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On Real and Complex Spectra in some Real C^*-Algebras and Applications

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Abstract. A real extension $\tilde{\mathcal{A}}$ of a complex C^*-algebra $\mathcal{A}$ by some element m which has a number of special properties is proposed. These properties allow us to introduce some suitable operations of addition, multiplication and involution on $\tilde{\mathcal{A}}$. After then we are able to study Moore-Penrose invertibility in $\tilde{\mathcal{A}}$. Because this notion strongly depends on the element m, we study under what conditions different elements m produce just the same involution on $\tilde{\mathcal{A}}$. It is shown that the set of all additive continuous operators $\mathcal{L}_{add}(\mathcal{H})$ acting in a complex Hilbert space $\mathcal{H}$ possesses unique involution only (in the sense defined below). In addition, we consider some properties of the real and complex spectra of elements belonging to $\tilde{\mathcal{A}}$, and show that whenever an operator sequence $\{\tilde{A}_n\} \subset \mathcal{L}_{add}(\mathcal{H})$ is weakly asymptotically Moore-Penrose invertible, then the real spectrum of $\tilde{A}_n^* \tilde{A}_n$ can be split in two special parts. This property has been earlier known for sequences of linear operators.

Keywords: *Real C^*-algebras, Moore-Penrose invertibility, singular integral equations with conjugation*

AMS subject classification: 65 R 20

1. Introduction

Let $\mathcal{A}$ be a real C^*-algebra with identity e. A usual method for introducing the notion of spectrum for elements of $\mathcal{A}$ is to consider the spectrum of the images of these elements in a special complex algebra which is called the complexification of $\mathcal{A}$. As a result, the spectra of elements of $\mathcal{A}$ are some sets in the complex plane $\mathbb{C}$. One of the reasons for such a definition is to make impossible the situation when the spectrum of an element may turn out to be empty. Another reason for using the complexification of $\mathcal{A}$ is closely connected with the previous one. Namely, it is possible that the initial algebra does not have the operation of multiplication by complex scalars. However, there are real algebras which possess this operation. For such algebras the spectra defined by complexification technique may be distinguished from the usual one (see [5] or [12: Remark 1.1.11]).

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In the present paper we deal with the investigation of Moore-Penrose invertibility in special real C^*-algebras. In complex algebras analogous problems were studied in [9, 10, 11, 15, 16, 18]. It turns out that the problems can be successfully treated by using spectral characteristics of some self-adjoint elements. However, self-adjoint elements of real algebras do not always possess the properties which such elements of complex algebras have. That is why we employ two different notions of the spectrum for the real C^*-algebras under consideration. Namely, we consider a real and a complex spectrum and the interdependency of these two objects. It is shown that the complex spectrum can be expressed by the real parts of the spectra of some elements belonging to an auxiliary complex C^*-algebra.

The notion of Moore-Penrose invertibility used in this work strongly depends on a special element m. It may happen that different elements m lead to different Moore-Penrose inverses. We give a description of a class of elements m such that they all generate just the same Moore-Penrose inverse for each a of our real algebra.

The results mentioned are used to study some problems of Moore-Penrose invertibility for additive operators, acting in a complex Hilbert space $\mathcal{H}$. In particular, we investigate the asymptotic Moore-Penrose invertibility in algebras appearing in spline approximation methods for singular integral equations with conjugation. Note that the stability problem in such algebras was studied in [2, 3, 7, 8, 14]. Finally, let us remark one technical detail. Definitions, examples, and remarks on the one hand and lemmata, theorems, and propositions on the other hand are numbered separately.

2. Algebra $\tilde{\mathcal{A}}$ and some its properties

Let $\mathcal{R}$ be a ring over the real number field $\mathbb{R}$. We assume that $\mathcal{R}$ contains a *complex* C^*-algebra $\mathcal{A}$ with identity e and an element $m \notin \mathcal{A}$, which are connected by the following relations:

- (A_1) For each $a \in \mathcal{A}$, the element mam is in $\mathcal{A}$.
- (A_2) $m^2 = e$ and $me = m$.
- (A_3) For each $\lambda \in \mathbb{C}$, $m(\lambda e) = \overline{\lambda} m$.
- (A_4) The null element 0 of the C^*-algebra $\mathcal{A}$ is also that of the ring $\mathcal{R}$.
- (A_5) For each $a \in \mathcal{A}$, $(mam)^* = ma^*m$, where "$*$" means an involution on $\mathcal{A}$. In the sequel, the set of all such elements m will be referred to as $\mathcal{M}(\mathcal{A})$.

Now, we consider a subset $\tilde{\mathcal{A}} \subset \mathcal{R}$ which consists of all elements $\tilde{a}$ having the form

$$\tilde{a} = b + cm \tag{1}$$

with $b, c \in \mathcal{A}$. It is easily seen that the set $\tilde{\mathcal{A}}$ is closed with respect to the operations of addition and multiplication. Moreover, one can define the operation of multiplication of elements of $\mathcal{A}$ by complex scalars as

$$\lambda\tilde{a} = \lambda(b + cm) = (\lambda b) + (\lambda c)m \qquad \forall \lambda \in \mathbb{C}.$$

The set $\tilde{\mathcal{A}}$ with the above operations becomes a real algebra (this is not a complex algebra because, in general, $\tilde{a}(\lambda\tilde{b}) \neq \lambda(\tilde{a}\tilde{b})$).

Let us note the following result.

Lemma 2.1. *Let assumptions* $(A_1) - (A_5)$ *hold. Then each element* $\tilde{a}$ *of* $\tilde{\mathcal{A}}$ *has a unique representation in the form* (1).

Proof. Indeed, if an element $\tilde{a}$ has two different representations $\tilde{a} = b_1 + c_1 m$ and $\tilde{a} = b_2 + c_2 m$, then we immediately get

$$(b_1 - b_2) + (c_1 - c_2)m = 0. \tag{2}$$

Multiplying this equality by ie from the right and by $(-i)$ from the left we get

$$-(b_1 - b_2) + (c_1 - c_2)m = 0. \tag{3}$$

Combining (2) with (3) and remembering assumption (A_4) we obtain our claim. ∎

As a next step we introduce an involution on $\tilde{\mathcal{A}}$. This can be done by using the previous lemma. Namely, for each $\tilde{a} \in \tilde{\mathcal{A}}$, $\tilde{a} = b + cm \quad (b, c \in \mathcal{A})$ we put

$$\tilde{a}^\star = b^* + mc^*. \tag{4}$$

It is a simply matter to show that the so defined involution possesses most of the basic properties we know for involutions on complex C^*-algebras. For instance, one has the following relations:

- For each $\tilde{a} \in \tilde{\mathcal{A}}$, the element $\tilde{a}^\star$ belongs to $\tilde{\mathcal{A}}$.
- For each $\tilde{a} \in \mathcal{A}$, $(\tilde{a}^\star)^\star = \tilde{a}$.
- For any $\tilde{a}, \tilde{b} \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$, $(\tilde{a}\tilde{b})^\star = \tilde{b}^\star\tilde{a}^\star$ and $(\alpha\tilde{a} + \beta\tilde{b})^\star = \tilde{a}^\star\overline{\alpha}e + \tilde{b}^\star\overline{\beta}e$.

In addition, we can see that $m^\star = m$ and, for each $\tilde{a} \in \mathcal{A}$, $\tilde{a}^\star = \tilde{a}^*$.

Therefore, it will cause no confusion if we will use the same symbol "$*$" for the notation of the involution on the set $\tilde{\mathcal{A}}$. We should only turn the attention of the reader to the fact that the above involution depends on the element m. In the sequel, we will study a number of algebras $\tilde{\mathcal{A}}$ which can be defined by different elements m. In these cases we will use special notations to make precise which involution is meant.

3. Real and complex spectra

Let $\mathcal{A}$ be a real C^*-algebra with identity e, and let $b \in \mathcal{A}$. Recall the definition of the real spectrum of b.

Definition 3.1. The real spectrum of b in $\mathcal{A}$, denoted by $\mathrm{sp}_{\mathcal{A}} b$, is the set of all real numbers λ such that $b - \lambda e$ is not invertible in $\mathcal{A}$.

