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# On Some Dimension Problems for Self-Affine Fractals

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**Abstract.** We deal with self-affine fractals in $\mathbb{R}^2$. We examine the notion of affine dimension of a fractal proposed in [26]. To this end, we introduce a generalized affine Hausdorff dimension related to a family of Borel sets. Among other results, we prove that for a suitable class of self-affine fractals (which includes all the so-called general Sierpiński carpets), under the "open set condition", the affine dimension of the fractal coincides – up to a constant – not only with its Hausdorff dimension arising from a non-isotropic distance $D_\theta$ in $\mathbb{R}^2$, but also with the generalized affine Hausdorff dimension related to the family of all balls in $(\mathbb{R}^2, D_\theta)$. We conclude the paper with a comparison between this assertion and results already known in the literature.

**Keywords:** *Self-affine fractals, Hausdorff measures, dimensions, homogeneous spaces*

**AMS subject classification:** Primary 28 A 80, secondary 28 A 78, 43 A 85

## Introduction

This paper deals with *self-affine fractals* in $\mathbb{R}^2$, that is, fractals arising from systems of contractive affinities. Even though not so intensively studied as the case of systems consisting of similitudes, the construction of fractals by affine maps has been widely investigated in recent years. A list of references on the subject can be found, for instance, in [17: Chapter 4, Remarks 4.15].

Our work is suggested by the recent book [26] by H. Triebel, where fractal geometry is studied in connection with Fourier analysis, function spaces on self-affine fractals and fractal differential operators.

Here we are interested in the geometrical background of the contents of this book. In the first chapter of [26] the author gives the definition of affine dimension of a self-affine fractal. This notion is the "affine counterpart" of the definition of the similarity dimension introduced by Mandelbrot for a system of similitudes: in the exponential equation defining the similarity dimension, one has to replace every similarity ratio with the square root of the corresponding affinity ratio. Here *square* roots appear, since the underlying space is $\mathbb{R}^2$. So the affine dimension of a fractal $\Gamma$ reduces to its similarity dimension if every affinity of the system defining $\Gamma$ is a similitude.

It is clear that *a priori* the similarity dimension refers to a system $\Sigma$ of similitudes, but – as is well known – under a suitable hypothesis of "minimal overlapping" it also

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coincides with the Hausdorff dimension of the fractal $\Gamma$ arising from $\Sigma$. Consequently, it turns out to be an intrinsic property of $\Gamma$.

This paper is motivated by the analogous question on the affine dimension, that is, by the desire to understand to what extent the affine dimension is not only a property of a system $\Psi$ of affinities, but a property of the fractal associated with $\Psi$. We shall prove that for a suitable class of self-affine fractals, still under conditions of "minimal overlapping", the affine dimension coincides – up to a constant – with the Hausdorff dimension arising from a distance in $\mathbb{R}^2$, which differs from the Euclidean one, but accords with the non-isotropicity of the situation. Also, if we agree to work with quasi-distances, we can choose a quasi-distance in $\mathbb{R}^2$ so that these two dimensions are equal.

Now we give a brief description of the content of the paper. Let us underline that for simplicity we work in $\mathbb{R}^2$; still, all the ideas of Sections 2 - 4, at least, apply equally in $\mathbb{R}^n$.

In Section 1 we present the basic material that will be needed later on.

In Section 2 we report a simple example, which in part motivated these investigations: we define an infinite family of systems of contractive affinities satisfying even stronger conditions than the usual one of "minimal overlapping". All these systems lead to the same fractal, but nevertheless any two of them have different affine dimensions.

In Sections 3 and 4 we provide an extension of the results proved by Hutchinson in [11] in the case of similitudes for the affine context. To this end, we need a new notion of dimension with which to compare the affine dimension. We therefore introduce in Section 3 the definitions of generalized affine Hausdorff measures and generalized affine Hausdorff dimension. These definitions depend on the choice of a family $\mathcal{F}$ of Borel sets: different families can lead to different dimensions for the same set. We obtain conditions which ensure that the affine dimension is an intrinsic property of the fractal, or at least of the fractal and the family $\mathcal{F}$ (see Proposition 4.1(ii) and (iii)).

In Section 5 we consider a subclass of the family of self-affine fractals that nevertheless is wide enough to contain all the so-called general Sierpiński carpets. For this class we can reinterpret the affinities of the system as similitudes with respect to a new non-isotropic distance. Therefore we are able to prove Theorem 5.1, which can be considered the main result of the paper: if a condition of "minimal overlapping" is satisfied, then the affine dimension of the fractal coincides – up to a constant – with the Hausdorff dimension arising from the new distance. Also, the affine dimension coincides with the generalized affine Hausdorff dimension arising from the family of all balls defined by the same non-isotropic distance.

Finally, in Section 6 we briefly compare the propositions of Section 5 with results already known about the calculation of the Hausdorff dimension – with respect to the usual distance – of self-affine fractals in $\mathbb{R}^2$.

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## 1. Notation and preliminaries

**1.1 Carathéodory's construction of measures.** We follow the exposition of [17: Chapters 4 and 5]. Another classical reference is the book [21].

Let $(X, D)$ be a metric space. For $E \subseteq X$ we denote by diam $E$ the diameter of $E$. Let $\mathcal{F}$ be a family of subsets of $X$ and $\zeta$ a non-negative function on $\mathcal{F}$, satisfying the following assumptions:

($\mathcal{F}1$) *For every $\delta > 0$ there are $F_1, F_2, \ldots \in \mathcal{F}$ such that $X = \bigcup_{i=1}^{\infty} F_i$ and* $\operatorname{diam} F_i \leq \delta$.

($\mathcal{F}2$) *For every $\delta > 0$ there is $F \in \mathcal{F}$ such that $\zeta(F) \leq \delta$ and* $\operatorname{diam} F \leq \delta$.

For $E \subseteq X$ and $\delta > 0$, we define

$$\beta_\delta(E) \;=\; \inf\left\{ \sum_{i=1}^{\infty} \zeta(F_i) \,\middle|\, E \subseteq \bigcup_{i=1}^{\infty} F_i \;,\; \operatorname{diam} F_i \leq \delta \;,\; F_i \in \mathcal{F} \right\},$$

and letting $\delta \to 0^+$ we define

$$\beta(E) \;=\; \lim_{\delta \to 0^+} \beta_\delta(E) \;.$$

So $\beta$ is a Borel (outer) measure in $X$ and, if the members of $\mathcal{F}$ are Borel sets, $\beta$ is Borel regular. If $\mathcal{F}$ consists of all subsets of $X$, $s$ is a non-negative real number, and $\zeta(E) = (\operatorname{diam} E)^s$, then the resulting (outer) measure is called the *$s$-dimensional Hausdorff measure* $\mathcal{H}^s$. If $\mathcal{F}$ consists of all balls in $X$ and $\zeta$ is defined as before, then the resulting (outer) measure is called the $s$-dimensional *spherical* Hausdorff measure $\mathcal{S}^s$. The two measures $\mathcal{H}^s$ and $\mathcal{S}^s$ are related by the inequalities $\mathcal{H}^s(E) \leq \mathcal{S}^s(E) \leq 2^s \mathcal{H}^s(E)$. The properties of the measures $\mathcal{H}^s$ allow us to define the Hausdorff dimension of a set $E \subseteq X$ as

$$\dim_H E \;=\; \sup\{s \mid \mathcal{H}^s(E) = \infty\}$$

and to prove that $\dim_H E \;=\; \inf\{t \mid \mathcal{H}^t(E) = 0\}$.

**1.2 Similitudes and affinities.** Let $(X, D)$ be a metric space. For a map $f : X \to X$ the terms *Lipschitz-continuous function*, *contraction* and *similitude* have the usual meaning (see, e.g., [11: p. 716 - 717]). We write $\operatorname{Lip} f$ for the Lipschitz constant of $f$. If $\sigma$ is a similitude, we denote $\operatorname{Lip}\sigma$ by $\rho_\sigma$ and call $\rho_\sigma$ the (similarity) *ratio* of $\sigma$.

Throughout the paper $X$ is the real space $\mathbb{R}^2$. In $\mathbb{R}^2$ we consider also distances $D$ different from the Euclidean one, which we denote by $D_e$. Nevertheless, all the distances we use give rise to a complete metric space, induce the Euclidean topology and therefore define the same family of Borel sets. We denote by $\mathcal{L}$ the usual Lebesgue (outer) measure. It is well known that, if $D = D_e$, then the Hausdorff measure $\mathcal{H}^2$ coincides – up to a constant – with $\mathcal{L}$.

