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Affinity Integral Manifolds for Impulsive Differential Equations

S. I. Kostadinov and G. T. Stamov

Abstract. Sufficient conditions on the existence of affinity integral manifolds of linear and nonlinear impulsive differential equations are obtained.

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1. Introduction

Impulsive differential equations represent a natural apparatus for mathematical simulation of real processes and phenomena studied in physics, biology, population dynamics, biotechnologies, control theory, economics, etc. For instance, if the population of a given species is regulated by some impulsive factors acting at certain moments, then we have no reason to expect that the process will be simulated by regular control. On the contrary, the solutions must have jumps at these moments and the jumps are given beforehand. Moreover, the mathematical theory of impulsive differential equations is much richer than the corresponding theory of equations without impulses. That is why in the recent years this theory has become an important area of numerous investigations [1 - 6].

In the present paper problems of the existence of affinity integral manifolds of linear and nonlinear systems of impulsive differential equations and some of their properties are considered.

2. Preliminary notes and definitions

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $\|\cdot\|$, and let $\mathbb{R}^+ = [0,\infty)$. Consider the system of impulsive differential equations

$$\left.\begin{aligned} \dot z &= A(t)z + F(t,z) && (t \neq \tau_k)\\ \Delta z(\tau_k) = z(\tau_k+0) - z(\tau_k) &= B_k z(\tau_k) + \Phi_k(z(\tau_k)) && (k \in \mathbb{N}) \end{aligned}\right\} \tag{1}$$

where

S. I. Kostadinov: Plovdiv Univ., Dept. Math., Plovdiv, Bulgaria
G. T. Stamov: Technical Univ. Sliven, Dept. Math., Sliven, Bulgaria

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(i) $t \in \mathbb{R}^+$, $z \in \mathbb{R}^{m+n}$, $A : \mathbb{R}^+ \to \mathbb{R}^{m+n} \times \mathbb{R}^{m+n}$
$B_k \in \mathbb{R}^{m+n} \times \mathbb{R}^{m+n}$, $F : \mathbb{R}^+ \times \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$, $\Phi_k : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$

(ii) $0 < \tau_1 < \tau_2 < \ldots$ and $\lim_{k\to\infty} \tau_k = \infty$.

Let $t_0 \in \mathbb{R}^+$ and $z_0 \in \mathbb{R}^{m+n}$. Denote by $z(t) = z(t; t_0, z_0)$ the solution of system (1) satisfying the initial condition $z(t_0 + 0) = z_0$. These solutions are piecewise continuous functions, with points of discontinuity of the first kind at which they are continuous from the left, i.e. at the moment τ_k the relations

$$\left.\begin{aligned} z(\tau_k - 0) &= z(\tau_k) \\ z(\tau_k + 0) &= z(\tau_k) + B_k z(\tau_k) + \Phi_k(z(\tau_k)) \end{aligned}\right\}$$

are satisfied.

Definition 1 (see [2: Definition 13.2]). An arbitrary set G in the extended phase space of system (1) is said to be an *integral manifold*, if for $t_0 \in \mathbb{R}^+$ and for arbitrary solution $z = z(t)$ of system (1) from $(t_0, z(t_0)) \in G$ it follows that $(t, z(t)) \in G$ for all $t > t_0$.

Definition 2. The integral manifold G is said to be an *affinity integral manifold* of system (1) if G is the graph of a function

$$\varphi : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^n, \qquad \varphi(t, x) = Q(t)x + \eta(t, x)$$

where the following conditions are satisfied:

a) Q is an $n \times m$ matrix-valued function with points of discontinuities of the first kind at the moments $t = \tau_k$ $(k \in \mathbb{N})$ at which Q is continuous from the left.

b) $\eta : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^n$ is a bounded function which is continuous with respect to x and with points of discontinuity of the first kind at the moments $t = \tau_k$ $(k \in \mathbb{N})$.

If this is satisfied, then $\varphi = \varphi(t, x)$ is said to be a *parameter function*.

