Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 2, 297–305

On the Low Wave Number Behavior of Two-Dimensional Scattering Problems for an Open Arc

R. Kress

Dedicated to Professor Lothar von Wolfersdorf

Abstract. The low wave number asymptotics for the solution of the Dirichlet problem for the two-dimensional Helmholtz equation in the exterior of an open arc is analyzed via a single-layer integral equation approach. It is shown that the solutions to the Dirichlet problem for the Helmholtz equation converge to a solution of the Dirichlet problem for the Laplace equation as the wave number tends to zero provided the boundary values converge.

Keywords: *Helmholtz equation, exterior boundary value problems, integral equation methods, low wave number limits, cosine substitution*

AMS subject classification: Primary 35 J 05, secondary 45 B 05

1. Introduction

The study of the low wave number behavior of solutions to exterior boundary value problems for the Helmholtz equation in two dimensions via integral equation methods has a long history. This is due to difficulties arising from the following two facts. Firstly, in two dimensions the fundamental solution of the Helmholtz equation does not converge to the fundamental solution of the Laplace equation if the wave number k tends to zero. Secondly, the fundamental solution of the Laplace equation is not bounded at infinity. For the Dirichlet problem in the exterior of a finite number of disjoint closed contours the limiting behavior was investigated by MacCamy [13], Werner [15] and the present author [10]. Here, the analysis can be based on reducing the boundary value problem to an integral equation of the second kind via a double-layer or a combined double- and single-layer approach.

In the present paper we want to examine the low wave number limit for the corresponding Dirichlet problem in the exterior of an open arc. In this case, as opposed to the case of closed contours, we need to work with an integral equation of the first kind via a single-layer approach. The existence and uniqueness of solutions to the Dirichlet problem in the exterior of an open arc is well established (c.f. [5] or the appendix in the Russian translation [2] of [1] and the references therein). More recently, the author [12] has suggested to base the analysis of the single-layer integral equation for an arc

R. Kress: Inst. für Num. und Ang. Math. der Univ., Lotzestr. 16-18, D - 37083 Göttingen

ISSN 0232-2064 / $ 2.50

on the cosine transformation which has been introduced by Multhopp [14] and by Yan and Sloan [16] for the corresponding integral equation in the potential theoretic case of the Laplace equation. Here, we will adopt this approach to investigate the low wave number limit. Our analysis will be in a Sobolev space setting instead of a Hölder space setting as in [12]. Clearly, the results can be extended to the Dirichlet problem in the exterior of a finite number of disjoint open arcs.

For corresponding investigations in two-dimensional elasticity we refer to Hsiao and Wendland [7, 8].

2. Scattering from an open arc

Assume that $\Gamma \subset \mathbb{R}^2$ is an arc of class C^2, that is, $\Gamma = \{z(s) : s \in [-1,1]\}$ where $z : [-1,1] \to \mathbb{R}^2$ is an injective and twice continuously differentiable function with $z'(s) \neq 0$ for all $s \in [-1,1]$. The mathematical treatment of the scattering of time-harmonic acoustic or electromagnetic waves from thin infinitely long cylindrical obstacles is modelled by the following boundary value problem:

Given a function $f_k \in C^1(\Gamma)$, find a solution $u_k \in C^2(\mathbb{R}^2 \setminus \Gamma) \cap C(\mathbb{R}^2)$ to the Helmholtz equation

$$\Delta u_k + k^2 u_k = 0 \qquad \text{in } \mathbb{R}^2 \setminus \Gamma \tag{2.1}$$

with wave number $k > 0$ satisfying the Dirichlet boundary condition

$$u_k = f_k \qquad \text{on } \Gamma \tag{2.2}$$

and the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\Big(\frac{\partial u_k}{\partial r} - iku_k\Big) = 0 \qquad (r = |x|) \tag{2.3}$$

uniformly in all directions $\frac{x}{|x|}$. Note that we do not explicitly assume any edge condition for the behavior of the solution at the two end points of Γ.