If $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is of the form $\psi(x) = Ax + h$, where $A \in \mathrm{GL}(2, \mathbb{R})$ and $h \in \mathbb{R}^2$, then $\psi$ is called an *affinity*. We briefly write $\psi = (A, h)$. We denote $|\det A|$ by $\alpha_\psi$ (or simply $\alpha$) and call $\alpha_\psi$ the (affinity) *ratio* of $\psi$. Recall that every affinity $\psi$ has the following properties: for any subset $E \subseteq \mathbb{R}^2$, $\mathcal{L}\big(\psi(E)\big) = \alpha_\psi\ \mathcal{L}(E)$; $\psi$ is a Lipschitz continuous function with $\alpha_\psi \leq (\text{Lip}\ \psi)^2$.

Finally, a one-parametric group $\{\delta_t \mid t > 0\}$ of non-isotropic dilations of $\mathbb{R}^2$ is defined as follows: we fix two strictly positive real numbers $\theta_1, \theta_2$ and we consider the family of maps

$$\delta_t(x) = \delta_t(x_1, x_2) = \big(t^{\theta_1} x_1, t^{\theta_2} x_2\big)$$

(see, e.g., [24: Chapter I, Example 2.3]).

**1.3 Systems of contractions and fractals.** We follow [11: §3, §4, and §5] for the following subsections (i) - (ii) and [26: Chapter I, Section 4] for subsection (iii).

(**i**) Let $\Phi = \{\varphi_1, \ldots, \varphi_N\}$ be a family of contractions of $\mathbb{R}^2$ (endowed with a distance $D$).

*There exists a unique (non-empty) compact set* $\Gamma$ *in* $\mathbb{R}^2$ *invariant with respect to* $\Phi$, *that is, such that* $\Gamma = \varphi_1(\Gamma) \cup \ldots \cup \varphi_N(\Gamma)$.

We call $\Gamma$ the *fractal associated with the system* $\Phi$. In addition, let $r = \{r_1, \ldots, r_N\}$ be a family of $N$ real numbers in $]0,1[$ with $\sum_{i=1}^N r_i = 1$. Then:

*There exists a unique Borel regular (outer) measure* $\mu$ *in* $\mathbb{R}^2$ *compactly supported and of total mass 1, such that* $\mu$ *is invariant with respect to* $(\Phi, r)$, *that is,*

$$\mu(E) = \sum_{i=1}^N r_i\ \mu\big(\varphi_i^{-1}(E)\big) \qquad (E \subseteq \mathbb{R}^2).$$

Moreover, *the support of* $\mu$ *is the fractal* $\Gamma$.

We say that the system $\Phi$ satisfies the *open set condition* (OSC) if there exists a bounded open set $\mathcal{O}$ in $\mathbb{R}^2$ such that

$$\left.\begin{array}{l} \varphi_i(\mathcal{O}) \subseteq \mathcal{O} \\ \varphi_i(\mathcal{O}) \cap \varphi_j(\mathcal{O}) = \emptyset \quad \text{ for } i \neq j \ \ (i = 1, 2, \ldots, N)\ . \end{array}\right\}$$

Stronger statements than OSC have been considered. The system $\Phi$ is said to satisfy the *strong open set condition* (SOSC) if there exists a bounded open set $\mathcal{O}$ such that OSC is verified by $\mathcal{O}$ and in addition $\mathcal{O} \cap \Gamma \neq \emptyset$, where $\Gamma$ is the fractal associated with $\Phi$ (see, e.g., [22]). The system $\Phi$ is said to satisfy the *separated open set condition* if there exists a bounded open set $\mathcal{O}$ such that OSC is verified by $\mathcal{O}$ and in addition the closures of the sets $\varphi_i(\mathcal{O})$ are disjoint (see, e.g., [25]).

(**ii**) Now we suppose furthermore that the contractions of $\Phi$ are similitudes. Then we write $\Phi = \Sigma = \{\sigma_1, \ldots, \sigma_N\}$. The number $d_\Sigma > 0$ uniquely defined by the relation $\sum_{i=1}^N \rho_{\sigma_i}^{d_\Sigma} = 1$ is called the *similarity dimension* of $\Sigma$. Let $\Gamma$ again be the fractal associated with the system $\Sigma$. By abuse of language $d_\Sigma$ is called the similarity dimension of $\Gamma$. Therefore we will write $d_\Sigma = d_\Sigma\Gamma$ and, when no confusion arises, we simply write $d_\Sigma\Gamma = d$.

Let $k$ be the Hausdorff dimension of $\Gamma$. The fractal $\Gamma$ is called *self-similar* (with respect to $\Sigma$) if $\mathcal{H}^k(\Gamma) > 0$ and $\mathcal{H}^k\big(\sigma_i(\Gamma) \cap \sigma_j(\Gamma)\big) = 0$ for $i \neq j$.

The following results are proved in [11: §5] for $D = D_e$:

(I) (i) *$\mathcal{H}^d(\Gamma) < +\infty$ and so $k \leq d$.*

(ii) *Let $\Gamma$ be self-similar with $\mathcal{H}^k(\Gamma) < +\infty$. Then $d = k$.*

(iii) *If $\mathcal{H}^d(\Gamma) > 0$, then $d = k$ and $\Gamma$ is self-similar.*

(II) *If the system $\Sigma$ satisfies OSC, then $\mathcal{H}^d(\Gamma) > 0$. Moreover, suppose $r_i = \rho_{\sigma_i}^d$ for $i = 1, \ldots, N$. Then the restriction of $\mathcal{H}^d$ to $\Gamma$ is – up to a multiplicative constant – the invariant measure with respect to $(\Sigma, r)$.*

The first statement in (II) can be made more precise as follows (cf. [22, 23]):

$$SOSC \iff OSC \iff \mathcal{H}^d(\Gamma) > 0.$$

Let us note that Hutchinson's results are extended in [16] to quasi-metric spaces of homogeneous type in the sense used by Coifman and Weiss in [3]. For a description of the frame within which the results of [16] are included we refer to [19, 20].

(**iii**) Now let us suppose that the contractions of $\Phi$ are affinities. Then we write $\Phi = \Psi = \{\psi_1, \ldots, \psi_N\}$.

Let $\Gamma$ be the fractal associated with $\Psi$. The real number $d_\Psi \Gamma > 0$ uniquely defined by the relation $\sum_{i=1}^N \alpha_{\psi_i}^{(d_\Psi \Gamma)/2} = 1$ is called the *affine dimension* of $\Gamma$ (see [26: Definition 4.12]). We will often write $d_\Psi \Gamma$ as $d_\Psi$. It should be observed that in the case of diagonal affinities $d_\Psi \Gamma$ was already considered in [15: Part I, Section 4] under the name of gap dimension. The number $d_\Psi \Gamma$ plays a key role in describing the asymptotic behaviour of the eigenvalues of some fractal differential operator in [26: Theorem 30.7] and [9].

A particular case deserves attention. Let $n_1, n_2$ be integers with $n_1 \geq 2$ and $n_2 \geq 2$, and let $T$ be the subset of $\mathbb{R}^2$ given by

$$T = \left\{ (t_1/n_1, t_2/n_2) \;\middle|\; t_1, t_2 \text{ integers}, \; 0 \leq t_1 < n_1, \; 0 \leq t_2 < n_2 \right\}.$$

Suppose that every element $\psi_i$ of the system $\Psi$ is of the form $(A, h_i)$, where the translation vectors $h_i$ are in $T$ and

$$A = \begin{pmatrix} 1/n_1 & 0 \\ 0 & 1/n_2 \end{pmatrix}.$$

Thus each $\psi_i$ maps the unit square $Q$ onto a rectangle contained in $Q$. The fractal associated with $\Psi$ is called *general Sierpiński carpet* in [18].