As it has been pointed out earlier real algebras often possess the operation of multiplication by complex scalars. When this is the case, one may consider the complex spectrum of b in $\mathcal{A}$, too. It is defined analogously, namely as the set of all complex numbers λ such that $b - \lambda e$ is not invertible in $\mathcal{A}$. In the sequel, the complex spectrum of b in $\mathcal{A}$ will be denoted by $\mathrm{Sp}_{\mathcal{A}} b$.

In this section we are going to describe some relations between the complex and the real spectra for the elements of the algebra $\tilde{\mathcal{A}}$ introduced earlier. To do this we have to mention some known results concerning real algebras.

Theorem 3.1 (see [13] App. 1). *Let b be a self-adjoint element of a unital (real or complex) C^*-algebra $\mathcal{A}$, and let $\mathcal{R}(b)$ be the smallest closed real C^*-subalgebra containing the elements b and e. Then $\mathcal{R}(b)$ is isometrically isomorphic to the algebra $C(\mathrm{sp}_{\mathcal{R}(b)} b)$ of all continuous real functions on $\mathrm{sp}_{\mathcal{R}(b)} b$.*

Lemma 3.2 (see [3]). *Let b be a self-adjoint element of a complex C^*-algebra $\mathcal{A}$. Then $\mathrm{sp}_{\mathcal{R}(b)} b = \mathrm{Sp}_{\mathcal{A}} b \quad (= \mathrm{sp}_{\mathcal{A}} b)$.*

Lemma 3.3 (see [3]). *Let $\mathcal{B}$ be a real C^*-subalgebra of a complex C^*-algebra $\mathcal{A}$. Then $\mathcal{B}$ is inverse closed in the algebra $\mathcal{A}$, i.e. if an element $b \in \mathcal{B}$ is invertible in $\mathcal{A}$, then it is also invertible in $\mathcal{B}$.*

Now we are in a position to give a description of the complex spectra for elements of $\tilde{\mathcal{A}}$ by real components of the spectra of some elements belonging to a known complex C^*-algebra.

Let $\tilde{a} \in \tilde{\mathcal{A}}$, and let $\mathcal{A}^{2\times 2}$ be the complex C^*-algebra of (2×2)-matrices with entries in $\mathcal{A}$. By Ψ, $\Psi : \tilde{\mathcal{A}} \to \mathcal{A}^{2\times 2}$ we denote the transformation

$$\Psi(\tilde{a}) = \Psi(b + cm) = \begin{pmatrix} b & c \\ mcm & mbm \end{pmatrix} \tag{5}$$

where $b, c \in \mathcal{A}$.

Theorem 3.4. *Let $\tilde{a} = b + cm \in \tilde{\mathcal{A}}$. Then*

$$\mathrm{Sp}_{\tilde{\mathcal{A}}} \tilde{a} = \bigcup_{\varphi \in [0,2\pi)} \{\mathrm{sp}^+_{\mathcal{A}^{2\times 2}} \Psi(\tilde{a}_\varphi)\}\, e^{i\varphi} \tag{6}$$

where

$$\tilde{a}_\varphi = e^{-i\varphi} b + e^{i\varphi} cm \qquad (\varphi \in [0, 2\pi)) \tag{7}$$

and

$$\mathrm{sp}^+_{\mathcal{B}} b = \{\mathrm{Sp}_{\mathcal{B}} b\} \cap \{\mathbb{R}^+ \cup 0\}.$$

Proof. Indeed, for each $\lambda = |\lambda|e^{i\varphi}$ we have

$$\tilde{a} - \lambda e = (\tilde{a}_\varphi - |\lambda|e)e^{i\varphi}.$$

Hence, the element $\tilde{a} - \lambda e$ is invertible in $\tilde{\mathcal{A}}$ if and only if all elements $\tilde{a}_\varphi - |\lambda|e$ for $\varphi \in [0, 2\pi)$ are so. We consider the subset $E^{2\times 2}_{\tilde{\mathcal{A}}}$ of $\mathcal{A}^{2\times 2}$ which consists of all matrices of $\mathcal{A}^{2\times 2}$ having the form (5). It is easily seen that $E^{2\times 2}_{\tilde{\mathcal{A}}}$ is a *real* subalgebra of $\mathcal{A}^{2\times 2}$. In addition, we can see that an element $\tilde{a}$ is invertible in $\tilde{\mathcal{A}}$ if and only if $\Psi(\tilde{a})$ is invertible in $E^{2\times 2}_{\tilde{\mathcal{A}}}$. Taking into account the relation

$$\Psi(\tilde{a}_\varphi - |\lambda|e) = \Psi(\tilde{a}_\varphi) - |\lambda|E \tag{8}$$

where $E = \left(\begin{smallmatrix} e & 0 \\ 0 & e \end{smallmatrix}\right)$, we obtain

$$\mathrm{Sp}_{\tilde{\mathcal{A}}}\tilde{a} = \bigcup_{\varphi\in[0,2\pi)} \Big\{\mathrm{sp}^+_{E^{2\times 2}_{\tilde{\mathcal{A}}}}\Psi(\tilde{a}_\varphi)\Big\}e^{i\varphi}.$$

To complete the proof we only need to remember Lemma 3.3. ∎

Corollary 3.5. *Let $\tilde{a} \in \tilde{\mathcal{A}}$. Then*

$$\mathrm{sp}_{\tilde{\mathcal{A}}}\tilde{a} = \mathrm{sp}_{\mathcal{A}^{2\times 2}}\Psi(\tilde{a}). \tag{9}$$

It should be noted here that such an equality is not always true for the complex spectrum $\mathrm{Sp}_{\tilde{\mathcal{A}}}\tilde{a}$ of $\tilde{a}$ (see Example 5.2 below).

We again consider the transformation $\Psi : \tilde{\mathcal{A}} \to \mathcal{A}^{2\times 2}$.

Lemma 3.6 (cf. [3: Lemma 5]). *The transformation Ψ is a *-isomorphism between the real algebras $\tilde{\mathcal{A}}$ and $E^{2\times 2}_{\tilde{\mathcal{A}}}$.*

Due to this lemma we can equip the algebra $\tilde{\mathcal{A}}$ with the norm

$$||\tilde{a}||_{\tilde{\mathcal{A}}} = ||\Psi(\tilde{a})||_{\mathcal{A}^{2\times 2}}.$$

With this norm $\tilde{\mathcal{A}}$ becomes a real C^*-algebra, i.e. it satisfies the condition

$$||\tilde{a}||^2_{\tilde{\mathcal{A}}} = ||\tilde{a}\tilde{a}^*||_{\tilde{\mathcal{A}}} \qquad \forall \tilde{a} \in \tilde{\mathcal{A}}. \tag{10}$$

Thus, throughout this paper the algebra $\tilde{\mathcal{A}}$ is assumed having the norm (10).

4. On the uniqueness of involution in $\tilde{\mathcal{A}}$

In Section 2 we defined an involution in the algebra $\tilde{\mathcal{A}}$ which, generally speaking, depends on the element m of the ring $\mathcal{R}$. Let us explain more precisely what this means.

Suppose we have two different elements m_1 and m_2 of $\mathcal{R}$ possessing the properties $(A_1) - (A_5)$ and producing just the same algebra $\tilde{\mathcal{A}}$. If $\tilde{a} \in \tilde{A}$, then we do not know whether the corresponding involutions

$$(\tilde{a})^*_{m_1} = (b_1 + c_1 m_1)^*_{m_1} = b_1^* + m_1 c_1^* \qquad (b_1, c_1 \in \mathcal{A})$$
$$(\tilde{a})^*_{m_2} = (b_2 + c_2 m_2)^*_{m_2} = b_2^* + m_2 c_2^* \qquad (b_2, c_2 \in \mathcal{A})$$

coincide. In this section we give some conditions when the equality $(\tilde{a})^*_{m_1} = (\tilde{a})^*_{m_2}$ is fulfilled for each $\tilde{a}$ of $\tilde{\mathcal{A}}$ provided that $\tilde{\mathcal{A}}_{m_1} = \tilde{\mathcal{A}}_{m_2}$ $(= \tilde{\mathcal{A}})$.

