Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 3, 569–584

Compactness and Existence Results for Ordinary Differential Equations in Banach Spaces

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Dedicated to Prof. P. P. Zabreĭko on the occasion of his 60th birthday

Abstract. We prove that the Picard-Lindelöf operator

$$Hx(t) = \int_{t_0}^{t} f(s, x(s))\, ds$$

with a vector function f is continuous and compact (condensing) in C, if f satisfies only a mild boundedness condition, and if $f(s, \cdot)$ is continuous and compact (resp. condensing). This generalizes recent results of the second author and immediately leads to existence theorems for local weak solutions of the initial value problem for ordinary differential equations in Banach spaces.

Keywords: *Ordinary differential equations in Banach spaces, nonlinear Volterra integral operators, Picard-Lindelöf operators, compactness, condensing operators, measures of non-compactness*

AMS subject classification: Primary 34 A 12, 34 G 20, 47 H 30, secondary 45 N 10, 45 P 05, 46 G 10

1. Introduction

We consider the Cauchy problem

$$\left.\begin{aligned} x' &= f(t, x) \\ x(t_0) &= u \end{aligned}\right\} \tag{1}$$

in a Banach space U. By a solution of problem (1) we mean a solution of the integral equation

$$x(t) = u + \int_{t_0}^{t} f(s, x(s))\, ds,$$

where the integral is understood in the sense of Bochner. The existence of solutions has been studied by many authors, e.g. by Krasnoselskiĭ and Krein [12], Ambrosetti

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ISSN 0232-2064 / $ 2.50

[2], and Sadovskii [17] (see also [6: §2.1 and §8.1], [1: Section 4.1], and the references therein). The usual assumption for such results is that f be uniformly continuous. In the article [19] we have replaced this condition by a condition on "uniform measurability". However, Mönch [14] and Mönch and von Harten [15] have found a related result where f is assumed to be only continuous. Their main ideas carry over to the case when f just satisfies a Carathéodory condition.

We employ these ideas in the proof of the basic Proposition 1.1 below. In Section 2 we extend that proposition to uncountable sets and to subsets of C. Finally, in Section 3 we apply our results to prove that the integral operator

$$Hx(t) = \int_{t_0}^{t} f(s, x(s))\, ds \tag{2}$$

is compact (or condensing) under the natural assumption that $f(s, \cdot)$ is compact (or condensing); only the Carathéodory condition and a very mild boundedness condition are required additionally. This result is apparently new and generalizes [19]. It immediately leads to existence results for problem (1).

2. Integration and compactness of sequences

Given a Banach space U and a subset $M \subseteq U$, we denote by $\chi(M)$, $\chi_i(M)$, $\alpha(M)$ the Hausdorff, inner Hausdorff, and the Kuratowski measures of non-compactness of the set M, respectively (see, e.g., [1]). Recall that $\chi(M)$ (resp. $\chi_i(M)$) is the infimum of all $\varepsilon > 0$ such that one can find a finite ε-net for M in U (resp. in M). We also write $\chi_U(M)$ instead of $\chi(M)$ to emphasize the dependence on U. Similarly, $\alpha(M)$ is the infimum of all $\delta > 0$ such that M can be covered by finitely many sets with diameter less than δ. Evidently,

$$\chi(M) \leq \chi_{U_0}(M) \leq \chi_i(M) \leq \alpha(M) \leq 2\chi(M) \tag{3}$$

for each closed subspace $U_0 \subseteq U$ with $M \subseteq U_0$. In contrast to the other measures of non-compactness, χ_i is neither monotone nor invariant under passing to the convex hull. Nevertheless, it is very natural and useful for the following considerations.

If $M \subseteq U$ is separable, then there exists always a countable dense subset $M_0 \subseteq M$, and so $\chi(M_0) = \chi(M)$. In general, one can only say the following (similar to [1: Theorem 1.4.5]):

Lemma 1.1. *For any set M in a Banach space U there exists a countable subset $M_0 \subseteq M$ such that $\alpha(M_0) \geq \chi_i(M)$.*

Proof. Without loss of generality, let $\chi_i(M) > 0$. Then there exists a sequence μ_n with $0 < \mu_n \uparrow \chi_i(M)$. Fix n for the moment and define a sequence $x_k \in M$ inductively as follows. If $x_1, \ldots, x_{k-1}$ are already defined, choose $x_k \in M$ such that $|x_k - x_j| > \mu_n$ for all $j < k$. Such an x_k exists, since otherwise $\{x_1, \ldots, x_{k-1}\}$ is a finite μ_n-net for M, contradicting $\mu_n < \chi_i(M)$. Now let M_n be the set of all x_k, and $M_0 = \bigcup M_n$. If M_0 is covered by finitely many sets $D_1, \ldots, D_N$, we find for each n some set D_k which contains infinitely many points of M_n. In particular, $\operatorname{diam} D_k > \mu_n$ ∎

Considering the Hausdorff measure χ, we lost the factor 2 in non-separable spaces in Lemma 1.1. But in general one can not do better, as can be seen from the following example which slightly simplifies the example in [1: Theorem 1.4.5]:

Example 1.1. Let I be an uncountable set, and let U consist of all bounded functions $x : I \to \mathbb{R}$ with at most countable support. Endowed with the sup-norm, U is a Banach space, since the support of a sequence $x_n \in U$ is countable, too. Assume that $M = \{x \in U : 0 \le x(t) \le 1\}$ has a finite ε-net $N \subseteq U$. Since N has at most countable support S, but I is uncountable, there is a function $x \in M$ with $x(t) = 1$ for some $t \notin S$. Hence $\varepsilon \ge \operatorname{dist}(x, N) \ge 1$, i.e. $\chi(M) \ge 1$. On the other hand, for each countable $M_0 \subseteq M$ we have $\chi(M_0) \le \frac{1}{2}$. Indeed, define $x_0 \in M$ by $x_0(t) = \frac{1}{2}$ on the support of M_0, and $x_0(t) = 0$ otherwise. Then $\{x_0\}$ is a finite $\frac{1}{2}$-net for M_0.