The definition of *regular anisotropic fractal* in [26: Chapter I, Subsection 4.18] is slightly more general. In fact, every $\psi_i$ is of the form $(A_i, k_i)$, where

$$A_i = \begin{pmatrix} \pm 1/n_1 & 0 \\ 0 & \pm 1/n_2 \end{pmatrix}$$

(the signs depend on $i$ and indicate a possible reflection); the vectors $k_i$ still have as components integer multiples of $1/n_1$ and $1/n_2$, and are chosen so that each $\psi_i(Q)$ is still contained in $Q$ and $\psi_i(Q)$ is different from $\psi_j(Q)$ if $i \neq j$. In [26] the matrices $A_i$ are written in the form

$$A_i = \begin{pmatrix} \pm(2^{-\kappa})^{a_1} & 0 \\ 0 & \pm(2^{-\kappa})^{a_2} \end{pmatrix} \tag{1.1}$$

with

$$a_1 = 2 \log n_1 \left(\log(n_1 n_2)\right)^{-1}, \quad a_2 = 2 \log n_2 \left(\log(n_1 n_2)\right)^{-1} \tag{1.2}$$

$$\kappa = \frac{1}{2} \log_2(n_1 n_2) . \tag{1.3}$$

Observe that $\kappa$ and $a_1, a_2$ are chosen so that $a_1 + a_2 = 2$.

## 2. An example as motivation

In this section we refer to the pathological phenomenon sketched in [26: Chapter I, Remark 5.12/p. 31], which is also connected with [6: Example 9.10/ p. 127 - 128].

Let $\eta$ be a real number in $]0,1[$. We consider the system $\Psi = \{\psi_1, \psi_2\}$ of affine contractions

$$\psi_1(x_1, x_2) = \Big(\frac{1}{2}x_1,\ \eta x_2\Big)$$
$$\psi_2(x_1, x_2) = \Big(\frac{1}{2}x_1 + \frac{1}{2},\ \eta x_2\Big).$$

The associated fractal set $\Gamma$ is $[0,1] \times \{0\}$; if $\eta^{-1}$ is an integer number, $\Gamma$ is a regular anisotropic fractal. An easy calculation shows that $d_\Psi \Gamma = 2\ (1 - \log_2 \eta)^{-1}$. So we reach the apparently surprising result that the fractal set $\Gamma$ is independent of $\eta$, whereas $d_\Psi \Gamma$ does depend on it and can even assume any value in $]0,2[$.

We will often return to this example and use it to illustrate certain situations. In Remark 5.9 we will try to reinterpret it in an appropriate context.

**Remark 2.1.** The system $\Psi$ satisfies OSC (e.g., with $\mathcal{O} = ]0,1[\times]0,1[$) and also SOSC (e.g., with $\mathcal{O} = ]0,1[\times]-1,1[$).

**Remark 2.2.** We can slightly modify the above construction and obtain as resulting fractal the set $\Gamma_q = [0,1] \times \{q\}$, for any $q \in [0,1]$. In fact, if we start with the two affinities

$$\psi_1(x_1, x_2) = \Big(\frac{1}{2}x_1,\ \eta x_2 + q(1-\eta)\Big)$$
$$\psi_2(x_1, x_2) = \Big(\frac{1}{2}x_1 + \frac{1}{2},\ \eta x_2 + q(1-\eta)\Big)$$

we reach the conclusion that $\Gamma_q = \psi_1(\Gamma_q) \cup \psi_2(\Gamma_q)$.

**Remark 2.3.** The previous example can be easily generalized. If $\eta \in ]0,1[$ and $n, N$ are integers such that $1 < N \leq n$, let us consider the system $\Psi = \{\psi_1, \ldots, \psi_N\}$, where

$\psi_i$ is of the form $(A, h_i)$ with $A = \begin{pmatrix} 1/n & 0 \\ 0 & \eta \end{pmatrix}$, and $h_i$ are $N$ translation vectors chosen in $\{(0,0), (1/n, 0), (2/n, 0), \dots, ((n-1)/n, 0)\}$. The associated fractal $\Gamma$ is of "Cantor set" type with $\dim_H \Gamma = \log_n N$ and $d_\Psi \Gamma = 2\,(1 - \log_n \eta)^{-1} \cdot \log_n N$. So the affine dimension can vary only in $]0,\ 2 \cdot \dim_H \Gamma[$. Note that the system $\Psi$ satisfies SOSC and, under a suitable choice of $N$ and $h_i$, also the separated open set condition.

## 3. Generalized affine Hausdorff measures

In this section we define a family of generalized Hausdorff measures which behaves in a natural way with respect to the affine maps of $\mathbb{R}^2$.

Let $\mathcal{F}$ denote a family of Borel sets in $\mathbb{R}^2$ satisfying conditions ($\mathcal{F}1$) and ($\mathcal{F}2$). We will fix $\mathcal{F}$ from time to time, since $\mathcal{F}$ has to be adapted to the specific situation.

**Definition 3.1.** Let $s \geq 0$. We denote by $\mathcal{A}^s_{\mathcal{F}}$ (and call it the *$s$-dimensional generalized affine Hausdorff measure*) the outer measure ensued from Carathéodory's construction, by setting $\zeta(F) = (\mathcal{L}F)^{s/2}$ for every $F \in \mathcal{F}$.

**Remark 3.1.** For $s = 0$ we interpret as usual $0^0 = 1$; therefore $\mathcal{A}^0_{\mathcal{F}}$ coincides with the counting measure.

**Remark 3.2.** Let us observe that if $\mathcal{F}$ is the family of all balls in $(\mathbb{R}^2, D_e)$, then – up to a constant – $\mathcal{A}^s_{\mathcal{F}} = \mathcal{S}^s$. Hence $\mathcal{A}^s_{\mathcal{F}}$ and $\mathcal{H}^s$ are both zero, positive and finite, or infinite. The same result holds if $\mathcal{F}$ is the family of all squares in $\mathbb{R}^2$, or of all squares with sides parallel to the axes. Note that Definition 3.1 and this remark already appeared in [13: Section 9.3: "Volume interpretation of the Hausdorff dimension"].

We list now some properties of $\mathcal{A}^s_{\mathcal{F}}$. Since the proofs are standard, we will omit them.

**Proposition 3.1.**

**(i)** *$\mathcal{A}^s_{\mathcal{F}}$ is a Borel regular measure.*

**(ii)** *Let $E$ be a subset of $\mathbb{R}^2$. If $0 \leq s < t$, then $\mathcal{A}^s_{\mathcal{F}}(E) < +\infty$ implies $\mathcal{A}^t_{\mathcal{F}}(E) = 0$.*

As usual, this result leads to the following definition.

**Definition 3.2.** Let $E$ be a subset of $\mathbb{R}^2$. We define

$$\dim_{\mathcal{F}} E = \sup \left\{ s \;\middle|\; \mathcal{A}^s_{\mathcal{F}}(E) = \infty \right\} .$$

We call $\dim_{\mathcal{F}} E$ the *generalized affine Hausdorff dimension* of the set $E$.

It follows that $\dim_{\mathcal{F}} E = \inf \left\{ t \;\middle|\; \mathcal{A}^t_{\mathcal{F}}(E) = 0 \right\}$ and that $\dim_{\mathcal{F}} E$ is invariant under the action of bi-Lipschitz transformations $f$, such that $f$ and $f^{-1}$ map $\mathcal{F}$ in $\mathcal{F}$.

**Remark 3.3.** Suppose $\mathcal{F} = \mathcal{B}$, where $\mathcal{B}$ is the family of all Borel sets in $\mathbb{R}^2$, and $D = D_e$. Let $B \in \mathcal{B}$ with $\mathcal{L}B = 0$. For every fixed $\delta > 0$ we can cover $B$ with a countable family $\{F_i\}$ of Borel sets with diam $F_i \leq \delta$ and $\mathcal{L}F_i = 0$ (take, for instance, $F_i = B \cap R_i$, where $\{R_i\}$ is a suitable family of rectangles). Therefore, if $s > 0$, then $\mathcal{A}^s_{\mathcal{B}}(B) = 0$. Now let $B \in \mathcal{B}$ with $\mathcal{L}B > 0$. Since $\mathcal{L}B = \mathcal{A}^2_{\mathcal{B}}(B)$, it follows that

$\mathcal{A}^s_{\mathcal{B}}(B) = \infty$ for $s \in ]0,2[$. Thus for every Borel set $B$ in $\mathrm{I\!R}^2$ we have $\dim_{\mathcal{B}} B = 0$ if $\mathcal{L}B = 0$ and $\dim_{\mathcal{B}} B = 2$ if $\mathcal{L}B > 0$. This is the reason why $\mathcal{F}$ will usually be a proper subset of $\mathcal{B}$.