The limiting case $k = 0$ corresponds to the following static boundary value problem: Given a function $f_0 \in C^1(\Gamma)$, find a solution $u_0 \in C^2(\mathbb{R}^2 \setminus \Gamma) \cap C(\mathbb{R}^2)$ to the Laplace equation

$$\Delta u_0 = 0 \qquad \text{in } \mathbb{R}^2 \setminus \Gamma \tag{2.4}$$

satisfying the Dirichlet boundary condition

$$u_0 = f_0 \qquad \text{on } \Gamma \tag{2.5}$$

and the boundedness condition

$$u_0(x) = O(1) \qquad (|x| \to \infty), \tag{2.6}$$

uniformly in all directions $\frac{x}{|x|}$. Again, no edge conditions are explicitly assumed.

Theorem 2.1. *The above exterior Dirichlet problems for an open arc both have at most one solution.*

Proof. As in the case of the Dirichlet problem for the exterior of a closed contour (see [11: Theorem 6.11]), uniqueness for the Laplace equation relies on the maximum-minimum principle for harmonic functions and the boundedness condition at infinity. For the Helmholtz equation, uniqueness follows from Rellich's lemma and an application of Green's theorem. The application of Green's theorem can be justified by using an approximation idea due to Heinz analogously to the case of the exterior of a closed contour (see [3: Theorem 3.7]) ∎

For functions $\varphi \in L^p(\Gamma)$ $(1 < p < \infty)$ we introduce the operator M by

$$M\varphi := \varphi - \frac{1}{|\Gamma|}\int_\Gamma \varphi\, ds.$$

Then, for $0 < k < 1$, we seek the solution of problem (2.1) - (2.3) in the form of a modified acoustic single-layer potential

$$u_k(x) = \int_\Gamma \Phi_k(x,y)\Big\{(M\varphi_k)(y) - \frac{2\pi}{\ln k}\,\varphi_k(y)\Big\}ds(y) \qquad (x \in \mathbb{R}^2) \tag{2.7}$$

with a density $\varphi_k \in L^p(\Gamma)$ and the fundamental solution to the Helmholtz equation in two dimensions given by

$$\Phi_k(x,y) = \frac{i}{4}\, H_0^{(1)}(k|x-y|) \qquad (x \neq y)$$

in terms of the Hankel function $H_0^{(1)}$ of order zero and of the first kind. Since the Hankel function has the asymptotic behavior

$$\frac{i}{4}\, H_0^{(1)}(t) = \frac{1}{2\pi}\,\ln\frac{2}{t} + \frac{i}{4} - \frac{1}{2\pi} - \frac{C}{2\pi} + O\Big(t^2 \ln\frac{1}{t}\Big) \qquad (t \to 0) \tag{2.8}$$

where $C = 0.5772\ldots$ denotes Euler's constant, the acoustic single-layer potential (2.7) has a limit as $k \to 0$ given by

$$u_0(x) = \int_\Gamma \{\Phi_0(x,y)(M\varphi_0)(y) + \varphi_0(y)\}ds(y) \qquad (x \in \mathbb{R}^2) \tag{2.9}$$

with the fundamental solution

$$\Phi_0(x,y) = \frac{1}{2\pi}\,\ln\frac{1}{|x-y|} \qquad (x \neq y)$$

to the Laplace equation. Note that in the difference $\Phi_k - \Phi_0$ the trouble making term $\ln k$ is eliminated due to $\int_\Gamma M\varphi_k\, ds = 0$.