The following lemma is a part of [14: Propostion 1.4] (see also [15: Proposition 2]):

Lemma 1.2. *Let U be a separable Banach space. Then there exist finite-dimensional subspaces $U_1 \subseteq U_2 \subseteq \ldots$ such that for any countable bounded $M \subseteq U$, $M = \{u_1, u_2, \ldots\}$, the equality*

$$\chi(M) = \lim_{k\to\infty} \limsup_{n\to\infty} \operatorname{dist}(u_n, U_k) \tag{4}$$

holds.

The idea to use Lemma 1.2 for the proof of the following proposition is apparently due to [14, 15]. However, the proof of [15: Proposition 3] (see also [14]) contains a minor mistake in the application of Fatou's lemma (lim sup instead of lim inf), which will be avoided in the following proof by applying Lebesgue's dominated convergence theorem instead. Moreover, we consider measurable functions instead of continuous functions and a much weaker boundedness condition.

Proposition 1.1. *Let U be a separable Banach space, T be some measure space, and $x_n : T \to U$ be (strongly) (Bochner) measurable. Then $t \mapsto \chi(\{x_1(t), x_2(t), \ldots\})$ is measurable. Moreover, if x_n is bounded in L_1 and of equicontinuous norm, i.e.*

$$\sup_n \int_T \|x_n(t)\|\, dt < \infty, \tag{5}$$

and

$$\lim_{k\to\infty} \sup_n \int_{D_k} \|x_n(t)\|\, dt = 0 \tag{6}$$

for each sequence $D_k \downarrow$ of measurable sets with $\bigcap D_k = \emptyset$, then

$$\chi\left(\left\{\int_T x_1(t)\, dt, \int_T x_2(t)\, dt, \ldots\right\}\right) \le \int_T \chi\big(\{x_1(t), x_2(t), \ldots\}\big)\, dt. \tag{7}$$

Proof. The set T_∞ of all $t \in T$ such that $\{x_1(t), x_2(t), \ldots\}$ is unbounded can be written as

$$T_\infty = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \{t : \|x_n(t)\| \ge k\}.$$

In particular, $T_\infty \cap E$ belongs to the Lebesgue extension of T for each set E of finite measure (see [9: Section III.6/Theorem 10]). Hence, to prove the measurability of $g(t) = \chi(\{x_1(t), x_2(t), \ldots\})$ (or (7)), we may assume that $T_\infty = \emptyset$. Similarly, we may assume that T is σ-finite (recall that measurability of g is equivalent to measurability on each σ-finite subset, and that integrable functions have σ-finite support).

We show now that for each integrable simple function x and each subspace $V \subseteq U$

$$\operatorname{dist}\left(\int_T x(t)\,dt, V\right) \le \int_T \operatorname{dist}(x(t), V)\,dt. \tag{8}$$

Indeed, if $x = \sum_{k=1}^K u_k \chi_{E_k}$ with pairwise disjoint sets E_k of finite measure μ_k, then (8) is equivalent to

$$\operatorname{dist}\left(\sum_{k=1}^K \mu_k u_k, V\right) \le \sum_{k=1}^K \mu_k \operatorname{dist}(u_k, V).$$

But this inequality is true, since $\operatorname{dist}(\cdot, V)$ is subadditive and positively homogeneous.

For each n there is a sequence of integrable simple functions $x_{n,j}$ such that $x_{n,j} \to x_n$, as $j \to \infty$, almost everywhere on T. We may additionally assume that $\|x_{n,j}(t)\| \le \|x_n(t)\|$ (see, e.g., [18: Lemma 4.1.1]). Choosing U_k as in Lemma 1.2, we have that each function $g_{n,k,j}(t) = \operatorname{dist}(x_{n,j}(t), U_k)$ is simple. Since, for almost all $t \in T$,

$$g_{n,k,j}(t) \to g_{n,k}(t) = \operatorname{dist}(x_n(t), U_k) \qquad (j \to \infty),$$

we may conclude that each $g_{n,k}$ is measurable. By (4), also

$$g(t) = \chi(\{x_1(t), x_2(t), \ldots\}) = \lim_{k\to\infty} \limsup_{n\to\infty} g_{n,k}(t)$$

is measurable.

To prove estimate (7), we first assume additionally that $\|x_n(t)\| \le y(t)$ for some integrable function y. By (8),

$$\operatorname{dist}\left(\int_T x_{n,j}(t)\,dt, U_k\right) \le \int_T g_{n,k,j}(t)\,dt.$$

Since the integrands are dominated by y, we may apply Lebesgue's dominated convergence theorem to both sides of this inequality and find for $j \to \infty$ that

$$\operatorname{dist}\left(\int_T x_n(t)\,dt, U_k\right) \le \int_T g_{n,k}(t)\,dt.$$

In particular, for each n and k,

$$\sup_{m\ge n} \operatorname{dist}\left(\int_T x_m(t)\,dt, U_k\right) \le \int_T \sup_{m\ge n} g_{m,k}(t)\,dt.$$

The integrand to the right is dominated by y and converges, as $n \to \infty$, almost everywhere on T to $\limsup_{n\to\infty} g_{n,k}(t)$. Applying Lebesgue's dominated convergence theorem two times (once for m and once for k), we arrive at

$$\lim_{k\to\infty} \limsup_{n\to\infty} \operatorname{dist}\left(\int_T x_n(t)\,dt, U_k\right) \le \int_T g(t)\,dt.$$

In view of (4), this is the desired inequality (8).

