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Mixed Boundary Value Problems for Nonlinear Elliptic Systems in n-Dimensional Lipschitzian Domains

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Abstract. Let $u : \Omega \to \mathbb{R}^N$ be the solution of the nonlinear elliptic system

$$-\sum_{i=1}^{n} \partial_i F_i(x, \nabla u) = f(x) + \sum_{i=1}^{n} \partial_i f_i(x),$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a piecewise smooth boundary (e.g., Ω is a polyhedron). It is assumed that a mixed boundary value condition is given. Global regularity results in Sobolev and in Nikolskii spaces are proven, in particular $[W^{s,2}(\Omega)]^N$-regularity $(s < \frac{3}{2})$ of u.

Keywords: *Mixed boundary value problems, piecewise smooth boundaries, Nikolskii spaces*

AMS subject classification: Primary 35 J 55, 35 J 65, secondary 35 J 25

0. Introduction

We treat the nonlinear elliptic system

$$\left.\begin{aligned} -\sum_{i=1}^{n} \partial_i F_i(x, \nabla u) &= f(x) + \sum_{i=1}^{n} \partial_i f_i(x) && \text{in } \Omega \\ u(x) &= 0 && \text{on } \Gamma_{\mathcal{D}} \\ -\sum_{i=1}^{n} F_i(x, \nabla u)\nu_i &= \sum_{i=1}^{n} f_i \nu_i && \text{on } \Gamma_{\mathcal{N}} \end{aligned}\right\} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is bounded, $u : \Omega \to \mathbb{R}^N$ is a vector-valued function, $\partial_i = \frac{\partial}{\partial x_i}$, $\partial\Omega = \Gamma_{\mathcal{D}} \cup \Gamma_{\mathcal{N}}$ where $\Gamma_{\mathcal{D}}$ is the Dirichlet boundary and $\Gamma_{\mathcal{N}}$ is the Neumann boundary, and ν is the outward normal of $\partial\Omega$. We suppose that $\partial\Omega$ is piecewise smooth (e.g., Ω is a polyhedron or has a Lipschitz boundary).

In this paper we investigate the regularity of the solution u of (0.1). Refining the method of [8] we obtain regularity results in Nikolskii spaces and in Sobolev spaces $[W^{s,2}(\Omega)]^N$, especially $[W^{s,2}(\Omega)]^N$-regularity $(s < \frac{3}{2})$ of u up to the boundary.

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Solutions of mixed boundary value problems in non-smooth domains may have singularities on the boundary at such points where the boundary condition is changing or where $\partial\Omega$ is not smooth.

In the case of a linear elliptic equation various authors have investigated the regularity of the solution. They have given a decomposition of the solution u into a regular and a singular part.In particular, for $\Omega \subset \mathbb{R}^2$ this provides an explicit description of the behaviour of u near the boundary (cf. [4, 7, 9, 11]). In the case when $\Omega \subset \mathbb{R}^n \quad (n \geq 3)$ there are difficulties by finding such a decomposition which describes all the singularities of u (see [2, 3, 10, 14, 17]).

In the case of nonlinear equations there are only few results. Semilinear Dirichlet problems on corner domains are treated in [12, 15] and in [5, 6], where results in weighted Sobolev spaces are given. Further, nonlinear mixed boundary value problems are investigated in [8]. Regularity results in Sobolev spaces are proven.

In this paper we generalize some results given in [8]. Let the boundary of Ω consist of smooth $(n-1)$-dimensional manifolds with piecewise smooth boundaries such that each boundary manifold is either a Dirichlet or a Neumann boundary manifold. Let us fix some point $P \in \partial\Omega$. Then we suppose that there is a ball $B(P)$ and a smooth mapping which maps Ω onto a domain $\hat{\Omega}$ such that $B(P) \cap \hat{\Omega}$ is the intersection of $B(P)$ and a polyhedron. In contrast to [8] we consider the case that $B(P) \cap \partial\hat{\Omega}$ contains more than one Dirichlet boundary manifold. Further, we admit that $B(P) \cap \hat{\Omega}$ is probably not convex. But we assume that each inner angle between a Dirichlet and a Neumann boundary manifold is not greater than π.

We suppose that there is a function $F(x,p)$ such that $F_i^r(x,p)$ is the partial derivative of $F(x,p)$ with respect to the component corresponding to p_i^r (here $F_i^r(x,p)$ denotes the r-th component of the vector $F_i(x,p)$). Hence, we deal with the variational case.

The aim of this paper is to show that $u \in [W^{s,2}(\Omega)]^N$ for $s < \frac{3}{2}$. This result is the best possible, for we admit that $\hat{\Omega}$ can be a polyhedron where the inner angle between a Dirichlet and a Neumann boundary manifold is equal to π. Otherwise, if all such angles are less than π, we prove that $u \in \mathcal{H}^{\frac{3}{2},2}(\Omega)$, where $\mathcal{H}^{s,p}(\Omega)$ denotes a Nikolskii space. Moreover, in the case when $N = 1$ the solution u of equation (0.1) is Hölder continuous. Then we show that $u \in L^p(\Omega)$ for some $p > 3$.

This paper is organized as follows. In Section 1 we state the assumptions on the data and the main results. Section 2 contains notations. In Section 3 the proofs of the main results are given. Finally, in Section 4 we explain the proofs with examples of tree-dimensional domains.

1. Assumptions on the data and main results

We need the following assumptions on the data.

(A1) $\Omega \subset \mathbb{R}^n \quad (n \geq 3)$ is a connected open domain with Lipschitz boundary.

(A2) $\partial\Omega = \bigcup_{1 \leq i \leq M} \overline{\Gamma_i}$, where Γ_i are open $(n-1)$-dimensional manifolds, and $\Gamma_i \cap \Gamma_j = \emptyset$ holds for $i \neq j$.

(A3) $\partial\Gamma_i \quad (1 \le i \le M)$ are $(n-2)$-dimensional Lipschitz continuous manifolds.

(A4) $\Gamma_1, \dots, \Gamma_\sigma \subset \Gamma_{\mathcal{D}}$ and $\Gamma_{\sigma+1}, \dots, \Gamma_M \subset \Gamma_{\mathcal{N}}$.

(A5) $P \in \bigcap_{i \in \Lambda} \partial\Gamma_i$ implies that $|\Lambda| \le n$.

(A6) To each point $P \in \partial\Omega$ there exists a mapping ϕ and a ball $B_R(\phi(P))$ such that:

(i) $B_R(\phi(P)) \cap \phi(\partial\Omega)$ is the intersection of $B_R(\phi(P))$ and a polyhedron.

(ii) $B_R(\phi(P)) \cap \phi(\partial\Omega)$ is simply connected.

(iii) $\phi, \phi^{-1} \in W^{2,\infty}_{loc}(\mathbb{R}^n)$ and the Jacobian of ϕ is positive definite.

(iv) If $\Gamma_i \in \Gamma_{\mathcal{D}}$, $\Gamma_j \in \Gamma_{\mathcal{N}}$, and $\partial\Gamma_i \cap \partial\Gamma_j \neq \emptyset$, then $\mathrm{angle}(\phi(\Gamma_i), \phi(\Gamma_j)) \le \pi$.

(v) At most one pair of boundary manifolds $\Gamma_i, \Gamma_j \quad (i \neq j, \partial\Gamma_i \cap \partial\Gamma_j \neq \emptyset)$ satisfies $\mathrm{angle}(\phi(\Gamma_i), \phi(\Gamma_j)) = \pi$.

Remark.