**Remark 3.4.** The generalized affine Hausdorff dimension $\dim_{\mathcal{F}} E$ of a subset $E$ in $\mathrm{I\!R}^2$ depends on the family $\mathcal{F}$. In Remark 3.3 we considered the case $\mathcal{F} = \mathcal{B}$. Now let us suppose that $\mathcal{F}$ consists of all balls with respect to $D_e$. By Remark 3.2, $\mathcal{A}^s_{\mathcal{F}}$ coincides – up to a constant – with $\mathcal{S}^s$ for every $s$. So, for $E \subseteq \mathrm{I\!R}^2$, $\dim_{\mathcal{F}} E = \dim_H E$.

The following property of $\mathcal{A}^s_{\mathcal{F}}$, with respect to the affinities leaving the family $\mathcal{F}$ invariant, is the exact counterpart of the behaviour of $\mathcal{H}^s$ with respect to the similitudes.

**Proposition 3.2.** *Let $\psi$ be an affinity of $\mathrm{I\!R}^2$ with ratio $\alpha$. Suppose that $\psi$ and $\psi^{-1}$ map the family $\mathcal{F}$ in $\mathcal{F}$. Then $\mathcal{A}^s_{\mathcal{F}}\big(\psi(E)\big) = \alpha^{s/2}\mathcal{A}^s_{\mathcal{F}}(E)$.*

**Remark 3.5.** From Proposition 3.2 it is clear that we are interested only in the subgroup of all the affinities $\psi$ such that $\psi$ and $\psi^{-1}$ leave $\mathcal{F}$ invariant. We will denote it by $G_{\mathcal{F}}$. It is natural to suppose, in addition to conditions ($\mathcal{F}1$) and ($\mathcal{F}2$) of Subsection 1.1, that $\mathcal{F}$ verifies the following condition:

($\mathcal{F}3$) *There are some contractive affinities in $G_{\mathcal{F}}$.*

Note that ($\mathcal{F}3$) is not implied by ($\mathcal{F}1$) and ($\mathcal{F}2$).

## 4. Strongly self-affine fractals and invariant measures

Now we extend some of the results proved in [11: §5] for similitudes and Hausdorff measures $\mathcal{H}^s$ to affinities and affine measures $\mathcal{A}^s_{\mathcal{F}}$.

Let $\mathcal{F}$ be a family of Borel sets satisfying conditions ($\mathcal{F}1$), ($\mathcal{F}2$), ($\mathcal{F}3$), and $G_{\mathcal{F}}$ be the corresponding group. Suppose that $\Psi = \{\psi_1, \ldots, \psi_N\}$ is a system of contractive affinities of $G_{\mathcal{F}}$ and let $\Gamma$ be the associated fractal. We shorten $\dim_{\mathcal{F}}\Gamma$ to $h$.

**Definition 4.1.** We say that $\Gamma$ is *strongly self-affine* (with respect to $\Psi$ and $\mathcal{F}$) if

**(i)** $\Gamma = \bigcup_{i=1}^N \psi_i(\Gamma)$.

**(ii)** $\mathcal{A}^h_{\mathcal{F}}(\Gamma) > 0$ and $\mathcal{A}^h_{\mathcal{F}}\big(\psi_i(\Gamma) \cap \psi_j(\Gamma)\big) = 0$ for $i \neq j$.

**Remark 4.1.** Definition 4.1 follows the pattern of the definition of self-similar fractal in [11: §5]. The term "self-affine" is used in the literature in a wider sense, namely to mean a fractal arising from a system of contractive affinities in $\mathrm{I\!R}^2$ with respect to the distance $D_e$. Therefore we must point out the difference.

**Remark 4.2.** Our definition of strong self-affinity is strictly related to the family $\mathcal{F}$. For instance, consider $(\mathrm{I\!R}^2, D_e)$. Let $\mathcal{F}$ be the family of all balls. Then $G_{\mathcal{F}}$ is the group of all similitudes. By Remarks 3.2 and 3.4, "strongly self-affine with respect to $\mathcal{F}$" has the same meaning as "self-similar".

On the contrary, let $\mathcal{F} = \mathcal{B}$ be the family of all Borel sets; therefore $G_{\mathcal{B}}$ is the whole affine group. Suppose that the fractal $\Gamma$ generated by $\Psi$ verifies $\mathcal{L}\Gamma = 0$. Since $\dim_{\mathcal{B}}\Gamma = 0$ and $\mathcal{A}^0_{\mathcal{B}}$ is the counting measure (recall Remarks 3.1 and 3.3), $\Gamma$ is strongly self-affine with respect to $\mathcal{B}$ if and only if $\psi_i(\Gamma) \cap \psi_j(\Gamma) = \emptyset$ for $i \neq j$.

So it is even possible that the same system $\Psi$ gives rise to a fractal $\Gamma$, which is strongly self-affine with respect to a family $\mathcal{F}_1$ but not with respect to a different family $\mathcal{F}_2$. For instance, recalling the first example given in Section 2, let us consider the system $\Psi$ with $\eta = 1/2$; consequently $\Gamma = [0,1] \times \{0\}$. If $\mathcal{F}_1$ is the family of all balls, then $\Gamma$ is strongly self-affine; on the contrary if $\mathcal{F}_2 = \mathcal{B}$, then $\Gamma$ is not strongly self-affine, since $\psi_1(\Gamma) \cap \psi_2(\Gamma) \neq \emptyset$. The latter result is still true if $\mathcal{F}_3 = \mathcal{R}$ is the family of all rectangles with sides parallel to the axes. Indeed, it is easy to prove that for every $s > 0$, $\mathcal{A}^s_{\mathcal{R}}(\Gamma) = 0$; so $\dim_{\mathcal{R}}\Gamma = 0$.

Our first goal is to compare $\dim_{\mathcal{F}}\Gamma$ with $d_\Psi\Gamma$. For the sake of simplicity we denote $\dim_{\mathcal{F}}\Gamma = h$ and $d_\Psi\Gamma = d_\Psi$. From now on let us suppose that the family $\mathcal{F}$ satisfies the following additional condition:

($\mathcal{F}4$) *Any compact set in* $\mathbb{R}^2$ *can be covered by a finite number of elements of* $\mathcal{F}$.

Observe that ($\mathcal{F}1$) is not implied by ($\mathcal{F}1$), ($\mathcal{F}2$) and ($\mathcal{F}3$). Then we can prove

**Proposition 4.1.**

**(i)** $\mathcal{A}^{d_\Psi}_{\mathcal{F}}(\Gamma) < +\infty$ *and so* $h \leq d_\Psi$.

**(ii)** *Let* $\Gamma$ *be strongly self-affine with* $\mathcal{A}^h_{\mathcal{F}}(\Gamma) < +\infty$. *Then* $d_\Psi = h$.

**(iii)** *If* $\mathcal{A}^{d_\Psi}_{\mathcal{F}}(\Gamma) > 0$, *then* $d_\Psi = h$ *and* $\Gamma$ *is strongly self-affine.*

**Proof.** (i): By ($\mathcal{F}4$), we can suppose $\Gamma \subseteq \cup_{i=1}^{\ell} F_i$ with $F_i \in \mathcal{F}$. Then, for every $p \in \mathbb{N}$,

$$\Gamma = \cup_{j_1,\dots,j_p}\psi_{j_1} \circ \dots \circ \psi_{j_p}(\Gamma) \ \subseteq\ \cup_{j_1,\dots,j_p} \cup_{i=1}^{\ell} \psi_{j_1} \circ \dots \circ \psi_{j_p}(F_i)$$

and diam $\psi_{j_1} \circ \dots \circ \psi_{j_p}(F_i) \to 0$ as $p \to \infty$. It follows that, for any $\delta > 0$, $\Gamma$ can be covered by a finite number of elements of the family $\mathcal{F}$ with diameter less than $\delta$. Therefore

$$\begin{aligned}
\mathcal{A}^{d_\Psi}_{\mathcal{F},\delta} &\leq \sum_{j_1,\dots,j_p} \sum_{i=1}^{\ell} \mathcal{L}\big(\psi_{j_1} \circ \dots \circ \psi_{j_p}(F_i)\big)^{d_\Psi/2} \\
&= \sum_{j_1,\dots,j_p} \sum_{i=1}^{\ell} \big(\alpha_{j_1} \dots \alpha_{j_p}\big)^{d_\Psi/2} \ \big(\mathcal{L}F_i\big)^{d_\Psi/2} \\
&= \sum_{i=1}^{\ell} \big(\mathcal{L}F_i\big)^{d_\Psi/2} \\
&< \infty
\end{aligned}$$

and $\mathcal{A}^{d_\Psi}_{\mathcal{F}}(\Gamma) < +\infty$ follows immediately.