To drop the assumption $\|x_n(t)\| \le y(t)$, we show first that (6) implies

$$\lim_{\delta\to 0} \sup_{\operatorname{mes} D<\delta} \sup_n \int_D |x_n(t)|\, dt = 0. \tag{9}$$

If this were false, there exist some $\varepsilon > 0$ and sequences E_k with $\operatorname{mes} E_k < k^{-2}$ and $n_k \in \mathbb{N}$ such that $\int_{E_k} |x_{n_k}(t)|\, dt \ge \varepsilon$. Putting $D_k = \bigcup_{j>k} E_j$, we have $\operatorname{mes} D_k \le \sum_{j>k} j^{-2} \to 0$, and get a contradiction to (6).

Now observe that the support S_n of x_n is σ-finite, and so is $S = \bigcup S_n$. Thus, there exist sets $T_k \uparrow$ of finite measure with $\bigcup T_k = S$. Let $D_k(n)$ denote the set of all $t \in T$ with $|x_n(t)| \ge k$. Then

$$\sup_n \operatorname{mes} D_k(n) = \sup_n \int_{D_k(n)} dt \le k^{-1} \sup_n \int_T \|x_n(t)\|\, dt \to 0 \tag{10}$$

as $k \to \infty$. Now define for each k a sequence $x_{k,n}(t) = x_n(t)$ for $t \in T_k \setminus D_k(n)$, and put $x_{k,n}(t) = 0$ for all other $t \in T$. Then

$$\left\| \int_T x_{k,n}(t)\, dt - \int_T x_n(t)\, dt \right\| \le \int_{S\setminus T_k} \|x_n(t)\|\, dt + \int_{D_k(n)} \|x_n(t)\|\, dt.$$

Both integrals tend to 0, uniformly in n, as $k \to \infty$: For the first integral apply (6) with $D_k = S \setminus T_k$, for the second integral use (9) and (10). Since χ is continuous with respect to the Hausdorff distance [1], we may conclude that

$$\chi\left(\left\{\int_T x_{k,1}(t)\, dt, \int_T x_{k,2}(t)\, dt, \ldots\right\}\right) \to \chi\left(\left\{\int_T x_1(t)\, dt, \int_T x_2(t)\, dt, \ldots\right\}\right)$$

as $k \to \infty$. But, for fixed k, the sequence $x_{k,n}$ is dominated by the integrable function $k\chi_{T_k}$, and so, by our previous proof,

$$\chi\left(\left\{\int_T x_{k,1}(t)\, dt, \int_T x_{k,2}(t)\, dt, \ldots\right\}\right) \le \int_T \chi(\{x_{k,1}(t), x_{k,2}(t), \ldots\})\, dt.$$

Since evidently $\chi(\{x_{k,1}(t), x_{k,2}(t), \ldots\}) \le \chi(\{x_1(t), x_2(t), \ldots\})$, a combination of these formulas gives (7) ∎

If T is a σ-finite measure space, condition (5) may be dropped, but the corresponding extension of the proof is rather technical. For our application Proposition 1.1 suffices: If T is σ-finite and without atoms, i.e. each set may be divided into finitely many sets of arbitrarily small measure, then (5) is a consequence of (6).

Indeed, let $T_1 \subseteq T_2 \subseteq \ldots$ be sets of finite measure with $\bigcup T_k = T$. Then (6) with $D_k = T \setminus T_k$ implies that for some set $E = T_k$ of finite measure we have

$$\sup_n \int_{T\setminus E} \|x_n(t)\|\, dt < \infty.$$

Now it remains to apply (9) by dividing E into a finite number of sets of measure less than δ.

2. Integration and compactness of sets

For a set M of functions $x : T \to U$ we introduce the shortcuts

$$M(t) := \{x(t) : x \in M\} \qquad (t \in T)$$

and

$$\int_T M(t)\,dt := \left\{ \int_T x(t)\,dt : x \in M \right\}.$$

Proposition 2.1. *Let U be a Banach space, T be a measure space, and M be a set of measurable functions $x : T \to U$. Assume that there is a measurable function q such that*

$$\chi_i(Q) \le q(t) \qquad (\text{for each countable } Q \subseteq M(t)). \tag{11}$$

Moreover, assume that M is bounded in L_1 and of absolutely continuous norm,

$$\sup_{x \in M} \int_T \|x(t)\|\,dt = 0 \tag{12}$$

and

$$\lim_{k\to\infty} \sup_{x \in M} \int_{D_k} \|x(t)\|\,dt = 0 \tag{13}$$

for each sequence $D_k \downarrow$ of measurable sets with $\bigcap D_k = \emptyset$. Then

$$\chi_i\left(\int_T M(t)\,dt\right) \le 2\int_T q(t)\,dt. \tag{14}$$

If M is countable or U is separable, we even have

$$\chi\left(\int_T M(t)\,dt\right) \le \int_T q(t)\,dt.$$

For separable U we may even additionally replace (11) *by*

$$\chi(Q) \le q(t) \qquad (\text{for each countable } Q \subseteq M(t)), \tag{15}$$

and for countable M we may additionally replace (11) *by*

$$\chi_i(M(t)) \le q(t). \tag{16}$$

If T is σ-finite and atomic free, then (12) *is a consequence of* (13).

