A New Minimal Point Theorem in Product Spaces

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Abstract. We derive a minimal point theorem for a subset $A$ in a cone in product spaces under a weak assumption concerning the boundedness of the considered set $A$. Using this result we improve two vectorial variants of Ekeland’s variational principle. Finally, a new characterization of well-based cones is given.

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Assume that $(X, d)$ is a complete metric space, $Y$ is a separated locally convex space, $Y^*$ is its topological dual, $K \subset Y$ is a convex cone, i.e. $K + K \subset K$ and $[0, \infty) \cdot K \subset K$, $K^+ = \{y^* \in Y^* : \langle y, y^* \rangle \geq 0 \text{ for all } y \in K\}$ is the dual cone of $K$ and $K^\# = \{y^* \in Y^* : \langle y, y^* \rangle > 0 \text{ for all } y \in K \setminus \{0\}\}$. In this note we suppose that $K$ is pointed, i.e. $K \cap (-K) = \{0\}$. The cone $K$ determines an order relation on $Y$, denoted in the sequel by $\leq_K$; so, for $y_1, y_2 \in Y$, $y_1 \leq_K y_2$ if $y_2 - y_1 \in K$. It is well known that $\leq_K$ is reflexive, transitive and antisymmetric. Let $k^0 \in K \setminus \{0\}$; using the element $k^0$ we introduce an order relation on $X \times Y$, denoted by $\leq_{k^0}$, in the following manner:

$$(x_1, y_1) \leq_{k^0} (x_2, y_2) \quad \text{iff} \quad y_1 + k^0 d(x_1, x_2) \leq_K y_2.$$ 

Note that $\leq_{k^0}$ is reflexive, transitive and antisymmetric. That is, our notations are those of [3].

The essential idea for the derivation of a minimal point theorem (cf. [2, 8]) in general product spaces $X \times Y$, as well as of the vectorial Ekeland principle, consists in including the ordering cone $K \subset Y$ in a “larger” cone $B \subset Y$: $K \setminus \{0\} \subset \text{int } B$. We will use $B$ to define a suitable functional $z_B : Y \to \mathbb{R}$. Moreover, we will replace the usual boundedness condition of the projection $P_Y A$ of $A$ onto $Y$ by a weaker one.
Theorem 1. Assume that there exists a proper convex cone $B \subset Y$ such that $K \setminus \{0\} \subset \text{int } B$. Suppose that the set $A \subset X \times Y$ satisfies the condition

$$(\text{H1})$$

for every $\leq k^0$-decreasing sequence $((x_n, y_n)) \subset A$ with $x_n \to x \in X$ there exists $y \in Y$ such that $(x, y) \in A$ and $(x, y) \leq k^0 (x_n, y_n)$ for every $n \in \mathbb{N}$

and that $P_Y(A) \cap (y - \text{int } B) = \emptyset$ for some $\bar{y} \in Y$. Then for every $(x_0, y_0) \in A$ there exists $(\bar{x}, \bar{y}) \in A$, minimal with respect to $\leq k^0$, such that $(\bar{x}, \bar{y}) \leq k^0 (x_0, y_0)$.

Proof. Let

$$z_B : Y \to \mathbb{R}, \quad z_B(y) = \inf\{t \in \mathbb{R} : y \in tk^0 - c1B\}.$$ 

By [3; Lemma 7], $z_B$ is a continuous sublinear function such that $z_B(y + tk^0) = z_B(y) + t$ for all $t \in \mathbb{R}$ and $y \in Y$, and for every $\lambda \in \mathbb{R}$

$$\{y \in Y : z_B(y) \leq \lambda\} = \lambda k^0 - c1B$$

$$\{y \in Y : z_B(y) < \lambda\} = \lambda k^0 - \text{int } B.$$ 

Moreover, if $y_2 - y_1 \in K \setminus \{0\}$, then $z_B(y_1) < z_B(y_2)$. Observe that for $(x, y) \in A$ we have that $z_B(y - \bar{y}) \geq 0$. Otherwise for some $(x, y) \in A$ we have $z_B(y - \bar{y}) < 0$. It follows that there exists $\lambda > 0$ such that $y - \bar{y} \in -\lambda k^0 - c1B$. Hence

$$y \in \bar{y} - (\lambda k^0 + c1B) \subset \bar{y} - (\text{int } B + c1B) \subset \bar{y} - \text{int } B$$

which is a contradiction. Since $0 \leq z_B(y - \bar{y}) \leq z_B(y) + z_B(-\bar{y})$, it follows that $z_B$ is bounded from below on $P_Y(A)$. Let us construct a sequence $((x_n, y_n))_{n \geq 0} \subset A$ as follows: having $(x_n, y_n) \in A$ we take $(x_{n+1}, y_{n+1}) \in A$, $(x_{n+1}, y_{n+1}) \leq k^0 (x_n, y_n)$, such that

$$z_B(y_{n+1}) \leq \inf\left\{z_B(y) : (x, y) \in A \text{ and } (x, y) \leq k^0 (x_n, y_n)\right\} + \frac{1}{n + 1}.$$ 