(ii) and (iii): By Proposition 3.2 the proof is the same as in [11: Proposition 5.1(4)] ∎

**Remark 4.3.** The inequality $h \leq d_\Psi$ (which appears in (i) of Proposition 4.1) shows that the affine dimension $d_\Psi$ is a natural upper bound for the generalized affine Hausdorff dimension $h = \dim_{\mathcal{F}}\Gamma$ for every family $\mathcal{F}$ such that $\Psi \subseteq G_{\mathcal{F}}$.

**Remark 4.4.** Let us investigate the role played by the "open set condition". We recall that for a system $\Sigma$ of similitudes in $(\mathbb{R}^2, D_e)$ with similarity dimension $d_\Sigma$ the following implications are true:

$$\begin{aligned} OSC &\iff \mathcal{H}^{d_\Sigma}(\Gamma) > 0 \\ OSC &\implies d_\Sigma = \dim_H \Gamma \end{aligned}$$

(see, e.g., [11: Theorem 5.3.(1)] and, for a more detailed discussion, [22]). The analogous assertions are not in general true in the affinity context, at least if the family $\mathcal{F}$ is not properly chosen. So it may happen that

OSC is verified and $\mathcal{A}_{\mathcal{F}}^{d_\Psi}(\Gamma) = 0$

OSC is verified and $d_\Psi \Gamma \neq \dim_{\mathcal{F}} \Gamma$.

To show this, we return again to the first example of Section 2 and choose $\mathcal{F} = \mathcal{R}$, that is the family of all rectangles with sides parallel to the axes. For every $\eta \in ]0,1[$, the system $\Psi$ satisfies OSC. Nevertheless, $\mathcal{A}_{\mathcal{R}}^{d_\Psi}(\Gamma) = 0$ and $0 = \dim_{\mathcal{R}} \Gamma \neq d_\Psi \Gamma$.

We conclude this section with some remarks on invariant measures. We refer again to [11] and we extend some of the results given there to contractive affinities. If $X$ is any subset of $\mathbb{R}^2$, we denote by $\mathcal{A}_{\mathcal{F}}^s | X$ the restriction of $\mathcal{A}_{\mathcal{F}}^s$ to $X$, that is $(\mathcal{A}_{\mathcal{F}}^s | X)(E) = \mathcal{A}_{\mathcal{F}}^s(X \cap E)$, for $E \subseteq \mathbb{R}^2$.

The following propositions are the "affine counterpart" of two theorems proved in [11] for similitudes.

**Proposition 4.2.** *Let $\psi$ be an affinity of $G_{\mathcal{F}}$ with ratio $\alpha$ and let $X$ be a subset of $\mathbb{R}^2$. Then, for every $s \geq 0$ and for every $E$ in $\mathbb{R}^2$,*

$$\big(\mathcal{A}_{\mathcal{F}}^s | \psi(X)\big)\ (E) \ = \ \alpha^{s/2} \big(\mathcal{A}_{\mathcal{F}}^s | X\big)\big(\psi^{-1}(E)\big)\ .$$

**Proof.** See [11: Subsection 2.6(2)] ∎

**Proposition 4.3.** *Let $\Psi = \{\psi_1, \dots, \psi_N\}$ be a family of $N$ contractive affinities of $G_{\mathcal{F}}$. Let $\alpha_i$ be the ratio of $\psi_i$ and $\Gamma$ the fractal associated to $\Psi$. We denote $d_\Psi \Gamma = d_\Psi$ and consider the $n$-ple $r = \{\alpha_1^{d_\Psi/2}, \dots, \alpha_N^{d_\Psi/2}\}$. If $\mathcal{A}_{\mathcal{F}}^{d_\Psi}(\Gamma) > 0$, then the unique $(\Psi, r)$-invariant measure of total mass 1 is $\big(\mathcal{A}_{\mathcal{F}}^{d_\Psi}(\Gamma)\big)^{-1} \cdot \mathcal{A}_{\mathcal{F}}^{d_\Psi} | \Gamma$.*

**Proof.** See [11: Theorem 5.3(1)] ∎

**Remark 4.5.** Let $\Psi$ and $\Gamma$ be as in Proposition 4.3. If $\mathcal{A}_{\mathcal{F}}^{d_\Psi}(\Gamma) > 0$, then the normalized invariant measure $\mu = \big(\mathcal{A}_{\mathcal{F}}^{d_\Psi}(\Gamma)\big)^{-1} \cdot \mathcal{A}_{\mathcal{F}}^{d_\Psi} | \Gamma$ obeys the scaling law $\mu\big(\psi_k(\Gamma)\big) = \alpha_k^{d_\Psi/2}$ for every $k = 1, \dots, N$. This is the natural extension of the scaling property for the invariant measure in the case of self-similar fractals. Let us also note that $\mu$ coincides with the Radon measure considered in [26: Theorem 4.15].

**Remark 4.6.** Proposition 4.1(iii), Proposition 4.3, and Remark 4.5 show that the condition $\mathcal{A}_{\mathcal{F}}^{d_\Psi}(\Gamma) > 0$ seems to be significant. However, it also seems rather hard to test. So it is natural to look for conditions which are simpler to check and are sufficient to imply it. In the next section we will come back to this problem and show that for a suitable, sufficiently large class of systems of contractive affinities, OSC still implies that the resulting fractal $\Gamma$ satisfies $\mathcal{A}_{\mathcal{F}}^{d_\Psi}(\Gamma) > 0$.

## 5. Regular anisotropic fractals and homogeneous spaces

In this section we limit ourselves to some specific systems of contractive affinities in $\mathbb{R}^2$. Firstly, we use only diagonal affinities that is those affinities $(A, h)$ such that $A$ is diagonal (with respect to the usual orthonormal base of $\mathbb{R}^2$). Next, our main assumption on the system $\Psi = \{\psi_1, \ldots, \psi_N\}$ is the following: there exist two positive numbers $\theta_1$ and $\theta_2$ such that every matrix $A_i$ (corresponding to the affinity $\psi_i$ of $\Psi$) can be written in the form

$$A_i = \begin{pmatrix} \pm\tau_i^{\theta_1} & 0 \\ 0 & \pm\tau_i^{\theta_2} \end{pmatrix} \qquad \text{with } \tau_i \in ]0,1[\ . \tag{5.1}$$

As we will show, these two conditions together with OSC guarantee that the affine dimension $d_\Psi\Gamma$ can be interpreted as an "intrinsic" number related to the fractal $\Gamma$ associated with $\Psi$, in the sense that $d_\Psi\Gamma$ depends only on $\Gamma$, $\theta_1$, and $\theta_2$. More precisely, this assertion has to be understood as follows: if the fractal $\Gamma$ arises from two different systems $\Psi$ and $\tilde{\Psi}$ of affinities, satisfying the previous two conditions and having the same exponents $\theta_1$ and $\theta_2$, then $d_\Psi\Gamma = d_{\tilde{\Psi}}\Gamma$.

**Remark 5.1.** Systems of diagonal affinities give rise to fractals which in [26: Chapter I, Subsection 4.17] are called PXT-fractals. Our second assumption leads to a smaller class of fractals. Nevertheless, this class strictly contains the class of the regular anisotropic fractals. To see this it is sufficient to recall formulae (1.1) - (1.3) in Subsection 1.3./(iii).

**Remark 5.2.** In (5.1) we can suppose $0 < \theta_1 \leq \theta_2$; otherwise it is sufficient to change the order of the orthonormal base in $\mathbb{R}^2$ (this is always possible, since we are not interested in graphs of functions).

**Remark 5.3.** Let us observe that a matrix $\left(\begin{smallmatrix} \pm\tau^{\theta_1} & 0 \\ 0 & \pm\tau^{\theta_2} \end{smallmatrix}\right)$ with $\tau \in ]0,1[$ can be written as $\left(\begin{smallmatrix} \pm\sigma^{\lambda_1} & 0 \\ 0 & \pm\sigma^{\lambda_2} \end{smallmatrix}\right)$ with $\sigma \in ]0,1[$, if and only if the pair $(\lambda_1, \lambda_2)$ is a multiple of $(\theta_1, \theta_2)$, that is, if and only if there exists $c > 0$ such that $\lambda_i = c\theta_i$ $(i = 1, 2)$.

So, for any matrix of the form (5.1) one can assume that $\theta_1$ and $\theta_2$ satisfy the property $\theta_1 + \theta_2 = 2$. This choice of $\theta_1$ and $\theta_2$ is usual in the theory of anisotropic function spaces (see [26: Chapter I, Subsection 4.17], [8: Subsection 2.2] and the references therein) and in the case of regular anisotropic fractals leads to formulae (1.1) and (1.2).