(i) By $\mathrm{angle}(\phi(\Gamma_i), \phi(\Gamma_j))$ we denote the inner angle between $\phi(\Gamma_i) \cap B_R(\phi(P))$ and $\phi(\Gamma_j) \cap B_R(\phi(P))$ where it is assumed that $\phi(\Gamma_i) \cap B_R(\phi(P)) \neq \emptyset$ and $\phi(\Gamma_j) \cap B_R(\phi(P)) \neq \emptyset$.

(ii) We assume that the inner angle between a boundary manifold of $\phi(\Gamma_{\mathcal{D}})$ and another one of $\phi(\Gamma_{\mathcal{N}})$ is not greater than π (cf. assumption (A6)/(ii)). But it is admitted that the inner angle between two boundary manifolds is greater than π if there is no change of the boundary value condition.

(iii) It is also possible to treat domains with a slit. Then instead of assumption (A6)/(v) we need the assumption that at most one pair of boundary manifolds $\Gamma_i, \Gamma_j \quad (i \neq j, \partial\Gamma_i \cap \partial\Gamma_j \neq \emptyset)$ satisfies $\mathrm{angle}(\phi(\Gamma_i), \phi(\Gamma_j)) = \mu\pi$, $\mu \in \{1, 2\}$.

Let $x \in \overline{\Omega}$ and $p \in \mathbb{R}^{nN}$ with components $x_i \quad (1 \le i \le n)$ and $p_i^r \quad (1 \le r \le N)$, respectively. We suppose that there is a C^2-function $F(x,p) : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ such that $\frac{\partial}{\partial p_i^r} F(x,p) = F_i^r(x,p)$ for all $1 \le i \le n$ and $1 \le r \le N$, where $F_i^r(x,p)$ denotes the r-th component of $F_i(x,p) \in \mathbb{R}^N$. We set

$$F_{x_i}(x,p) = \frac{\partial}{\partial x_i} F(x,p), \quad F_{i,x_k}(x,p) = \frac{\partial}{\partial x_k} F_i(x,p), \quad F_{i,k}^{rs}(x,p) = \frac{\partial}{\partial p_k^s} F_i^r(x,p)$$

for $1 \le i, k \le n$ and $1 \le r, s \le N$. Furthermore, we suppose that there are functions g_0, g_{x_i}, g_i and $g_{i,x_k} \quad (1 \le i, k \le n)$ such that:

(H1) $c_0 + c_0' |p|^2 \le F(x,p) \le g_0(x) + c|p|^2$ for $g_0 \in L^\infty(\Omega)$ and $c_0' > 0$.

(H2) $|F_{x_i}(x,p)| \le g_{x_i}(x) + c|p|^2$ for $g_{x_i} \in L^1(\Omega)$.

(H3) $|F_i(x,p)| \le g_i(x) + c|p|$ for $g_i \in L^2(\Omega)$.

(H4) $|F_{i,x_k}(x,p)| \le g_{i,x_k}(x) + c|p|$ for $g_{i,x_k} \in L^2(\Omega)$.

(H5) $|F_{i,k}^{rs}(x,p)| \le c$.

(H6) There is a constant $k_0 > 0$ independent of x and p such that for all $\xi \in \mathbb{R}^{nN}$

$$k_0 |\xi|^2 \le \sum_{r,s=1}^{N} \sum_{i,k=1}^{n} F_{i,k}^{rs}(x,p) \xi_i^r \xi_k^s.$$

(H7) $f^r(x) \in L^2(\Omega)$ and $f_i^r(x) \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ for $1 \le i \le n$ and $1 \le r \le N$.

Remark. Hypothesis (H6) can be replaced by the weaker condition

(H6′) There are constants $k_0 > 0$ and k_1 independent of x and p such that for all $\xi \in [H^1(\Omega)]^N$

$$k_0 \int_\Omega |\nabla \xi|^2 dx - k_1 \int_\Omega |\xi|^2 dx \le \int_\Omega \sum_{r,s=1}^N \sum_{i,k=1}^n F_{i,k}^{rs}(x, \nabla u) \partial_i \xi^r \partial_k \xi^s dx.$$

Let us note that the changes to be made in the proofs are obvious.

Under the above hypotheses there exists a unique weak solution $u \in [W^{1,2}(\Omega)]^N$ of problem (0.1) (see [16]).

We use the usual Sobolev spaces $W^{s,p}(\Omega)$ and the Nikolskii spaces $\mathcal{H}^{s,p}(\Omega)$ (cf. [1]). In detail, let s be no integer, let $z \in \mathbb{R}^n$, $s = m + \sigma$ where $0 < \sigma < 1$ and m is an integer, $\Omega_\eta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \eta\}$, and $1 \le p < \infty$. The spaces $W^{s,p}(\Omega)$ and $\mathcal{H}^{s,p}(\Omega)$ consist of all functions u for which the norms

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|_{W^{m,p}(\Omega)}^p + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{n+p\sigma}} dxdy \right)^{\frac{1}{p}}$$

and

$$\|u\|_{\mathcal{H}^{s,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)}^p + \sum_{|\alpha|=m} \sup_{\substack{\eta>0 \\ 0<|z|<\eta}} \int_{\Omega_\eta} \frac{|\partial^\alpha u(x+z) - \partial^\alpha u(x)|^p}{|z|^{\sigma p}} dx \right)^{\frac{1}{p}}$$

are finite.

We will prove the following results:

Theorem 1.1.

a) *The solution u of equation* (0.1) *satisfies*

$$u \in [W^{s,2}(\Omega)]^N \qquad \text{for all } s < \frac{3}{2}. \tag{1.1}$$

b) *If angle$(\Gamma_i, \Gamma_j) \ne \pi$ for each pair of boundary manifolds Γ_i, Γ_j $(i \ne j, \partial\Gamma_i \cap \partial\Gamma_j \ne \emptyset)$, then*

$$u \in [\mathcal{H}^{\frac{3}{2},2}(\Omega)]^N \tag{1.2}$$

holds.

Remark.

(i) By assumption we consider the case when $n \ge 3$. But our proofs of (1.1) and (1.2) also hold when $n = 2$.

(ii) angle$(\Gamma_i, \Gamma_j) \ne \pi$ implies that angle$(\phi(\Gamma_i), \phi(\Gamma_j)) \ne \pi$, for ϕ is smooth.

Using the Sobolev imbedding theorem and (1.1) we get $u \in [W^{1,s}(\Omega)]^N$ for $s < \frac{2n}{n-1}$. Let us note that $s < 3$ for $n \ge 3$. The next theorem improves this result in the case when $N = 1$.

Theorem 1.2. *Let $N = 1$ and let the functions g_{x_i}, g_i, g_{i,x_k}, f and f_k given in hypotheses* (H1) - (H7) *satisfy*

$$g_i \in L^{\frac{n}{1-\delta}}(\Omega), \qquad g_{x_i}, g_{i,x_k}, f, \partial_i f_k \in L^{\frac{2n}{3-\delta}}(\Omega) \tag{1.3}$$

for $1 \le i, k \le n$ and some $\delta > 0$. Then there exists a constant $\varepsilon_0 > 0$ independent of n such that the solution u of equation (0.1) *satisfies*

$$\nabla u \in L^s(\Omega) \qquad \textit{for} \;\; s = 3 + \varepsilon_0. \tag{1.4}$$

Remark. The results of Theorem 1.1 and Theorem 1.2 also hold for solutions $u(x,t)$ of parabolic systems. Let $u(x,0) \in [W^{1,2}(\Omega)]^N$. Then we get the results (1.1), (1.2), and (1.4) in the spaces $\left[L^2\big(0,T;W^{s,p}(\Omega)\big)\right]^N$ and $\left[L^2\big(0,T;\mathcal{H}^{s,p}(\Omega)\big)\right]^N$.