Theorem 4.1. *Let $\tilde{\mathcal{A}}_{m_1}$ and $\tilde{\mathcal{A}}_{m_2}$ be the real algebras generated by a complex C^*-algebra $\mathcal{A}$ and by elements m_1 and m_2, respectively. Then the following assertions are equivalent:*

1. $\tilde{\mathcal{A}}_{m_1} = \tilde{\mathcal{A}}_{m_2}$ $(= \tilde{\mathcal{A}})$ *and, for each $\tilde{a} \in \tilde{\mathcal{A}}$,*

$$(\tilde{a})^*_{m_1} = (\tilde{a})^*_{m_2}. \tag{11}$$

2. *The element $m_1 m_2$ belongs to the algebra $\mathcal{A}$ and*

$$(m_1 m_2)^* = m_2 m_1. \tag{12}$$

Proof. *Necessity part*: Let the algebra $\tilde{\mathcal{A}}_{m_1}$ coincide with $\tilde{\mathcal{A}}_{m_2}$, and let they both have just the same involution. Then there exist $f, g \in \mathcal{A}$ such that $m_1 = f + g m_2$. Following the proof of Lemma 2.1 we get $m_1 = -f + g m_2$. Hence, $m_1 = g m_2$, or

$$m_1 m_2 = g, \tag{13}$$

i.e. $m_1 m_2 \in \mathcal{A}$.

Let us show equality (12). To do this we compute each of the involutions for the element m_1. Firstly, we have

$$(m_1)^*_{m_1} = m_1 \tag{14}$$

and, secondly,

$$(m_1)^*_{m_2} = (g m_2)^*_{m_2} = m_2 g^*. \tag{15}$$

Comparing (14) with (15) we obtain $m_1 = m_2 g^*$. Multiplying this equality by m_2 from the left and remembering (13) we get our claim.

Sufficiency part. Let us suppose that the element $m_1 m_2$ belongs to the algebra $\tilde{\mathcal{A}}$ and satisfies equality (12). First of all, we show that the algebras $\tilde{\mathcal{A}}_{m_1}$ and $\tilde{\mathcal{A}}_{m_2}$ coincide. Let $\tilde{a} \in \tilde{\mathcal{A}}_{m_1}$. Then there exist $b, c \in \mathcal{A}$ such that $\tilde{a} = b + c m_1$. Hence,

$$\tilde{a} = b + c m_1 = b + c m_1 \cdot m_2^2 = b + c \cdot (m_1 m_2) m_2 = b + c_1 m_2,$$

i.e. $\tilde{a} \in \tilde{\mathcal{A}}_{m_2}$ because $b, c_1 \in \mathcal{A}$. The inclusion $\tilde{\mathcal{A}}_{m_2} \subset \tilde{\mathcal{A}}_{m_1}$ is proved analogously. Now we take an element $\tilde{a}$ of $\tilde{\mathcal{A}}$ and find its involutions generated by each of the elements m_1 and m_2. We have

$$(\tilde{a})^*_{m_1} = (b + cm_1)^*_{m_1} = b^* + m_1 c^*$$

and

$$\begin{aligned}(\tilde{a})^*_{m_2} &= (b + cm_1)^*_{m_2} = (b + c(m_1 m_2) m_2)^*_{m_2} = b^* + m_2 (c(m_1 m_2))^* \\ &= b^* + m_2 (m_1 m_2)^* c^* = b^* + m_2 (m_2 m_1) c^* = b^* + m_1 c^*,\end{aligned}$$

i.e. $(\tilde{a})^*_{m_1} = (\tilde{a})^*_{m_2}$ ∎

Corollary 4.2. *Let both the elements $m_1, m_2 \in \mathcal{R}$ satisfy assumptions $(A_1) - (A_5)$. If there exists a unitary element $a \in \mathcal{A}$ such that $m_2 = m_1 a$, then m_1 and m_2 define just the same algebra $\tilde{\mathcal{A}}$, and for each $\tilde{a} \in \tilde{\mathcal{A}}$ one has $(\tilde{a})^*_{m_1} = (\tilde{a})^*_{m_2}$.*

Indeed, since $m_1 m_2 = a \in \mathcal{A}$ we only need to prove equality (12). But this is almost apparent, because $(m_1 m_2)^* = (m_1 m_1 a)^* = a^* = a^{-1} = m_2 m_1$, which proves the claim.

The following two assertions give us some methods to construct elements of $\mathcal{R}$ which would have properties $(A_1) - (A_5)$ and would produce an algebra $\tilde{\mathcal{A}}$ with just the same involution.

Corollary 4.3. *Let m satisfy assumptions $(A_1) - (A_5)$, and let q be a unitary element of $\mathcal{A}$ such that*

$$q^* = mqm. \tag{16}$$

Then:

1. *The element $m_q = mq$ satisfies all conditions $(A_1) - (A_5)$.*
2. $\tilde{\mathcal{A}}_m = \tilde{\mathcal{A}}_{m_q}$ $(= \tilde{\mathcal{A}})$.
3. *For each $\tilde{a} \in \tilde{\mathcal{A}}$, $(\tilde{a})^*_m = (\tilde{a})^*_{m_q}$.*

Proof. In view of the previous corollary we only have to show the first claim of this assertion. Note that the relations $(A_1), (A_3), (A_4)$ are evident, and (A_2) immediately follows from (16). Moreover, for the element $(m_q b\, m_q)^*$ $(b \in \mathcal{A})$ we have

$$\begin{aligned}(m_q b m_q)^* &= (mqbmq)^* = q^*(mqbm)^* = q^*(mb^*q^*m) \\ &= m(mq^*m)b^*m(mq^*m) = mq \cdot b^* \cdot mq = m_q b^* m_q.\end{aligned}$$

This yields the proof. ∎

Corollary 4.4. *Let m satisfy assumptions $(A_1) - (A_5)$, and let $q \in \mathcal{A}$ be a self-adjoint element such that $q^2 = e$. Then for the element $m_q = qmq$ all assertions of Corollary 4.3 are true.*

Proof. The validity of the relations $(A_1) - (A_5)$ can be proved by straightforward computation. Here we only show the validity of equality (A_5). For each $b \in \mathcal{A}$ we have

$$(m_q b m_q)^* = (qmqbqmq)^* = q^* m (qbq)^* m q^* = qmqb^*qmq = m_q b^* m_q.$$

It remains to prove the second assertion of Theorem 4.1. Indeed, the element $mm_q = (mqm)q$ evidently belongs to the algebra $\mathcal{A}$, and

$$(mm_q)^* = ((mqm)q)^* = q^* m q^* m = qmqm = m_q m.$$

This finishes the proof ∎

Now we would like to give some examples of complex C^*-algebras and its extensions generated by different elements possessing properties $(A_1) - (A_5)$.

Example 4.1. Let Γ be the unit circle in the complex plane $\mathbb{C}$, i.e. $\Gamma = \{t \in \mathbb{C} : |t| = 1\}$. By $L_2 = L_2(\Gamma)$ we denote the set of all complex-valued Lebesgue-measurable functions φ on Γ such that $\int_\Gamma |\varphi(t)|^2 |dt| < \infty$. Provided with the scalar product

$$(\varphi, \psi) = \frac{1}{2\pi} \int_\Gamma \varphi(t)\overline{\psi(t)}\,|dt| \qquad \big(\varphi, \psi \in L_2(\Gamma)\big)$$

this set actually becomes a Hilbert space. By M we denote the operator of complex conjugation in $L_2(\Gamma)$, i.e.

$$(M\varphi)(t) = \overline{\varphi(t)} \qquad \forall \varphi \in L_2(\Gamma).$$

As is shown in [3], the extension of the C^*-algebra $\mathcal{A} = \mathcal{L}(L_2(\Gamma))$ by the operator M leads to the algebra of all additive continuous operators $\tilde{\mathcal{A}} = \mathcal{L}_{add}(L_2(\Gamma))$ acting on the space $L_2(\Gamma)$. Due to the fact that the operator M satisfies relations $(A_1) - (A_5)$, we can introduce a "good" involution on this set.