Proof. Recall that each integrable function is essentially separably valued (see [9: Section III.6/Theorem 10]). By Proposition 1.1, (7) and (11) we have, for each countable $M_0 \subseteq M$,

$$\chi_{U_0}\left(\int_T M_0(t)\,dt\right) \le \int_T \chi_{U_0}(M_0(t))\,dt \le \int_T q(t)\,dt,$$

where U_0 is the closed linear hull of the essential range of the functions in M_0. If M is countable, we may choose $M_0 = M$, and are done. In general, we have by (3) in particular

$$\alpha\left(\int_T M_0(t)\,dt\right) \le 2\int_T q(t)\,dt.$$

Now (14) follows by Lemma 1.1. If U is separable, we may choose $U_0 = U$ and observe that the set $\int_T M(t)\,dt$ contains a countable subset with the same Hausdorff measure of non-compactness (recall the remark before Lemma 1.1). The final statement is proved analogously to the countable case ∎

A sufficient condition for (11) is of course

$$\alpha(M(t)) \leq q(t).$$

Be aware that (16) is in general *not* sufficient for (11), since χ_i is not monotone.

We give now an example which shows that the factor 2 in (14) is best possible in general, even if we replace χ_i by χ in (14) and even when $T = [0,1]$, if we assume the continuum hypothesis.

Example 2.1. We assume that there is a linear ordering $\leq$ on $T = [0,1]$ such that the predecessor set $P(t) = \{s \in T : s \leq t\}$ is at most countable for all $t \in T$. (If we assume the continuum hypothesis, we even find a well-ordering with this property, namely the order inherited by a bijection of the first uncountable ordinal $\aleph_1$ onto T). Define U as in Example 1.1 for $I = T$. For $s \in T$ we define an element $u_s \in U$ as the characteristic function of the set $P(s)$. Moreover, we define $x_s : T \to U$ by

$$x_s(t) = \begin{cases} u_s & \text{if } s \leq t \\ 0 & \text{otherwise} \end{cases}$$

and put $M = \{x_s : s \in T\} \cup \{2x_s : s \in T\}$. On the one hand,

$$M(t) = \{x(t) : x \in M\} = \{0\} \cup \{u_s : s \leq t\} \cup \{2u_s : s \leq t\}.$$

Hence the set $\{u_t\} \subseteq M(t)$ provides a finite 1-net for $M(t)$, i.e. $\chi_i(M(t)) \leq 1$. On the other hand, each function x_s takes almost everywhere the value u_s (except for the countable set $\{t \in T : t < s\}$), and so

$$\int_T M(t)\,dt = \left\{ \int_T x(t)\,dt : x \in M \right\} = \{u_s : s \in T\} \cup \{2u_s : s \in T\}.$$

If $N \subseteq U$ is a finite ε-net for $\int_T M(t)\,dt$, observe that N has at most countable support. Since I is uncountable, there is some $s \in T$ such that all $y \in N$ satisfy $y(s) = 0$. For this s we have $\operatorname{dist}(2u_s, N) \geq 2$; consequently, $\chi(\int_T M(t)\,dt) \geq 2 \geq 2\chi_i(M(t))$.

A result which is similar to Proposition 1.1 (and Proposition 2.1 for countable M) can be found in [11]. There one can also find quite sophisticated examples showing that the results considered there are sharp in a certain sense. In particular, [11: Example 4.4] shows that one may not replace (11) by (15) even for countable M without loosing the factor 2 (in that example one even has $T = [0,1]$ and M consists of uniformly bounded countinuous functions).

With an argument of [14, 15] we may enlarge the class of spaces U for which (11) may be replaced by the weaker assumption (15):

Definition 2.1. We say that a Banach space U has the *retraction property*, if for each separable subspace $U_0 \subseteq U$ there exists a non-expanding mapping $P : U \to U$ (i.e. $\|Pu - Pv\| \leq \|u - v\|$) with separable range and $Pu = u$ on U_0.

The retraction property is important for us in view of the following observation:

Lemma 2.1. *Let U have the retraction property, and let $U_0 \subseteq U$ be a separable subspace. Then there exists a separable closed subspace $U_1 \subseteq U$ with $U_0 \subseteq U_1$ such that $\chi_{U_1}(M) = \chi(M)$ for each $M \subseteq U_0$.*

Proof. Choose P as in Definition 2.1, and let U_1 denote the closed linear hull of $P(U_0)$. If $N \subseteq U$ is a finite ε-net for $M \subseteq U_0$, then $P(N) \subseteq U_1$ is a finite ε-net for $P(M) = M$, and so $\chi_{U_1}(M) \leq \chi(M)$ ∎

Of course, any separable Banach space U has the retraction property. Moreover, any Hilbert space has the retraction property: Choose P as the orthogonal projection of U onto $\overline{U}_0$. More general, if we assume the axiom of choice, any weakly compactly generated Banach space has the retraction property:

Recall that a Banach space U is called *weakly compactly generated*, if there is some weakly compact set $K \subseteq U$ whose linear span is dense in U. All separable spaces and all reflexive spaces are weakly compactly generated. For weakly compactly generated spaces U, there exists even a projection P of U onto a separable subspace $U_1 \supseteq U_0$ with $\|P\| = 1$ [8: Chapter 5, §2/ Theorem 3] (however, the proof makes essential use of the axiom of choice).