The potentials (2.7) and (2.9) satisfy the Helmholtz and Laplace equations in $\mathbb{R}^2 \setminus \Gamma$, respectively. Furthermore, (2.7) fulfills the radiation condition and (2.9) is bounded at infinity because of $\int_\Gamma M\varphi_0\, ds = 0$. For densities in $L^p(\Gamma)$ the single-layer potentials

(2.7) and (2.9) are continuous throughout $\mathbb{R}^2$ (see [6: p. 276]). Hence, (2.7) solves the Dirichlet problem (2.1) - (2.3) provided the density $\varphi_k \in L^p(\Gamma)$ is a solution to the integral equation

$$\int_\Gamma \Phi_k(x,y)\Big\{(M\varphi_k)(y) - \frac{2\pi}{\ln k}\,\varphi_k(y)\Big\}ds(y) = f_k(x) \qquad (x \in \Gamma). \tag{2.10}$$

Analogously, (2.9) solves the Dirichlet problem (2.4) - (2.6) provided $\varphi_0 \in L^p(\Gamma)$ is a solution to the integral equation

$$\int_\Gamma \{\Phi_0(x,y)(M\varphi_0)(y) + \varphi_0(y)\}\, ds(y) = f_0(x) \qquad (x \in \Gamma). \tag{2.11}$$

Substituting $s = \cos t$ into $\Gamma = \{z(s) : s \in [-1,1]\}$, we define operators A and B which map functions on Γ into functions on the interval $[0,\pi]$ by setting

$$\left.\begin{aligned} (A\varphi)(t) &= \sin t\,|z'(\cos t)|\,\varphi(z(\cos t)) \\ (Bf)(t) &= f(z(\cos t)) \end{aligned}\right\}$$

for $0 \le t \le \pi$. The operator N given by

$$N\psi = \psi - \left[\int_0^\pi A1\,d\tau\right]^{-1} A1 \int_0^\pi \psi\,d\tau$$

is related to the operator M by $NA = AM$. By this substitution, the integral equation (2.10) is equivalently transformed into

$$\int_0^\pi H_k(t,\tau)\Big\{(N\psi_k)(\tau) - \frac{2\pi}{\ln k}\,\psi_k(\tau)\Big\}d\tau = g_k(t) \qquad (t \in [0,\pi]), \tag{2.12}$$

where $\psi_k = A\varphi_k$ and $g_k = Bf_k$ and where the kernel is given by

$$H_k(t,\tau) = \Phi_k(z(\cos t), z(\cos\tau)) \qquad (t \ne \tau).$$

Analogously, the parametrized version of (2.11) is given by

$$\int_0^\pi \{H_0(t,\tau)(N\psi_0)(\tau) + \psi_0(\tau)\}\, d\tau = g_0(t) \qquad (t \in [0,\pi]), \tag{2.13}$$

where

$$H_0(t,\tau) := \frac{1}{2\pi}\ln\frac{1}{|z(\cos t) - z(\cos\tau)|} \qquad (t \ne \tau).$$

We will look for solutions of equations (2.12) and (2.13) in $L^2[0,\pi]$. For the corresponding density φ on Γ, related through $\psi = A\varphi$, we then have

$$\int_\Gamma |\varphi(x)|^p ds(x) = \int_0^\pi \frac{|\psi(t)|^p}{|z'(\cos t)|^{p-1}\sin^{p-1} t}\,dt$$

and, with the aid of Hölder's inequality, can estimate

$$\int_\Gamma |\varphi(x)|^p ds(x) \le \frac{1}{\inf_{s\in[-1,1]} |z'(s)|^{p-1}} \left[\int_0^\pi \sin^{\frac{1-p}{1-\frac{p}{2}}} t\, dt\right]^{1-\frac{p}{2}} \left[\int_0^\pi |\psi(t)|^2 dt\right]^{\frac{p}{2}} \tag{2.14}$$

where the first integral on the right-hand side is finite provided $1 < p < \frac{4}{3}$, that is, $\varphi \in L^p(\Gamma)$ for $1 < p < \frac{4}{3}$ which ensures continuity of the single-layer potentials.