2. Notations

Let $B_R(x) = \{y \in \mathbb{R}^n : |x - y| < R\}$. The boundary of Ω is piecewise smooth. By assumption to each point $P \in \partial\Omega$ there is a constant $R_0 > 0$ and a $W^{2,\infty}$-mapping

$$\phi^* : x \to \hat{x}$$

such that $B_{R_0}(\hat{P}) \cap \hat{\Omega}$ is the intersection of $B_{R_0}(\hat{P})$ and a polyhedron. (We use the denotations $\hat{P} = \phi^*(P)$, $\hat{\Omega} = \phi^*(\Omega)$ etc. and we will write B_R instead of $B_R(\hat{P})$.)

In the sequel we suppose that $\hat{P}$ and $R_0 \in (0,1]$ are fixed such that $\hat{P}$ is the only vertex of $B_{R_0}(\hat{P}) \cap \partial\hat{\Omega}$ or that there is no vertex of $\partial\hat{\Omega}$ in $B_{R_0}(\hat{P})$. Further, let $\hat{P} \in \partial\hat{\Gamma}_k$ for some $k \in \{1, \dots, M\}$.

We need appropriate basis vectors $\{\zeta^1, \dots, \zeta^n\}$ in $B_{R_0}(\hat{P})$. Let Λ_1, Λ_2, and Λ_3 be disjoint index sets (some of them possibly empty) such that $\cup_{i=1}^3 \Lambda_i = \{1, \dots, n\}$. Let $\alpha^* > 0$, $|\zeta^i| = 1$ for $1 \le i \le n$, and angle$(\zeta^i, \zeta^j) \ge \alpha^*$ for $1 \le i < j \le n$. We assume the following:

1) $y + s\zeta^i \in (\hat{\Omega} \cup \partial\hat{\Omega})$ for $y \in (\partial\hat{\Omega} \cap B_{R_0})$, $0 < s < R_0$, and $1 \le i \le n$.

2) If $\hat{\Gamma}_{\mathcal{D}} \cap B_{R_0} \neq \emptyset$, then ζ^i $\;(i \in \Lambda_1)$ is parallel to $\hat{\Gamma}_{\mathcal{D}} \cap B_{R_0}$.

3) If $\hat{\Gamma}_{\mathcal{D}} \cap B_{R_0} = \emptyset$, then $\Lambda_1 = \{1, \dots, n\}$.

4) If $i \in \Lambda_1$, $y \in (\hat{\Gamma}_{\mathcal{D}} \cap B_{R_0})$, $s > 0$, and $y + s\zeta^i \in B_{R_0}$, then $y + s\zeta^i \in \hat{\Gamma}_{\mathcal{D}}$.

5) If $\hat{\Gamma}_{\mathcal{N}} \cap B_{R_0} \neq \emptyset$, then ζ^i $\;(i \in \Lambda_2)$ is parallel to $\hat{\Gamma}_{\mathcal{N}} \cap B_{R_0}$.

6) If $\hat{\Gamma}_{\mathcal{N}} \cap B_{R_0} = \emptyset$, then $\Lambda_2 = \{1, \dots, n\}$.

7) ζ^i $\;(i \in \Lambda_2)$ satisfies

i) angle$(\zeta^i, \hat{\Gamma}_{\mathcal{D}} \cap B_{R_0}) \ge \alpha^*$

ii) $y - s\zeta^i \notin (\hat{\Omega} \cup \partial\hat{\Omega})$ for $y \in (\hat{\Gamma}_{\mathcal{D}} \cap B_{R_0})$, and $0 < s < R_0$.

8) If angle$(\hat{\Gamma}_i, \hat{\Gamma}_j) = \pi$ $\;(i \neq j, \hat{\Gamma}_i \cap \hat{\Gamma}_j \cap B_{R_0} \neq \emptyset)$, then $\Lambda_3 = \{n\}$, otherwise $\Lambda_3 = \emptyset$.

9) ζ^n $\;(n \in \Lambda_3)$ satisfies angle$(\zeta^n, (\hat{\Gamma}_i \cup \hat{\Gamma}_j) \cap B_{R_0}) \ge \alpha^*$ where i, j are given in Assumption 8).

Remark.

i) Let us note that there is such a basis. Some examples how to choose the basis vectors are given in Section 4.

ii) We can find a constant α^* depending only on n and on the geometry of $\partial\Omega$.

In the sequel let $h > 0$. We define $E_i^\sigma y = y + \sigma\zeta^i$, $E_i^\sigma f(y) = f(y + \sigma\zeta^i)$,

$$D_i^h f(y) = \frac{E_i^h f(y) - f(y)}{h} \qquad \text{and} \qquad D_i^{-h} f(y) = \frac{f(y) - E_i^{-h} f(y)}{h}$$

and we will write $E_i^\sigma f(y)g(y)$ instead of $(E_i^\sigma f(y))g(y)$.

We set $R = \frac{R_0}{8}$, $B = B_R \cap \hat{\Omega}$, $B' = B_{4R} \cap \hat{\Omega}$, and

$$\hat{\Omega}_i^h = \Big\{ y \in B_{R_0} : y \neq x + h\zeta^i, x \in B_{R_0} \Big\}$$

$$\hat{\Omega}_i^{-h} = \Big\{ y \in B_{R_0} \setminus \hat{\Omega} : y = x - h\zeta^i, x \in B_{R_0} \cap \hat{\Omega} \Big\}.$$

Let τ_0 be a cut-off function with $\tau_0 \equiv 1$ in B, $\operatorname{supp} \tau_0 = B_{4R}$, and $|\nabla\tau_0| \leq c$, where c depends only on R_0. By τ we denote the restriction of τ_0 onto $\hat{\Omega} \cup \partial\hat{\Omega}$.

Moreover, we need appropriate extensions of functions into $\hat{\Omega}_i^{-h}$ for $i \in \Lambda_2$. Let the function $g(y)$ be defined on $\hat{\Omega}$. Let $z_0 \in \partial\hat{\Omega} \cap B_{R_0}$ and $z_0 - \lambda\zeta^i \in \hat{\Omega}_i^{-h}$ for $0 < \lambda \leq h$. Then we set

$$g(z_0 - \lambda\zeta^i) = g(z_0 + \lambda\zeta^i). \tag{2.1}$$

This is an $W^{1,2}$-extension if $g \in W^{1,2}(\hat{\Omega})$. In particular, it holds that $\|g\|_{W^{1,2}(\hat{\Omega}_i^{-h})} \leq c\|g\|_{W^{1,2}(\hat{\Omega})}$, where the constant c depends only on the data, for α^* depends only on n and on the geometry of $\partial\Omega$.

Next, we define an appropriate extension of $v = u \circ (\phi^*)^{-1}$ into $\hat{\Omega}_i^{-h}$ for $i \in \Lambda_2$. Let $y \in \partial\hat{\Omega} \cap \partial\hat{\Omega}_i^{-h}$, $0 < \lambda \leq h$, and $y - \lambda\zeta^i \in \hat{\Omega}_i^{-h}$. We set

$$v(y - \lambda\zeta^i) = 0. \tag{2.2}$$

This provides an $W^{1,2}$-extension of v, for $i \in \Lambda_2$ implies that $(\partial\hat{\Omega} \cap \partial\hat{\Omega}_i^{-h}) \subset \hat{\Gamma}_{\mathcal{D}}$. In particular, it holds for $1 \leq r \leq N$ that

$$\|v^r\|_{\mathcal{H}^{\frac{3}{2},2}(\hat{\Omega}_i^{-h})} \leq c\|v^r\|_{\mathcal{H}^{\frac{3}{2},2}(\hat{\Omega})}$$

where c and c' depend only on the data and v^r is the r-th component of v. Thus, extension (2.2) is an $\mathcal{H}^{\frac{3}{2},2}$-extension (cf. [8]).

