad \text{in } Q_T \tag{1.2}$$

where f is an external force. Here ϱ is a smooth and strictly positive function.

An inverse problem which typically arises in applications regards the possibility of determining the relaxation kernel k through measurements related to u. A possible

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formulation of this problem takes advantage of equation (1.2). More precisely, if a set of initial and boundary conditions is associated with (1.2), then an additional condition is considered (e.g., u is known for some $x_0 \in [0, L]$ and any $t \in [0, T]$) in order to identify k. Consequently, the identification problem consists in finding a pair (u, k) which satisfies equation (1.2) and fulfils the conditions mentioned above. When φ and ψ are both linear with respect to u_x, this kind of inverse problem has been extensively studied by several authors during these last years (cf. [3, 5, 12 - 16, 18]; see also [1, 2, 5, 6] for multi-dimensional models). On the contrary, the most difficult nonlinear case has received much less attention. In this respect, the only result we are aware of concerns the case in which φ is linear with respect to u_x (see [16]). There, by assuming Neumann homogeneous boundary conditions (that is, free ends) and u known at some point $x_0 \in [0, L]$, for any $t \in [0, T]$, as additional condition, the author proves local (in time) existence, uniqueness, and continuous dependence on the data. Here we want to show that similar results can be obtained under weaker assumptions by using an alternative approach which allows us to deal directly with classical (and not variational) solutions and by changing the boundary and the additional conditions. Of course, the most interesting (and hard) case, namely when φ is nonlinear with respect to u_x as well, remains open. However, it is worth recalling that, when k is known, there are some results about the well-posedness of initial and boundary value problems for (1.2) (see, e.g., [17] and references therein). Several results are also available for the direct problem in our simpler setting (see [7, 8, 10, 11]).

On account of what we have just observed, we assume

$$\varphi(x,z) = \varphi_0(x) z \qquad \big((x,z) \in [0,L] \times \mathbb{R}\big) \tag{1.3}$$

where φ_0 is a smooth function with strictly positive derivative. Then equation (1.2) can be rewritten in this form

$$\varrho(x) u_{tt}(x,t) - \left(\varphi_0(x) u_x(x,t) + \int_0^t k(\tau)\psi\big(x, u_x(x,t-\tau)\big)d\tau\right)_x = f(x,t) \tag{1.4}$$

for $(x,t) \in Q_T$. Then we introduce the usual initial data

$$\left.\begin{aligned} u(x,0) &= u_0(x) \\ u_t(x,0) &= u_1(x) \end{aligned}\right\} \qquad (x \in [0,L]) \tag{1.5}$$

and we take Dirichlet boundary conditions (not necessarily homogenenous)

$$\left.\begin{aligned} u(0,t) &= \alpha(t) \\ u(L,t) &= \beta(t) \end{aligned}\right\} \qquad (t \in [0,T]). \tag{1.6}$$

We further suppose that the traction exerted at one end is known, that is

$$\varphi_0(0) u_x(0,t) + \int_0^t k(\tau)\psi\big(0, u_x(0,t-\tau)\big)\, d\tau = g(t) \tag{1.7}$$

for all $t \in [0,T]$. Here, u_0, u_1, α, β and g are given functions.

Therefore, what we want to study in this paper is the following

Problem (P). Find a pair (u, k) satisfying (1.4) - (1.7).

Note that, due to the assumption on φ_0, (1.4) is a nonlinear Volterra integro-differential equation of hyperbolic type. To solve problem (P), we first write down the corresponding problem for the pair (v, k), where $v = u_t$. Then by differentiating equation (1.7) we observe that k has to solve a Volterra integral equation involving the traces of u_x and v_x at $x = 0$. This fact allows us to formulate a further equivalent problem for the triplet (u, v, k), provided that the initial datum satisfies a suitable non-vanishing condition. More precisely, we shall deal with an initial and boundary value problem for a system of two nonlinear Volterra integro-differential hyperbolic equations coupled with a Volterra integral equation of the second kind. We can uniquely solve that problem for T sufficiently small. Moreover, continuous dependence on the data can be established. The plan of the paper goes as follows. The main results are stated in the next section. Section 3 is basically devoted to establish the equivalence result mentioned above. Sections 4 and 5 contain the proofs of the main theorems, while in Section 6 we report the proof of a technical lemma.

2. Assumptions and main results

Let us suppose the following:

$$\varrho \in C^1([0,L]) \quad \text{with} \quad \varrho(x) \geq \varrho_0 > 0 \quad \text{for all} \quad x \in [0,L] \tag{2.1}$$

$$\varphi_0 \in C^1([0,L]) \quad \text{with} \quad \varphi_0(x) \geq c_0 > 0 \text{ for all } \ x \in [0,L] \tag{2.2}$$

$$\psi \in C^1([0,L] \times \mathbb{R}) \tag{2.3}$$

$$\psi_{xz}, \psi_{zz} \in C^0([0,L] \times \mathbb{R}) \tag{2.4}$$

$$\begin{gathered}\text{For all} \quad M > 0 \quad \text{there exists} \quad \Lambda_0(M) > 0 \text{ such that} \\ |\psi_{zz}(x,z_1) - \psi_{zz}(x,z_2)| + |\psi_{xz}(x,z_1) - \psi_{xz}(x,z_2)| \leq \Lambda_0(M)|z_1 - z_2| \\ \text{for all} \quad x \in [0,L] \quad \text{and all} \quad z_1, z_2 \in \mathbb{R} \quad \text{with} \quad |z_1| + |z_2| \leq M\end{gathered} \tag{2.5}$$

$$f \in W^{2,1}\big(0,T; C^0([0,L])\big) \tag{2.6}$$

$$u_0, u_1 \in C^2([0,L]) \tag{2.7}$$

$$\varphi_0 u_0'' + f(\cdot, 0) \in C^1([0,L]) \tag{2.8}$$

$$\alpha, \beta \in W^{4,1}(0,T) \tag{2.9}$$

$$g \in W^{2,1}(0,T). \tag{2.10}$$

In addition, we assume the following compatibility relations:

$$\alpha(0) = u_0(0) \quad \text{and} \quad \beta(0) = u_0(L) \tag{2.11}$$

$$\alpha'(0) = u_1(0) \quad \text{and} \quad \beta'(0) = u_1(L) \tag{2.12}$$

$$\varrho(0)\alpha''(0) = \varphi_0(0)u_0''(0) + \varphi''(0)u'(0) + f(0,0) \tag{2.13}$$

$$\varrho(L)\beta''(0) = \varphi_0(L)u_0''(L) + \varphi''(L)u'(L) + f(L,0) \tag{2.14}$$

$$g(0) = \varphi_0(0)u_0'(0). \tag{2.15}$$

Then we introduce a rigorous formulation of problem (P).

Problem (P). Find a pair (u,k) satisfying

$$u, u_t \in C^2(\bar{Q}_T) \tag{2.16}$$
$$k \in W^{1,1}(0,T) \tag{2.17}$$

and (1.4) - (1.7).

Our first result regarding local (in time) existence and uniqueness is given by

Theorem 2.1. *Let* (2.1) – (2.15) *hold and set*

$$\gamma = \psi(0, u_0'(0)). \tag{2.18}$$

If

$$\gamma \neq 0 \tag{2.19}$$

and the compatibility conditions

$$\begin{aligned} \varrho(0)\alpha^{(3)}(0) = \varphi_0(0)u_1''(0) + \varphi_0'(0)u_1'(0) + f_t(0,0) \\ + k_0\big[\psi_z(0,u_0'(0))u_0''(0) + \psi_x(0,u_0'(0))\big] \end{aligned} \tag{2.20}$$

$$\begin{aligned} \varrho(L)\beta^{(3)}(0) = \varphi_0(L)u''(L) + \varphi_0'(L)u_1'(L) + f_t(L,0) \\ + k_0\big[\psi_z(L,u_0'(L))u_0''(L) + \psi_x(L,u_0'(L))\big] \end{aligned} \tag{2.21}$$

hold where

$$k_0 = \gamma^{-1}\left[g'(0) - \varphi_0(0)u_1'(0)\right], \tag{2.22}$$

then there exists $T_0 \in (0,T]$ *such that problem* (P) *has a unique solution.*

The proof of this theorem will be given in Section 4. Moreover, in Section 5 we will prove that the solution to problem (P) continuously depends on the data. Indeed, we have the following