Our perspective draws us to conclude that, since the underlying space is $\mathbb{R}^2$, it is more convenient to assume $\theta_1 = 1$ and $\theta_2 = \theta \geq 1$. This choice leads to simpler notation and, as will be seen, to the advantage that we can define in $\mathbb{R}^2$ a new distance related to $\theta$, and not only a quasi-distance related to $\theta_1$ and $\theta_2$. For these reasons we work with systems $\Psi = \{\psi_1, \ldots, \psi_N\}$ of affinities such that the matrix $A_i$ of $\psi_i$ has the form

$$\begin{pmatrix} \pm t_i & 0 \\ 0 & \pm t_i^\theta \end{pmatrix} \quad , \tag{5.2}$$

where $t_i \in ]0,1[$ and $\theta \geq 1$ is a fixed real number. Nevertheless, we often return to the case $\theta_1 + \theta_2 = 2$.

Corresponding to any system of affinities, whose matrices are as in (5.2), let us consider the group $G_\theta$ of all the affinities $(A, h)$ such that $h \in \mathbb{R}^2$ and $A$ varies between the matrices of the form $\left(\begin{smallmatrix} \pm t & 0 \\ 0 & \pm t^\theta \end{smallmatrix}\right)$ (now clearly with $t \in ]0, +\infty[$). Any affinity in $G_\theta$ can be regarded as the composition of the non-isotropic dilation

$$\delta_t(x) = \delta_t(x_1, x_2) = \bigl(tx_1, t^\theta x_2\bigr) \qquad (t \in ]0, +\infty[) , \tag{5.3}$$

the translation given by the vector $h$, and in some cases a reflection.

**Definition 5.1.** For $\theta \geq 1$, let $N_\theta : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$N_\theta(x) = N_\theta(x_1, x_2) = \max\bigl\{|x_1|, |x_2|^{1/\theta}\bigr\} \tag{5.4}$$

and let $D_\theta$ be defined by

$$D_\theta(x, y) = N_\theta(x - y) . \tag{5.5}$$

**Proposition 5.1.** *$N_\theta$ satisfies the following conditions:*

$N_\theta(x) \geq 0$

$N_\theta(x) = 0$ *if and only if* $x = 0$

$N_\theta(-x) = N_\theta(x)$

$N_\theta(x + y) \leq N_\theta(x) + N_\theta(y)$

$N_\theta\bigl(\delta_t(x)\bigr) = t\ N_\theta(x)$.

*Therefore, $D_\theta$ is a distance in $\mathbb{R}^2$, and $\delta_t$ is a similitude in $(\mathbb{R}^2, D_\theta)$ with ratio $t$, that is $D_\theta\bigl(\delta_t(x), \delta_t(y)\bigr) = t\ D_\theta(x, y)$. Moreover, $(\mathbb{R}^2, D_\theta)$ is a complete metric space.*

**Proof.** The triangle inequality follows from the formula $(a + b)^p \leq a^p + b^p$ for $a, b \geq 0$ and $p \in ]0, 1]$. All the rest is straightforward ∎

**Remark 5.4.** An element of $G_\theta$ is a contraction in $\mathbb{R}^2$ with respect to $D_e$ if and only if it is a contraction with respect to $D_\theta$, that is, if and only if $t \in ]0, 1[$.

**Remark 5.5.** $N_\theta$ satisfies all the conditions of homogeneous norm except for the smoothness condition in $\mathbb{R}^2 \backslash \{0\}$ (see [10: Chapter 1, Section A] and [12]).

**Remark 5.6.** If we substitute the pair $(1, \theta)$ with one of its multiple $(\theta_1, \theta_2) = (c, c\theta)$ (where $c > 0$), then we can define $N_{\theta_1\theta_2}(x) = \max\bigl\{|x_1|^{1/\theta_1}, |x_2|^{1/\theta_2}\bigr\}$. $N_{\theta_1\theta_2}$ gives rise to the quasi-distance $D_{\theta_1\theta_2}(x, y) = N_{\theta_1\theta_2}(x - y)$. This quasi-distance becomes a distance if and only if $\theta_1 \geq 1$ and $\theta_2 \geq 1$.

We recall now that a system $\Psi = \{\psi_1, \ldots, \psi_N\} \subseteq G_\theta$ can be seen as a system of affinities in $\mathbb{R}^2$ with $\alpha_{\psi_i} = t_i^{1+\theta}$, and as a system of similitudes in $(\mathbb{R}^2, D_\theta)$ with $\rho_{\psi_i} = t_i$. This remark suggests a comparison between the affine dimension and the similarity dimension in $(\mathbb{R}^2, D_\theta)$. To this aim, let us first introduce further notation.

In $(\mathbb{R}^2, D_\theta)$ we denote

by $\operatorname{diam}_\theta E$ the diameter of a subset $E \subseteq \mathbb{R}^2$

by $\mathcal{H}_\theta^l$ the $l$-dimensional Hausdorff measure

by $\mathcal{S}_\theta^l$ the $l$-dimensional spherical Hausdorff measure by $\dim_{H,\theta}$ the Hausdorff dimension.

We also denote

by $\mathcal{R}_\theta$ the family of all balls in $(\mathbb{R}^2, D_\theta)$.

Let us remark that if $F \in \mathcal{R}_\theta$ is a ball in $(\mathbb{R}^2, D_\theta)$ with radius $r$, then from the Euclidean point of view $F$ is a rectangle with sides parallel to the axes and of length $2r$ and $2r^\theta$. It follows that the group $G_\theta$ introduced in this section coincides with the group $G_{\mathcal{R}_\theta}$ (see Remark 3.5). Finally, if $\Psi = \{\psi_1, \ldots, \psi_N\}$ is a system of similitudes with respect to $D_\theta$, we denote by $d_\theta$ the similarity dimension of $\Psi$.

Using the notation just introduced, let us compare the affine dimension $d_\Psi \Gamma$ with the similarity dimension $d_\theta$.

**Proposition 5.2.** *Let $\Psi = \{\psi_1, \ldots, \psi_N\}$ be a family of contractive affinities in $G_\theta$ and let $\Gamma$ be the associated fractal. Then $d_\Psi \Gamma = 2\ (1+\theta)^{-1}\ d_\theta$.*

**Proof.** The affine dimension $d_\Psi = d_\Psi \Gamma$ and the similarity dimension $d_\theta$ are uniquely determined by

$$1 = \sum_{i=1}^N \left(t_i^{1+\theta}\right)^{d_\Psi/2} = \sum_{i=1}^N t_i^{(1+\theta)d_\Psi/2}$$

$$1 = \sum_{i=1}^N t_i^{d_\theta}\ .$$

By the uniqueness of $\gamma$ such that $\sum_{i=1}^N t_i^\gamma = 1$ we have $d_\theta = (1+\theta)2^{-1} d_\Psi$ ∎

**Corollary 5.1.** *Let $\Gamma$ be a regular anisotropic fractal in the sense of Subsection* 1.3.(iii). *Let $a_1, a_2$ be as in* (1.2) *with $a_1 \le a_2$. Then $d_\Psi \Gamma = a_1\ d_\theta$.*

**Remark 5.7.** The constant $2\ (1+\theta)^{-1}$, which appears in Proposition 5.2, depends on the choice of the exponents $(1,\theta)$ in (5.2). A different normalization of the exponents would lead to a different constant. In fact, let us substitute $(1,\theta)$ with $(\theta_1, \theta_2) = (c, c\theta)$ (where $c > 0$) and let us define consequently the quasi-distance $D_{\theta_1\theta_2}$ in $\mathbb{R}^2$ (see Remark 5.6). Then we obtain $d_{\theta_1\theta_2} = c d_\theta$, where $d_{\theta_1\theta_2}$ is the new similarity dimension of the system $\Psi$, and therefore $d_\Psi \Gamma = 2\ (1+\theta)^{-1}\ c^{-1}\ d_{\theta_1\theta_2}$.

The choice of $\theta_1$ and $\theta_2$ in the framework of anisotropic function spaces leads to the following result.

**Corollary 5.2.** *Let $\Psi = \{\psi_1, \ldots, \psi_N\}$ be a system of contractive affinities $\psi_i = (A_i, h_i)$, where $A_i$ is of the form* (5.1) *and $\theta_1 + \theta_2 = 2$. Then $d_\Psi \Gamma = d_{\theta_1\theta_2}$.*

Now we want to state the main assertions of this section. To this end, let us first prove a lemma, which compares $\mathcal{A}^s_{\mathcal{R}_\theta}$ with $\mathcal{S}^l_\theta$.