In what follows we will write $\sum_{i,k,l}$ and $\sum_{r,s}$ instead of $\sum_{i,k,l=1}^n$ and $\sum_{r,s=1}^N$, respectively. Further, ∇v is an $\mathbb{R}^{nN}$-vector and $|\nabla v|^2 = \sum_r \sum_i |\partial_i v^r|^2$. The point $\cdot$ denotes the Euclidean scalar product and c denotes a constant which will be allowed to vary from equation to equation.

3. The regularity of the solution

In this section we prove Theorem 1.1 and Theorem 1.2.

Let A be the matrix whose elements are defined by $a_{ik} = \frac{\partial}{\partial x_i}(\phi^{*k})$, where ϕ^{*k} denotes the k-th component of $\phi^*(x)$. Let $y = \hat{x}$. In the sequel we only deal with functions defined onto $\hat{\Omega}$. For simplicity we will write $f(y)$ instead of $f((\phi^*)^{-1}(y))$ etc. The function $v = u \circ (\phi^*)^{-1}$ is the weak solution of

$$-\sum_i \widetilde{\partial}_i F_i(y, \widetilde{\nabla} v) = f(y) + \sum_i \widetilde{\partial}_i f_i(y) \tag{3.1}$$

where $\tilde{\partial}_i v(y) = \sum_k a_{ik}(y) \partial_k v(y)$.

In detail, A is positive definite, the smallest eigenvalue $\lambda_0 > 0$ depends only on the geometry of $\partial\Omega$, and

$$a_{ik}(y) \in W^{1,\infty}(\hat{\Omega}) \tag{3.2}$$

holds. Further, let us note that $v(y) \in [W^{1,2}(\hat{\Omega})]^N$.

We need several propositions.

Proposition 3.1. *It holds that*

$$\sup_{0<h<4R} \int_{B'} \tau h |D_i^h \nabla v|^2 dy \le c \qquad \textit{for} \;\; i \in \Lambda_1 \tag{3.3}$$

where the constant c depends only on R_0 and the data.

Proof. Let $0 < h < 4R$. First, we suppose that $1 \in \Lambda_1$ and we prove (3.3) for $i = 1$. The Taylor expansion of $F(y,p)$ $(p \in \mathbb{R}^{nN})$ entails

$$\begin{aligned} \sum_r \sum_i (p'-p)_i^r F_i^r(y,p) &= F(y,p') - F(y,p) \\ &- \sum_{r,s} \sum_{i,k} (p'-p)_i^r (p'-p)_k^s \int_0^1 (1-t) F_{i,k}^{rs}\big(y, tp' + (1-t)p\big)\, dt. \end{aligned} \tag{3.4}$$

Let

$$m_{ik}^{rs}(h) = \int_0^1 (1-t) F_{i,k}^{rs}(y, tE_1^h \widetilde{\nabla} v + (1-t)\widetilde{\nabla} v)\, dt$$

for $1 \le i,k \le n$ and $1 \le r,s \le N$. We set $p = \widetilde{\nabla} v$ and $p' = E_1^h \widetilde{\nabla} v$. Thus, $(p'-p)_i^r = hD_1^h \tilde{\partial}_i v^r \equiv \sum_l hD_1^h(a_{il}\partial_l v^r)$ and

$$\begin{aligned} \sum_r \sum_{i,l} F_i^r(y, \widetilde{\nabla} v) D_1^h(a_{il}\partial_l v^r) &= \frac{F(y, E_1^h \widetilde{\nabla} v) - F(y, \widetilde{\nabla} v)}{h} \\ &- \sum_{r,s} \sum_{i,k} h \left(\sum_l D_1^h(a_{il}\partial_l v^r) \right) \left(\sum_l D_1^h(a_{kl}\partial_l v^s) \right) m_{ik}^{rs}(h). \end{aligned} \tag{3.5}$$

The function $\varphi = \tau D_1^h v$ is an admissible test function. Multiplying (3.1) by φ yields

$$\sum_{i,l} \int_{B'} F_i(y, \widetilde{\nabla} v) \cdot \partial_l(a_{il}\tau) D_1^h v + \sum_{i,l} \int_{B'} F_i(y, \widetilde{\nabla} v) \cdot (a_{il}\tau) \partial_l D_1^h v$$

$$= \int_{B'} \tau f \cdot D_1^h v - \sum_{i,l} \int_{B'} f_i \cdot \partial_l \left(a_{il} \tau D_1^h v \right)$$

where the point $\cdot$ denotes the Euclidean scalar product in $\mathbb{R}^N$. Applying (3.5) we obtain

$$\begin{aligned}
(I) &= \int_{B'} \tau \sum_{r,s} \sum_{i,k} h \left(\sum_l D_1^h (a_{il} \partial_l v^r) \right) \left(\sum_l D_1^h (a_{kl} \partial_l v^s) \right) m_{ik}^{rs}(h) \\
&= \int_{B'} \tau \frac{F(y, E_1^h \widetilde{\nabla} v) - F(y, \widetilde{\nabla} v)}{h} - \sum_{i,l} \int_{B'} \tau F_i(y, \widetilde{\nabla} v) \cdot D_1^h a_{il} \partial_l E_1^h v \\
&\quad + \sum_{i,l} \int_{B'} F_i(y, \widetilde{\nabla} v) \cdot \partial_l (a_{il}\tau) D_1^h v - \int_{B'} \tau f \cdot D_1^h v + \sum_{i,l} \int_{B'} f_i \cdot \partial_l \left(a_{il} \tau D_1^h v \right) \\
&= (II) + \ldots + (VI).
\end{aligned}$$

The identity $D_1^h (g\tilde{g}) = D_1^h g E_1^h \tilde{g} + g D_1^h \tilde{g}$ yields

$$\begin{aligned}
(I) = \int_{B'} \tau \sum_{r,s} \sum_{i,k} h & \left(\sum_l \left(D_1^h a_{il} \partial_l E_1^h v^r + a_{il} D_1^h \partial_l v^r \right) \right) \\
& \times \left(\sum_l \left(D_1^h a_{kl} \partial_l E_1^h v^s + a_{kl} D_1^h \partial_l v^s \right) \right) m_{ik}^{rs}(h).
\end{aligned}$$