Theorem 2.2. *Let* $\{f_j, u_{0j}, u_{1j}, \alpha_j, \beta_j, g_j\}$ $(j=1,2)$ *be two sets of functions satisfying* (2.1) – (2.15) *and* (2.19 – (2.21). *Denote by* (u_j, k_j) *the corresponding solution to problem* (P) *and consider two positive constants* C_1, C_2 *such that*

$$\begin{aligned} &\max_{j\in\{1,2\}} \Big\{ \|(f_j)_{tt}\|_{L^1(0,T;C^0([0,L]))}, \|(f_j)_t(\cdot,0)\|_{C^0([0,L])}, \\ &\qquad \|f_j(\cdot,0)\|_{C^0([0,L])}, \|u_{0j}\|_{C^2([0,L])}, \|u_{1j}\|_{C^2([0,L])}, \\ &\qquad \|\varphi_0 u_{0j}'' + f_j(\cdot,0)\|_{C^1([0,L])}, |g_j'(0)|, \|\alpha_j^{(4)}\|_{L^1(0,T)}, \|\beta_j^{(4)}\|_{L^1(0,T)} \Big\} \\ &\leq C_1 \end{aligned} \tag{2.23}$$

and

$$\max_{j\in\{1,2\}} \|k_j'\|_{L^1(0,T)} \leq C_2 \tag{2.24}$$

where k_{0j} *is defined by* (2.22) *with* g *and* u_1' *replaced by* g_j *and* u_{1j}', *respectively, and* γ *substituted with* γ_j *defined by* (2.18) *with* u_0 *in place of* u_{0j} $(j=1,2)$. *Assume that*

$$|\psi_x(x,z)| \leq c_1 + c_2|z| \qquad \big((x,z) \in (0,L)\times\mathbb{R}\big) \tag{2.25}$$

for some positive constants c_1 *and* c_2, *and*

$$\psi_z \in L^\infty((0,L)\times\mathbb{R}). \tag{2.26}$$

Then there exists a function $\Lambda_1 \in C^0\big((0,+\infty)^4;(0,+\infty)\big)$ *such that*

$$\begin{aligned}
&\|u_1-u_2\|_{C^2(\bar Q_T)} + \|(u_1-u_2)_t\|_{C^2(\bar Q_T)} + \|k_1-k_2\|_{W^{1,1}(0,T)}\\
&\quad\le \Lambda_1(\mu,C_1,C_2,T)\Big\{\|(f_1-f_2)_{tt}\|_{L^1(0,T;C^0([0,L]))}\\
&\qquad + \|(f_1-f_2)_t(\cdot,0)\|_{C^0([0,L])} + \|(f_1-f_2)(\cdot,0)\|_{C^0([0,L])}\\
&\qquad + \|u_{01}-u_{02}\|_{C^2([0,L])} + \|u_{11}-u_{12}\|_{C^2([0,L])}\\
&\qquad + \|\varphi_0(u_{01}''-u_{02}'') + (f_1-f_2)(\cdot,0)\|_{C^1([0,L])}\\
&\qquad + \|(\alpha_1-\alpha_2)^{(4)}\|_{L^1(0,T)} + \|(\beta_1-\beta_2)^{(4)}\|_{L^1(0,T)}\\
&\qquad + |(g_1-g_2)'(0)| + \|(g_1-g_2)''\|_{L^1(0,T)}\Big\}
\end{aligned} \tag{2.27}$$

where $\mu = \min\{|\gamma_1|^{-1}, |\gamma_2|^{-1}\}$. *Moreover,* Λ_1 *is non-decreasing in each of its variables and also depends on* $L, \varrho, \varrho_0, \varphi_0, \psi, c_0, c_1, c_2$.

Remark 2.1. Assumptions (2.25) - (2.26) allow to obtain a bound for u_j and its time derivative in $C^2(\bar Q_T)$ taking advantage of (2.23) - (2.24). In place of (2.25) - (2.26) we can suppose to have an *a priori* bound on $(u_j)_x$ in $C^0(\bar Q_T)$ (see below Section (5.12) - (5.14)).

3. An equivalent problem and a preliminary lemma

Let us assume that problem (P) has a solution (u,k). Then differentiate equations (1.4) and (1.7) with respect to time. Setting

$$v = u_t \tag{3.1}$$

we obtain

$$\begin{aligned}
&\varrho(x)v_{tt}(x,t) - (\varphi_0(x)v_x(x,t))_x\\
&\qquad - \int_0^t k(\tau)\Big[\psi_{xz}(x,u_x(x,t-\tau))v_x(x,t-\tau)\\
&\qquad + \psi_{zz}(x,u_x(x,t-\tau))u_{xx}(x,t-\tau)v_x(x,t-\tau)\\
&\qquad + \psi_z(x,u_x(x,t-\tau))v_{xx}(x,t-\tau)\Big]d\tau\\
&\quad = f_t(x,t) + k(t)\Big[\psi_z(x,u_0'(x))u_0''(x) + \psi_x(x,u_0'(x))\Big]
\end{aligned} \tag{3.2}$$

for all $(x,t)\in Q_T$ and

$$\begin{aligned}
&\varphi_0(0)v_x(0,t) + \psi(0,u_0'(0))k(t)\\
&\qquad + \int_0^t k(\tau)\psi_z(0,u_x(0,t-\tau))v_x(0,t-\tau)\,d\tau = g'(t)
\end{aligned} \tag{3.3}$$

for all $t \in [0,T]$. Then, from (1.5) and (1.4) with $t=0$, we derive the initial conditions

$$\left.\begin{aligned} v(x,0) &= v_0(x) \\ v_t(x,0) &= v_1(x) \end{aligned}\right\} \qquad (x \in [0,L]) \tag{3.4}$$

where

$$\left.\begin{aligned} v_0(x) &= u_1(x) \\ v_1(x) &= (\varrho(x))^{-1}\big[\varphi_0(x)u_0''(x) + \varphi_0'(x)u_0'(x) + f(x,0)\big] \end{aligned}\right\} \tag{3.5}$$

for any $x \in [0,L]$. Note that, recalling (2.1) - (2.2) and (2.5) - (2.7),

$$\left.\begin{aligned} v_0 &\in C^2(\bar{Q}_T) \\ v_1 &\in C^1(\bar{Q}_T) \end{aligned}\right\} \tag{3.6}$$

follows. Regarding the boundary conditions, on account of (1.6) we have

$$\left.\begin{aligned} v(0,t) &= \tilde{\alpha}(t) \\ v(L,t) &= \tilde{\beta}(t) \end{aligned}\right\} \qquad (t \in [0,T]) \tag{3.7}$$

where

$$\left.\begin{aligned} \tilde{\alpha}(t) &= \alpha'(t) \\ \tilde{\beta}(t) &= \beta'(t) \end{aligned}\right\} \qquad (t \in [0,T]). \tag{3.8}$$

Then due to (2.8) observe that

$$\tilde{\alpha}, \tilde{\beta} \in W^{3,1}(0,T). \tag{3.9}$$

On the other hand, from equation (3.3) we infer (cf. (2.2))

$$k(0) = k_0. \tag{3.10}$$

Moreover, thanks to (2.19), equation (3.3) can be rewritten in the form

$$k = \gamma^{-1}\big[k * N_1(u,v) + N_2(v) + g'\big] \qquad \text{in } [0,T] \tag{3.11}$$

where $*$ denotes the time convolution product over $(0,t)$ and

$$N_1(\tilde{u},\tilde{v})(t) = -\psi_z(0,\tilde{u}_x(0,t))\tilde{v}_x(0,t) \tag{3.12}$$

$$N_2(\tilde{v})(t) = -\varphi_0(0)\tilde{v}_x(0,t) \tag{3.13}$$

for any $t \in [0,T]$ and any $\tilde{u},\tilde{v} \in C^1(\bar{Q}_T)$. Set now

$$h = k' \qquad \text{a.e. in } (0,T) \tag{3.14}$$

and note that (cf. (3.10))

$$k = k_0 + 1 * h \qquad \text{in } [0,T]. \tag{3.15}$$

Then equation (3.11) becomes

$$k_0 + 1 * h = \gamma^{-1}\big[(k_0 + 1 * h) * N_1(u,v) + N_2(v) + g'\big] \qquad \text{in } \ [0,T],$$

and differentiating it with respect to time we obtain

$$h = \gamma^{-1}\big[k_0 N_1(u,v) + h * N_1(u,v) + N_2(v_t) + g''\big] \qquad \text{a.e. in } \ [0,T]. \tag{3.16}$$

Taking advantage of (3.15), we can write down equations (1.4) and (3.2) this way as

$$u_{tt} - au_{xx} - bu_x = (k_0 + 1 * h) * \mathcal{R}_1(u) + F \tag{3.17}$$

$$v_{tt} - av_{xx} - bv_x = (k_0 + 1 * h) * \mathcal{R}_2(u,v) + (1 * h)c + k_0 c + F_t \tag{3.18}$$

in Q_T, where a, b, c, F are defined by, respectively,

$$a(x) := (\varrho(x))^{-1}\varphi_0(x), \quad b(x) = (\varrho(x))^{-1}\varphi_0'(x) \quad \forall\, x \in [0,L] \tag{3.19}$$

$$c(x) := (\varrho(x))^{-1}[\psi_z(x, u_0'(x))u_0''(x) + \psi_x(x, u_0'(x))] \quad \forall\, x \in [0,L] \tag{3.20}$$

$$F(x,t) = (\varrho(x))^{-1} f(x,t) \quad \forall\,(x,t) \in \bar{Q}_T \tag{3.21}$$

for all $(x,t) \in \bar{Q}_T$, while $\mathcal{R}_1, \mathcal{R}_2$ are given by

$$\mathcal{R}_1(\tilde{u})(x,t) = (\varrho(x))^{-1}\Big[\psi_x(x, \tilde{u}_x(x,t)) + \psi_z(x, \tilde{u}_x(x,t))\tilde{u}_{xx}(x,t)\Big] \tag{3.22}$$

$$\begin{aligned}\mathcal{R}_2(\tilde{u},\tilde{v})(x,t) = (\varrho(x))^{-1}\Big[&\psi_{xz}(x, \tilde{u}_x(x,t))\tilde{v}_x(x,t) + \psi_{zz}(x, \tilde{u}_x(x,t)) \\ &\times \tilde{u}_{xx}(x,t)\tilde{v}_x(x,t) + \psi_z(x, \tilde{u}_x(x,t))\tilde{v}_{xx}(x,t)\Big]\end{aligned} \tag{3.23}$$

for any $(x,t) \in \bar{Q}_T$ and all $\tilde{u}, \tilde{v} \in C^2(\bar{Q}_T)$.