**Lemma 5.1.** *Let $s > 0$. Then $\mathcal{A}^s_{\mathcal{R}_\theta} = 2^{\frac{1-\theta}{2}\cdot s}\ \mathcal{S}_\theta^{\frac{1+\theta}{2}\cdot s}$.*

**Proof.** Let $F \in \mathcal{R}_\theta$ be a ball centered at a point $(x_1, x_2)$ of $\mathbb{R}^2$ with radius $r$ and therefore $\operatorname{diam}_\theta F = 2r$. ¿From the Euclidean point of view we have $\mathcal{L}F = 4r^{1+\theta}$ and $\operatorname{diam} F = 2\left(r^2 + r^{2\theta}\right)^{1/2}$, where $\operatorname{diam} F$ stands for the Euclidean diameter of $F$. Suppose $0 < \delta \le 1$; if $\operatorname{diam} F \le \delta$, then $\operatorname{diam}_\theta F \le \delta$, and if $\operatorname{diam}_\theta F \le \delta$, then $\operatorname{diam} F \le \sqrt{2}\delta$. By standard arguments the lemma follows ∎

Now we can prove the following theorem. We use the notation introduced after Remark 5.3 and simply write $\dim_{H,\theta}\Gamma$ as $h_\theta$.

**Theorem 5.1.** *Let $\Psi = \{\psi_1, \dots, \psi_N\}$ be a system of contractive affinities in $G_\theta$ satisfying OSC. Then $d_\Psi\Gamma = \dim_{\mathcal{R}_\theta}\Gamma = 2\,(1+\theta)^{-1}\,h_\theta$ and $\Gamma$ is strongly self-affine (with respect to $\Psi$ and $\mathcal{R}_\theta$).*

*Moreover, let $r = (\alpha_1^{d_\Psi/2}, \dots, \alpha_N^{d_\Psi/2})$, where $\alpha_i = t_i^{1+\theta}$ is the affinity ratio of $\psi_i$. Then the $(\Psi, r)$-invariant measure is*

$$\left(\mathcal{A}^{d_\Psi}_{\mathcal{R}_\theta}(\Gamma)\right)^{-1} \cdot \mathcal{A}^{d_\Psi}_{\mathcal{R}_\theta}|\Gamma \;=\; \left(\mathcal{H}^{h_\theta}_{\theta}(\Gamma)\right)^{-1} \cdot \mathcal{H}^{h_\theta}_{\theta}|\Gamma.$$

**Proof.** $(\mathbb{R}^2, D_\theta)$ is a homogeneous space in the sense used by Coifman and Weiss in [3]. Therefore we can apply the results of [16] recalled in Subsection 1.3/(ii). Since $\Psi$ is a system of similitudes in $(\mathbb{R}^2, D_\theta)$ with similarity dimension $d_\theta$ and $\Psi$ satisfies OSC, it follows that $d_\theta = h_\theta$. By Proposition 5.2 we have $d_\Psi = 2(1+\theta)^{-1}h_\theta$. ¿From OSC, it follows that $\mathcal{H}^{d_\theta}_\theta(\Gamma) > 0$ and therefore $\mathcal{S}^{d_\theta}_\theta(\Gamma) = \mathcal{S}_\theta^{\frac{1+\theta}{2}\cdot d_\Psi}(\Gamma) > 0$ (as usually $d_\Psi$ stands for $d_\Psi\Gamma$). By Lemma 5.1 we have $\mathcal{A}^{d_\Psi}_{\mathcal{R}_\theta}(\Gamma) > 0$. Proposition 4.1/(iii) implies that $d_\Psi = \dim_{\mathcal{R}_\theta}\Gamma$ and $\Gamma$ is strongly self-affine, so the first part of the proposition is proved. Proposition 4.3 shows that $\left(\mathcal{A}^{d_\Psi}_{\mathcal{R}_\theta}(\Gamma)\right)^{-1} \cdot \mathcal{A}^{d_\Psi}_{\mathcal{R}_\theta}|\Gamma$ is a $(\Psi, r)$-invariant measure of total mass 1. By the uniqueness result $\left(\mathcal{A}^{d_\Psi}_{\mathcal{R}_\theta}(\Gamma)\right)^{-1} \cdot \mathcal{A}^{d_\Psi}_{\mathcal{R}_\theta}|\Gamma$ has to coincide with $\left(\mathcal{H}^{d_\theta}_\theta(\Gamma)\right)^{-1} \cdot \mathcal{H}^{d_\theta}_\theta|\Gamma$. Since $d_\theta = h_\theta$, the proposition is proved ∎

**Corollary 5.3.** *Let $\Psi$ be as in Corollary 5.2. Suppose that $\Psi$ satisfies OSC. Then $d_\Psi\Gamma = h_{\theta_1\theta_2}$, where $h_{\theta_1\theta_2}$ stands for the Hausdorff dimension of $\Gamma$ in $(\mathbb{R}^2, D_{\theta_1\theta_2})$.*

**Remark 5.8.** Also the other assertions of Theorem 5.1 can be easily extended to the quasi-metric space $(\mathbb{R}^2, D_{\theta_1\theta_2})$. One has only to observe that the family of all balls in $(\mathbb{R}^2, D_\theta)$ coincides with the family of all balls in $(\mathbb{R}^2, D_{\theta_1\theta_2})$ if $(\theta_1, \theta_2) = (c, c\theta)$.

**Remark 5.9.** We return once more to the first example of Section 2 and try to explain it in this new context. Now we have to fix $\theta$. Therefore, since $t = 1/2$, only one $\eta$ is possible, namely $\eta = 2^{-\theta}$. It follows that the affine dimension is $d_\Psi\Gamma = 2\,(1 - \log_2\eta)^{-1} = 2\,(1+\theta)^{-1}$. This result concurs with Proposition 5.2, since one can easily check that the similarity dimension $d_\theta$ of $\Gamma$ with respect to $D_\theta$ is 1.

## 6. Final remarks

In this section we compare our previous considerations with results already known about the Hausdorff dimension of some self-affine fractals.

Firstly, let us look at the relationship between the usual Hausdorff dimension $\dim_H$ in $(\mathbb{R}^2, D_e)$ and the Hausdorff dimension $\dim_{H,\theta}$ in $(\mathbb{R}^2, D_\theta)$. To this end it is convenient to consider in $\mathbb{R}^2$ the norm $N_1(x) = \max\{|x_1|, |x_2|\}$ and the induced distance $D_1$ (notice that $N_1$ and $D_1$ are a particular case of $N_\theta$ and $D_\theta$). The identity map from $\mathbb{R}^2$ with the Euclidean distance to $\mathbb{R}^2$ with the distance $D_1$ is a bi-Lipschitz continuous transformation; so we have $\dim_H E = \dim_{H,1} E$, for every $E \subseteq \mathbb{R}^2$.

Now we can state the following proposition.

**Proposition 6.1.** *For any $E \subseteq \mathbb{R}^2$ the inequalities*

$$\frac{1}{\theta}\,\dim_{H,\theta} E \quad \le \quad \dim_H E \quad \le \quad \dim_{H,\theta} E \tag{6.1}$$

*hold.*

**Proof.** It is easy to prove that if $N_1(x) \le 1$, then $\big(N_\theta(x)\big)^\theta \le N_1(x) \le N_\theta(x)$, and that for $E \subseteq \mathbb{R}^2$ the assertion $\operatorname{diam}_1 E \le 1$ is equivalent to $\operatorname{diam}_\theta E \le 1$. If $\operatorname{diam}_1 E \le 1$, the above inequalities lead to $(\operatorname{diam}_\theta E)^\theta \le \operatorname{diam}_1 E \le \operatorname{diam}_\theta E$. This result allows us to prove by standard methods that $\mathcal{H}_\theta^{s\theta}(E) \le \mathcal{H}_1^s(E) \le \mathcal{H}_\theta^s(E)$, for any $s > 0$. Now (6.1) is an easy consequence ∎

From Theorem 5.1 and Proposition 6.1 we immediately obtain a relation between the affine dimension of a fractal and its usual Hausdorff dimension.