Corollary 2.1. *If U has the retraction property, then we may replace* (11) *by* (15) *in Proposition* 2.1.

Proof. The proof is almost the same as for separable spaces: Just replace U_0 by the space $U_1 \supseteq U_0$ of Lemma 2.1 ∎

We now consider $T = I$ with a (not necessarily compact) interval I on the extended real line. By $C(I,U)$ we denote the set of all continuous functions $x : I \to U$ which have a continuous extension to the compact closure $\overline{I}$ of I; we equip $C(I,U)$ with the usual sup-norm.

We need an extension of the classical Arzelà-Ascoli criterion. For the Kuratowski measure of non-compactness similar results as the following lemma have been proved by several authors (see, e.g., [2, 10, 16]). However, if we consider the Kuratowski measure of non-compactness, we loose again the factor 2 in our applications.

Lemma 2.2. *For any $M \subseteq C(I,U)$ we have*

$$\chi(M) \geq \chi\Big(\bigcup_{x \in M} x(I)\Big) \geq \sup_{t \in I} \chi(M(t)) \tag{17}$$

$$\chi_i(M) \geq \sup_{t \in I} \chi_i(M(t)) \tag{18}$$

$$\chi_i(M) \geq \chi_i\Big(\bigcup_{x \in M} x(I)\Big) \tag{19}$$

$$\alpha(M) \geq \sup_{t \in I} \alpha(M(t)). \tag{20}$$

Moreover, if M has an equicontinuous extension to the closure $\overline{I}$ of I in the extended real line, then

$$\chi(M) = \sup_{t \in I} \chi(M(t)) = \max_{t \in \overline{I}} \chi(M(t)) = \chi\Big(\bigcup_{x \in M} x(I)\Big). \tag{21}$$

Proof. The estimate (20) is a special case of a result proved in [16]. The second estimate (17) follows from the monotonicity of χ, and the first has been proved in [19]; however the proof is analogous to that of (19).

Let $c > \chi_i(M)$, and $N \subseteq M$ be a finite c-net for M. Then $N(t) \subseteq M(t)$ is a finite c-net for $M(t)$, and (18) follows. Since $\overline{I}$ is compact, $x(I) \subseteq x(\overline{I})$ is precompact for each $x \in N$. Given $\varepsilon > 0$, we thus find a finite ε-net for each set $x(I)$ in $x(I)$. The union of these nets evidently is a finite $(c+\varepsilon)$-net for $\bigcup_{x\in M} x(I)$. Hence, (19) is established.

To prove (21), let $\varepsilon > 0$ and $c > \sup_t \chi_i(M(t))$ be given. Choose a finite partition $t_1 < \ldots < t_n$ of $\overline{I}$ such that $\|x(t) - x(s)\| < \varepsilon$ for all t, s in the same section of this partition and all $x \in M$. For each $k = 1, \ldots, n$ there exists a finite c-net N_k for $M(t_k)$. Let N be the set of all functions which are linear in each of the sections and which pass at t_k through a point of N_k. Evidently, N is a finite $(c+2\varepsilon)$-net for M. Hence

$$\chi(M) \leq \sup_{t\in\overline{I}} \chi(M(t)).$$

The function $t \mapsto \chi(M(t))$ is continuous, since χ and M both are continuous with respect to the Hausdorff distance (recall that M is equicontinuous on $\overline{I}$). Hence the supremum is even a maximum on the compact set $\overline{I}$, and we have

$$\chi(M) \leq \sup_{t\in I} \chi(M(t)) \leq \bigcup_{x\in M} x(I).$$

The converse estimate follows by (17) ∎

We say that a sequence of measurable functions x_n converges to a function x *in measure on sets of finite measure*, if the restrictions $x_n|_E$ converge to $x|_E$ in measure for each set E of finite measure. If the underlying measure space is σ-finite, this notion of convergence is generated by a metric. Observe that in particular each a.e.-convergent sequence of measurable functions also converges with respect to this metric.

With Proposition 2.1 and Lemma 2.2 we can prove now:

Theorem 2.1. *Let U be a Banach space, I a (not necessarily compact) interval, and M a set of measurable functions $x : I \to U$, equipped with the above metric. Let $t_0 \in \overline{I}$, and let J denote the integration operator*

$$Jx(t) = \int_{t_0}^{t} x(s)\, ds. \tag{22}$$

Assume that M satisfies

$$\lim_{k\to\infty} \sup_{x\in M} \int_{D_k} \|x(t)\|\, dt = 0 \tag{23}$$

for each sequence $D_k \downarrow$ of measurable sets in I with $\bigcap D_k = \emptyset$. Then $J : M \to C(I, U)$ is defined and continuous. The range $J(M)$ has an equicontinuous extension to $\overline{I}$. Moreover, if there is a measurable function $q : I \to [0, \infty]$ with

$$\chi_i(Q) \leq q(t) \qquad (\textit{for each countable } Q \subseteq M(t)), \tag{24}$$

then

$$\chi(J(M)) \leq 2 \int_I q(t)\, dt. \tag{25}$$

If M is countable or U is separable, we may drop the factor 2 in (25). If M is countable, we may additionally replace (24) by $\chi_i(M(t)) \leq q(t)$. If U has the retraction property, we may always replace χ_i in (24) by χ.