By (3.2) and hypothesis (H5) it follows that

$$\left| \int_{B'} \tau \sum_{r,s} \sum_{i,k} h \left(\sum_l D_1^h a_{il} \partial_l E_1^h v^r \right) \left(\sum_l D_1^h a_{kl} \partial_l E_1^h v^s \right) m_{ik}^{rs}(h) \right|$$

$$\leq ch \|\nabla E_1^h v\|_{L^2(B')}^2$$

and

$$\left| \int_{B'} \tau \sum_{r,s} \sum_{i,k} h \left(\sum_l D_1^h a_{il} \partial_l E_1^h v^r \right) \left(\sum_l a_{kl} D_1^h \partial_l v^s \right) m_{ik}^{rs}(h) \right|$$

$$\leq \frac{ch}{\eta} \|\nabla E_1^h v\|_{L^2(B')}^2 + \eta h \int_{B'} \tau \left| D_1^h \nabla v \right|^2$$

for $\eta > 0$. Hypothesis (H6) entails

$$
\begin{aligned}
&\int_{B'} \tau \sum_{r,s} \sum_{i,k} h \left(\sum_l a_{il} D_1^h \partial_l v^r \right) \left(\sum_l a_{kl} D_1^h \partial_l v^s \right) m_{ik}^{rs}(h) \\
&\quad \geq \frac{k_0}{2} \int_{B'} \tau \sum_r \sum_i h \left(\sum_l a_{il} D_1^h \partial_l v^r \right)^2 \\
&\quad = \frac{k_0}{2} \int_{B'} \tau \sum_r h D_1^h \nabla v^r \cdot \left(A^T A\right) D_1^h \nabla v^r \\
&\quad \geq \frac{k_0 \lambda_0^2}{2} \int_{B'} \tau h \left| D_1^h \nabla v \right|^2 .
\end{aligned}
$$

Altogether we obtain

$$
(I) \geq c \int_{B'} \tau h \left| D_1^h \nabla v \right|^2 - ch
$$

for a sufficiently small $\eta > 0$. Further, using Taylor expansion and summation by parts we get

$$
\begin{aligned}
(II) &= \int_{B'} \tau \frac{F(y, E_1^h \widetilde{\nabla} v) - F(E_1^h y, E_1^h \widetilde{\nabla} v)}{h} + \int_{B'} \tau D_1^h F(y, \widetilde{\nabla} v) \\
&= \int_{B'} \tau \sum_k \zeta^{1k} \int_0^1 F_{x_k} \Big(ty + (1-t) E_1^h y, E_1^h \widetilde{\nabla} v \Big) dt dy \\
&\quad + \int_{B'} D_1^h (\tau F(y, \widetilde{\nabla} v)) - \int_{B'} D_1^h \tau F(E_1^h y, E_1^h \widetilde{\nabla} v) \\
&= (II)_1 + (II)_2 + (II)_3
\end{aligned}
$$

where ζ^{1k} denotes the k-th component of the basis vector ζ^1. Hypotheses (H2) and (H1) entail

$$
\begin{aligned}
|(II)_1| &\leq c \left(\sum_k \sup_{0 \leq t \leq 1} \| g_{x_k}(y + th\zeta^1) \|_{L^1(B')} + \| E_1^h \widetilde{\nabla} v \|_{L^2(B')}^2 \right) \leq c \\
(II)_2 &= -h^{-1} \int_{\hat{\Omega}_1^h} \tau F(y, \widetilde{\nabla} v) \qquad (3.6) \\
|(II)_3| &\leq c \int_{B'} \left(\left| E_1^h g_0 \right| + \left| E_1^h \widetilde{\nabla} v \right|^2 \right) \leq c.
\end{aligned}
$$

By (3.2) and hypotheses (H3) and (H7) we get

$$
\begin{aligned}
|(III)| &\leq c \left(\sum_i \| g_i \|_{L^2(B')}^2 + \| \widetilde{\nabla} v \|_{L^2(B')}^2 + \| \nabla E_1^h v \|_{L^2(B')}^2 \right) \leq c \\
|(IV)| &\leq c \left(\sum_i \| g_i \|_{L^2(B')}^2 + \| \widetilde{\nabla} v \|_{L^2(B')}^2 + \| D_1^h v \|_{L^2(B')}^2 \right) \leq c \\
|(V)| &\leq c \left(\| f \|_{L^2(B')}^2 + \| D_1^h v \|_{L^2(B')}^2 \right) \leq c.
\end{aligned}
$$

Next, summation by parts yields

$$(VI) = \sum_{i,l} \int_{B'} f_i \cdot \partial_l(\tau a_{il}) D_1^h v - \sum_{i,l} \int_{B'} D_1^h(\tau a_{il} f_i) \cdot \partial_l E_1^h v + \sum_{i,l} \int_{B'} D_1^h(\tau a_{il} f_i \cdot \partial_l v)$$

$$= (VI)_1 + (VI)_2 + (VI)_3.$$

Due to (3.2) and hypothesis (H7) we obtain

$$|(VI)_1| \le c \left(\sum_i \|f_i\|^2_{L^2(B')} + \|D_1^h v\|^2_{L^2(B')} \right) \le c$$

$$|(VI)_2| \le c \left(\sum_i \|f_i\|^2_{L^2(B')} + \sum_i \|D_1^h f_i\|^2_{L^2(B')} + \|\nabla E_1^h v\|^2_{L^2(B')} \right) \le c.$$

Applying hypothesis (H1) we get for $\eta > 0$

$$\begin{aligned} |(VI)_3| &= \left| \frac{1}{h} \sum_{i,l} \int_{\hat\Omega_1^h} \tau a_{il} f_i \cdot \partial_l v \right| \\ &\le \frac{c}{\eta h} \left|\hat\Omega_1^h\right| \sum_i \|f_i\|^2_{L^\infty(\hat\Omega_1^h)} + \frac{\eta}{h} \int_{\hat\Omega_1^h} \tau |\widetilde\nabla v|^2 \qquad (3.7) \\ &\le c + \frac{\eta}{c_0' h} \int_{\hat\Omega_1^h} \tau F(y, \widetilde\nabla v). \end{aligned}$$

Let $\eta = \frac{c_0'}{2}$. Then (3.6), (3.7), and hypothesis (H1) yield

$$(II)_2 + |(VI)_3| \le c - \frac{1}{2h} \int_{\hat\Omega_1^h} \tau F(y, \widetilde\nabla v) \le c - \frac{c_0}{2h} \left|\hat\Omega_1^h\right| \le c.$$

Altogether we obtain assertion (3.3) for $i = 1$. Finally, let us note that the proof of (3.3) for arbitrary $i \in \Lambda_1$ follows in the same way ∎

Proposition 3.2. *There exists a constant c depending only on R_0 and the data such that*

$$\sup_{0<h<4R} \int_{B'} \tau h \left|D_i^{-h} \nabla v\right|^2 dy \le c \qquad \textit{for } \ i \in \Lambda_2. \qquad (3.8)$$

Proof. Let $0 < h < 4R$. We give the proof of (3.8) for some fixed number $i \in \Lambda_2$, say $i = 1$.

First, we extend v into $\hat\Omega_1^{-h}$ by using (2.2), and the functions $F(\cdot, p)$, g_0, τ, a_{ik} $(1 \le i, k \le n)$ by using (2.1). Now, let us verify that $\varphi = -\tau D_1^{-h} v$ is an admissible test function. The conditions on ζ^i $(i \in \Lambda_2)$ imply that $y - h\zeta^1 \notin \hat\Omega \cup \partial\hat\Omega$ for $y \in \hat\Gamma_{\mathcal D} \cap B'$. Hence, the extension (2.2) yield

$$v(y - h\zeta^1) = 0 \qquad \text{for } y \in \hat\Gamma_{\mathcal D} \cap B',$$

thus

$$\varphi(y) = \tau h^{-1}\big(v(y-h\zeta^1) - v(y)\big) = 0 \qquad \text{for } y \in \hat{\Gamma}_{\mathcal{D}} \cap B'.$$