We write system (1) in the form

$$\left.\begin{aligned} \dot{x} &= A^{11}(t)x + A^{12}(t)y + f(t, x, y) && (t \neq \tau_k) \\ \Delta x(\tau_k) &= B_k^{11} x(\tau_k) + B_k^{12} y(\tau_k) + I_k(x(\tau_k), y(\tau_k)) && (k \in \mathbb{N}) \\ \dot{y} &= A^{21}(t)x + A^{22}(t)y + g(t, x, y) && (t \neq \tau_k) \\ \Delta y(\tau_k) &= B_k^{21} x(\tau_k) + B_k^{22} y(\tau_k) + J_k(x(\tau_k), y(\tau_k)) && (k \in \mathbb{N}) \end{aligned}\right\} \tag{2}$$

where

(i) $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, $(f, g) = F$, $(I_k, J_k) = \Phi_k$ $(k \in \mathbb{N})$

(ii) $A^{11} : \mathbb{R}^+ \to \mathbb{R}^{m+m}$, $A^{12} : \mathbb{R}^+ \to \mathbb{R}^{m+n}$, $A^{21} : \mathbb{R}^+ \to \mathbb{R}^{n+m}$, $A^{22} : \mathbb{R}^+ \to \mathbb{R}^{n+n}$

(iii) $B_k^{11} \in \mathbb{R}^{m+m}$, $B_k^{12} \in \mathbb{R}^{m+n}$, $B_k^{21} \in \mathbb{R}^{n+m}$, $B_k^{22} \in \mathbb{R}^{n+n}$.

Introduce the following conditions:

(H1) The matrix-valued function A is continuous.

(H2) $\det(E_m + B_k^{11}) \neq 0$ $(k \in \mathbb{N})$ where $E_m \in \mathbb{R}^{m+m}$ is the identity matrix.

Recall (see [1: p. 46]) that if $U_k(t,s)$ is the Cauchy matrix for the equation

$$\dot{x} = A^{11}(t)x \qquad (\tau_{k-1} < t < \tau_k)$$

and conditions (H1) and (H2) hold, then the Cauchy matrix for the equation

$$\left.\begin{aligned} \dot{x} &= A^{11}(t)x && (t \neq \tau_k) \\ \Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k) &= B_k^{11} x(\tau_k) && (k \in \mathbb{N}) \end{aligned}\right\} \tag{3}$$

is

$$W(t,s) = \tag{4}$$

$$\begin{cases} U_k(t,s) & \text{if } \tau_k < s \le t \le \tau_{k+1} \\ U_{k+1}(t,\tau_k+0)(E_m + B_k^{11})U_k(\tau_k,s) & \text{if } \tau_{k-1} < s \le \tau_k < t \le \tau_{k+1} \\ U_k(t,\tau_k)(E_m + B_k^{11})^{-1}U_{k+1}(\tau_k+0,s) & \text{if } \tau_{k-1} < t \le \tau_k < s \le \tau_{k+1} \\ \left.\begin{array}{l} U_{k+1}(t,\tau_k+0)\prod_{j=0}^{k-i-1}(E_m + B_{k-j}^{11}) \\ U_{k-j}(\tau_{k-j},\tau_{k-j-1}+0)(E_m + B_i^{11})U_i(\tau_i,s) \end{array}\right\} & \text{if } \tau_{i-1} < s \le \tau_i \le \tau_k < t \le \tau_{k+1} \\ U_i(t,\tau_i)\prod_{j=i}^{k-1}(E_m + B_j^{11})^{-1}U_{k+1}(\tau_k+0,s) & \text{if } \tau_{i-1} < s \le \tau_i \le \tau_k < t \le \tau_{k+1}. \end{cases}$$