Just the same involution on $\mathcal{L}_{add}(L_2(\Gamma))$ can be also obtained by using the operator $M_1 = SMS$, where S is the singular integral operator of Cauchy, i.e.

$$(S\varphi) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)\,d\tau}{\tau - t} \qquad (\varphi \in L_2(\Gamma))$$

because $S^* = S$ and $S^2 = S$ (cf. [6], Chapter 1 and Corollary 4.4).

Example 4.2. Let us consider the Hilbert space $L_2(\Gamma)$ again. We take a continuously differentiable function $\alpha = \alpha(t)$ $(t \in \Gamma)$ such that

$$\alpha^2(t) = \alpha(\alpha(t)) = t \qquad \forall t \in \Gamma. \tag{17}$$

It is also supposed that

$$|\alpha'(t)| = 1 \qquad \forall t \in \Gamma. \tag{18}$$

For example, one can choose the function $\alpha(t) = \frac{1}{t}$ $(t \in \Gamma)$. We denote by W_α the operator of the Carleman shift

$$(W_\alpha\varphi)(t) = \varphi(\alpha(t)) \qquad (\varphi \in L_2(\Gamma),\, t \in \Gamma)$$

and by B_a the operator of multiplication by a continuous function a, i.e.

$$(B_a\varphi)(t) = a(t)\varphi(t) \qquad (\varphi \in L_2(\Gamma),\, t \in \Gamma).$$

Let $\mathcal{A}$ be the smallest closed C^*-subalgebra of the C^*-algebra $\mathcal{L}(L_2(\Gamma))$ containing the operator W_α and all operators B_a with $a \in C(\Gamma)$. We consider the extensions of this algebra by the elements M and $M_1 = MW_\alpha$.

It is easily seen that M satisfies all assumptions $(A_1)-(A_5)$. Therefore, we only have to check the conditions of Corollary 4.3 for the operator W_α. Indeed, it follows from (17) and (18) that $W_\alpha^2 = I$ and $W_\alpha^* = B_{|\alpha|}W_\alpha = W_\alpha$. In addition, a straightforward computation gives

$$MW_\alpha M = W_\alpha = W_\alpha^*,$$

i.e. all conditions of Corollary 4.3 are fulfilled. Therefore, the elements M and M_1 define just the same algebra $\tilde{\mathcal{A}}$ and just the same involution on this algebra.

Example 4.3. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}_{add}(\mathcal{H})$ be the set of all additive continuous operators acting in $\mathcal{H}$. As it was established in [3] the algebra $\mathcal{L}_{add}(\mathcal{H})$ can be considered as an extension of the C^*-algebra $\mathcal{L}(\mathcal{H})$ by any continuous additive operator M with the following properties:

- $M^2 = I$, where I is the identical operator of $\mathcal{L}(\mathcal{H})$.
- $(M\varphi, \psi) = \overline{(\varphi, M\psi)}$ for all $\varphi, \psi \in \mathcal{H}$.

The set of all such operators will be denoted by $\mathcal{L}_M(\mathcal{H})$. Now we are in a position to show that all such operators produce just the same involution on $\mathcal{L}_{add}(\mathcal{H})$. First of all, we note that each operator M satisfying the above conditions is an anti-linear one, i.e.

$$M(\lambda x) = \overline{\lambda} Mx \qquad \forall \lambda \in \mathbb{C} \text{ and } \forall x \in \mathcal{H}.$$

Hence, for arbitrary $M_1, M_2 \in \mathcal{L}_M(\mathcal{H})$, the operator$M_1M_2$ belongs to $\mathcal{L}(\mathcal{H})$. In addition, for any $\varphi, \psi \in \mathcal{H}$ one finds

$$(M_1M_2\varphi, \psi) = \overline{(M_2\varphi, M_1\psi)} = (\varphi, M_2M_1\psi).$$

Thus $(M_1M_2)^* = M_2M_1$, and using Theorem 4.1 finishes the proof.

We see that a wide class of anti-linear operators produces just the same involution on $\mathcal{L}_{add}(\mathcal{H})$ (in the sense of definition (4)). Moreover, in the last case it turns out that all possible involutions (4) coincide!

Below, we are going to prove this claim. However, before we have to introduce some additional notations.

Let $\mathcal{H}_\mathbb{R}$ be the same Hilbert space $\mathcal{H}$ considered as a real space. As a scalar product on $\mathcal{H}_\mathbb{R}$ we will use the form

$$\langle x, y\rangle = \langle x, y\rangle_{\mathcal{H}_\mathbb{R}} = \mathrm{Re}(x, y) \qquad (x, y \in \mathcal{H}_\mathbb{R}). \tag{19}$$

For each $A \in \mathcal{L}_{add}(\mathcal{H})$ we refer to as $A_\mathbb{R}^*$ such an operator $B \in \mathcal{L}_{add}(\mathcal{H})$ which satisfies the relation

$$\langle Ax, y\rangle = \langle x, By\rangle \qquad \forall x, y \in \mathcal{H}.$$

We list without proofs some elementary properties of these operators:

1) For each $A \in \mathcal{L}_{add}(\mathcal{H})$, the operator $A_\mathbb{R}^*$ exists and is uniquely determined;
2) For any $A, B \in \mathcal{L}_{add}(\mathcal{H})$, $(A + B)_\mathbb{R}^* = A_\mathbb{R}^* + B_\mathbb{R}^*$ and $(AB)_\mathbb{R}^* = B_\mathbb{R}^*A_\mathbb{R}^*$;
3) For each $A \in \mathcal{L}_{add}(\mathcal{H})$, $(A_\mathbb{R}^*)_\mathbb{R}^* = A$;
4) For each $A \in \mathcal{L}(\mathcal{H})$, $A_\mathbb{R}^* = A^*$;
5) For each $M \in \mathcal{M}(\mathcal{L}(\mathcal{H}))$, the operator $MM_\mathbb{R}^*$ belongs to $\mathcal{L}(\mathcal{H})$.

Theorem 4.5. *Let $\mathcal{H}$ be a complex Hilbert space. Then any operators $M_1, M_2 \in \mathcal{M}(\mathcal{L}(\mathcal{H}))$ produce just the same involution on $\mathcal{L}_{add}(\mathcal{H})$.*

Proof. First of all, we suppose that $\dim\mathcal{H} > 1$ and consider an arbitrary operator $M \in \mathcal{M}(\mathcal{L}(\mathcal{H}))$. Let $A \in \mathcal{L}(\mathcal{H})$. On account of properties 1) – 4) one can write

$$(MA^*M)^*_{\mathbb{R}} = M^*_{\mathbb{R}} A M^*_{\mathbb{R}}. \tag{20}$$

On the other hand,

$$(MA^*M)^*_{\mathbb{R}} = (MA^*M)^* = MAM.$$

Therefore, according to condition (A_2) we get

$$(MM^*_{\mathbb{R}})A = A(MM^*_{\mathbb{R}}) \qquad \forall A \in \mathcal{L}(\mathcal{H}).$$

Since $\dim\mathcal{H} > 1$ the algebra $\mathcal{L}(\mathcal{H})$ is irreducible. Hence, the operator $MM^*_{\mathbb{R}} \in \mathcal{L}(\mathcal{H})$ is an operator of multiplication by some scalar (cf.,[1], Proposition 2.3.1) i.e. $MM^*_{\mathbb{R}} = \lambda I$ for some $\lambda \in \mathbb{C}$. However, the operator $MM^*_{\mathbb{R}}$ is self-adjoint. This implies $\lambda \in \mathbb{R}$, and applying axiom (A_2) once again we obtain $\lambda = \pm 1$. Thus, there may exist only two situations, $M^*_{\mathbb{R}} = M$ or $M^*_{\mathbb{R}} = -M$.