Multiplying (3.1) by φ and integrating over $\hat{\Omega}$ we get

$$\begin{aligned}
-\int_{B'} \tau f \cdot D_1^{-h} v + \sum_{i,l} \int_{B'} f_i \cdot \partial_l(a_{il}\tau D_1^{-h} v) + \sum_{i,l} \int_{B'} F_i(y, \widetilde{\nabla} v) \cdot \partial_l(a_{il}\tau) D_1^{-h} v \\
= -\sum_{i,l} \int_{B'} F_i(y, \widetilde{\nabla} v) \cdot (\tau a_{il}) \partial_l D_1^{-h} v \\
= \sum_{i,l} \int_{B'} \tau F_i(y, \widetilde{\nabla} v) \cdot \big[-D_1^{-h}(a_{il}\partial_l v) + D_1^{-h} a_{il} E_1^{-h} \partial_l v \big]
\end{aligned} \tag{3.9}$$

where we have used the identity $D_1^{-h}(g\tilde{g}) = D_1^{-h} g E_1^{-h} \tilde{g} + g D_1^{-h} \tilde{g}$. The Taylor expansion of $F(y, \cdot)$ yields

$$\begin{aligned}
\sum_r \sum_i (p'-p)_i^r F_i^r(y,p) \\
= F(y,p') - F(y,p) \\
- \sum_{r,s} \sum_{i,k} (p'-p)_i^r (p'-p)_k^s \int_0^1 (1-t) F_{i,k}^{rs}\big(y, tp' + (1-t)p\big)\, dt.
\end{aligned}$$

We set

$$m_{ik}^{rs}(-h) = \int_0^1 (1-t) F_{i,k}^{rs}\big(y, tE_1^{-h}\widetilde{\nabla} v + (1-t)\widetilde{\nabla} v\big)\, dt$$

for $1 \le i,k \le n$ and $1 \le r,s \le N$. Let us put $p = \widetilde{\nabla} v$ and $p' = E_1^{-h}\widetilde{\nabla} v$. Then we obtain

$$\begin{aligned}
-\sum_r \sum_{i,l} F_i^r(y, \widetilde{\nabla} v) D_1^{-h}\,(a_{il}\partial_l v^r) \\
= \tfrac{1}{h}\big(F(y, E_1^{-h}\widetilde{\nabla} v) - F(y, \widetilde{\nabla} v)\big) \\
- \sum_{r,s} \sum_{i,k} h \Big(\sum_l D_1^{-h}(a_{il}\partial_l v^r) \Big) \Big(\sum_l D_1^{-h}(a_{kl}\partial_l v^s) \Big) m_{ik}^{rs}(-h).
\end{aligned}$$

Thus, (3.9) yields

$$(I) = \int_{B'} \tau h \sum_{r,s} \sum_{i,k} \Big(\sum_l D_1^{-h}(a_{il}\partial_l v^r) \Big) \Big(\sum_l D_1^{-h}(a_{kl}\partial_l v^s) \Big) m_{ik}^{rs}(-h)$$

$$
\begin{aligned}
&= \int_{B'} \tau h^{-1}\big(F(y, E_1^{-h}\widetilde{\nabla} v) - F(y, \widetilde{\nabla} v)\big) \\
&\quad + \sum_{i,l} \int_{B'} \tau F_i(y, \widetilde{\nabla} v) \cdot D_1^{-h} a_{il} \partial_l E_1^{-h} v \\
&\quad - \sum_{i,l} \int_{B'} F_i(y, \widetilde{\nabla} v) \cdot \partial_l (a_{il}\tau) D_1^{-h} v \\
&\quad + \int_{B'} \tau f \cdot D_1^{-h} v \\
&\quad - \sum_{i,l} \int_{B'} f_i \cdot \partial_l (a_{il} \tau D_1^{-h} v) \\
&= (III) + \ldots + (VI).
\end{aligned}
$$

Hypothesis (H6) entails

$$
(I) \geq \frac{k_0}{2} \int_{B'} \tau h D_1^{-h} \widetilde{\nabla} v \cdot D_1^{-h} \widetilde{\nabla} v = \frac{k_0}{2} \int_{B'} \tau h \sum_r \left| D_1^{-h}(A \nabla v^r) \right|^2 .
$$

We use

$$
\int_{B'} \tau h A D_1^{-h} \nabla v^r \cdot A D_1^{-h} \nabla v^r \geq \lambda_0^2 \int_{B'} \tau h \left| D_1^{-h} \nabla v^r \right|^2
$$

$$
\int_{B'} \tau h (D_1^{-h} A) \nabla E_1^{-h} v^r \cdot (D_1^{-h} A) \nabla E_1^{-h} v^r \leq c \int_{B'} \tau h \left| \nabla E_1^{-h} v^r \right|^2 \leq c
$$

$$
2 \int_{B'} \tau h (D_1^{-h} A) \nabla E_1^{-h} v^r \cdot A D_1^{-h} \nabla v^r \leq \frac{c}{\eta} \int_{B'} \tau h \left| \nabla E_1^{-h} v^r \right|^2 + \eta \int_{B'} \tau h \left| D_1^{-h} \nabla v^r \right|^2
$$

for $\eta > 0$. Putting $\eta = \frac{k_0 \lambda_0^2}{4}$ it follows that

$$
(I) \geq \frac{k_0 \lambda_0^2}{4} \int_{B'} \tau h \left| D_1^{-h} \nabla v \right|^2 - c.
$$

Next,

$$
\begin{aligned}
(II) &= - \int_{B'} \tau D_1^{-h} F(y, \widetilde{\nabla} v) + \int_{B'} \tau h^{-1} \Big(F(y, E_1^{-h} \widetilde{\nabla} v) - F(E_1^{-h} y, E_1^{-h} \widetilde{\nabla} v) \Big) \\
&= (II)_1 + (II)_2.
\end{aligned}
$$

Summation by parts entails

$$
\begin{aligned}
(II)_1 &= - \int_{B' \cup B''} \tau D_1^{-h} F(y, \widetilde{\nabla} v) \\
&= - \int_{B' \cup B''} D_1^{-h} \big(\tau F(y, \widetilde{\nabla} v) \big) + \int_{B' \cup B''} D_1^{-h} \tau F(E_1^{-h} y, E_1^{-h} \widetilde{\nabla} v) \\
&= (II)_{11} + (II)_{12}
\end{aligned}
$$

where

$$B'' = \Big\{ y \in B_{R_0} \setminus B' : y = x + h\zeta^1, x \in B' \Big\}.$$

The extensions (2.1) and (2.2) entail

$$|(II)_{11}| = \tfrac{1}{h} \left| \int_{\hat\Omega_1^{-h}} \tau F(y, \widetilde\nabla v) \right| \le \tfrac{1}{h} \int_{\hat\Omega_1^{-h}} |g_0| \le \|g_0\|_{L^\infty(\hat\Omega_1^h)} \tfrac{1}{h} |\hat\Omega_1^{-h}| \le c.$$

Further, using hypothesis (H1) we obtain

$$|(II)_{12}| \le c \int_{B'} |F(E_1^{-h} y, E_1^{-h} \widetilde\nabla v)| \le c \int_{B'} \left(|E_1^{-h} g_0| + |E_1^{-h} \widetilde\nabla v|^2 \right) \le c.$$

Let ζ^{1k} be the k-th component of the basis vector ζ^1. Hypothesis (H2) and the Taylor expansion entail

$$\begin{aligned}
|(II)_2| &\le \int_{B'} \tau \sum_k |\zeta^{1k}| \int_0^1 \left| F_{x_k}(ty + (1-t) E_1^{-h} y, E_1^{-h} \widetilde\nabla v) \right| dt dy \\
&\le c \bigg(\sum_k \sup_{0 \le t \le 1} \|g_{x_k}(y - th\zeta^1)\|_{L^1(B')} + \|E_1^{-h} \widetilde\nabla v\|^2_{L^2(B')} \bigg) \\
&\le c.
\end{aligned}$$