It is easy to verify that the relations

$$\left.\begin{aligned} W(t,t) &= E_m \\ W(\tau_k - 0, \tau_k) = W(\tau_k, \tau_k - 0) &= E_m \\ W(\tau_k + 0, s) &= (E_m + B_k^{11})W(\tau_k, s) \\ W(s, \tau_k + 0) &= W(\tau_k, s)(E_m + B_k^{11})^{-1} \\ \frac{\partial W(t,s)}{\partial t} &= A^{11}(t)W(t,s) \quad (t \neq \tau_k) \\ \frac{\partial W(t,s)}{\partial s} &= -W(t,s)A^{11}(s) \end{aligned}\right\} \tag{5}$$

are valid. Introduce the condition

(H3) $\det(E_n + B_k^{22}) \neq 0 \ \ (k \in \mathbb{N})$ where $E_n \in \mathbb{R}^{n+n}$ is the identity matrix.

We denote by $Y = Y(t)$, where $Y(t_0) = E_n \ \ (t_0 \in (0,\tau_1))$ the fundamental matrix of the system

$$\left.\begin{aligned} \dot{x} &= A^{22}(t)x && (t \neq \tau_k) \\ \Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k) &= B_k^{22} x(\tau_k) && (k \in \mathbb{N}). \end{aligned}\right\} \tag{6}$$

Definition 3. Let P be a projector (i.e. $P^2 = P$) in $\mathbb{R}^n$. The function

$$G(t,s) = \begin{cases} Y(t)PY^{-1}(t) & \text{for } t \ge s \\ Y(t)(P - E_n)Y^{-1}(s) & \text{for } s \ge t \end{cases}$$

is said to be the *Green function* of system (6).

It is easy to verify that the relations

$$\begin{aligned} \frac{\partial G(t,s)}{\partial t} &= A^{22}(t)G(t,s) && (t \neq s) \\ \frac{\partial W(t,s)}{\partial s} &= -G(t,s)A^{22}(s) && (t \neq s) \\ G(\tau_k+0,t) &= (E_n + B_k^{22})G(\tau_k,t) && (t \neq \tau_k) \\ G(t,\tau_k+0) &= G(t,\tau_k)(E_n + B_k^{22})^{-1} && (t \neq \tau_k) \\ G(t+0,t) - G(t,t-0) &= E_n && (t \neq \tau_k) \\ G(t,t+0) - G(t,t-0) &= -E_n && (t \neq \tau_k) \\ G(\tau_k+0,\tau_k+0) &= (E_n + B_k^{22})G(\tau_k,\tau_k+0) && (k \in \mathbb{N}) \end{aligned} \tag{7}$$

are valid. Introduce the following conditions.

(H4) $0 < t_0 < \tau_1$, and there exist constants $p > 0$ and $\varepsilon > 0$ such that

$$i(s,t) \leq p(t-s) + \varepsilon$$

where $i(s,t)$ is the number of the points τ_k lying in the interval (s,t).

(H5) The inequalities

$$\left. \begin{aligned} \|W(t,s)\| &\leq K e^{\alpha|t-s|} \\ \|G(t,s)\| &\leq N e^{\alpha|t-s|} \end{aligned} \right\} \qquad (t,s \in \mathbb{R}^+)$$

hold where $K, N, \Delta > 0$ and $0 < \alpha < \delta$.

Lemma 1 (see [2: Lemma 3.4]). *Let the inequality*

$$u(t) \leq \int_{t_0}^{t} u(s)v(s)\,ds + F(t) + \sum_{t_0<\tau_k<t} \beta_k u(\tau_k) + \sum_{t_0<\tau_k<t} \alpha_k(t)$$

hold, where u is a piecewise continuous function with points of discontinuity of the first kind τ_k $(k \in \mathbb{N})$, v a locally integrable function, $F(t)$ and $\alpha_k(t)$ non-decreasing for $t \geq t_0$ and $\alpha_k(t), \beta_k \geq 0$ $(k \in \mathbb{N})$. Then

$$u(t) \leq \Big(F(t) + \sum_{t_0<\tau_k<t} \alpha_k(t)\Big) \prod_{t_0<\tau_k<t} (1+\beta_k) \exp\Big(\int_{t_0}^{t} v(s)\,ds\Big).$$