Let us now show that the second case is impossible. Indeed, if we suppose that $M^*_{\mathbb{R}} = -M$, then $I + MM^*_{\mathbb{R}} = 0$. Therefore, for any $x \in \mathcal{H}$ we have

$$\begin{aligned} ||x||^2 &\le (x, x) + (M^*_{\mathbb{R}}x, M^*_{\mathbb{R}}x)\langle x, x\rangle + \langle M^*_{\mathbb{R}}x, M^*_{\mathbb{R}}x\rangle \\ &= \langle x, x\rangle + \langle MM^*_{\mathbb{R}}x, x\rangle \\ &= \langle (I + MM^*_{\mathbb{R}})x, x\rangle \\ &\le ||(I + MM^*_{\mathbb{R}})x||\,||x|| \end{aligned}$$

(using the Cauchy-Schwarz inequality), whence

$$||x|| \le ||(I + MM^*_{\mathbb{R}})x|| = 0 \qquad \forall x \in \mathcal{H},$$

which is impossible. Hence, for each $M \in \mathcal{M}(\mathcal{L}(\mathcal{H}))$ we obtain $M^*_{\mathbb{R}} = M$.

It remains to use the necessary and sufficient condition (12). Namely, for any $M_1, M_2 \in \mathcal{M}(\mathcal{L}(\mathcal{H}))$ we can write

$$(M_1M_2)^* = (M_1M_2)^*_{\mathbb{R}} = (M_2)^*_{\mathbb{R}}(M_1)^*_{\mathbb{R}} = M_2M_1.$$

Thus if $\dim\mathcal{H} > 1$, then all the operators of $\mathcal{M}(\mathcal{L}(\mathcal{H}))$ produce just the same involution on $\mathcal{L}_{add}(\mathcal{H})$. Let us now consider the situation $\dim\mathcal{H} = 1$. In this case the algebra $\mathcal{L}(\mathcal{H})$ consists of the operator of multiplication by complex scalars only, and it is a simple matter to check that

$$\mathcal{M}(\mathcal{L}(\mathcal{H})) = \left\{ e^{i\varphi}M : \varphi \in [0, 2\pi) \right\}$$

where M is the operator of complex conjugation. However, all such elements produce just the same involution on $\mathcal{L}_{add}(\mathcal{H})$ ∎

Remark 4.4. Theorem 4.5 can be generalized on further complex C^*-algebras. Properties 1) - 5) show what assumptions one needs in order to guarantee that the corresponding results remain true.

All the previous considerations have dealt with the cases when different elements of $\mathcal{M}(\mathcal{A})$ produced just the same algebra $\tilde{\mathcal{A}}$. However, it may also occur another situation. Now we give an example when different elements m_1 and m_2 generate non-coinciding algebras $\tilde{\mathcal{A}}_{m_1}$ and $\tilde{\mathcal{A}}_{m_1}$, respectively.

Example 4.5. Let a continuous function α satisfy relation (17) (but optionally relation (18)), and let B_a, W_α, M and M_1 be the same operators as in Example 4.2. By $\mathcal{A}$ we now denote the smallest closed C^*-subalgebra of $\mathcal{L}(L_2(\Gamma))$ containing all operators B_a, with function a being in $C(\Gamma)$. Because

$$M_1 B_a M_1 = B_{\overline{a\circ\alpha}} \qquad \text{and} \qquad (M_1 B_a M_1)^* = B_{a\circ\alpha} = M_1 B_a^* M_1$$

the element M_1 generates the algebra $\tilde{\mathcal{A}}_{M_1}$. However, it is easily seen that $\tilde{\mathcal{A}}_{M_1} \neq \tilde{\mathcal{A}}_M$.

5. Moore-Penrose invertibility in the algebra $\tilde{\mathcal{A}}$

After introducing operations of multiplication and involution on the algebra $\tilde{\mathcal{A}}$ we may also consider the notion of Moore-Penrose invertibility in this algebra. To do this we suppose that the algebra $\tilde{\mathcal{A}}$ is generated by a complex C^*-algebra $\mathcal{A}$ and by an element m with properties $(A_1) - (A_5)$.

Definition 5.1. An element $\tilde{a} \in \tilde{\mathcal{A}}$ is said to be *m-Moore-Penrose invertible* in the algebra $\tilde{\mathcal{A}}$ if there exists an element $\tilde{b} \in \tilde{\mathcal{A}}$ such that the relations

$$\tilde{a}\tilde{b}\tilde{a} = \tilde{a}, \qquad \tilde{b}\tilde{a}\tilde{b} = \tilde{b}, \qquad (\tilde{a}\tilde{b})^*_m = \tilde{a}\tilde{b}, \qquad (\tilde{b}\tilde{a})^*_m = \tilde{b}\tilde{a}. \tag{21}$$

are true.

If such an element $\tilde{b}$ exists, then it is called an m-Moore-Penrose inverse of $\tilde{a}$ and denoted by $\tilde{a}^+_m$.

Lemma 5.1. *Let the elements m_1 and m_2 produce just the same algebra $\tilde{\mathcal{A}}$. If $(m_1m_2)^* = m_2m_1$, then each element $\tilde{a} \in \tilde{\mathcal{A}}$ is m_1-Moore-Penrose invertible if and only if it is m_2-Moore-Penrose invertible and if $\tilde{a}^+_{m_1} = \tilde{a}^+_{m_2}$.*

Proof. If $(m_1m_2)^* = m_2m_1$, then the m_1-involution on $\tilde{\mathcal{A}}$ coincides with the m_2-involution. It implies the validity of the first assertion of Lemma 5.1. The calculation

$$\begin{aligned}
\tilde{a}^+_{m_1} &= \tilde{a}^+_{m_1}\tilde{a}\tilde{a}^+_{m_1} = \tilde{a}^+_{m_1}(\tilde{a}\tilde{a}^+_{m_1})^*_{m_1} = \tilde{a}^+_{m_1}(\tilde{a}^+_{m_1})^*_{m_1}\tilde{a}^*_{m_2} \\
&= \tilde{a}^+_{m_1}(\tilde{a}^+_{m_1})^*_{m_1}\tilde{a}^*_{m_2}(\tilde{a}^+_{m_2})^*_{m_2}\tilde{a}^*_{m_2} = \tilde{a}^+_{m_1}((\tilde{a}^+_{m_1})^*_{m_1}\tilde{a}^*_{m_1})(\tilde{a}^+_{m_2})^*_{m_2}\tilde{a}^*_{m_2} \\
&= \tilde{a}^+_{m_1}\tilde{a}\tilde{a}^+_{m_1}(\tilde{a}\tilde{a}^+_{m_2})^*_{m_2} = \tilde{a}^+_{m_1}\tilde{a}\tilde{a}^+_{m_2} = (\tilde{a}^+_{m_1}\tilde{a})^*_{m_1}\tilde{a}^+_{m_2} = \tilde{a}^*_{m_1}(\tilde{a}^+_{m_1})^*_{m_1}\tilde{a}^+_{m_2} \\
&= \tilde{a}^*_{m_2}(\tilde{a}^+_{m_2})^*_{m_2}\tilde{a}^*_{m_2}(\tilde{a}^+_{m_1})^*_{m_1}\tilde{a}^+_{m_2} = (\tilde{a}^+_{m_2}\tilde{a})^*_{m_2}(\tilde{a}^+_{m_1}\tilde{a})^*_{m_1}\tilde{a}^+_{m_2} \\
&= \tilde{a}^+_{m_2}(\tilde{a}\tilde{a}^+_{m_1}\tilde{a})\tilde{a}^+_{m_2} = \tilde{a}^+_{m_2}\tilde{a}\tilde{a}^+_{m_2} = \tilde{a}^+_{m_2}
\end{aligned}$$

gives the second assertion ∎

Corollary 5.2. *Let $\mathcal{H}$ be a complex Hilbert space, and let $M_1, M_2 \in \mathcal{M}(\mathcal{L}(\mathcal{H}))$. Then an element $A \in \mathcal{L}_{add}(\mathcal{H})$ is M_1-Moore-Penrose invertible if and only if it is M_2-Moore-Penrose invertible and $A^+_{M_1} = A^+_{M_2}$ $(A \in \mathcal{L}_{add}(\mathcal{H}))$.*

The proof of this corollary immediately follows from Theorem 4.5.

Therefore, in what follows we will suppose that either the element m is fixed or all the elements of $\mathcal{M}(\mathcal{A})$ produce just the same involution, and we will write $\tilde{a}^*$ and $\tilde{a}^+$ instead of $\tilde{a}^*_m$ and of $\tilde{a}^+_m$, respectively.