By (3.2) and Hypotheses (H3) and (H7) we get

$$\begin{aligned}
|(III)| &\le c \bigg(\sum_i \|g_i\|^2_{L^2(B')} + \|\widetilde\nabla v\|^2_{L^2(B')} + \|\nabla E_1^{-h} v\|^2_{L^2(B')} \bigg) \le c \\
|(IV)| &\le c \bigg(\sum_i \|g_i\|^2_{L^2(B')} + \|\widetilde\nabla v\|^2_{L^2(B')} + \|D_1^{-h} v\|^2_{L^2(B')} \bigg) \le c \\
|(V)| &\le c \left(\|f\|^2_{L^2(B')} + \|D_1^{-h} v\|^2_{L^2(B')} \right) \le c.
\end{aligned}$$

Next,

$$\begin{aligned}
(VI) &= -\sum_{i,l} \int_{B'} f_i \cdot \partial_l (a_{il} \tau) D_1^{-h} v - \sum_{i,l} \int_{B'} \tau a_{il} f_i \cdot D_1^{-h} \partial_l v \\
&= (VI)_1 + (VI)_2.
\end{aligned}$$

Due to (3.2) and Hypothesis (H1)

$$|(VI)_1| \le c \bigg(\sum_i \|f_i\|^2_{L^2(B')} + \|D_1^{-h} v\|^2_{L^2(B')} \bigg) \le c$$

follows. Using summation by parts we obtain

$$\begin{aligned}
(VI)_2 &= -\sum_{i,l} \int_{B' \cup B''} \tau a_{il} f_i \cdot D_1^{-h} \partial_l v \\
&= \sum_{i,l} \int_{B' \cup B''} D_1^{-h} (\tau a_{il} f_i) \partial_l E_1^{-h} v - \sum_{i,l} \int_{B' \cup B''} D_1^{-h} (\tau a_{il} f_i \partial_l v) \\
&= (VI)_3 + (VI)_4.
\end{aligned}$$

In view of hypothesis (H7) we get

$$\begin{aligned}|(VI)_3| &= \sum_{i,l}\int_{B'}\Big(D_1^{-h}(\tau a_{il})f_i + E_1^{-h}(\tau a_{il})D_1^{-h}f_i\Big)\partial_l E_1^{-h}v\\ &\le c\Big(\sum_i \|f_i\|^2_{L^2(B')} + \sum_i \|D_1^{-h}f_i\|^2_{L^2(B')} + \|\nabla E_1^{-h}v\|^2_{L^2(B')}\Big)\\ &\le c.\end{aligned}$$

The extension (2.2) yields $\partial_l v = 0$ in $\hat\Omega_1^{-h}$. This implies that

$$(VI)_4 = \frac{1}{h}\sum_{i,l}\int_{\hat\Omega_1^{-h}} \tau a_{il} f_i \partial_l v = 0.$$

Thus, the assertion follows ∎

Proposition 3.3. *Let $\Lambda_3 = \{n\}$ and $0 < \delta < \frac{1}{2}$. Then there exists a constant c depending only on R_0, δ, and the data such that*

$$\sup_{0<h<4R}\int_B h^{1+\delta}|D_n^h \nabla v|^2 dy \le c. \tag{3.10}$$

The proof of this proposition follows as in [8] using (3.1), (3.3), (3.8), and Fourier series.

Now, we are able to prove the main results.

Proof of Theorem 1.1. a) Recall that $\Omega_\eta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \eta\}$ and note that the basis vectors ζ^i fulfil angle$(\zeta^i, \zeta^j) \ge \alpha^*$ for $1 \le i < j \le n$, where the constant α^* depends only on the geometry of $\partial\Omega$. It holds that $\tau \equiv 1$ in B. Thus, (3.3), (3.8), and (3.10) yield for all $\delta \in (0, \frac{1}{2})$

$$\sup_{\substack{\eta>0\\ 0<|z|<\eta}}\int_{((\phi^*)^{-1}(B))_\eta}\frac{|\nabla u(x+z) - \nabla u(x)|^2}{|z|^{1-\delta}}\,dx \le c \tag{3.11}$$

where the constant c depends only on the data, δ, and on R_0. Further, let us note that R_0 depends only on the shape of $\partial\Omega$.

Next, there are a finite set of points $\{\hat P_1, \dots, \hat P_k\}$ and a set of balls $B_{R_i}(\hat P_i)$ such that

$$\partial\Omega \subset \bigcup_{i=1}^k (B^i \cap \partial\Omega), \qquad \text{where } B^i = (\phi^*)^{-1}(B_{R_i}(\hat P_i)),$$

and $\hat P_i$ is the only vertex of $\partial\hat\Omega$ in $B_{R_i}(\hat P_i)$ or $B_{R_i}(\hat P_i) \cap \partial\hat\Omega$ contains no vertex of $\partial\hat\Omega$. Further, the radii R_i $(1 \le i \le k)$ depend only on the data, for they are determined by the geometry of Ω. Thus,

$$u \in \big[\mathcal{H}^{\frac{3}{2}-\frac{\delta}{2},2}(\Omega)\big]^N \qquad \text{for } \delta \in \big(0, \tfrac{1}{2}\big)$$

follows. The imbedding theorem of Nikolskii spaces into Sobolev spaces (cf. [1])

$$\mathcal{H}^{s,p}(\Omega) \to W^{s-\varepsilon,p}(\Omega) \qquad \text{for } \varepsilon > 0$$

entails $u \in [W^{s,2}(\Omega)]^N$ for all $s < \frac{3}{2}$. This yields assertion (1.1).

b) Using (3.3) and (3.8) we get (3.11) for $\delta = 0$. Proceeding as above we obtain $u \in \big[\mathcal{H}^{\frac{3}{2},2}(\Omega)\big]^N$ ∎

Proof of Theorem 1.2. We only sketch the proof. Assumption (1.3) yields $f \in L^q(\Omega)$ and $f_i \in L^{2q}(\Omega)$ for some $q > \frac{n}{2}$. Now, $N = 1$ holds. Following [13] we see that $u \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha > 0$. Thus, we can proceed as in [8]. The Hölder continuity and the equation yield

$$\int_{B_r(y_0)\cap\hat{\Omega}} \frac{|\nabla v(y)|^2}{|y-y_0|^{n-2+2\varepsilon}}\, dy \le c$$

for some $\varepsilon > 0$. Replacing the test functions φ by $r^{-\varepsilon}\varphi$ in Propositions 3.1 and 3.2 and recalling the proof of Proposition 3.3 we get

$$\int_{B_r(\hat{P})\cap\hat{\Omega}} r^{3-\varepsilon-n} \left| h^{\frac{1+\delta}{2}} D_i^h \nabla v \right|^2 \le c$$

for $1 \le i \le n$, $0 < r \le \frac{R_0}{8}$ and $0 < \delta < \frac{1}{2}$. Applying an imbedding theorem of Morrey-Nikolskii type we obtain the assertion ∎

4. Examples

In this section we give some explicit examples of the index sets Λ_1, Λ_2, Λ_3, and the basis vectors $\zeta^1, \dots, \zeta^n$.

Let $\Omega \subset \mathbb{R}^3$ be a polyhedron. We consider three typical situations: an edge of $\partial\Omega$ (Example 1), the case when angle$(\Gamma_{\mathcal{D}}, \Gamma_{\mathcal{N}}) = \pi$ (Example 2), and a corner point (Example 3).