3. Low wave number limit

For the further discussion we rewrite equations (2.12) and (2.13) in operator notation as

$$K_k\psi = f_k$$

with

$$(K_k\psi)(t) = \int_0^\pi H_k(t,\tau)\Big\{(N\psi)(\tau) - \frac{2\pi}{\ln k}\,\psi(\tau)\Big\}d\tau \qquad (t\in[0,\pi])$$

for $0 < k < 1$ and

$$(K_0\psi)(t) = \int_0^\pi \{H_0(t,\tau)(N\psi)(\tau) + \psi(\tau)\}\, d\tau \qquad (t\in[0,\pi]).$$

In addition we introduce the integral operator

$$(L\psi)(t) = -\frac{1}{4\pi}\int_0^\pi \ln\Big(\frac{4}{e^2}\,[\cos t - \cos\tau]^2\Big)\psi(\tau)\, d\tau \qquad (t\in[0,\pi]).$$

In the sequel, by $H^1_e[0,\pi]$ we denote the subspace of even functions in $H^1[-\pi,\pi]$.

Lemma 3.1. *The operator $L : L^2[0,\pi] \to H^1_e[0,\pi]$ is bounded and has a bounded inverse $L^{-1} : H^1_e[0,\pi] \to L^2[0,\pi]$. The operator $K_0 - L : L^2[0,\pi] \to H^1_e[0,\pi]$ is compact.*

Proof. From the identity

$$\ln\Big(\frac{4}{e^2}\,[\cos t - \cos\tau]^2\Big) = \ln\Big(\frac{4}{e}\,\sin^2\frac{t-\tau}{2}\Big) + \ln\Big(\frac{4}{e}\,\sin^2\frac{t+\tau}{2}\Big)$$

it follows that

$$\int_0^\pi \ln\Big(\frac{4}{e^2}\,[\cos t - \cos\tau]^2\Big)\cos m\tau\, d\tau = \int_0^{2\pi} \ln\Big(\frac{4}{e}\,\sin^2\frac{t-\tau}{2}\Big)\cos m\tau\, d\tau.$$

Therefore, for $f_m(t) = \cos mt$, from [9: p. 88] or [11: Lemma 8.21] we have that $Lf_m = \lambda_m f_m$ where $\lambda_m = \frac{1}{2\max(1,m)}$ $(m \in \mathbb{N}_0)$. Hence L maps the function $\psi \in L^2[0,\pi]$ with Fourier series $\psi = \sum_{m=0}^{\infty} a_m f_m$ into $L\psi = \sum_{m=0}^{\infty} \lambda_m a_m f_m$. Since $\{f_m\}_{\mathbb{N}_0}$ forms a complete orthogonal system for $L^2[0,\pi]$ and for $H_e^1[0,\pi]$, this implies boundedness of $L : L^2[0,\pi] \to H_e^1[0,\pi]$ and invertibility, with the bounded inverse L^{-1} given in terms of the Fourier series $L^{-1}g = \sum_{m=0}^{\infty} \frac{1}{\lambda_m} b_m f_m$ for $g \in H_e^1[0,\pi]$ with $g = \sum_{m=0}^{\infty} b_m f_m$. By Taylor's formula

$$z(s) - z(\sigma) = (s-\sigma) \int_0^1 z'(\sigma + \lambda(s-\sigma))\, d\lambda,$$

using the fact that $z \in C^2[-1,1]$ with $z'(s) \neq 0$ for $s \in [-1,]$, it can be verified that

$$H_{0,1}(t,\tau) := \frac{1}{2\pi} \ln \frac{1}{|z(\cos t) - z(\cos\tau)|} + \frac{1}{4\pi} \ln\left(\frac{4}{e^2}\,[\cos t - \cos\tau]^2\right) \quad (t \neq \tau)$$

can be extended in a continuously differentiable manner on $[0,\pi] \times [0,\pi]$. Hence, the operator $K_{0,1} : L^2[0,\pi] \to H_e^1[0,\pi]$ defined by