3. Main results

We consider the linear part of the system

$$\left.\begin{aligned} \dot{x} &= A^{11}(t)x + A^{12}(t) && (t \neq \tau_k) \\ \Delta x(\tau_k) &= B_k^{11}x(\tau_k) + B_k^{12}y(\tau_k) && (k \in \mathbb{N}) \\ \dot{y} &= A^{21}(t)x + A^{22}(t) && (t \neq \tau_k) \\ \Delta y(\tau_k) &= B_k^{21}x(\tau_k) + B_k^{22}y(\tau_k) && (k \in \mathbb{N}). \end{aligned}\right\} \tag{8}$$

Let G be the affinity integral manifold of system (8) in the form

$$G = \Big\{(t,x,y) : \; y = Q(t)x \text{ for } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^m\Big\}. \tag{9}$$

Along with G we consider the system

$$\left.\begin{aligned} &\dot{Q} + Q(t)A^{11}(t) + Q(t)A^{12}(t)Q(t) \\ &\qquad = A^{21}(t) + A^{22}(t)Q(t) \qquad (t \neq \tau_k) \\ &\Delta Q(\tau_k) + Q(\tau_k) + Q(\tau_k+0)B_k^{11} + Q(\tau_k+0)B_k^{12}Q(\tau_k) \\ &\qquad = B_k^{21} + B_k^{22}Q(\tau_k) \qquad (k \in \mathbb{N}). \end{aligned}\right\} \tag{10}$$

Lemma 2. *The manifold* (9) *is an affinity integral manifold of system* (8) *if and only if $Q = Q(t)$ is a bounded solution of system* (10).

Proof. Lemma 2 can be proved by straightforward calculations and we thus suppress the proof ∎

Lemma 3. *Let conditions* (H1) - (H5) *be fulfilled. Further, assume that the following is true:*

1. $A^{12}(t) = 0$ *and* $B_k^{12} = 0$ *for all* $t \in \mathbb{R}^+$ *and* $k \in \mathbb{N}$.

2. $\sup_{t\in\mathbb{R}^+} \|A^{21}(t)\| \le \delta$ *and* $\sup_{k\in\mathbb{N}} \|B_k^{21}(t)\| \le \delta$ *for some* $\delta > 0$.

Then for system (8) *there exists an affinity integral manifold in the form* (9).

Proof. Let $x(t) = x(t;t_0,x_0)$ be the solution of the Cauchy problem for system (3) where $x(t_0) = x_0$. Then from [1: p. 46] it follows that $x(t_0) = W(t,t_0)\,x_0$, and for the system

$$\left.\begin{aligned} \dot{x} &= A^{22}(t)y + A^{21}(t)W(t,t_0)x && (t \neq \tau_k) \\ \Delta y(\tau_k) &= B_k^{22}y(\tau_k) + B_k^{21}W(\tau_k,t_0)x(\tau_k) && (k \in \mathbb{N}) \end{aligned}\right\}$$

there exists only a bounded solution in the form

$$\begin{aligned} y(t) = &\int_{t_0}^{\infty} G(t,s)A^{21}(s)W(s,t_0)\,x_0\,ds \\ &+ \sum_{k=1}^{\infty} G(t,\tau_k+0)B_k^{21}W(\tau_k,t_0)\,x_0. \end{aligned}$$

If the graph of the solution $(x(t), y(t))$ $(t > t_0)$ is from the affinity integral manifold, then

$$Q(t)W(t,s)\,x_0 = \int_{t_0}^{\infty} G(t,s)A^{21}(s)W(s,t_0)\,x_0\,ds + \sum_{k=1}^{\infty} G(t,\tau_k+0)B_k^{21}W(\tau_k,t_0)\,x_0.$$