Of course, the sequence $((x_n, y_n))$ is $\leq k^0$-decreasing. It follows that

$$y_{n+p} + k^0 d(x_{n+p}, x_n) \leq k y_n \quad \forall \ n, p \in \mathbb{N}^*$$

so that

$$d(x_{n+p}, x_n) \leq z_B(y_n) - z_B(y_{n+p}) \leq \frac{1}{n} \quad \forall \ n, p \in \mathbb{N}^*.$$ 

It follows that $(x_n)$ is a Cauchy sequence in the complete metric space $(X, d)$, and so $(x_n)$ is convergent to some $\bar{x} \in X$. By condition (H1) there exists $\bar{y} \in Y$ such that $(\bar{x}, \bar{y}) \in A$ for every $n \in \mathbb{N}$.

Let us show that $(\bar{x}, \bar{y})$ is the desired element. Indeed, $(\bar{x}, \bar{y}) \leq k^0 (x_0, y_0)$. Suppose that $(x', y') \in A$ is such that $(x', y') \leq k^0 (\bar{x}, \bar{y})$ (\leq k^0 (x_n, y_n) for every $n \in \mathbb{N}$). Thus $z_B(y') + d(x', \bar{x}) \leq z_B(\bar{y})$, whence

$$d(x', \bar{x}) \leq z_B(\bar{y}) - z_B(y') \leq z_B(y_n) - z_B(y') \leq \frac{1}{n} \quad \forall \ n \geq 1.$$ 

It follows that $d(x', \bar{x}) = z_B(\bar{y}) - z_B(y') = 0$. Hence $x' = \bar{x}$ As $y' \leq k \bar{y}$, if $y' \neq \bar{y}$, then $\bar{y} - y' \in K \setminus \{0\}$, whence $z_B(y') < z_B(\bar{y})$, which is a contradiction. Therefore $(x', y') = (\bar{x}, \bar{y})$. \square
Comparing with [3; Theorem 4], note that the present condition on \( K \) is stronger (because in this case \( K^* \neq \emptyset \)), while the condition on \( A \) is weaker (\( A \) may be not contained in a half-space). Note that when \( K \) and \( k^0 \) are as in Theorem 1, Corollaries 2 and 3 from [3] may be improved. In the next result \( Y^* = Y \cup \{ \infty \} \) with \( \infty \notin Y \); we consider that \( y \leq_K \infty \) for every \( y \in Y \). We consider also a function \( f : X \to Y^* \) and \( \text{dom} \ f = \{ x \in X : f(x) \neq \infty \} \).

In the following corollary we derive a variational principle of Ekeland’s type for objective functions which take values in a general space \( Y \) (cf. [2, 3, 5 - 7]) under a weaker assumption with respect to the usual lower semicontinuity. For the case \( Y = \mathbb{R} \), assumption (H4) in Corollary 2 is fulfilled for decreasingly semicontinuous real-valued functions as in the paper [4].

**Corollary 2.** Let \( f : X \to Y^* \). Assume that there exists a proper convex cone \( B \subset Y \) such that \( K \setminus \{ 0 \} \subset \text{int} B \) and \( f(X) \cap (y - B) = \emptyset \) for some \( y \in Y \). Also, suppose that

\[ (H3) \quad \{ x' \in X : f(x') + k^0 d(x', x) \leq_K f(x) \} \text{ is closed for every } x \in X \]

or

\[ (H4) \quad \text{for every sequence } (x_n) \subset \text{dom} f \text{ with } x_n \to x \text{ and } (f(x_n)) \leq_K \text{decreasing, } f(x) \leq_K f(x_n) \text{ for every } n \in \mathbb{N}, \text{ and } K \text{ is closed in the direction } k^0. \]

Then for every \( x_0 \in \text{dom} f \) there exists \( \overline{x} \in X \) such that

\[ f(\overline{x}) + k^0 d(\overline{x}, x_0) \leq_K f(x_0) \]

and

\[ \forall x \in X : \ f(x) + k^0 d(\overline{x}, x) \leq_K f(\overline{x}) \implies x = \overline{x}. \]

We say that \( K \) is closed in the direction \( k^0 \) if \( K \cap (y - \mathbb{R}_+ k^0) \) is closed for every \( y \in K \). The proof of Corollary 2 is similar to those of Corollaries 2 and 3 in [3].