We have thus shown that the triplet (u,v,h) is a solution to the following

Problem (P1). Find a triplet $(u,v,h) \in (C^2(\bar{Q}_T))^2 \times L^1(0,T)$ solving equations (3.16) - (3.18) and fulfilling conditions (1.5) - (1.6), (3.4) and (3.7).

Conversely, taking the compatibility relations (2.13) - (2.15) and (2.20) - (2.21) into account, one can also prove that if (u,v,h) solves problem (P1), then (u,k) is a solution to problem (P), where k is given by (3.15).

Summing up, we have

Proposition 3.1. *Let* (2.1) – (2.15) *and* (2.19) – (2.21) *hold. Then problem* (P) *has a unique solution if and only if problem* (P1) *has a unique solution.*

We conclude this section by reporting for the reader's convenience a quite standard result which is a slight generalization of [7: Theorem 2.3], namely

Lemma 3.1. *Let*

$$\varepsilon \in C^1([0,L]) \quad \text{with} \quad \varepsilon(x) \geq \varepsilon_0 > 0 \quad \text{for all} \quad x \in [0,L] \tag{3.24}$$
$$\eta \in C^0([0,L]) \tag{3.25}$$
$$\ell \in W^{1,1}(0,T;C^0([0,L])) \tag{3.26}$$
$$w_0 \in C^2([0,L]) \quad \text{and} \quad w_1 \in C^1([0,L]) \tag{3.27}$$
$$p, q \in W^{3,1}(0,T) \tag{3.28}$$
$$p(0) = w_0(0) \quad \text{and} \quad q(0) = w_0(L) \tag{3.29}$$
$$p'(0) = w_1(0) \quad \text{and} \quad q'(0) = w_1(L) \tag{3.30}$$
$$p''(0) = \varepsilon(0)w_0''(0) + \eta(0)w_0'(0) + \ell(0,0) \tag{3.31}$$
$$q''(0) = \varepsilon(L)w_0''(L) + \eta(L)w_0'(L) + \ell(L,0). \tag{3.32}$$

Then there exists a unique $w \in C^2(\bar{Q}_T)$ *such that*

$$w_{tt} - \varepsilon w_{xx} - \eta w_x = \ell \qquad \text{in} \quad Q_T \tag{3.33}$$

and the initial condition

$$\left.\begin{aligned} w(x,0) &= w_0(x) \\ w_t(x,0) &= w_1(x,0) \end{aligned}\right\} \qquad (x \in [0,L]) \tag{3.34}$$

as well as the boundary condition

$$\left.\begin{aligned} w(0,t) &= p(t) \\ w(L,t) &= q(t) \end{aligned}\right\} \qquad (t \in [0,T]) \tag{3.35}$$

are fulfilled. Moreover, there exists a positive constant C_3 *which only depends on* $L, \|\varepsilon\|_{C^1([0,L])}, \varepsilon_0$ *and* $\|\eta\|_{C^0([0,L])}$ *such that, for any* $t \in [0,T]$,

$$\begin{aligned} \|w\|_{C^2(\bar{Q}_t)} \leq C_3 \Big\{ (1+t) \Big[\|\ell_t\|_{L^1(0,t;C^0([0,L]))} + \|\ell(\cdot,0)\|_{C^0([0,L])} \Big] \\ + \|w_0\|_{C^2([0,L])} + \|w_1\|_{C^1([0,L])} + \|p^{(3)}\|_{L^1(0,t)} + \|q^{(3)}\|_{L^1(0,t)} \Big\}. \end{aligned} \tag{3.36}$$

This lemma, whose proof is given in Section 6, will be very useful in the sequel.

4. Proof of Theorem 2.1

We are going to solve problem (P1) locally in time by using the Contraction Mapping Principle. Let us set

$$X_T = (C^2(\bar{Q}_T))^2 \times L^1(0,T).$$

We endow X_T with the norm

$$\|(\tilde{u}, \tilde{v}, \tilde{h})\|_{X_T} = \|\tilde{u}\|_{C^2(\bar{Q}_T))} + \|\tilde{v}\|_{C^2(\bar{Q}_T))} + \|\tilde{h}\|_{L^1(0,T)} \tag{4.1}$$

and introduce the bounded subset of X_T

$$B(E,T) = \left\{(\tilde{u},\tilde{v},\tilde{h}) \in X(T) \,\middle|\, \|(\tilde{u},\tilde{v},\tilde{h})\|_{X_T} \le E\right\}$$

for some positive constant E. This set is clearly a complete metric space with respect to the metric induced by the norm of X_T. Fix $(\tilde{u},\tilde{v},\tilde{h}) \in X_T$ and set (cf. (3.16))

$$h = \mathcal{H}(\tilde{u},\tilde{v},\tilde{h}) := \gamma^{-1}\big[k_0 N_1(\tilde{u},\tilde{v}) + \tilde{h} * N_1(\tilde{u},\tilde{v}) + N_2(\tilde{v}_t) + g''\big] \tag{4.2}$$

a.e. in $[0,T]$. Recalling (2.3), (2.10) and (3.12) - (3.13), one easily realizes that

$$h \in L^1(0,T). \tag{4.3}$$

Consider then the following problem (cf. (3.17) - (3.18)).

Problem (P2). Find a pair $(u,v) \in (C^2(\bar{Q}_T))^2$ satisfying (1.5) - (1.6), (3.4) and (3.6) and such that

$$u_{tt} - au_{xx} - bu_x = \mathcal{U}(\tilde{u},\tilde{v},\tilde{h}) \tag{4.4}$$

$$v_{tt} - av_{xx} - bv_x = \mathcal{V}(\tilde{u},\tilde{v},\tilde{h}) \tag{4.5}$$

where

$$\mathcal{U}(\tilde{u},\tilde{v},\tilde{h}) = \big(k_0 + 1 * \mathcal{H}(\tilde{u},\tilde{v},\tilde{h})\big) * \mathcal{R}_1(\tilde{u}) + F \tag{4.6}$$

$$\mathcal{V}(\tilde{u},\tilde{v},\tilde{h}) = \big(k_0 + 1 * \mathcal{H}(\tilde{u},\tilde{v},\tilde{h})\big) * \mathcal{R}_2(\tilde{u},\tilde{v}) + \big(1 * \mathcal{H}(\tilde{u},\tilde{v},\tilde{h})\big)c + k_0 c + F_t \tag{4.7}$$

in Q_T.

Observe that (cf. (2.1) - (2.4), (2.6) - (2.7) and (3.19) - (3.23))

$$a \in C^1([0,L]) \quad \text{and} \quad b, c \in C^0([0,L]) \tag{4.8}$$

$$F, F_t \in W^{1,1}\big(0,T; C^0([0,L])\big) \tag{4.9}$$

$$\mathcal{R}_1(\tilde{u}), \mathcal{R}_2(\tilde{u},\tilde{v}) \in C^2(\bar{Q}_T). \tag{4.10}$$

Consequently, we have (cf. also (4.3))

$$\mathcal{U}(\tilde{u},\tilde{v},\tilde{h}), \mathcal{V}(\tilde{u},\tilde{v},\tilde{h}) \in W^{1,1}\big(0,T; C^0([0,L])\big). \tag{4.11}$$

On account of (2.7) - (2.9), (2.11) - (2.14), (2.20) - (2.21), (3.4) - (3.9) and (4.11) we are in a position to apply Lemma 3.1 which ensures that problem (P2) has a unique solution $(u,v) \in C^2(\bar{Q}_T)$. Also, estimate (3.36) entails that, for any $t \in (0,T]$,

$$\begin{aligned}
&\|u\|_{C^2(\bar{Q}_t)} + \|v\|_{C^2(\bar{Q}_t)} \\
&\quad \le C_3\Big\{(1+t) \\
&\qquad \times \Big[\|(\mathcal{U}(\tilde{u},\tilde{v},\tilde{h}))_t\|_{L^1(0,t;C^0([0,L]))} + \|(\mathcal{V}(\tilde{u},\tilde{v},\tilde{h}))_t\|_{L^1(0,t;C^0([0,L]))} \\
&\qquad + \|\mathcal{U}(\tilde{u},\tilde{v},\tilde{h})(\cdot,0)\|_{C^0([0,L])} + \|\mathcal{V}(\tilde{u},\tilde{v},\tilde{h})(\cdot,0)\|_{C^0([0,L])}\Big] \\
&\qquad + \|u_0\|_{C^2([0,L])} + \|v_0\|_{C^2([0,L])} + \|u_1\|_{C^1([0,L])} + \|v_1\|_{C^1([0,L])} \\
&\qquad + \|\alpha^{(3)}\|_{L^1(0,t)} + \|\tilde{\alpha}^{(3)}\|_{L^1(0,t)} + \|\beta^{(3)}\|_{L^1(0,t)} + \|\tilde{\beta}^{(3)}\|_{L^1(0,t)}\Big\}
\end{aligned} \tag{4.12}$$