As it has been mentioned earlier, the Moore-Penrose invertibility in complex C^*-algebras can be handled by using some spectral characteristics of suitable self-adjoint elements. However, considering a real algebra we may meet situations when the spectra of self-adjoint elements possess unusual properties, for instance, they can contain complex points.

Example 5.2. Let $\mathcal{H} = \mathbb{C}$, $\mathcal{A}$ be the algebra of all operators of multiplications by $a \in \mathbb{C}$, and let $M(x) = \overline{x}$ for all $x \in \mathbb{C}$. Then

$$\tilde{\mathcal{A}} = \{b + cM : b, c \in \mathcal{A}\}.$$

We consider a self-adjoint element $\tilde{a}$ of $\tilde{\mathcal{A}}$. It is clear that $\tilde{a} = b + cM$ is self-adjoint if and only if $\overline{b} = b$. If we again exploit transformation (5), we obtain that $\mathrm{Sp}_{\mathcal{A}}\tilde{a}$ is the circle of the radius $|c|$ with the center at the point b, i.e. if $c \neq 0$, then the spectrum of $\tilde{a}$ contains complex points, as well.

In the proposition below there are collected some results we need to continue the investigation of the Moore-Penrose invertibility in the algebra $\tilde{\mathcal{A}}$.

Proposition 5.3. *Let $\tilde{\mathcal{A}}$ be generated by an element m and a C^*-algebra $\mathcal{A}$ with identity e. If $\tilde{a} \in \tilde{\mathcal{A}}$, then the following assertions are equivalent:*

1) *$\tilde{a}$ is Moore-Penrose invertible.*

2) *$\tilde{a}^*\tilde{a}$ is invertible or 0 is an isolated point of the real spectrum of $\tilde{a}^*\tilde{a}$.*

3) *There exists a projection $\tilde{p}$ in $\mathcal{R}_{\tilde{\mathcal{A}}}(\tilde{a}^*\tilde{a})$ such that $\tilde{a}^*\tilde{a}\tilde{p} = 0$ and $\tilde{a}^*\tilde{a} + \tilde{p}$ is invertible.*

4) *There exists a projection $\tilde{q}$ in $\tilde{\mathcal{A}}$ such that $\tilde{a}\tilde{q} = 0$ and $\tilde{a}^*\tilde{a} + \tilde{q}$ is invertible.*

If one of these conditions is fulfilled, then $\tilde{q}$ is uniquely determined and $\tilde{a}^+ = (\tilde{a}^*\tilde{a} + \tilde{q})^{-1}\tilde{a}^*$.

The proof of this proposition follows from [3: Propositions 7 and 8, Corollaries 11 and 15] and from Corollary 3.5 of the present article.

Following [3] we will say that a C^*-subalgebra $\mathcal{B}$ of the C^*-algebra $\mathcal{A}$ is *m-closed* if for each $b \in \mathcal{B}$ the element mbm is in $\mathcal{B}$ again.

Proposition 5.4 (cf. [3: Corollary 13]). *Let $\mathcal{A}$ be a complex C^*-algebra with identity e and $\mathcal{B}$ be an m-closed C^*-subalgebra of $\mathcal{A}$ containing e. Then the element $\tilde{b}$ of $\tilde{\mathcal{B}}$ is Moore-Penrose invertible in $\tilde{\mathcal{B}}$ if and only if it is Moore-Penrose invertible in $\tilde{\mathcal{A}}$.*

From now on, we are going to restrict ourselves to considering a special real C^*-algebra $\tilde{\mathcal{F}}$. Namely, let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{F}$ be the set of all bounded sequences $\{A_n\}$ of bounded linear operators A_n acting in $\mathcal{H}$. This set equipped with the norm

$$||\{A_n\}||_{\mathcal{F}} = \sup_n ||\{A_n\}||$$

and with the natural operations of addition, multiplication, scalar multiplication and with the involution $\{A_n\}^* = \{A_n^*\}$ becomes a C^*-algebra with identity. Considering the extension of $\mathcal{F}$ by any element M with properties $(A_1)-(A_5)$ we obtain a real algebra $\tilde{\mathcal{F}}$ consisting of all the sequences $\{\tilde{A}_n\}$ the members of which are additive continuous operators acting on $\mathcal{H}$. Let $\tilde{\mathcal{G}}$ be the set of all sequences of $\tilde{\mathcal{F}}$ tending to zero in the operator norm. It is easily seen that $\tilde{\mathcal{G}}$ is an ideal of $\tilde{\mathcal{F}}$. In addition, it is worth noticing that the operator norm on $\tilde{\mathcal{F}}$ is equivalent to the norm the set $\tilde{\mathcal{F}}$ is equipped with as the extension of $\mathcal{F}$ by M (see [3]).

Definition 5.3. A sequence $\{\tilde{A}_n\} \in \tilde{\mathcal{F}}$ is said to be *weakly* asymptotically Moore-Penrose invertible if there exists a sequence $\{\tilde{B}_n\} \in \tilde{\mathcal{F}}$ such that the sequences

$$\{\tilde{A}_n\tilde{B}_n\tilde{A}_n - \tilde{A}_n\},\ \{\tilde{B}_n\tilde{A}_n\tilde{B}_n - \tilde{B}_n\},\ \{(\tilde{A}_n\tilde{B}_n)^* - \tilde{A}_n\tilde{B}_n\}, \{(\tilde{B}_n\tilde{A}_n)^* - \tilde{B}_n\tilde{A}_n\}$$

belong to the ideal $\tilde{\mathcal{G}}$ or, in other words, if the coset $\{\tilde{A}_n\}+\tilde{\mathcal{G}}$ is Moore-Penrose invertible in the quotient algebra $\tilde{\mathcal{F}}/\tilde{\mathcal{G}}$.

It has been shown in [17] that whenever a sequence $\{A_n\} \in \mathcal{F}$ is weakly asymptotically Moore-Penrose invertible, then the spectrum $\mathrm{Sp}(A_n^*A_n)$ of $A_n^*A_n$ can be split in two parts one of which is bounded from zero by a positive constant (independent of n), while the other part tends to zero if n tends to infinity. We now intend to show that this property remains true for elements of the real algebra $\tilde{\mathcal{F}}$ as well. However, the spectrum $\mathrm{Sp}(\tilde{A}_n^*\tilde{A}_n)$ (which coincides with the real spectrum of $\tilde{A}_n^*\tilde{A}_n$ in that case) should be replaced by the real spectrum $\mathrm{sp}(\tilde{A}_n^*\tilde{A}_n)$ of $\tilde{A}_n^*\tilde{A}_n$.

Proposition 5.5. *A sequence $\{\tilde{A}_n\}$ is weakly asymptotically Moore-Penrose invertible in $\tilde{\mathcal{F}}$ if and only if there exist non-negative numbers d and r_n with $d > 0$ and $r_n \to 0$ as $n \to \infty$ such that*

$$\mathrm{sp}(\tilde{A}_n^*\tilde{A}_n) \subseteq [0, r_n] \cup [d, \infty] \tag{22}$$

for all sufficiently large n.

To prove this assertion one can exploit the scheme of the proof of the corresponding result for complex C^*-algebras (see [17]). However, one has to be sure in the validity of auxiliary results used in [17]. We are now going to give brief comments for some propositions we need.

In the theory of complex C^*-algebras an "almost projection" lemma is well-known (see [19: Lemma 5.1.6]). A careful consideration of its proof shows that this lemma remains also true for real C^*-algebras. More precisely, we have the following lemma.

Lemma 5.6. *Let $\mathcal{B}$ be a real C^*-algebra with identity, and let $b \in \mathcal{B}$ be a self-adjoint element with $||b^2 - b|| < \frac{1}{4}$. Then there exists a self-adjoint element $g \in \mathcal{B}$ such that $b + g$ is a projection and $||g|| \leq 2||b^2 - b||$.*

For the proof of this result in the real situation one can use the isometrical isomorphism between the real algebra $\mathcal{R}(b)$ and $C_{\mathbb{R}}(\mathrm{sp}\mathcal{R}(b))$ (see Theorem 3.1), if one observes that the corresponding proof in the complex situation deals with real functions only.