$$(K_{0,1}\psi)(t) = \int_0^\pi H_{0,1}(t,\tau)(N\psi)(\tau)\, d\tau \qquad (t \in [0,\pi])$$

is compact. The operator $K_{0,2} : L^2[0,\pi] \to H_e^1[0,\pi]$ defined by

$$(K_{0,2}\psi)(t) = [LA1 + 1] \int_0^\pi \psi(\tau)\, d\tau \qquad (t \in [0,\pi])$$

is bounded and has finite-dimensional range and therefore also is compact. Now compactness of $K_0 - L : L^2[0,\pi] \to H_e^1[0,\pi]$ follows since $K_0 - L = K_{0,1} + K_{0,2}$ ∎

Theorem 3.2. *There exist constants $c > 0$ and $0 < k_0 < 1$ such that for all $0 \le k \le k_0$ the inverse operators $K_k^{-1} : H_e^1[0,\pi] \to L^2[0,\pi]$ exist and satisfy*

$$\|K_k^{-1} - K_0^{-1}\| \le \frac{c}{\ln|k|} .$$

Proof. We first show that $K_0 : L^2[0,\pi] \to H_e^1[0,\pi]$ has a bounded inverse. For a given $g \in H_e^1[0,\pi]$, solving $K_0\psi = g$ for $\psi \in L^2[0,\pi]$ is equivalent to solving the equation of the second kind

$$\psi + L^{-1}(K_0 - L)\psi = L^{-1}g$$

where the operator $L^{-1}(K_0 - L) : L^2[0,\pi] \to L^2[0,\pi]$ is compact by the previous Lemma 3.1. Hence, by the Riesz-Fredholm theory, it suffices to show injectivity of K_0 in order to establish bijectivity of K_0 and boundedness of the inverse.

Let $\psi_0 \in L^2[0,\pi]$ be a solution to the homogeneous equation $K_0\psi_0 = 0$. Then the corresponding potential u_0 with density $\varphi_0 \in L^p(\Gamma)$ $(0 < p < \frac{4}{3})$ defined by (2.9) solves the homogeneous boundary value problem (2.4) - (2.6). Therefore by Theorem 2.1 we have $u_0 = 0$ in $\mathbb{R}^2$. From (2.9) we have the asymptotic behavior

$$u_0(x) = \int_\Gamma \varphi_0 ds + o(1) \qquad (|x| \to \infty)$$

uniformly for all directions. Therefore $u_0 = 0$ in $\mathbb{R}^2$ implies that $\int_\Gamma \varphi_0 ds = 0$. Now, in view of $\operatorname{grad} u_0 = 0$ in $\mathbb{R}^2 \setminus \Gamma$, the potential theoretic jump relations yield

$$\varphi_0(x) = \lim_{h\to 0} \Big[\nu(x) \cdot \operatorname{grad} u_0(x - h\nu(x)) - \nu(x) \cdot \operatorname{grad} u_0(x + h\nu(x))\Big] = 0$$

for almost all $x \in \Gamma$. Here, ν denotes the unit normal vector on Γ (chosen with one of the two possible orientations). That the jump relations for the gradient of the single-layer potential are valid for L^p densities can be seen by relating them to the Sokhotski–Plemelj formulae for the Cauchy integral (see [11: p. 116]). However, the latter are valid for L^p densities (see [4: p. 50]). Finally, $\varphi_0 = 0$ almost everywhere on Γ implies $\psi_0 = 0$ almost everywhere in $[0, \pi]$ and the injectivity of K_0 is proven.