Lemma 3 will be proved if the function

$$Q(t) = \int_{t_0}^{\infty} G(t,s)A^{21}(s)W(s,t)\,ds + \sum_{k=1}^{\infty} G(t,\tau_k+0)B_k^{21}W(\tau_k,t) \tag{11}$$

is the bounded solution of system (8) for which $A^{12} \equiv 0$ and $B_k^{12} \equiv 0$ $(k \in \mathbb{N})$. From (5) and (7) for $t \neq \tau_k$ we obtain

$$\begin{aligned}
\dot{Q} &= \frac{d}{dt}\Bigg(\int_{t_0}^{t} G(t,s)A^{21}(t)W(s,t)\,ds \\
&\quad + \int_{t_0}^{\infty} G(t,s)A^{21}(t)W(s,t)\,ds \\
&\quad + \sum_{k=1}^{\infty} G(t,\tau_k+0)B_k^{21}W(\tau_k,t)\Bigg) \\
&= G(t,t-0)A^{21}(t)W(t-0,t) - G(t,t+0)A^{21}(t)W(t+0,t) \\
&\quad + \int_{t_0}^{\infty} A^{22}(t)G(t,s)A^{21}(s)W(s,t)\,ds \\
&\qquad - \int_{t_0}^{\infty} G(t,s)A^{21}(s)W(s,t)A^{11}(t)\,ds \\
&\quad + \sum_{k=1}^{\infty} A^{22}(t)G(t,\tau_k+0)B_k^{21}W(\tau_k,t) \\
&\qquad - \sum_{k=1}^{\infty} G(t,\tau_k+0)B_k^{21}W(\tau_k,t)A^{11}(t) \\
&= A^{21}(t) + A^{22}(t)Q(t) - Q(t)A^{11}(t).
\end{aligned}$$

For $t = \tau_i$ $(i \in \mathbb{N})$ it follows

$$\begin{aligned}
&\Delta Q(\tau_i) + Q(\tau_k+0)B_i^{11} \\
&\quad = \int_{t_0}^{\infty} G(\tau_i+0,s)A^{21}(s)W(s,\tau_i+0)(E_m + B_i^{11})\,ds \\
&\qquad + \sum_{k=1}^{\infty} G(\tau_i+0,\tau_k+0)B_k^{21}W(\tau_k,\tau_i+0)(E_m + B_k^{11}) \\
&\qquad - \int_{t_0}^{\infty} G(\tau_i,s)A^{21}(s)W(s,\tau_i)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{k=1}^{\infty} G(\tau_i, \tau_k + 0) B_k^{21} W(\tau_i, \tau_k) \\
&= \int_{t_0}^{\infty} (E_n + B_i^{22}) G(\tau_i, s) A^{21}(s) W(s, \tau_i)\, ds \\
&\quad + \sum_{k=1}^{\infty} (E_n + B_i^{22}) G(\tau_i, \tau_k + 0) B_k^{21} W(\tau_k, \tau_k) + B_i^{22} \\
&\quad - \int_{t_0}^{\infty} G(\tau_i, s) A^{21}(s) W(s, \tau_i)\, ds \\
&\quad - \sum_{k=1}^{\infty} G(\tau_i, \tau_k + 0) B_k^{21} W(\tau_i, \tau_k) \\
&= B_i^{22} + B_i^{22} Q(\tau_i).
\end{aligned}
$$

Hence (11) is solution of system (10). On the other hand, for $t > t_0$

$$
\|Q(t)\| \leq \int_{t_0}^{\infty} KNe^{-(\Delta-\alpha)|t-s|}\delta ds + \sum_{k=1}^{\infty} KNe^{-(\Delta-\alpha)|t-\tau_k|}\delta. \tag{12}
$$

From assumption (H4) $\sum_{k=1}^{\infty} e^{-(\Delta-\alpha)|t-\tau_k|} < C < \infty$ follows where C depend only on $(\Delta - \alpha)$ and the sequence $\{\tau_k\}$. Then from (12) it follows that $Q = Q(t)$ is a bounded solution of system (10) ∎