Thus we can define a mapping $\mathbf{J} : X_T \to X_T$ by setting

$$\mathbf{J}(\tilde{u}, \tilde{v}, \tilde{h}) = (u, v, h) \qquad \big(\tilde{u}, \tilde{v}, \tilde{h}) \in X_T\big). \tag{4.13}$$

We are now going to show that $\mathbf{J}$ has a unique fixed point in $B(E_0, T_0)$ for some $(E_0, T_0) \in (0, +\infty) \times (0, T]$. Of course, on account of (3.16) - (3.18) this is equivalent to say that problem (P1) has a unique solution for $T = T_0$. Recalling (2.3) and (3.12) - (3.13), from (4.2) we easily deduce

$$\|\mathcal{H}(\tilde{u}, \tilde{v}, \tilde{h})\|_{L^1(0,t)} \le |\gamma|^{-1} \left[C_4(E + E^2)t + \|g''\|_{L^1(0,t)}\right] \tag{4.14}$$

for any $(\tilde{u}, \tilde{v}, \tilde{h}) \in B(E, t)$ $(t \in (0, T])$ where C_4 is a positive constant depending only on $k_0, \varphi_0(0)$ and $\|\psi_z\|_{L^\infty((0,L)\times(-E,E))}$. Observe now that (cf. (4.6) - (4.7))

$$\mathcal{U}(\tilde{u}, \tilde{v}, \tilde{h})(\cdot, 0) := F(\cdot, 0) \tag{4.15}$$

$$\mathcal{V}(\tilde{u}, \tilde{v}, \tilde{h})(\cdot, 0) := k_0 c + F_t(\cdot, 0) \tag{4.16}$$

$$(\mathcal{U}(\tilde{u}, \tilde{v}, \tilde{h}))_t := F_t + \big(k_0 + \mathcal{H}(\tilde{u}, \tilde{v}, \tilde{h}) * \big)\mathcal{R}_1(\tilde{u}) \tag{4.17}$$

$$(\mathcal{V}(\tilde{u}, \tilde{v}, \tilde{h}))_t := \mathcal{H}(\tilde{u}, \tilde{v}, \tilde{h})c + F_{tt} + \big(k_0 + \mathcal{H}(\tilde{u}, \tilde{v}, \tilde{h}) * \big)\mathcal{R}_2(\tilde{u}, \tilde{v}). \tag{4.18}$$

Hence, thanks to (2.1), (2.3) - (2.4) and (3.22) - (3.23), from (4.15) - (4.18) we derive (cf. also (4.8) - (4.9))

$$\begin{aligned}
&\|(\mathcal{U}(\tilde{u}, \tilde{v}, \tilde{h}))_t\|_{L^1(0,t;C^0([0,L]))} + \|(\mathcal{V}(\tilde{u}, \tilde{v}, \tilde{h}))_t\|_{L^1(0,t;C^0([0,L])}\\
&\qquad + \|\mathcal{U}(\tilde{u}, \tilde{v}, \tilde{h})(\cdot, 0)\|_{C^0([0,L])} + \|\mathcal{V}(\tilde{u}, \tilde{v}, \tilde{h})(\cdot, 0)\|_{C^0([0,L])}\\
&\quad \le \|F_t\|_{W^{1,1}(0,t;C^0([0,L]))} + \|F(\cdot, t)\|_{C^0([0,L])}\\
&\qquad + \|F_t(\cdot, t)\|_{C^0([0,L])} + |k_0|\|c\|_{C^0([0,L])}\\
&\qquad + C_5(E^2 + E)t\big\{1 + \|\mathcal{H}(\tilde{u}, \tilde{v}, \tilde{h})\|_{L^1(0,t)}\big\}\\
&\qquad + \|c\|_{C^0([0,L])}\|\mathcal{H}(\tilde{u}, \tilde{v}, \tilde{h})\|_{L^1(0,t)}
\end{aligned} \tag{4.19}$$

for any $(\tilde{u}, \tilde{v}, \tilde{h}) \in B(E, t)$. Here C_5 is a positive constant which only depends on k_0, ϱ_0, u_0 and on the L^∞-norms of $\psi_x, \psi_z, \psi_{xz}, \psi_{zz}$ on $(0, L) \times (-E, +E)$. Then combining (4.14) and (4.19), we obtain for any $t \in (0, T]$

$$\begin{aligned}
&\|(\mathcal{U}(\tilde{u}, \tilde{v}, \tilde{h}))_t\|_{L^1(0,t;C^0([0,L]))} + \|(\mathcal{V}(\tilde{u}, \tilde{v}, \tilde{h}))_t\|_{L^1(0,t;C^0([0,L])}\\
&\qquad + \|\mathcal{U}(\tilde{u}, \tilde{v}, \tilde{h})(\cdot, 0)\|_{C^0([0,L])} + \|\mathcal{V}(\tilde{u}, \tilde{v}, \tilde{h})(\cdot, 0)\|_{C^0([0,L])}\\
&\quad \le \|F_t\|_{W^{1,1}(0,t;C^0([0,L]))} + \|F(\cdot, t)\|_{C^0([0,L])}\\
&\qquad + \|F_t(\cdot, t)\|_{C^0([0,L])} + |k_0|\|c\|_{C^0([0,L])}\\
&\qquad + C_5(E^2 + E)t\left\{1 + |\gamma|^{-1}\left[C_4(E + E^2)t + \|g''\|_{L^1(0,t)}\right]\right\}\\
&\qquad + |\gamma|^{-1}\|c\|_{C^0([0,L])}\left[C_4(E + E^2)t + \|g''\|_{L^1(0,t)}\right].
\end{aligned} \tag{4.20}$$

Taking advantage of (4.12), (4.14), (4.20), we can find a polynomial function $P(y_1, y_2)$ such that $P(y_1, y_2) > 0$ for any $y_1, y_2 > 0$ and $P(y_1, 0) = 0$ for any $y_1 > 0$, and a pair of positive constants C_6, C_7 such that (cf. (4.13))

$$\|\mathbf{J}(\tilde{u}, \tilde{v}, \tilde{h})\|_{X_t} \le C_6 P(E, t) + C_7 \tag{4.21}$$

for any $(\tilde{u}, \tilde{v}, \tilde{h}) \in B(E,t)$. We note that C_6 and C_7 depend on $L, \varphi_0, c_0, k_0, \gamma^{-1}, \varrho, \varrho_0$ and on the norms of $u_0, u_1, f, \alpha, \beta, g$. Also, C_6 depends on the L^∞-norms of $\psi_z, \psi_x, \psi_{xz}, \psi_{zz}$ on $(0,L) \times (-E,+E)$.

Choosing for instance $E_0 = 2C_7$ and, consequently, $T_0 \in (0,T]$ such that $0 < C_6 P(E_0,T_0) \le C_7$ we have from (4.21) that $\mathbf{J}$ maps $B(E_0,T_0)$ into itself. Let us now prove that $\mathbf{J}^n$ is a contraction for $n \in \mathbb{N}$ large enough. This suffices to conclude by means of the generalized Contraction Mapping Principle (see, e.g., [4: Theorem 2.2/p. 88]).

Let $(\tilde{u}^i, \tilde{v}^i, \tilde{h}^i) \in B(E_0,T_0)$ and consider $(u^i, v^i, h^i) = \mathbf{J}(\tilde{u}^i, \tilde{v}^i, \tilde{h}^i)$ $(i = 1,2)$. Observe that (cf. (4.2))

$$\begin{aligned} h^1 - h^2 = \gamma^{-1}\Big[&k_0(N_1(\tilde{u}^1,\tilde{v}^1) - N_1(\tilde{u}^2,\tilde{v}^2)) + (\tilde{h}^1 - \tilde{h}^2) * N_1(\tilde{u}^1,\tilde{v}^1) \\ &+ \tilde{h}^2 * \big(N_1(\tilde{u}^1,\tilde{v}^1) - N_1(\tilde{u}^2,\tilde{v}^2)\big) + N_2(\tilde{v}^1_t - \tilde{v}^2_t)\Big] \end{aligned} \tag{4.22}$$

a.e. in $[0,T]$. Also, setting $U = u^1 - u^2$ and $V = v^1 - v^2$, on account of problem (P2) we easily deduce that

$$U_{tt} - aU_{xx} - bU_x = \mathcal{U}(\tilde{u}^1,\tilde{v}^1,\tilde{h}^1) - \mathcal{U}(\tilde{u}^2,\tilde{v}^2,\tilde{h}^2) \tag{4.23}$$

$$V_{tt} - aV_{xx} - bV_x = \mathcal{V}(\tilde{u}^1,\tilde{v}^1,\tilde{h}^1) - \mathcal{V}(\tilde{u}^2,\tilde{v}^2,\tilde{h}^2) \tag{4.24}$$

in Q_{T_0}. In addition, U and V satisfy homogeneous initial and boundary conditions. From (2.3) - (2.4) and (3.12) we infer

$$\big\|N_1(\tilde{u}^1,\tilde{v}^1) - N_1(\tilde{u}^2,\tilde{v}^2)\big\|_{C^0([0,t])} \le C_8 \left(\|\tilde{u}^1 - \tilde{u}^2\|_{C^1(\bar{Q}_t)} + \|\tilde{v}^1 - \tilde{v}^2\|_{C^1(\bar{Q}_t)}\right) \tag{4.25}$$

for any $t \in (0,T_0]$, where $C_8 > 0$ depends on E_0 and on the L^∞-norms of ψ_z, ψ_{zz} on $(0,L) \times (-E_0,+E_0)$. Hence, thanks to (4.25) from (4.22) we deduce, for any $t \in (0,T_0]$,

$$\begin{aligned} &\|h^1 - h^2\|_{L^1(0,t)} \\ &\quad \le C_9 \int_0^t \Big(\|\tilde{h}^1 - \tilde{h}^2\|_{L^1(0,\tau)} + \|\tilde{u}^1 - \tilde{u}^2\|_{C^1(\bar{Q}_\tau)} + \|\tilde{v}^1 - \tilde{v}^2\|_{C^2(\bar{Q}_\tau)}\Big)\, d\tau. \end{aligned} \tag{4.26}$$