Proof. Exhausting $\overline{I}$ by closed and bounded intervals I_k, we find, for each $\varepsilon > 0$, some k with

$$\sup_{x \in M} \int_{I \setminus I_k} \|x(t)\|\, dt < \varepsilon.$$

Since, analogously to (9),

$$\lim_{\operatorname{mes} D \to 0} \sup_{x \in M} \int_D \|x(t)\|\, dt = 0,$$

we may conclude that $J(M)$ is defined and has an equicontinuous extension to $\overline{I}$. Moreover, M is bounded in L_1. If $x_n \in M$ converges to $x \in M$ in measure on sets of finite measure, then we have for all $t \in I$ that

$$\|Jx(t) - Jx_n(t)\| \leq \int_I \|x(s) - x_n(s)\|\, ds \to 0 \qquad (n \to \infty),$$

where the last expression converges to 0 by Vitali's convergence theorem (see, e.g., [9: Section III.6/Theorem 15] or [18: Theorem 3.3.3]). To prove (25), apply Proposition 2.1 for $T = [t_0, t]$ resp. $T = [t, t_0]$ to find that

$$\chi\left(\int_{t_0}^t M(s)\, ds\right) \leq 2 \int_T q(s)\, ds \leq 2 \int_I q(s)\, ds$$

holds for all $t \in I$. Now apply (21). The proof of the last statements is analogous ∎

3. Compactness and existence of solutions

Let I be an interval of the extended real line, and $t_0 \in \overline{I}$. We consider the Picard-Lindelöf operator

$$Hx(t) = \int_{t_0}^t f(s, x(s))\, ds. \tag{26}$$

Let f be defined on a subset of $I \times U$ with a Banach space U and take values in a Banach space V. The operator (26) may be written as the composition $H = JF$ of the linear integration operator (22) studied in the last section and of the nonlinear superposition operator

$$Fx(s) = f(s, x(s)). \tag{27}$$

Theorem 3.1. *Let $f : I \times U \to V$, and let B be a set of measurable functions $x : I \to U$ with the property that* (27) *is a.e. defined on I and measurable for each $x \in B$. Moreover, assume that*

$$\lim_{k\to\infty} \sup_{x\in B} \int_{D_k} \|f(s,x(s))\|\, ds = 0 \qquad (D_k \downarrow \emptyset). \tag{28}$$

Then (26) *maps B into a subset of $C(I,V)$ which has an equicontinuous extension to $\overline{I}$. If almost all $f(t,\cdot)$ are continuous on their set $D(t)$ of definition, this mapping is continuous (where B is equipped with the metric of convergence in measure on sets of finite measure). Moreover, if $B \subseteq C(I,U)$ and*

$$\chi_i(f(\{t\} \times D_0)) \le q(t,\gamma(D_0)) \qquad (\text{for each countable } D_0 \subseteq D(t)) \tag{29}$$

holds for almost all $t \in I$ where $\gamma \in \{\alpha,\chi,\chi_i\}$ and q is measurable in the first argument and non-decreasing in the second, then

$$\chi(H(B_0)) \le \int_I q(t,\gamma(B_0))\, dt \qquad (\text{for each countable } B_0 \subseteq B). \tag{30}$$

If we have either $\gamma \ne \chi_i$ or even

$$\alpha(f(\{t\} \times D_0)) \le q(t,\gamma(D_0)) \qquad (\text{for each countable } D_0 \subseteq D(t)), \tag{31}$$

then also

$$\chi(H(B_0)) \le 2\int_I q(t,\gamma(B_0))\, dt \qquad (B_0 \subseteq B). \tag{32}$$

If V has the retraction property, we may replace χ_i in (29) *and α in* (31) *by χ; if V is even separable, we may additionally drop the factor* 2 *in* (32).

Proof. The first statement follows immediately from Theorem 2.1 and the decomposition $H = JF$. For the continuity, it suffices to prove that if a sequence $x_n \in B$ converges in measure on sets of finite measure to $x \in B$, then there is a subsequence such that $Hx_{n_k} \to Hx$ uniformly on I. However, since I is σ-finite, there is a subsequence with $x_{n_k} \to x$ a.e. on I (see, e.g., [18: Lemma 2.2.3]). Then $Fx_{n_k} \to Fx$ a.e. on I, and it remains to apply Theorem 2.1.

We now prove (30) and (32) for $\gamma \in \{\alpha,\chi\}$. For almost all $t \in I$ we may argue as follows: For each countable $Q \subseteq f(\{t\} \times B_0(t))$ we have $Q = f(\{t\} \times B_1(t))$ for some countable $B_1 \subseteq B_0$. Since Lemma 2.2 implies that $\gamma(B_1(t)) \le \gamma(B_1) \le \gamma(B_0)$, we find

$$\chi_i(Q) = \chi_i(f(\{t\} \times B_1(t))) \le q(t,\gamma(B_1(t))) \le q(t,\gamma(B_0)).$$

Now apply Theorem 2.1. To prove (30) for $\gamma = \chi_i$ we argue analogously, using the estimate

$$\chi_i(Q) \le \alpha(Q) \le \alpha(f(\{t\} \times B_0(t))) \le q(t,\gamma(B_0(t))) \le q(t,\gamma(B_0)).$$

Replacing everywhere χ_i and α by χ, we get the result if V has the retraction property ∎

The question under which conditions (27) maps measurable functions into measurable functions is a delicate problem, even in the scalar case (see [3: Chapter 1]). Product-measurability of f is neither necessary nor sufficient.