From (2.8) and

$$\frac{i}{4} \frac{d}{dt} H_0^{(1)}(t) = -\frac{1}{2\pi t} + O\Big(t \ln \frac{1}{t}\Big)$$

for $t \to 0$ it can be deduced that $\|K_k - K_0\| = O(1/\ln|k|)$ $(k \to 0)$ for the difference $K_k - K_0 : L^2[0, \pi] \to H_e^1[0, \pi]$. Now the statement of the Theorem follows from the invertibility of K_0 and the boundedness of the inverse K_0^{-1} by writing $K_k = K_0[I + K_0^{-1}(K_k - K_0)]$ and using the Neumann series ∎

Corollary 3.3. *For* $0 \le k \le k_0$ *the Dirichlet problem* (2.1) – (2.3) *is uniquely solvable.*

Proof. This follows from Theorems 2.1 and 3.2 by using the single-layer approach (2.7) ∎

Now we are in a position to state the main and final result of this paper.

Theorem 3.4. *Assume that* $\|f_k - f_0\|_{C^1(\Gamma)} \to 0$ *as* $k \to 0$. *Then, for* $k \to 0$, *the solution* u_k *of the Dirichlet problem* (2.1) – (2.3) *for the Helmholtz equation converges uniformly on compact subsets of* $\mathbb{R}^2$ *to the solution* u_0 *of the Dirichlet problem* (2.4) – (2.6) *for the Laplace equation.*

Proof. We represent u_k and u_0 in the form of the potentials (2.7) and (2.9) with densities φ_k and φ_0, respectively. For the corresponding solutions $\psi_k = A\varphi_k$ of the integral equations (2.12) and (2.13), that is, for $\psi_k = K_k^{-1} g_k$, using Theorem 3.2 and the triangle inequality, we can conclude that

$$\|\psi_k - \psi_0\|_{L^2[0,\pi]} \to 0 \qquad (k \to 0). \tag{3.1}$$

Here, we have used that

$$\|g_k - g_0\|_{H_e^1[0,\pi]} = \|B(f_k - f_0)\|_{H_e^1[0,\pi]} \le c_1 \|f_k - f_0\|_{C^1(\Gamma)}$$

for some positive constant c_1. In view of (2.14), the convergence (3.1) implies that

$$\|\varphi_k - \varphi_0\|_{L^p(\Gamma)} \to 0 \qquad (k \to 0) \tag{3.2}$$

for $1 < p < \frac{4}{3}$.

We split the difference $u_k - u_0 = v_k + w_k$ where, for $x \in \mathbb{R}$,

$$v_k(x) = \int_\Gamma \Big\{\Phi_0(x,y)(M[\varphi_k - \varphi_0])(y) + [\varphi_k(y) - \varphi_0(y)]\Big\}ds(y)$$

and

$$w_k(x) = \int_\Gamma \Big\{[\Phi_k(x,y) - \Phi_0(x,y)](M\varphi_k)(y) - \Big[\frac{2\pi}{\ln k}\Phi_k(x,y) + 1\Big]\varphi_k(y)\Big\}ds(y).$$

For v_k, from the boundedness of $K_0 : L^2[0,\pi] \to H_e^1[0,\pi]$ and (3.1), we have that

$$\begin{aligned} \|Bv_k\|_\infty &= \|K_0(\psi_k - \psi_0)\|_\infty \\ &\leq c_2\|K_0(\psi_k - \psi_0)\|_{H_e^1[0,\pi]} \\ &\leq c_3\|\psi_k - \psi_0\|_{L^2[0,\pi]} \\ &\to 0 \end{aligned} \qquad (k \to 0)$$

for some constants $c_2 > 0$ and $c_3 > 0$, that is $\|v_k\|_{\infty,\Gamma} \to 0$ as $k \to 0$. For sufficiently large $R > 0$, by the Hölder inequality and (3.2), we can estimate

$$|v_k(x)| \leq c_4\,\|\varphi_k - \varphi_0\|_{L^p(\Gamma)} \to 0 \qquad (k \to 0)$$

for $|x| = R$ and some constant $c_4 > 0$ depending on R. Now, by the maximum-minimum principle for harmonic functions from the two preceding estimates we derive that $v_k \to 0$ $(k \to 0)$ uniformly on the disk $\{x \in \mathbb{R}^2 : |x| \leq R\}$.