Here C_9 is a positive constant only depending on $k_0, \varphi_0(0), T_0, E_0$ and on the L^∞-norms of ψ_z, ψ_{zz} on $(0,L) \times (-E_0,+E_0)$. On the other hand, recalling (2.1), (2.3) - (2.4) and (3.22), we have

$$\|\mathcal{R}_1(u^1) - \mathcal{R}_1(u^2)\|_{C^0(\bar{Q}_t)} \le (\varrho_0)^{-1} C_{10} \|\tilde{u}^1 - \tilde{u}^2\|_{C^2(\bar{Q}_t)} \tag{4.27}$$

for any $t \in (0,T_0]$, where C_{10} is a positive constant which only depends on the L^∞-norms of $\psi_z, \psi_{xz}, \psi_{zz}$ on $(0,L) \times (-E_0,+E_0)$. Moreover, thanks to (2.1) and (2.3) - (2.5), from (3.23) we derive, for any $t \in (0,T_0]$,

$$\begin{aligned} &\big\|\mathcal{R}_2(u^1,v^1) - \mathcal{R}_2(u^2,v^2)\big\|_{C^0(\bar{Q}_t)} \\ &\qquad \le (\varrho_0)^{-1} C_{11} \left(\|\tilde{u}^1 - \tilde{u}^2\|_{C^2(\bar{Q}_t)} + \|\tilde{v}^1 - \tilde{v}^2\|_{C^2(\bar{Q}_t)}\right). \end{aligned} \tag{4.28}$$

Here $C_{11} > 0$ only depends on E_0 and on the L^∞-norms of $\psi_z, \psi_{zz}, \psi_{xz}$ on $(0,L) \times (-E_0, +E_0)$. On account of (4.27) - (4.28) and recalling (4.17) - (4.18) we then obtain

$$\begin{aligned} &\|(\mathcal{U}(\tilde{u}^1, \tilde{v}^1, \tilde{h}^1) - \mathcal{U}(\tilde{u}^2, \tilde{v}^2, \tilde{h}^2))_t\|_{L^1(0,t;C^0([0,L]))} \\ &\quad + \|(\mathcal{V}(\tilde{u}^1, \tilde{v}^1, \tilde{h}^1) - \mathcal{V}(\tilde{u}^2, \tilde{v}^2, \tilde{h}^2))_t\|_{L^1(0,t;C^0([0,L]))} \\ &\quad \le C_{12}\Big\{ \int_0^t \big(\|\tilde{u}^1 - \tilde{u}^2\|_{C^2(\bar{Q}_\tau)} + \|\tilde{v}^1 - \tilde{v}^2\|_{C^2(\bar{Q}_\tau)}\big)\, d\tau \\ &\quad + (1+t)\|h^1 - h^2\|_{L^1(0,t)} \Big\} \end{aligned} \tag{4.29}$$

for all $t \in (0, T_0]$ where $C_{12} > 0$ depends on $\varrho_0, \varphi_0', k_0$ and on the same quantities as C_{11} does. Using estimate (3.36) and taking advantage of (4.29), we infer

$$\begin{aligned} &\|U\|_{C^2(\bar{Q}_t)} + \|V\|_{C^2(\bar{Q}_t)} \\ &\quad \le C_3 C_{12}\Big\{ \int_0^t \big(\|\tilde{u}^1 - \tilde{u}^2\|_{C^2(\bar{Q}_\tau)} + \|\tilde{v}^1 - \tilde{v}^2\|_{C^2(\bar{Q}_\tau)}\big)\, d\tau \\ &\quad + (1+t)\|h^1 - h^2\|_{L^1(0,t)} \Big\} \end{aligned} \tag{4.30}$$

for all $t \in (0, T_0]$). Finally, thanks to (4.26) and (4.30), we can find a positive constant C_{13} depending on $L, k_0, \varphi_0, c_0, \varrho, \varrho_0, T_0, E_0$ and on the L^∞-norms of $\psi_z, \psi_{xz}, \psi_{zz}$ on $(0,L) \times (-E_0, +E_0)$ such that, for any $t \in (0, T_0]$,

$$\big\|\mathbf{J}(\tilde{u}^1, \tilde{v}^1, \tilde{h}^1) - \mathbf{J}(\tilde{u}^2, \tilde{v}^2, \tilde{h}^2)\big\|_{X_t} \le C_{13} \int_0^t \big\|(\tilde{u}^1, \tilde{v}^1, \tilde{h}^1) - (\tilde{u}^2, \tilde{v}^2, \tilde{h}^2)\big\|_{X_\tau} d\tau. \tag{4.31}$$

Inequality (4.31) entails that $\mathbf{J}^n$ is a contraction of $B(E_0, T_0)$ into itself provided that $n \in \mathbb{N}$ is large enough. This completes the proof.

5. Proof of Theorem 2.2

Let us set (cf. (3.1)) $v_j = (u_j)_t$ $(j = 1, 2)$. Moreover, on account of (3.14), set $h_j = k_j'$. Then, recalling Section 3, we easily realize that (u_j, v_j) solves (cf. (4.4) - (4.5))

$$(u_j)_{tt} - a(u_j)_{xx} - b(u_j)_x = \mathcal{U}_j(u_j, v_j, h_j) \tag{5.1}$$
$$(v_j)_{tt} - a(v_j)_{xx} - b(v_j)_x = \mathcal{V}_j(u_j, v_j, h_j) \tag{5.2}$$

where (cf. (4.6) - (4.7))

$$\mathcal{U}_j(u_j, v_j, h_j) = (k_{0j} + 1 * h_j) * \mathcal{R}_1(u_j) + F_j \tag{5.3}$$
$$\mathcal{V}_j(u_j, v_j, h_j) = (k_{0j} + 1 * h_j) * \mathcal{R}_2(u_j, v_j) + (1 * h_j)c_j + k_{0j}c_j + (F_j)_t \tag{5.4}$$

in Q_T and (cf. (2.22) and (3.20) - (3.21))

$$k_{0j} = \gamma^{-1}\left[g_j'(0) - \varphi_0(0)u_{1j}'(0)\right] \tag{5.5}$$

$$c_j(x) = (\varrho(x))^{-1}\left[\psi_z(x, u_{0j}'(x))u_{0j}''(x) + \psi_x(x, u_{0j}'(x))\right] \quad (x \in [0, L]) \tag{5.6}$$

$$F_j(x, t) = (\varrho(x))^{-1} f_j(x, t) \quad ((x, t) \in \bar{Q}_T). \tag{5.7}$$

Also, (u_j, v_j) fulfills the initial and boundary conditions (cf. (1.5) - (1.6), (3.4) and (3.7) - (3.8))

$$u_j(x, 0) = u_{0j}(x) \qquad (u_j)_t(x, 0) = u_{1j}(x, 0) \qquad (x \in [0, L]) \tag{5.8}$$

$$v_j(x, 0) = v_{0j}(x) \qquad (v_j)_t(x, 0) = v_{1j}(x, 0) \qquad (x \in [0, L]) \tag{5.9}$$

$$u(0, t) = \alpha_j(t) \qquad u(L, t) = \beta_j(t) \qquad (t \in [0, T]) \tag{5.10}$$

$$v(0, t) = \alpha_j'(t) \qquad v(L, t) = \beta_j'(t) \qquad (t \in [0, T]). \tag{5.11}$$

Here v_{0j} and v_{1j} are defined by (3.5) with u_0, u_1, f replaced by u_{0j}, u_{1j}, f_j, respectively. Applying estimate (3.36) to the Cauchy-Dirichlet problem (5.1), (5.8), (5.10), we deduce that, for any $t \in (0, T]$,

$$\begin{aligned} &\|u_j\|_{C^2(\bar{Q}_t)} \\ &\quad \le C_3\Big\{\|(\mathcal{U}_j(u_j, v_j, h_j))_t\|_{L^1(0,t;C^0([0,L]))} + \|\mathcal{U}_j(u_j, v_j, h_j)(\cdot, 0)\|_{C^0([0,L])} \\ &\qquad + \|u_{0j}\|_{C^2([0,L])} + \|u_{1j}\|_{C^1([0,L])} + \|\alpha_j^{(3)}\|_{L^1(0,t)} + \|\beta_j^{(3)}\|_{L^1(0,t)}\Big\}. \end{aligned} \tag{5.12}$$