**Corollary 6.1.** *Let $\Psi$ be a system of contractive affinities in $G_\theta$ satisfying OSC and let $\Gamma$ be the associated fractal. Then*

$$\frac{1}{\theta}\,d_\theta \quad \le \quad \dim_H \Gamma \quad \le \quad d_\theta \tag{6.2}$$

*and therefore*

$$\frac{1+\theta}{2\theta}\,d_\Psi\Gamma \quad \le \quad \dim_H \Gamma \quad \le \quad \frac{1+\theta}{2}\,d_\Psi\Gamma\,. \tag{6.3}$$

*In particular, let $\Gamma$ be a regular anisotropic fractal in the sense of Subsection* 1.3./(iii)*. Let $a_1, a_2$ be as in* (1.2)*, with $a_1 \le a_2$. Then*

$$\frac{1}{a_2}\,d_\Psi\Gamma \quad \le \quad \dim_H\Gamma \quad \le \quad \frac{1}{a_1}\,d_\Psi\Gamma\,. \tag{6.4}$$

It remains to be shown that the inequalities in (6.1) - (6.4) are sharp. It would be sufficient to consider line segments on the $x_1$-axis and the $x_2$-axis, respectively. However, we prefer to achieve the result as a by-product of the next comparisons between our propositions and the ones proved in the papers [14, 18].

In [18] the author studies general Sierpiński carpets (see Subsection 1.3/(iii)) and proves the following formula for the Hausdorff dimension of the fractal $\Gamma$:

$$\dim_H \Gamma \;=\; \log_m \left(\sum_{j=1}^{m} r_j^{\log_n m}\right). \tag{6.5}$$

Here $1 < m < n$ and $r_j$ (with $1 \le j \le m$) are positive integers with the following meaning: $m$ and $n$ appear in the matrix $A$, that is, $A = \begin{pmatrix} m^{-1} & 0 \\ 0 & n^{-1} \end{pmatrix}$, and $r_j$ denotes the number of rectangles located at the $j$-th column. The names of the axes are interchanged from those in [18], in accordance with our previous notation. Let us remark that a result similar to the one of [18] is obtained in [2] by different methods.

If we reinterpret the above system of affinities as in Section 5, that is as a system of similitudes in $(\mathbb{R}^2, D_\theta)$, we easily obtain $\dim_{H,\theta}\Gamma = d_\theta = \log_m N$, where $N = \sum_{j=1}^m r_j$.

Hence from Theorem 5.1 we have $d_\Psi\Gamma = 2(\theta+1)^{-1}\log_m N$, where $\theta = \log_m n$. So formula (6.1) immediately leads to the estimate

$$\frac{1}{\theta}\log_m N \quad \le \quad \dim_H \Gamma \quad \le \quad \log_m N\,, \tag{6.5}$$

which is equivalent to

$$\log_m\left(\sum_{j=1}^m r_j^{\frac{1}{\theta}}\right) \quad \le \quad \log_m N \quad \le \quad \theta\,\log_m\left(\sum_{j=1}^m r_j^{\frac{1}{\theta}}\right)\,. \tag{6.6}$$

By an elementary calculation we can conclude:

**(i)** $\dim_H \Gamma = \log_m N$ *if and only if* $r_j \le 1$ *for every* $j$*, that is at most one rectangle is chosen in every column.*

**(ii)** $\log_m N = \theta \dim_H \Gamma$ *if and only if there exists* $j_0$ *such that* $r_{j_0} = N$ *(hence* $r_j = 0$ *for* $j \ne j_0$*), that is all the chosen rectangles belong to the same column.*

In the proof of the "only if" part in (ii) we used the inequality $(a+b)^p > a^p + b^p$ for $a, b > 0$ and $p > 1$. So it is additionaly shown that the inequalities in (6.1) and (6.2) are sharp.

These remarks can be extended to some of the fractals considered in the paper [14]. Let us underline the fact that not every fractal of [14] can be obtained via the construction explained in Section 5, and viceversa. Therefore we describe only those systems of affinities which both satisfy the hypotheses of [14] and are contained in $G_\theta$ for a suitable $\theta$. They are precisely the systems with which [14: Remark 2/p.549] is concerned; as observed in [14], all Sierpiński carpets are of this type. Again, with respect to [14] we interchange the names of the axes.

Thus we consider systems $\Psi = \{\psi_{j\ell}\}$, where $1 \le j \le m$ and for $j$ fixed, $1 \le \ell \le r_j$. Here $\psi_{j\ell} = (A_j, h_{j\ell})$, with $A_j = \left(\begin{smallmatrix} t_j & 0 \\ 0 & t_j^\theta \end{smallmatrix}\right)$ and $h_{j\ell} = \left(\begin{smallmatrix} c_j \\ d_{j\ell} \end{smallmatrix}\right)$; in addition we suppose $N = \sum_{j=1}^m r_j > 1$. If we denote by $Q$ the unit square of $\mathbb{R}^2$, the further hypotheses stated in [14: at p. 534] guarantee that the $N$ open rectangles $\psi_{j\ell}(\mathrm{Int}\ Q)$ are pairwise disjoint subsets of $Q$. Note that the rectangles are arranged in $m$ columns; moreover, all the $r_j$ rectangles of the $j$-th column are congruent: they have width $t_j$ and, since $\Psi \subseteq G_\theta$, height $t_j^\theta$.

In [14: Remark 2] the authors show that $\dim_H \Gamma$ is the unique positive real number $\delta$ defined by

$$\sum_{j=1}^m r_j^{1/\theta}\, t_j^\delta \;=\; 1\,.$$

If we reinterpret the system $\Psi$ as in Section 5, we obtain that $d_\theta$ is the unique positive number such that

$$\sum_{j=1}^m r_j\, t_j^{d_\theta} \;=\; 1\,.$$

So, in order to compare $\dim_H \Gamma$ and $d_\theta$, it is natural to consider the functions $f_1$ and $f_2$ defined as

$$f_1(\xi) = \sum_{j=1}^{m} r_j^{1/\theta}\, t_j^\xi \qquad \text{and} \qquad f_2(\xi) = \sum_{j=1}^{m} r_j\, t_j^\xi\ .$$

Let us suppose $\theta > 1$. It is easy to prove the following equivalences:

**(i)$'$** *The three conditions*

$\dim_H \Gamma = d_\theta$

$f_1 = f_2$

$r_j = 1$ *for all* $j = 1, \ldots, m$, *that is, again, each column contains only one rectangle*

*are equivalent.*

**(ii)$'$** *The three conditions*

$d_\theta = \theta \dim_H \Gamma$

$f_2(\theta\xi) = \big(f_1(\xi)\big)^\theta$ *for all* $\xi$

$m = 1$, *that is, again, all the rectangles are located on the same column*

*are equivalent.*

**Remark 6.1.** To conclude this section, let us come back to the normalization used in the theory of anisotropic function spaces (see Remark 5.3). So $\theta_1$ and $\theta_2$ in (5.1) satisfy the condition $\theta_1 + \theta_2 = 2$; as before we suppose $\theta_1 \le \theta_2$, therefore we have $\theta_1 \le 1 \le \theta_2$. If we consider in $\mathbb{R}^2$ the quasi-distance $D_{\theta_1\theta_2}$ induced by $N_{\theta_1\theta_2}$ (see Remark 5.6), we can easily prove that Proposition 6.1 now assumes the following form:

*For any* $E \subseteq \mathbb{R}^2$, *the inequalities*

$$\frac{1}{\theta_2} \dim_{H,\theta_1,\theta_2} E \ \le\ \dim_H E \ \le\ \frac{1}{\theta_1} \dim_{H,\theta_1,\theta_2} E \tag{6.1$'$}$$

*hold where* $\dim_{H,\theta_1,\theta_2}$ *stands for the Hausdorff dimension in* $(\mathbb{R}^2, D_{\theta_1\theta_2})$.

To prove this assertion it is sufficient to observe that $N_1(x) \le 1$ implies

$$\big(N_{\theta_1\theta_2}(x)\big)^{\theta_2} \le N_1(x) \le \big(N_{\theta_1\theta_2}(x)\big)^{\theta_1}\ ,$$

and then to proceed as in the final part of Proposition 6.1.

Also Corollary 6.1 can be rewritten in this new framework. In fact, under the same hypotheses on the system $\Psi$, formula (6.3) takes the form

$$\frac{1}{\theta_2}\, d_\Psi \Gamma \ \le\ \dim_H \Gamma \ \le\ \frac{1}{\theta_1}\, d_\Psi \Gamma\ , \tag{6.3$'$}$$

which for regular anisotropic fractals coincides with (6.4).

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