Let $P = (0,0,0)^T$, $B_{R_0} = \{y : |y| < \frac{1}{2}\}$, and let e_k $(1 \le k \le 3)$ be the k-th unit vector in $\mathbb{R}^3$.

Example 1. Let

$$\Gamma_*^1 = \Big\{ y \in B_{R_0} : y_1 = 0, y_3 > 0 \Big\}$$
$$\Gamma_*^2 = \Big\{ y \in B_{R_0} : y_3 = 0, y_1 > 0 \Big\}$$

and

$$\Omega \cap B_1 = \Big\{ y \in B_1 : y_1 > 0, y_3 > 0 \Big\}.$$

Case 1: $\Gamma_{\mathcal{D}} \cap B_{R_0} = \overline{\Gamma_*^1}$ and $\Gamma_{\mathcal{N}} \cap B_{R_0} = \Gamma_*^2$. Let us put $\zeta^1 = e_2$ and $\zeta^2 = e_3$. Then ζ^1 and ζ^2 are parallel to $\Gamma_{\mathcal{D}} \cap B_{R_0}$, thus, $\Lambda_1 = \{1,2\}$. Next, we put $\Lambda_2 = \{3\}$. We must choose ζ^3 such that ζ^3 is parallel to $\Gamma_{\mathcal{N}} \cap B_{R_0}$ and angle$(\zeta^3, \Gamma_{\mathcal{D}} \cap B_{R_0}) \ge \alpha^*$ for some suitable large constant $\alpha^* > 0$ (i.e., $\alpha^* \sim$ angle(Γ_*^1, Γ_*^2)). Thus, let $\zeta^3 = e_3$.

Case 2: $\Gamma_{\mathcal{D}} \cap B_{R_0} = \emptyset$ and $\Gamma_{\mathcal{N}} \cap B_{R_0} = \overline{\Gamma_*^1} \cup \overline{\Gamma_*^2}$. It holds that $\Lambda_1 = \{1,2,3\}$. We must choose ζ^i $(1 \le i \le 3)$ such that

i) $y + s\zeta^i \in \overline{\Omega}$ for $y \in \partial\Omega \cap B_{R_0}$ and $0 < s < R_0$

ii) angle$(\zeta^i, \zeta^j) \ge \alpha^*$ for $1 \le i < j \le 3$ and some suitable constant $\alpha^* > 0$.

Thus, let $\zeta^i = e_i$ for $1 \le i \le 3$.

Case 3: $\Gamma_{\mathcal{D}} \cap B_{R_0} = \overline{\Gamma_*^1} \cup \overline{\Gamma_*^2}$ and $\Gamma_{\mathcal{N}} \cap B_{R_0} = \emptyset$. Now, it holds that $\Lambda_2 = \{1,2,3\}$. The basis vectors ζ^i $(1 \le i \le 3)$ must fulfil

i) $y + s\zeta^i \in \overline{\Omega}$ for $y \in \partial\Omega \cap B_{R_0}$ and $0 < s < R_0$

ii) $\text{angle}(\zeta^i, \zeta^j) \geq \alpha^*$ for $1 \leq i < j \leq 3$ and $\alpha^* > 0$

iii) $\text{angle}(\zeta^i, \Gamma_{\mathcal{D}} \cap B_{R_0}) \geq \alpha^*$

where $\alpha^* > 0$ is suitable. Thus, let $\zeta^1 = \frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_3$, $\zeta^2 = \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_3$, and $\zeta^3 = \frac{1}{3}e_2 + \frac{2}{3}(e_1 + e_3)$.

Example 2. Let

$$\Omega \cap B_{R_0} = \left\{ y \in B_{R_0} : y_3 > 0 \right\}$$

and

$$\Gamma_{\mathcal{D}} \cap B_{R_0} = \left\{ y \in B_{R_0} : y_3 = 0, y_1 \geq 0 \right\}$$
$$\Gamma_{\mathcal{N}} \cap B_{R_0} = \left\{ y \in B_{R_0} : y_3 = 0, y_1 < 0 \right\}.$$

We choose $\zeta^1 = e_1$ and $\zeta^2 = e_2$. Then $y + s\zeta^i \in \Gamma_{\mathcal{D}} \cap B_{R_0}$ holds for $y \in \Gamma_{\mathcal{D}} \cap B_{R_0}$, $s > 0$, and $y + s\zeta^i \in B_{R_0}$. Thus, $\Lambda_1 = \{1, 2\}$. Further, $\Lambda_2 = \emptyset$ and $\Lambda_3 = \{3\}$. Let us put $\zeta^3 = e_3$.

Example 3. Let $\Omega = [0, 1]^3$.

Case 1: $\Gamma_{\mathcal{D}} = \{y \in \partial\Omega : y_3 = 0\}$ and $\Gamma_{\mathcal{N}} = \partial\Omega \setminus \Gamma_{\mathcal{D}}$. The two vectors e_1 and e_2 are parallel to $\Gamma_{\mathcal{D}} \cap B_{R_0}$ and e_3 is parallel to $\Gamma_{\mathcal{N}} \cap B_{R_0}$. Thus, let $\Lambda_1 = \{1, 2\}$, $\zeta^1 = e_1$, $\zeta^2 = e_2$, $\Lambda_2 = \{3\}$, and $\zeta^3 = e_3$.

Case 2: $\Gamma_{\mathcal{D}} = \{y \in \partial\Omega : y_2 = 0 \vee y_3 = 0\}$ and $\Gamma_{\mathcal{N}} = \partial\Omega \setminus \Gamma_{\mathcal{D}}$. Now, e_1 is parallel to $\Gamma_{\mathcal{D}} \cap B_{R_0}$, thus, $\Lambda_1 = \{1\}$ and $\zeta^1 = e_1$. Further, the two vectors e_2 and e_3 are parallel to $\Gamma_{\mathcal{N}} \cap B_{R_0}$, thus, $\Lambda_2 = \{2, 3\}$. We must choose ζ^i $(i = 2, 3)$ such that

i) $\text{angle}(\zeta^i, \Gamma_{\mathcal{D}} \cap B_{R_0}) \geq \alpha^*$

ii) $\text{angle}(\zeta^2, \zeta^3) \geq \alpha^*$

for some suitable constant $\alpha^* > 0$. Thus, let $\zeta^2 = \frac{\sqrt{3}}{2}e_2 + \frac{1}{2}e_3$ and $\zeta^3 = \frac{1}{2}e_2 + \frac{\sqrt{3}}{2}e_3$.

Case 3: $\Gamma_{\mathcal{D}} = \emptyset$ and $\Gamma_{\mathcal{N}} = \partial\Omega$. It holds that $\Lambda_1 = \{1, 2, 3\}$. Let $\zeta^i = e_i$ for $1 \leq i \leq 3$.

Case 4: $\Gamma_{\mathcal{D}} = \partial\Omega$ and $\Gamma_{\mathcal{N}} = \emptyset$. Now, it holds that $\Lambda_2 = \{1, 2, 3\}$. We choose ζ^i $(1 \leq i \leq 3)$ such that

i) $\text{angle}(\zeta^i, \Gamma_{\mathcal{D}} \cap B_{R_0}) \geq \alpha^*$

ii) $\text{angle}(\zeta^i, \zeta^j) \geq \alpha^*$ for $1 \leq i < j \leq 3$ and $\alpha^* > 0$

iii) $y + s\zeta^i \in \Omega$ for $y \in \partial\Omega \cap B_{R_0}$ and $0 < s < R_0$

where $\alpha^* > 0$ is suitable.

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