On the other hand, from (2.23) - (2.26), (3.22) and (5.3) we infer (cf. also (4.15) and (4.17))

$$\begin{aligned} &\|(\mathcal{U}_j(u_j, v_j, h_j))_t\|_{L^1(0,t;C^0([0,L]))} + \|\mathcal{U}_j(u_j, v_j, h_j)(\cdot, 0)\|_{C^0([0,L])} \\ &\qquad \le \Lambda_2(C_1, C_2, T)\Big\{1 + \int_0^t \|u_j\|_{C^2(\bar{Q}_\tau)} d\tau\Big\} \end{aligned} \tag{5.13}$$

for any $t \in (0, T]$. Here and in the sequel of the proof, Λ_r $(r \in \mathbb{N})$ denotes a positive and continuous function which is non-decreasing in each of its variables and depends on $\varrho, \varrho_0, \varphi_0, \psi, c_0, c_1, c_2$ at most. Then, combining (5.12) with (5.13) and using the Gronwall lemma, we obtain

$$\|u_j\|_{C^2(\bar{Q}_T)} \le \Lambda_3(C_1, C_2, T). \qquad (j = 1, 2) \tag{5.14}$$

Consider now a Cauchy-Dirichlet problem for v_j, namely (5.2), (5.9), (5.11). Estimate (3.36) yields again

$$\begin{aligned} \|v_j\|_{C^2(\bar{Q}_t)} \le C_3\Big\{&\|(\mathcal{V}_j(u_j, v_j, h_j))_t\|_{L^1(0,t;C^0([0,L]))} \\ &+ \|\mathcal{V}_j(u_j, v_j, h_j)(\cdot, 0)\|_{C^0([0,L])} + \|v_{0j}\|_{C^2([0,L])} \\ &+ \|v_{1j}\|_{C^1([0,L])} + \|\alpha_j^{(4)}\|_{L^1(0,t)} + \|\beta_j^{(4)}\|_{L^1(0,t)}\Big\}. \end{aligned} \tag{5.15}$$

Recalling (2.3) - (2.4), (3.23) and (5.4) - (5.7) and using (2.23) - (2.24) and (5.14), we obtain (cf. also (4.16) and (4.18))

$$\|\mathcal{V}_j(u_j,v_j,h_j)\|_{W^{1,1}(0,t;C^0([0,L]))} \leq \Lambda_4(C_1,C_2,T)\left\{1+\int_0^t \|v_j\|_{C^2(\bar{Q}_\tau)}d\tau\right\}. \tag{5.16}$$

A combination of (5.15) and (5.16) yields, via the Gronwall lemma,

$$\|v_j\|_{C^2(\bar{Q}_T)} \leq \Lambda_5(C_1,C_2,T) \qquad (j=1,2). \tag{5.17}$$

Thanks to bounds (5.14) and (5.17), we can now proceed to get estimate (2.26). Set

$$\mathbf{u} = u_1 - u_2 \qquad \mathbf{v} = v_1 - v_2 \qquad \mathbf{h} = h_1 - h_2 \tag{5.18}$$

$$\mathbf{u}_0 = u_{01} - u_{02} \qquad \mathbf{u}_1 = u_{11} - u_{12} \qquad \mathbf{f} = f_1 - f_2 \tag{5.19}$$

$$\mathbf{g} = g_1 - g_2 \qquad \mathbf{a} = \alpha_1 - \alpha_2 \qquad \mathbf{b} = \beta_1 - \beta_2. \tag{5.20}$$

Taking (4.2) into account, we deduce (cf. also (5.18) and (5.20))

$$\begin{aligned}\mathbf{h} = (\gamma_1-\gamma_2)^{-1}&\Big[k_{01}N_1(\tilde{u}_1,\tilde{v}_1) + h_1 * N_1(u_1,v_1) + N_2((v_1)_t) + g_1''\Big]\\ &+ \gamma_2\Big[(k_{01}-k_{02})N_1(u_1,v_1)\\ &+ k_{02}\big(N_1(u_1,v_1) - N_1(u_2,v_2)\big) + \mathbf{h} * N_1(u_1,v_1)\\ &+ h_2 * \big(N_1(u_1,v_1) - N_1(u_2,v_2)\big) + N_2((v_1-v_2)_t) + \mathbf{g}''\Big]\end{aligned} \tag{5.21}$$

a.e. in $[0,T]$. Recalling (2.3), (2.18) and (2.22) - (2.23) we easily get

$$\begin{aligned}&|(\gamma_1-\gamma_2)^{-1}| + |k_{01}-k_{02}|\\ &\qquad \leq \Lambda_6(\mu,C_1)\Big\{\|\mathbf{u}_0'\|_{C^0([0,L])} + \|\mathbf{u}_1'\|_{C^0([0,L])} + |\mathbf{g}'(0)|\Big\}\end{aligned} \tag{5.22}$$

where $\mu = \min\{|\gamma_1|^{-1}, |\gamma_2|^{-1}\}$. On the other hand, taking (2.4), (3.12) - (3.13) and (5.14) into account, we have, for any $t \in [0,T]$,

$$\begin{aligned}&\big|N_1(u_1,v_1)(t) - N_1(u_2,v_2)(t)\big| + \big|N_2((v_1-v_2)_t)(t)\big|\\ &\qquad \leq \Lambda_7(C_1)\left\{\|\mathbf{u}_x\|_{C^0(\bar{Q}_t)} + \|\mathbf{v}_x\|_{C^1(\bar{Q}_t)}\right\}.\end{aligned} \tag{5.23}$$

Using now (2.3), (2.23) - (2.24), (5.14), (5.17) and (5.22) - (5.23), from (5.21) we derive the inequality

$$\begin{aligned}\|\mathbf{h}\|_{L^1(0,t)} \leq \Lambda_8(\mu,C_1,C_2,T)&\Big\{\|\mathbf{u}_0'\|_{C^0([0,L])}\\ &+ \|\mathbf{u}_1'\|_{C^0([0,L])} + |\mathbf{g}'(0)| + \|\mathbf{g}''\|_{L^1(0,t)}\\ &+ \int_0^t \left[\|\mathbf{u}_x\|_{C^0(\bar{Q}_\tau)} + \|\mathbf{v}_x\|_{C^1(\bar{Q}_\tau)}\right]d\tau + \int_0^t \|\mathbf{h}\|_{L^1(0,\tau)}d\tau\Big\}\end{aligned} \tag{5.24}$$

for any $t \in [0,T]$. An application of the Gronwall lemma to (5.24) gives

$$\begin{aligned} \|\mathbf{h}\|_{L^1(0,t)} \le \Lambda_9(\mu, C_1, C_2, T)\Big\{ & \|\mathbf{u}_0'\|_{C^0([0,L])} \\ & + \|\mathbf{u}_1'\|_{C^0([0,L])} + |\mathbf{g}'(0)| + \|\mathbf{g}''\|_{L^1(0,t)} \qquad (t \in [0,T]). \\ & + \int_0^t \left[\|\mathbf{u}_x\|_{C^0(\bar{Q}_\tau)} + \|\mathbf{v}_x\|_{C^1(\bar{Q}_\tau)}\right] d\tau \Big\} \end{aligned} \tag{5.25}$$

We can now observe that the pair $(\mathbf{u}, \mathbf{v})$ solves the Cauchy-Dirichlet problem (cf. (5.1) - (5.2), (5.8) - (5.11) and (5.18-20))

$$\left.\begin{aligned} & \mathbf{u}_{tt} - a\mathbf{u}_{xx} - b\mathbf{u}_x = \mathcal{U}_1(u_1, v_1, h_1) - \mathcal{U}_2(u_2, v_2, h_2) \\ & \mathbf{v}_{tt} - a\mathbf{v}_{xx} - b\mathbf{v}_x = \mathcal{V}_1(u_1, v_1, h_1) - \mathcal{V}_2(u_2, v_2, h_2) \\ & \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{u}_t(x,0) = \mathbf{u}_1(x,0) \qquad (x \in [0,L]) \\ & \mathbf{v}(x,0) = \mathbf{v}_0(x), \quad \mathbf{v}_t(x,0) = \mathbf{v}_1(x,0) \qquad (x \in [0,L]) \\ & \mathbf{u}(0,t) = \mathbf{a}(t), \quad \mathbf{u}(L,t) = \mathbf{b}(t) \qquad (t \in [0,T]) \\ & \mathbf{v}(0,t) = \mathbf{a}'(t), \quad \mathbf{v}(L,t) = \mathbf{b}'(t) \qquad (t \in [0,T]). \end{aligned}\right\} \tag{5.26}$$