Let $\{\tilde{\mathcal{A}}_n\}$ be in $\tilde{\mathcal{F}}$. By $\{\tilde{\mathcal{A}}_n\}^\circ$ we denote the coset of $\tilde{\mathcal{F}}/\tilde{\mathcal{G}}$ which contains the sequence $\{\tilde{\mathcal{A}}_n\}$, i.e. $\{\tilde{\mathcal{A}}_n\}^\circ := \{\tilde{\mathcal{A}}_n\} + \tilde{\mathcal{G}}$. Suppose that $\{\tilde{\mathcal{A}}_n\}$ is weakly asymptotically Moore-Penrose invertible. Due to Proposition 5.4 there is a sequence $\{\tilde{P}_n\} \in \tilde{\mathcal{F}}$ such that $(\{\tilde{P}_n\}^\circ)^2 = \{\tilde{P}_n\}^\circ$ and $(\{\tilde{P}_n\}^\circ)^* = \{\tilde{P}_n\}^\circ$. It is possible, therefore, to pick up a sequence $\{\tilde{P}_n\} \in \{\tilde{P}_n\}^\circ$ such that $\tilde{P}_n^* = \tilde{P}_n$ and $||\tilde{P}_n^2 - \tilde{P}_n|| \to 0$ as $n \to \infty$. Together with Lemma 5.6 this implies that one can choose a sequence $\{\tilde{\Pi}_n\} \in \{\tilde{P}_n\}^\circ$ such that each member of it is a projection, i.e. $\tilde{\Pi}_n^2 = \tilde{\Pi}_n$ and $\tilde{\Pi}_n^* = \tilde{\Pi}_n$ for all $n \in \mathbb{N}$.

Combining this result with Proposition 5.3 we obtain

Lemma 5.7. *A sequence $\{\tilde{A}_n\} \in \tilde{\mathcal{F}}$ is weakly asymptotically Moore-Penrose invertible if and only if there exists a sequence $\{\tilde{\Pi}_n\}$ of projections on $\mathcal{H}$ such that the sequence $\{\tilde{A}_n^*\tilde{A}_n + \tilde{\Pi}_n\}$ is stable and $||\tilde{A}_n\tilde{\Pi}_n|| \to 0$ as $n \to \infty$. The sequence $\{\tilde{\Pi}_n\}$ is unique modulo $\tilde{\mathcal{G}}$ and $\{(\tilde{A}_n^*\tilde{A}_n + \tilde{\Pi}_n)^{-1}A_n^*\}^\circ = \{\{\tilde{A}_n\}^+\}^\circ$.*

It should be noted that in place of $\tilde{q}$ in assertion 4) of Proposition 5.3 the element $\tilde{a}^\sqcap = e - \tilde{a}^+\tilde{a}$ can be chosen. According to this designation we refer to as $\Pi\{\tilde{A}_n\}$ the set of all projections in $(\{\tilde{A}_n\}^\circ)^\sqcap$.

Lemma 5.8. *Let $\{\tilde{A}_n\} \in \tilde{\mathcal{F}}$ be weakly asymptotically Moore-Penrose invertible. Then there is a sequence $\{\tilde{\Pi}_n\} \in \Pi\{\tilde{A}_n\}$ such that $\tilde{\Pi}_n \in \mathcal{R}_{\mathcal{L}_{add}(\mathcal{H})}(\tilde{A}_n^*\tilde{A}_n)$ for all $n \in \mathbb{N}$. The sequence $\{\tilde{\Pi}_n\}$ is unique in the following sense: If $\{\tilde{\Pi}_n^{(1)}\}, \{\tilde{\Pi}_n^{(2)}\} \in \mathcal{R}_{\mathcal{L}_{add}(\mathcal{H})}(\tilde{A}_n^*\tilde{A}_n)$ for all $n > n_0$, then $\{\tilde{\Pi}_n^{(1)}\} = \{\tilde{\Pi}_n^{(2)}\}$ for all sufficiently large n.*

The proof of this lemma also follows the corresponding proof of Theorem 3 in [17], and the only additional result we need here is the Gelfand-Naimark theorem for commutative real C^*-algebras. However, this can be found in [5].

Proof of Proposition 5.5. After having proved Lemmas 5.7 and 5.8 we can use Theorem 3.1 to obtain relation (22). However, as is shown before, in real algebras the spectral properties of self-adjoint elements can be different from those in complex algebras. Therefore, we first show that the real spectrum of $\tilde{A}_n^*\tilde{A}_n$ does not contain negative points. Indeed, as it follows from Corollary 3.5,

$$\mathrm{sp}_{\mathcal{L}_{add}(\mathcal{H})}(\tilde{A}_n^*\tilde{A}_n) = \mathrm{sp}_{\mathcal{L}^{2\times 2}(\mathcal{H})}(\Psi(\tilde{A}_n)^*\Psi(\tilde{A}_n)).$$

However, the self-adjoint operator $\Psi(\tilde{A}_n)^*\Psi(\tilde{A}_n)$ does not have any negative points in its spectrum since it belongs to the complex C^*-algebra $\mathcal{L}^{2\times 2}(\mathcal{H})$. Hence

$$\mathrm{sp}_{\mathcal{L}_{add}(\mathcal{H})}(\tilde{A}_n^*\tilde{A}_n) \subset [0, +\infty].$$

Let $\{\tilde{A}_n\}$ be weakly asymptotically Moore-Penrose invertible. Due to Lemmas 5.7 and 5.8 we can take a sequence of projections $\{\tilde{\Pi}_n\}$ such that $\{\tilde{A}_n\} \in \mathcal{R}(\tilde{A}_n^*\tilde{A}_n)$ $\quad (n \in \mathbb{N})$

and $||\tilde{A}_n^* \tilde{A}_n \tilde{\Pi}_n|| \to 0$ as $n \to \infty$ and the sequence $\{\tilde{A}_n^* \tilde{A}_n + \tilde{\Pi}_n\}$ is stable. Then there exists a number $d > 0$ such that

$$||(\{\tilde{A}_n^* \tilde{A}_n + \tilde{\Pi}_n\}^\circ)^{-1}||_{\tilde{\mathcal{F}}/\mathcal{G}} < \frac{1}{d},$$

and we put

$$r_n := ||\tilde{A}_n^* \tilde{A}_n \tilde{\Pi}_n||. \tag{23}$$

Using the definition of the norm in $\tilde{\mathcal{F}}/\tilde{\mathcal{G}}$ we can choose a number n_0 such that

$$||\{\tilde{A}_n^* \tilde{A}_n + \tilde{\Pi}_n\}^{-1}||_{\mathcal{L}_{add}(\mathcal{H})} < \frac{1}{d} \qquad (n > n_0). \tag{24}$$

Now we fix some $n > n_0$. Due to the isometrical isomorphism between $\mathcal{R} := \mathcal{R}(\tilde{A}_n^* \tilde{A}_n)$ and $C(\mathrm{sp}_{\mathcal{R}}(\tilde{A}_n^* \tilde{A}_n))$ (see Theorem 3.1) we can identify the element $\tilde{A}_n^* \tilde{A}_n$ with the function $x \to x$, and the projection $\tilde{\Pi}_n$ with the function $x \to p_n(x)$, where p_n takes the values 0 and 1 only. Then it follows from (23) and (24) that

$$x + p_n(x) > d \qquad \text{and} \qquad x p_n(x) < r_n \qquad (x \in \mathrm{sp}_{\mathcal{R}}(\tilde{A}_n^* \tilde{A}_n)).$$

Analyzing the last two inequalities we obtain our claim. ∎

Now we consider the notion of asymptotical Moore-Penrose invertibility for sequences of $\tilde{\mathcal{F}}$. We recall that a sequence $\{\tilde{A}_n\} \in \tilde{\mathcal{F}}$ is said to be asymptotically Moore-Penrose invertible if there is an n_0 such that the operators $\tilde{A}_n$ are Moore-Penrose invertible for all $n > n_0$ and if $\sup_{n>n_0} ||\tilde{A}_n^+|| < +\infty$.