As mentioned in [3], condition (H1) is verified if \( K \) is a well based convex cone, \( Y \) is a Banach space and \( A \) is closed. As usually (cf. [1]), a convex set \( S \) is said to be a base for a convex cone \( K \subset Y \)

\[ K = \mathbb{R}_+ S = \{ \lambda y : \lambda \geq 0 \text{ and } y \in S \} \quad \text{and} \quad 0 \notin \text{cl} S. \]

The cone \( K \) is called well based if \( K \) has a bounded base \( S \). Concerning well based convex cones in normed spaces we have the following characterization.

**Proposition 3.** Let \( Y \) be a normed vector space and \( K \subset Y \) a proper convex cone. Then \( K \) is well based if and only if there exist \( k^0 \in K \) and \( z^* \in K^* \) such that \( \langle k^0, z^* \rangle > 0 \) and

\[ K \cap S_1 \subset k^0 + \{ y \in Y : \langle y, z^* \rangle > 0 \} \]

where \( S_1 = \{ y \in Y : ||y|| = 1 \} \) is the unit sphere in \( Y \).

**Proof.** Suppose first that \( K \) is well based with bounded base \( S \); therefore \( 0 \notin \text{cl} S \) and \( K = [0, \infty) \cdot S \). Then there exists \( z^* \in Y^* \) such that \( 1 \leq \langle y, z^* \rangle \) for all \( y \in S \). Consider \( S := \{ k \in K : \langle k, z^* \rangle = 1 \} \). It follows that \( S \) is a base of \( K \); moreover, since
$\bar{S} \subset [0, 1] \cdot S$, $\bar{S}$ is also bounded. Taking $k^1 \in K \setminus \{0\}$ we have $K \cap S_1 \subset \lambda k^1 + B_+$ for some $\lambda > 0$, where $B_+ = \{y \in Y : \langle y, z^* \rangle > 0\}$. Otherwise

$$\forall n \in \mathbb{N}^* \exists k_n \in K \cap S_1 : \quad k_n \notin \frac{1}{n} k^1 + B_+.$$ 

Therefore $\langle k_n, z^* \rangle \leq \frac{1}{n} \langle k^1, z^* \rangle$ for every $n \geq 1$. But, because $\bar{S}$ is a base, $k_n = \lambda_b b_n$ with $\lambda_n > 0$ and $b_n \in \bar{S}$; it follows that $1 = ||k_n|| = \lambda_n ||b_n|| \leq \lambda_n M$ with $M > 0$ (because $\bar{S}$ is bounded). Therefore

$$M^{-1} \leq \lambda_n = \langle \lambda_b b_n, z^* \rangle = \langle k_n, z^* \rangle \leq n^{-1} \langle k^1, z^* \rangle \quad \forall n \in \mathbb{N}^*$$

whence $M^{-1} \leq 0$, which is a contradiction. Thus there exists $\lambda > 0$ such that $K \cap S_1 \subset \lambda k^1 + B_+$. Taking $k^0 := \lambda k^1$ the conclusion follows.

Suppose now that $K \cap S_1 \subset k^0 + B_+$ for some $k^0 \in K$ and $z^* \in K^+$ with $\langle k^0, z^* \rangle = c > 0$, where $B_+$ is defined as above. Consider $S = \{k \in K : \langle k, z^* \rangle = 1\}$. Let $k \in K \setminus \{0\}$. Then $||k||^{-1} k = k^0 + y$ for some $y \in B_+$. It follows that $\langle k, z^* \rangle > c ||k|| > 0$; therefore $z^* \in K^+$ and so $k \in (0, \infty) \cdot S$. Since $\cl S \subset \{y \in Y : \langle k, z^* \rangle = 1\}$, we have that $S$ is a base of $K$. Let now $y \in S \subseteq K$. Then $||y||^{-1} y \in K \cap S_1$. There exists $z \in B_+$ such that $||y||^{-1} y = k^0 + z$. We get

$$1 = \langle y, z^* \rangle = ||y|| \langle k^0 + z, z^* \rangle \geq c ||y||$$

whence $||y|| \leq c^{-1}$. Therefore $S$ is bounded, and so $K$ is well-based. \hfill $\blacksquare$

References


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