In the most important case that the sets $D(t)$ are independent of t, i.e. if $f: I \times D \to V$ for some $D \subseteq U$, it is sufficient that f satisfies the Carathéodory condition, i.e. that $f(\cdot, u)$ is measurable for all $u \in D$ and that $f(s, \cdot)$ is continuous for almost all $s \in I$.

Indeed, if $x : I \to U$ is measurable with $x(I) \subseteq D$, there exists a sequence of measurable functions $x_n : I \to U$ which converges a.e. on I to x and satisfies $x_n(I) \subseteq D$ (consider, e.g., the proof of [9: Section III.6./Theorem 10] to see this). Evidently, Fx_n is measurable and converges a.e. on I to Fx.

Corollary 3.1. *Let $f : I \times U \to V$ satisfy the Carathéodory condition and the boundedness condition* (28) *for $B = C(I, D)$. If $f(t, \cdot)$ is compact, then the operator* (26) *defines a compact and continuous mapping from $C(I, D)$ into $C(I, V)$.*

Corollary 3.1 is somewhat surprising, since under its assumptions, $f(I \times D)$ is usually not contained in a compact set.

In connection with Darbo's fixed point theorem [5], Theorem 3.1 immediately gives an existence result for the Cauchy problem (1). However, since we have a better estimate (30) for countable subsets, we may gain the factor 2, if we use the following fixed point theorem of Daher [4] instead:

Theorem 3.2. *Let K be a non-empty, bounded, closed and convex subset of a Banach space, and let $A : K \to K$ be continuous. Assume that for all countable $C \subseteq K$ we have $\chi(A(C)) < \chi(C)$, if C is not precompact. Then A has a fixed point in K.*

Theorem 3.2 is a special case of [14: Theorem 2.1] (see also [7: Theorem 18.2]).

Corollary 3.2. *Let $V = U$, and let $D \subseteq U$ contain some closed ball with radius $r > 0$ around the initial value u. Assume that $f : I \times D \to V$ satisfies the Carathéodory condition and the boundedness condition* (28) *with $B = \{x \in C(I, U) : x(I) \subseteq D\}$. Moreover, assume that there is a non-degenerate interval $I_0 \subseteq I$ with $t_0 \in \overline{I}_0$ and $\varepsilon > 0$ such that*

$$\int_{I_0} q(t, \lambda)\, dt < \lambda \qquad (0 < \lambda \leq \varepsilon) \tag{33}$$

for some function q measurable in the first argument and non-decreasing in the second such that for almost all $t \in I$

$$\chi_i(f(\{t\} \times D_0)) \leq q(t, \chi(D_0)) \qquad (\textit{for each countable } D_0 \subseteq D). \tag{34}$$

Then there exists a local solution of the Cauchy problem (1).

More precisely, if $\varepsilon \leq r$, and B_ε denotes the set of all functions $x \in C(I, U)$ which satisfy $\|x(t) - u\| \leq \varepsilon$, then there exists a non-degenerate interval $J_0 \subseteq I_0$ such that $t_0 \in \overline{J}_0$,

$$\sup_{t,s \in J_0} \sup_{x \in B_\varepsilon} \|Hx(t) - Hx(s)\| \leq \varepsilon, \tag{35}$$

and problem (1) *has a solution on J_0 in B_ε.*

Proof. Estimate (35) is a consequence of the equicontinuity of $H(B)$. We consider B_ε as a subset of $X = C(J_0, U)$. By Theorem 3.1, the mapping $Ax = Hx + u$ is defined and continuous from B_ε into X, even into B_ε by (35) and $Hx(t_0) = 0$, which satisfies

$$\chi(A(B_0)) = \chi(H(B_0)) \leq \int_{J_0} q(t, \chi(B_0))\, dt \qquad \text{(for each countable } B_0 \subseteq B_\varepsilon).$$

Hence, in view of (33) and Theorem 3.2, $A : B_\varepsilon \to B_\varepsilon$ has a fixed point $x \in B_\varepsilon$ ∎

If U has the retraction property, we may even replace χ_i in (34) by χ.

The crucial condition (33) holds in particular for some I_0 (and each $\varepsilon > 0$) if we have an estimate of the form

$$\alpha(f(t, D_0)) \leq q(t)\alpha(D_0) \qquad (D_0 \subseteq D)$$

with an integrable function q (consider $\tilde{q}(t, u) = q(t)u$). For D being the ball with center u and radius $\varepsilon > 0$, this is the case, if almost all $f(t, \cdot)$ can be written in the form

$$f(t, u) = L(t, u) + K(t, u),$$

where $L(t, \cdot)$ satisfies a Lipschitz condition

$$\|L(t, u_1) - L(t, u_2)\| \leq q(t)\|u_1 - u_2\| \qquad (\|u_i - u\| \leq \varepsilon),$$

and $K(t, \cdot)$ is a compact mapping. This generalizes the classical result of Krasnoselskiĭ and Krein [12] where it is assumed that $K(I \times D)$ be precompact.

Observe that Corollary 3.2 also covers the terminal value problem, since the cases $t_0 = \pm\infty$ are not excluded. We remark, however, that Corollary 3.2 does not cover all cases that are contained in [15]. For example, for $I = [0, 1]$ and $t_0 = 0$, the Nagumo function $q(t, \lambda) = \frac{\lambda}{t}$ does not satisfy (33).