Theorem 5.9. *A sequence $\{\tilde{A}_n\} \in \tilde{\mathcal{F}}$ is weakly asymptotically Moore-Penrose invertible if and only if it can be represented as a sum of an asymptotically Moore-Penrose invertible sequence and a sequence of $\tilde{\mathcal{G}}$.*

Proof. *Necessity part*: Let $\{\tilde{A}_n\} \in \tilde{\mathcal{F}}$ be weakly asymptotically Moore-Penrose invertible, and let $\{\tilde{\Pi}_n\}$ be the projection sequence defined in the proof of Proposition 5.5. By $\{\tilde{B}_n\}$ we denote the operator

$$\tilde{B}_n = \tilde{A}_n(I - \tilde{\Pi}_n). \tag{25}$$

Then, as it follows from the proof of Proposition 5.5, the real spectrum of $\tilde{B}_n^* \tilde{B}_n + \tilde{\Pi}_n$ is contained in the set $\{1\} \cup [d, +\infty)$. Therefore, the operator $\tilde{B}_n$ is Moore-Penrose invertible, $\tilde{B}_n^+ = (\tilde{B}_n^* \tilde{B}_n + \tilde{\Pi}_n)^{-1} \tilde{B}_n^*$ and $\|\tilde{B}_n^+\| \le 1/\sqrt{d}$. Henceforth, the sequence $\{\tilde{B}_n\}$ is asymptotically Moore-Penrose invertible. In addition, we have $\|\tilde{A}_n \tilde{\Pi}_n\| \to 0$ as $n \to \infty$ (cf. Lemma 28). Finally, from (25) we obtain

$$\tilde{A}_n = \tilde{B}_n + \tilde{G}_n \qquad \text{with } \|\tilde{G}_n\| \to 0 \text{ as } n \to \infty. \tag{26}$$

The proof of the sufficiency is evident. ∎

The above results are of practical interest and can be applied to the Moore-Penrose regularization of approximation methods for equations with additive operators. Below we consider this problem for spline approximation methods for singular integral equations with conjugation.

In the space $L_2(\Gamma)$ (see Example 4.1) we consider the equation

$$\begin{aligned}(Ax)(t) \equiv a(t)x(t) &+ \frac{b(t)}{\pi i}\int_\Gamma \frac{x(\tau)d\tau}{\tau - t} + c(t)\overline{x(t)} + \frac{d(t)}{\pi i}\int_\Gamma \frac{\overline{x(\tau)}d\tau}{\tau - t} \\ &+ \int_\Gamma k_1(t,\tau)x(\tau)\,d\tau + \int_\Gamma k_2(t,\tau)\overline{x(\tau)}\,d\tau \qquad (t \in \Gamma) \\ &= g(t)\end{aligned} \tag{27}$$

where a, b, c, d are supposed to be piecewise continuous functions on Γ and $k_1, k_2 \in C(\Gamma \times \Gamma)$. As was shown in [2] the investigation of spline-approximation methods for equation (27) can be reduced to studying an algebra of operator sequences. This algebra is generated by a special sequence of anti-linear operators $\{M_n\}$, by sequences tending to zero in the operator norm as well as by diagonal sequences ϕ_n^ε and by sequences of circulants $\hat{\psi}_n$ formed by given transformations of piecewise continuous functions ϕ and ψ, respectively. More precisely, the sequences which appear in spline approximation methods for the equation (27) can be represented in the form

$$\{\tilde{A}_n\} = \big\{a_n^\varepsilon \hat{\alpha}_n + b_n^\varepsilon \hat{\beta}_n + (c_n^\varepsilon \hat{\gamma}_n + d_n^\varepsilon \hat{\theta}_n)M_n + G_n\big\}, \quad ||G_n|| \to 0 \text{ as } n \to \infty \tag{28}$$

where $a_n^\varepsilon, b_n^\varepsilon, c_n^\varepsilon, d_n^\varepsilon$ are related with the coefficients a, b, c, d of (27) whereas $\hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n, \hat{\theta}_n$ reflect the method used. For the form of these terms we refer the reader to [2]. In [3] the sequence (28) was studied with respect to the weak asymptotic Moore-Penrose invertibility. Now we are able to say something more about the asymptotic behaviour of this sequence. However, we need additional notations.

Thus, let P and Q stand for the operators $\frac{1}{2}(I+S)$ and $\frac{1}{2}(I-S)$, respectively, where S was defined in Example 4.1, and let a, b, c, d and $\alpha, \beta, \gamma, \theta$ be piecewise continuous functions. We form the matrices

$$\begin{aligned} A(t) &= \begin{pmatrix} a(t) & 0 \\ 0 & \overline{a(t)} \end{pmatrix} & B(t) &= \begin{pmatrix} b(t) & 0 \\ 0 & \overline{b(t)} \end{pmatrix} & C(t) &= \begin{pmatrix} c(t) & 0 \\ 0 & \overline{c(t)} \end{pmatrix} \\ D(t) &= \begin{pmatrix} d(t) & 0 \\ 0 & \overline{d(t)} \end{pmatrix} & \alpha^\star(t) &= \begin{pmatrix} \alpha(t) & 0 \\ 0 & \overline{\alpha(\bar{t})} \end{pmatrix} & \beta^\star(t) &= \begin{pmatrix} \beta(t) & 0 \\ 0 & \overline{\beta(\bar{t})} \end{pmatrix} \\ \gamma^\star(t) &= \begin{pmatrix} \gamma(t) & 0 \\ 0 & \overline{\gamma(\bar{t})} \end{pmatrix} & \theta^\star(t) &= \begin{pmatrix} \theta(t) & 0 \\ 0 & \overline{\theta(\bar{t})} \end{pmatrix} & \Lambda &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Corollary 5.10. *Let $a, b, c, d \in PC(\Gamma)$ as well as $\alpha, \beta, \gamma, \theta \in PC(\Gamma)$. The sequence (28) can be represented in the form (26) if and only if all the operators $W_u, W^u \in \mathcal{L}(L_2^2(\Gamma))$ $(u \in \Gamma)$,*

$$\begin{aligned} W^u = &\Big[A(t)\alpha^\star(u+0) + B(t)\beta^\star(u+0) + \big(C(t)\gamma^\star(u+0) + D(t)\theta^\star(u+0)\big)\Lambda\Big]P \\ &+ \Big[A(t)\alpha^\star(u-0) + B(t)\beta^\star(u-0) + (C(t)\gamma^\star(u-0) + D(t)\theta^\star(u-0))\Lambda\Big]Q \end{aligned}$$

and

$$W_u = P\Big[A(u+0)\alpha^\star(\bar t) + B(u+0)\beta^\star(\bar t) + \big(C(u+0)\gamma^\star(\bar t) + D(u+0)\theta^\star(\bar t)\big)\Lambda\Big]$$
$$+ Q\Big[A(u-0)\alpha^\star(\bar t) + B(u-0)\beta^\star(\bar t) + \big(C(u-0)\gamma^\star(\bar t) + D(u-0)\theta^\star(\bar t)\big)\Lambda\Big]$$

are normally solvable, and the norms of their Moore-Penrose inverses are uniformly bounded with respect to $u \in \Gamma$.

The proof of this corollary follows from Theorem 5.9 and from [3: Theorem 25]. Note that the Fredholmness of the operators W_u and W^u is sufficient for the uniform boundedness of their Moore-Penrose inverses (cf. [15]).

Remark 5.4. In order to regularize the sequence $\{\tilde A_n\}$ of (28) practically, we have to consider the matrix sequence $(\{\Psi(\tilde A_n)\}$,

$$\Psi(\tilde A_n) = \begin{pmatrix} a_n^\varepsilon \hat\alpha_n + b_n^\varepsilon \hat\beta_n & c_n^\varepsilon \hat\gamma_n + d_n^\varepsilon \hat\theta_n \\ M_n(c_n^\varepsilon \hat\gamma_n + d_n^\varepsilon \hat\theta_n)M_n & M_n(a_n^\varepsilon \hat\alpha_n + b_n^\varepsilon \hat\beta_n)M_n \end{pmatrix}.$$

These matrices can be regularized (in the sense of (26)) using their singular value decompositions (see [17]). After then we can apply the mapping Ψ^{-1} to regularized matrices in order to get the corresponding regularization for the initial sequence (28).

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