Then, estimate (3.36) applied to (5.26) entails

$$\begin{aligned} & \|\mathbf{u}\|_{C^2(\bar{Q}_t)} + \|\mathbf{v}\|_{C^2(\bar{Q}_t)} \\ & \quad \le C_3\Big\{ \big\|(\mathcal{U}_1(u_1, v_1, h_1) - \mathcal{U}_2(u_2, v_2, h_2))_t\big\|_{L^1(0,t;C^0([0,L]))} \\ & \qquad + \big\|(\mathcal{V}_1(u_1, v_1, h_1) - \mathcal{V}_2(u_2, v_2, h_2))_t\big\|_{L^1(0,t;C^0([0,L]))} \\ & \qquad + \big\|(\mathcal{U}_1(u_1, v_1, h_1) - \mathcal{U}_2(u_2, v_2, h_2))(\cdot, 0)\big\|_{C^0([0,L])} \\ & \qquad + \big\|(\mathcal{V}_1(u_1, v_1, h_1) - \mathcal{V}_2(u_2, v_2, h_2))(\cdot, 0)\big\|_{C^0([0,L])} \\ & \qquad + \|\mathbf{u}_0\|_{C^2([0,L])} + \|\mathbf{v}_0\|_{C^2([0,L])} + \|\mathbf{u}_1\|_{C^1([0,L])} + \|\mathbf{v}_1\|_{C^1([0,L])} \\ & \qquad + \|\mathbf{a}^{(3)}\|_{W^{1,1}(0,t)} + \|\mathbf{b}^{(3)}\|_{W^{1,1}(0,t)} \Big\} \end{aligned} \tag{5.27}$$

for any $t \in [0,T]$. From (5.3) - (5.4) we infer

$$\begin{aligned} \mathcal{U}_1(u_1, v_1, h_1) - \mathcal{U}_2(u_2, v_2, h_2) &= b\mathbf{u}_x + (\mathbf{k}_0 + 1 * \mathbf{h}) * \mathcal{R}_1(u_1) \\ & \quad + (k_{02} + 1 * h_2) * (\mathcal{R}_1(u_1) - \mathcal{R}_1(u_2)) + \mathbf{F} \\ \mathcal{V}_1(u_1, v_1, h_1) - \mathcal{V}_2(u_2, v_2, h_2) &= b\mathbf{v}_x + (\mathbf{k}_0 + 1 * \mathbf{h}) * \mathcal{R}_2(u_1, v_1) \\ & \quad + (k_{02} + 1 * h_2) * (\mathcal{R}_2(u_1, v_1) - \mathcal{R}_2(u_2, v_2)) \\ & \quad + \mathbf{c}(1 * h_1) + c_2(1 * \mathbf{h}) + \mathbf{c}k_{01} + c_2\mathbf{k}_0 + \mathbf{F}_t \end{aligned}$$

in Q_T, where

$$\mathbf{k}_0 = k_{01} - k_{02}, \qquad \mathbf{F} = F_1 - F_2, \qquad \mathbf{c} = c_1 - c_2. \tag{5.28}$$

Hence (cf. (4.15) - (4.18))

$$\left.\begin{aligned}
&\big(\mathcal{U}_1(u_1,v_1,h_1)-\mathcal{U}_2(u_2,v_2,h_2)\big)(\cdot,0)=\mathbf{F}(\cdot,0)\\
&\big(\mathcal{V}_1(u_1,v_1,h_1)-\mathcal{V}_2(u_2,v_2,h_2)\big)(\cdot,0)=k_{01}\mathbf{c}(\cdot)+\mathbf{k}_0c_2(\cdot)+\mathbf{F}_t(\cdot,0)\\
&\big(\mathcal{U}_1(u_1,v_1,h_1)-\mathcal{U}_2(u_2,v_2,h_2)\big)_t\\
&\quad=(\mathbf{k}_0+\mathbf{h}*)\mathcal{R}_1(u_1)+(k_{02}+h_2*)(\mathcal{R}_1(u_1)-\mathcal{R}_1(u_2))+\mathbf{F}_t\\
&\big(\mathcal{V}_1(u_1,v_1,h_1)-\mathcal{V}_2(u_2,v_2,h_2)\big)_t\\
&\quad=(\mathbf{k}_0+\mathbf{h}*)\mathcal{R}_2(u_1,v_1)\\
&\qquad+(k_{02}+h_2*)(\mathcal{R}_2(u_1,v_1)-\mathcal{R}_2(u_2,v_2))+\mathbf{c}h_1+c_2\mathbf{h}+\mathbf{F}_{tt}.
\end{aligned}\right\}\tag{5.29}$$

Recalling (3.22) - (3.23), and (2.1), (2.3), (2.5), (5.14), (5.15) and (5.28), standard computations lead to the estimate

$$\begin{aligned}
&\big\|(\mathcal{R}_1(u_1)-\mathcal{R}_1(u_2))(\cdot,t)\big\|_{C^0([0,L])}+\big\|(\mathcal{R}_2(u_1,v_1)-\mathcal{R}_2(u_2,v_2))(\cdot,t)\big\|_{C^0([0,L])}\\
&\qquad\le\Lambda_{10}(C_1,C_2,T)\left\{\|\mathbf{u}(\cdot,t)\|_{C^2([0,L])}+\|\mathbf{v}(\cdot,t)\|_{C^2([0,L])}\right\}
\end{aligned}\tag{5.30}$$

for any $t\in[0,T]$. Moreover, from (5.5) - (5.6), owing to (2.4) and (2.23), we infer

$$\begin{aligned}
&\|\mathbf{c}\|_{C^0([0,L])}+\|\mathbf{F}_t\|_{W^{1,1}(0,T;C^0([0,L]))}\\
&\quad\le\Lambda_{11}(C_1,T)\Big\{\|\mathbf{u}_0\|_{C^2([0,L])}+\|\mathbf{f}_{tt}\|_{L^1(0,T;C^0([0,L]))}+\|\mathbf{f}_t(\cdot,0)\|_{C^0([0,L])}\Big\}.
\end{aligned}\tag{5.31}$$

Thanks to estimates (5.22), (5.25) and (5.30) - (5.31), on account of (5.29) it is not hard to prove that (cf. also (5.28))

$$\begin{aligned}
&\big\|(\mathcal{U}_1(u_1,v_1,h_1)-\mathcal{U}_2(u_2,v_2,h_2))_t\big\|_{L^1(0,t;C^0([0,L]))}\\
&\qquad+\big\|(\mathcal{V}_1(u_1,v_1,h_1)-\mathcal{V}_2(u_2,v_2,h_2))_t\big\|_{L^1(0,t;C^0([0,L]))}\\
&\qquad+\big\|(\mathcal{U}_1(u_1,v_1,h_1)-\mathcal{U}_2(u_2,v_2,h_2))(\cdot,0)\big\|_{C^0([0,L])}\\
&\qquad+\big\|(\mathcal{V}_1(u_1,v_1,h_1)-\mathcal{V}_2(u_2,v_2,h_2))(\cdot,0)\big\|_{C^0([0,L])}\\
&\quad\le\Lambda_{12}(\mu,C_1,C_2,T)\Big\{\|\mathbf{f}_{tt}\|_{L^1(0,T;C^0([0,L]))}\\
&\qquad+\|\mathbf{f}_t(\cdot,0)\|_{C^0([0,L])}+\|\mathbf{f}(\cdot,0)\|_{C^0([0,L])}\\
&\qquad+\|\mathbf{u}_0\|_{C^2([0,L])}+\|\mathbf{u}_1\|_{C^1([0,L])}+|\mathbf{g}'(0)|+\|\mathbf{g}''\|_{L^1(0,T)}\\
&\qquad+\int_0^t\Big(\|\mathbf{u}\|_{C^0([0,\tau];C^2([0,L]))}+\|\mathbf{v}\|_{C^0([0,\tau];C^2([0,L]))}\Big)d\tau\Big\}
\end{aligned}\tag{5.32}$$

for any $t \in [0,T]$. Hence inequalities (5.27) and (5.32) yield

$$\begin{aligned}
&\|\mathbf{u}\|_{C^2(\bar{Q}_t)} + \|\mathbf{v}\|_{C^2(\bar{Q}_t)} \\
&\quad \le \Lambda_{13}(\mu, C_1, C_2, T)\Big\{\|\mathbf{u}_0\|_{C^2([0,L])} \\
&\qquad + \|\mathbf{v}_0\|_{C^2([0,L])} + \|\mathbf{u}_1\|_{C^1([0,L])} + \|\mathbf{v}_1\|_{C^1([0,L])} \\
&\qquad + \|\mathbf{f}_{tt}\|_{L^1(0,T;C^0([0,L]))} + \|\mathbf{f}_t(\cdot,0)\|_{C^0([0,L])} + \|\mathbf{f}(\cdot,0)\|_{C^0([0,L])} \\
&\qquad + \|\mathbf{a}^{(3)}\|_{W^{1,1}(0,t)} + \|\mathbf{b}^{(3)}\|_{W^{1,1}(0,t)} + |\mathbf{g}'(0)| + \|\mathbf{g}''\|_{L^1(0,T)} \\
&\qquad + \int_0^t \Big(\|\mathbf{u}\|_{C^0([0,\tau];C^2([0,L]))} + \|\mathbf{v}\|_{C^0([0,\tau];C^2([0,L]))}\Big)\,d\tau\Big\}
\end{aligned}$$

for all $t \in [0,T]$ and the Gronwall lemma entails

$$\begin{aligned}
&\|\mathbf{u}\|_{C^2(\bar{Q}_T)} + \|\mathbf{v}\|_{C^2(\bar{Q}_T)} \\
&\quad \le \Lambda_{14}(\mu, C_1, C_2, T)\Big\{\|\mathbf{u}_0\|_{C^2([0,L])} \\
&\qquad + \|\mathbf{v}_0\|_{C^2([0,L])} + \|\mathbf{u}_1\|_{C^1([0,L])} + \|\mathbf{v}_1\|_{C^1([0,L])} \\
&\qquad + \|\mathbf{f}_{tt}\|_{L^1(0,T;C^0([0,L]))} + \|\mathbf{f}_t(\cdot,0)\|_{C^0([0,L])} + \|\mathbf{f}(\cdot,0)\|_{C^0([0,L])} \\
&\qquad + \|\mathbf{a}^{(3)}\|_{W^{1,1}(0,T)} + \|\mathbf{b}^{(3)}\|_{W^{1,1}(0,T)} + |\mathbf{g}'(0)| + \|\mathbf{g}''\|_{L^1(0,T)}\Big\}.
\end{aligned} \tag{5.33}$$

Finally, taking (2.4), (2.20) - (2.21), (2.23), (3.5), (3.14) - (3.15), (5.17) - (5.20) and (5.22) into account, inequality (2.27) follows from (5.25) and (5.33).