Finally, for the function w_k, using (2.8) and the Hölder inequality, we can estimate

$$|w_k(x)| \leq \frac{c_5}{\ln|k|}\,\|\varphi_k\|_{L^p(\Gamma)} \to 0 \qquad (k \to 0)$$

for $|x| \leq R$ and some constant $c_5 > 0$ depending on R. This completes the proof ∎

In the case where $f_0 = 1$ we have that $u_0 = 1$ in all of $\mathbb{R}^2$. On the other hand, the radiation condition implies that $u_k(x) \to 0$ as $|x| \to \infty$ for all $k > 0$. Therefore, in Theorem 3.4, we cannot have uniform convergence in $\mathbb{R}^2$.

From the proof of Theorem 3.4 it is obvious that we also have uniform convergence of the derivatives of arbitraty order on compact subsets of $\mathbb{R}^2 \setminus \Gamma$.

References

[1] Colton, D. and R. Kress: *Integral Equation Methods in Scattering Theory.* New York: Wiley-Intersci. Publ. 1983.

[2] Colton, D. and R. Kress: *Integral Equation Methods in Scattering Theory* (in Russian). Moscow: Mir 1987 (Russian translation of [1]).

[3] Colton, D. and R. Kress: *Inverse Acoustic and Electromagnetic Scattering Theory.* 2nd edition. Berlin: Springer-Verlag 1998.

[4] Gohberg, I. and N. V. Krupnik: *Einführung in die Theorie der eindimensionalen singulären Integraloperatoren.* Basel: Birkhäuser Verlag 1979.

[5] Hayashi, Y.: *The Dirichlet problem for the two-dimensional Helmholtz equation for an open boundary.* J. Math. Anal. Appl. 44 (1973), 489 – 530.

[6] Henrici, P.: *Applied and Computational Complex Analysis.* Vol 3. New York: Wiley-Intersci. Publ. 1986.

[7] Hsiao, G. C. and W. L. Wendland: *On the low frequency asymptotics of the exterior 2-D Dirichlet problem in dynamic elasticity.* In: Inverse and Ill-Posed Problems (eds.: H.W. Engl and C.W. Groetsch). Boston: Acad. Press 1987, pp. 461 – 482.

[8] Hsiao, G. C. and W. L. Wendland: *On the low frequency asymptotics for exterior elasticity problems.* In: Integral Equations and Inverse Problems (Pitman Res. Notes Math. Series: Vol. 235; eds.: V. Petkov and R. Lazarov). Burnt Hill: Longman 1991, pp. 115 – 127.

[9] Kirsch, A.: *An Introduction to the Mathematical Theory of Inverse Problems.* Berlin – Heidelberg – New York: Springer-Verlag 1996.

[10] Kress, R.: *On the low wave number asymptotics for the two-dimensional exterior Dirichlet problem for the reduced wave equation.* Math. Meth. Appl. Sci. 9 (1987), 335 – 341.

[11] Kress, R.: *Linear Integral Equations.* 2nd edition. Berlin – Heidelberg – New York: Springer-Verlag 1999.

[12] Kress, R.: *Inverse scattering from an open arc.* Math. Meth. Appl. Sci. 18 (1995), 267 – 293.

[13] MacCamy, R. C.: *Low frequency acoustic oszillations.* Quart. Appl. Math. 23 (1965), 247 – 256.

[14] Multhopp, H.: *Die Berechnung der Auftriebsverteilung von Tragflügeln.* Luftfahrt-Forschung 4 (1938), 153 – 169.

[15] Werner, P.: *Low frequency asymptotics for the reduced wave equation in two-dimensional exterior spaces.* Math. Meth. Appl. Sci. 8 (1986), 134 – 156.

[16] Yan, Y. and I. H. Sloan: *On integral equations of the first-kind with logarithmic kernels.* J. Int. Equ. Appl. 1 (1988), 549 – 579.

Received 11.12.1997