Example 3.1. Consider the nonlinear Barbashin equation

$$\frac{\partial x(t, s)}{\partial t} = c(t, s, x(t, s)) + \int_a^b k(t, s, \sigma, x(t, \sigma))\, d\sigma \quad (t \in [0, \infty),\ s \in [a, b]) \tag{36}$$

under the initial value condition $x(0, s) = \varphi(s)$. Assume that $c(t, \cdot, \cdot)$, $k(t, \cdot, \cdot, \cdot)$, and φ are continuous and c is even Lipschitz continuous with respect to the last argument (for simplicity with a global Lipschitz constant). We consider x as a function from $[0, \infty)$ into $X = C([a, b])$ by putting $x(t)(s) = x(t, s)$. To prove that (36) has a local solution (for t near 0), it suffices to prove that the initial value problem in the space X,

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= C(t, x(t)) + K(t, x(t)) \\ x(0) &= \varphi \end{aligned}\right\} \tag{37}$$

has a local solution. Here, C and K are defined by

$$C(t, u)(s) = c(t, s, u(s))$$

and

$$K(t,u)(s) = \int_a^b k(t,s,\sigma,u(\sigma))\,d\sigma.$$

Observe that $C(t,\cdot)$ satisfies a Lipschitz condition, and $K(t,\cdot)$ is continuous and compact. If c and k do not depend continuously on the first parameter t, one can not expect a classical solution of (37). The results in the cited literature do not apply for this case. However, if $C(\cdot,u)$ and $K(\cdot,u)$ are at least measurable for $u \in C([a,b])$, it makes sense to confine ourselves to a weak solution of (37), i.e. to a solution of the integrated form of (37). Corollary 3.2 implies that such a local solution exists if, e.g., c and k are uniformly bounded.

Let us finally state a parameter-dependent version of Theorem 3.1 which we will need in a forthcoming paper:

Let $f : D \subseteq I \times U \times \Lambda \to V$ with I, U, V as before, and Λ being a non-empty set. Consider the operator

$$H(x,\lambda)(t) = \int_{t_0}^t f(s,x(s),\lambda)\,ds \tag{38}$$

and the corresponding superposition operator

$$F(x,\lambda)(t) = f(t,x(t),\lambda). \tag{39}$$

By $D(t)$ we denote the set of all u such that $(t,u,\lambda) \in D$ for all $\lambda \in \Lambda$.

Theorem 3.3. *Let $f : D \subseteq I\times U\times\Lambda \to V$, and B be a set of measurable functions x satisfying $x(t) \in D(t)$ for almost all $t \in I$. Let (39) be measurable for all $(x,\lambda) \in B\times\Lambda$. Moreover, assume that*

$$\lim_{k\to\infty} \sup_{(x,\lambda)\in B\times\Lambda} \int_{D_k} \|f(s,x(s),\lambda)\|\,ds = 0 \qquad (D_k \downarrow \emptyset).$$

Then (38) maps $B\times\Lambda$ into a subset of $C(I,V)$ which has an equicontinuous extension to $\overline{I}$. Moreover, if $B \subseteq C(I,U)$ and

$$\alpha(f(\{t\}\times D_0\times\Lambda_0)) \le q(t,\gamma(D_0)) \quad (\text{each countable } D_0 \subseteq D(t), \Lambda_0 \subseteq \Lambda) \tag{40}$$

holds for almost all $t \in I$, where $\gamma \in \{\alpha,\chi\}$ and q is measurable in the first argument and nondecreasing in the second, then

$$\chi(H(B_0\times\Lambda)) \le 2\int_I q(t,\gamma(B_0))\,dt \qquad (\text{for each } B_0 \subseteq B) \tag{41}$$

and for each countable $\Lambda_0 \subseteq \Lambda$ even

$$\chi(H(B_0\times\Lambda_0)) \le \int_I q(t,\gamma(B_0))\,dt \qquad (\text{for each countable } B_0 \subseteq B).$$

If V has the retraction property, we may replace α by χ in (40); if V is even separable, we may additionally drop the factor 2 in (41).

Proof. Observe that $H(B \times \Lambda) = JF(B \times \Lambda)$. By Theorem 2.1, $H(B \times \Lambda)$ thus is equicontinuous. For the second part, apply Theorem 2.1 to the set $F(B_0 \times \Lambda)$. If $Q \subseteq F(B_0 \times \Lambda)(t)$ is countable, we find countable $B_1 \subseteq B_0$, $\Lambda_0 \subseteq \Lambda$ with $Q \subseteq F(B_1 \times \Lambda_0)(t)$. Since, by Lemma 2.2, we have $\gamma(B_1(t)) \leq \gamma(B_1) \leq \gamma(B_0)$, we may conclude that, for almost all $t \in I$,

$$\chi_i(Q) \leq \alpha\big(\{f(t, x(t), \lambda) : (x, \lambda) \in B_1 \times \Lambda_0\}\big) \leq 2q(t, \gamma(B_1(t))) \leq 2q(t, \gamma(B_0)),$$

as required. If V has the retraction property, we may everywhere replace χ_i and α by χ ∎

If (40) holds even for arbitrary $D_0 \subseteq D(t)$ (not only for countable), we may also allow $\gamma = \chi_i$, since also in this case

$$\alpha\big(\{f(t, x(t), \lambda) : (x, \lambda) \in B_1 \times \Lambda_0\}\big) \leq 2q(t, \gamma(B_0(t))) \leq 2q(t, \gamma(B_0)).$$

Immediately after this text was finished, the article [13] appeared which partly overlaps with our results: Our Proposition 1.1 is the same as [13: Corollary 3.1] (but with a different proof) and our Proposition 2.1/Corollary 2.1 is only slightly more general than [13: Theorem 3.12].

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Received 11.05.1998