6. Proof of Lemma 3.1

Assume for the moment that $p = q = 0$. Suppose that $w \in C^2(\bar{Q}_T)$ solves (3.33) - (3.35) and formulate an equivalent problem for a first-order system. Let us set

$$\left.\begin{aligned}
w^1 &= \frac{1}{2}\big(w_t + \sqrt{\varepsilon}\, w_x\big) \\
w^2 &= \frac{1}{2}\big(w_t - \sqrt{\varepsilon}\, w_x\big)
\end{aligned}\right\} \quad \text{in } \; Q_T. \tag{6.1}$$

Then it is straightforward to check that (w^1, w^2) solves the system

$$\left.\begin{aligned}
w^1_t - \sqrt{\varepsilon} w^1_x &= \frac{1}{2}\big[\ell + \lambda(w^1 - w^2)\big] \\
w^2_t + \sqrt{\varepsilon} w^2_x &= \frac{1}{2}\big[\ell + \lambda(w^1 - w^2)\big]
\end{aligned}\right\} \tag{6.2}$$

in Q_T and fulfils the initial conditions

$$\left.\begin{aligned}
w^1(x,0) &= \frac{1}{2}\big(w_1(x) + \sqrt{\varepsilon(x)}\, w_0'(x)\big) \\
w^2(x,0) &= \frac{1}{2}\big(w_1(x) - \sqrt{\varepsilon(x)}\, w_0'(x)\big)
\end{aligned}\right\} \qquad (x \in [0,L]) \tag{6.3}$$

and boundary conditions

$$\left.\begin{aligned} w^1(0,t)+w^2(0,t)=0\\ w^1(L,t)+w^2(L,t)=0 \end{aligned}\right\} \qquad (t\in[0,T]). \tag{6.4}$$

where $\lambda = \frac{2\eta-\varepsilon'}{4\sqrt{\varepsilon}}$ in $[0,L]$. Let us introduce the change of variable

$$y=\zeta(x)=\int_0^x \frac{d\xi}{\sqrt{\varepsilon(\xi)}} \qquad (x\in[0,L])$$

and define

$$\tilde{w}^i(y)=w^i(\zeta^{-1}(y)) \qquad (y\in[0,\tilde{L}];\ i=1,2) \tag{6.5}$$

where $\tilde{L}=\zeta(L)$. Then the Cauchy-Dirichlet problem (6.2) - (6.4) can be rewritten as

$$\left.\begin{aligned} \tilde{w}^1_t-\tilde{w}^1_y=\frac{1}{2}\big[\tilde{\ell}+\tilde{\lambda}(\tilde{w}^1-\tilde{w}^2)\big]\\ w^2_t+\tilde{w}^2_y=\frac{1}{2}\big[\tilde{\ell}+\tilde{\lambda}(\tilde{w}^1-\tilde{w}^2)\big] \end{aligned}\right\} \qquad \text{in } R_T \tag{6.6}$$

with initial conditions

$$\left.\begin{aligned} \tilde{w}^1(y,0)=\tilde{w}^1_0=\frac{1}{2}\big(\tilde{w}_1(y)+\tilde{w}_0'(y)\big)\\ \tilde{w}^2(y,0)=\tilde{w}^2_0=\frac{1}{2}\big(\tilde{w}_1(y)-\tilde{w}_0'(y)\big) \end{aligned}\right\} \qquad (y\in[0,\tilde{L}]) \tag{6.7}$$

and boundary condition

$$\left.\begin{aligned} \tilde{w}^1(0,t)+\tilde{w}^2(0,t)=0\\ \tilde{w}^1(\tilde{L},t)+\tilde{w}^2(\tilde{L},t)=0 \end{aligned}\right\} \qquad (t\in[0,T]) \tag{6.8}$$

where $R_T=(0,\tilde{L})\times(0,T)$ and

$$\begin{aligned} \tilde{\ell}(y,t)&=\ell(\zeta^{-1}(y),t) \\ \tilde{w}_0(y)&=w_0(\zeta^{-1}(y)) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \tilde{\lambda}(y)&=\lambda(\zeta^{-1}(y)) \\ \tilde{w}_1(y)&=w_1(\zeta^{-1}(y)) \end{aligned} \tag{6.9}$$

for $y\in[0,\tilde{L}]$ and $t\in[0,T]$. We now define the metric space

$$W(T)=\Big\{(z^1,z^2)\in(C^1(\bar{R}_T))^2\,\Big|\, z^1(y,0)=z^2(y,0)=\tilde{w}^1_0 \text{ in } [0,\tilde{L}]\Big\}$$

endowed with the metric induced by the norm

$$\|(z^1,z^2)\|_{W(T)}=\max\big\{\|z^1\|_{C^1(\bar{R}_T)},\|z^2\|_{C^1(\bar{R}_T)}\big\}.$$

Of course, $W(T)$ is complete.

Let $(\tilde{z}^1, \tilde{z}^2) \in W(T)$ be given. Thanks to [7: Theorem 2.2], we can find a unique $(z^1, z^2) \in W(T)$ which solves the system

$$\left.\begin{aligned} z_t^1 - z_y^1 &= \frac{1}{2}\big[\tilde{\ell} + \tilde{\lambda}(\tilde{z}^1 - \tilde{z}^2)\big] \\ z_t^2 + z_y^2 &= \frac{1}{2}\big[\tilde{\ell} + \tilde{\lambda}(\tilde{z}^1 - \tilde{z}^2)\big] \end{aligned}\right\}$$

in R_T and satisfies the initial and boundary conditions (6.7) - (6.8). Moreover, owing to [7: Formulas (2.15) - (2.16)], there exists a positive constant C_{14}, only depending on $\|\tilde{\lambda}\|_{C^0([0,\tilde{L}])}$, such that

$$\begin{aligned} &\|(z^1, z^2)\|_{W(t)} \\ &\quad \le C_{14}\Big\{(1+t)\Big[\|\tilde{\ell}_t\|_{L^1(0,t;C^0([0,\tilde{L}]))} + \|\tilde{\ell}(\cdot,0)\|_{C^0([0,\tilde{L}])}\Big] \\ &\qquad + \|\tilde{w}_0\|_{C^2([0,\tilde{L}])} + \|\tilde{w}_1\|_{C^1([0,\tilde{L}])} + (1+t)\int_0^t \|(\tilde{z}^1, \tilde{z}^2)\|_{W(\tau)} d\tau\Big\} \end{aligned} \tag{6.10}$$

for all $t \in [0,T]$. Consider the mapping $\mathcal{W}: W(T) \to W(T)$ defined by $\mathcal{W}(\tilde{z}^1, \tilde{z}^2) = (z^1, z^2)$. Taking advantage of estimate (6.10), we obtain

$$\big\|\mathcal{W}(\tilde{z}_1^1, \tilde{z}_1^2) - \mathcal{W}(\tilde{z}_2^1, \tilde{z}_2^2)\big\|_{W(t)} \le C_{13}(1+t)\int_0^t \|(\tilde{z}_1^1, \tilde{z}_1^2) - (\tilde{z}_2^1, \tilde{z}_2^2)\|_{W(\tau)} d\tau \tag{6.11}$$

for any $t \in [0,T]$ and any $(\tilde{z}_1^1, \tilde{z}_1^2), (\tilde{z}_2^1, \tilde{z}_2^2) \in W(T)$. From here we deduce that $\mathcal{W}^n$ is a contraction of $W(T)$ into itself for some $n \in \mathbb{N}$. Thus the generalized Contraction Principle yields that $\mathcal{W}$ has a unique fixed point in $W(T)$, that is, there exists a unique solution $(\tilde{w}^1, \tilde{w}^2) \in C^1(\bar{R}_T)$ to the Cauchy-Dirichlet problem (6.6) - (6.8). This is clearly equivalent to say that problem (3.33) - (3.35) admits a unique solution $w \in C^2(\bar{Q}_T)$ with $p = q = 0$, by virtue of (6.1) and (6.5). Also, from (6.10) and the Gronwall lemma we derive the bound

$$\begin{aligned} \|(\tilde{w}^1, \tilde{w}^2)\|_{C^1(\bar{R}_t)} \le C_{15}\Big\{&(1+t)\big[\|\tilde{\ell}_t\|_{L^1(0,t;C^0([0,\tilde{L}]))} + \|\tilde{\ell}(\cdot,0)\|_{C^0([0,\tilde{L}])}\big] \\ &+ \|\tilde{w}_0\|_{C^2([0,\tilde{L}])} + \|\tilde{w}_1\|_{C^1([0,\tilde{L}])}\Big\} \end{aligned} \tag{6.12}$$

for all $t \in [0,T]$, where C_{15} is a positive constant only depending on T and $\|\tilde{\lambda}\|_{C^0([0,\tilde{L}])}$. On account of (6.1), (6.5) and (6.9), from (6.12) we infer (3.36) with $p = q = 0$.

For non-homogeneous boundary data we can arguing exactly as in [7: Theorem 2.4], taking the compatibility conditions (3.29) - (3.32) into account.

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