Theorem 1. *Let conditions* (H1) - (H5) *be fulfilled. Further, let there exists* $\delta > 0$ *such that*

$$
\sup_{t\in\mathbb{R}^+} \|A^{12}(t)\| \leq \delta, \quad \sup_{k\in\mathbb{N}} \|B_k^{12}\| \leq \delta, \quad \sup_{t\in\mathbb{R}^+} \|A^{21}(t)\| \leq \delta, \quad \sup_{k\in\mathbb{N}} \|B_k^{22}\| \leq \delta.
$$

Then there exists $\delta_0 > 0$ *such that, for any* $\delta \in (0, \delta_0]$ *and* $t > t_0$, *for system* (8) *there exists an affinity integral manifold in the form* (9).

Proof. We shall obtain the parameter function $\varphi(t, x)$ by the method of consistent approach. Set

$$
\left.
\begin{aligned}
\varphi_0 &= 0 \\
\varphi_n &= Q_n(t)x \quad (n \in \mathbb{N})
\end{aligned}
\right\}
$$

where

$$
Q_n(t) = \int_{t_0}^{\infty} G(t, s) A^{21}(s) W_{n-1}(s, t) ds + \sum_{k=1}^{\infty} G(t, \tau_k + 0) B_k^{21} W_{n-1}(\tau_k, t)
$$

and $W_{n-1}(t, s)$ be the Cauchy matrix of the system

$$
\left.
\begin{aligned}
\dot{x} &= (A^{11}(t) + A^{12}(t) Q_{n-1}(t)) x && (t \neq \tau_k) \\
\Delta x(\tau_k) &= (B_k^{11} + B_k^{12}(\tau_k) Q_{n-1}(\tau_k)) x(\tau_k) && (k \in \mathbb{N}).
\end{aligned}
\right\} \tag{13}
$$

Consider the system

$$\left.\begin{aligned} \dot{x} &= (A^{11}(t) + A^{12}(t)Q(t))x && (t \neq \tau_k) \\ \Delta x(\tau_k) &= (B_k^{11} + B_k^{12}Q(\tau_k))x(\tau_k) && (k \in \mathbb{N}) \\ \dot{y} &= A^{22}(t)y + A^{21}(t)x && (t \neq \tau_k) \\ \Delta y(\tau_k) &= B_k^{22}y(\tau_k) + B_k^{21}x(\tau_k) && (k \in \mathbb{N}). \end{aligned}\right\} \tag{14}$$

We shall proof that $\{Q_n(t)\}_{n=1}^{\infty}$ is a uniformly bounded sequence for any $t > 0$. For $n = 1$ and $A^{12} \equiv 0, B_k^{12} \equiv 0$ system (13) coincide with system (8), and the matrix $W_0(t,s)$ coincide with matrix the $W(t,s)$. From Lemma 3 it follows that there exists $q > 0$ such that $\|Q_1(t)\| \leq q$. Let $\|Q_n(t)\| \leq q$ for arbitrary n. Then

$$\begin{aligned} \|Q_{n+1}(t)\| \leq \int_{t_0}^{\infty} \|G(t,s)\| \, \|A^{21}(s)\| \, \|W_n(s,t)\| \, ds \\ + \sum_{k=1}^{\infty} \|G(t,\tau_k + 0)\| \, \|B_k^{21}\| \, \|W(\tau_k, t)\|. \end{aligned} \tag{15}$$

From (13) for $t > s$ we obtain

$$\begin{aligned} W_n(t,s) = W(t,s) \\ &+ \int_s^t W(t,\tau)A^{12}(\tau)Q_n(\tau)W_n(\tau,s)\, d\tau \\ &+ \sum_{s<\tau_k<t} W(t,\tau_k)B_k^{12}Q_n(\tau_k)W_n(\tau_k,s). \end{aligned}$$

Then

$$\|W(t,s)\| \leq Ke^{\alpha(t-s)} + \int_s^t Kq\delta e^{\alpha(t-\tau)} d\tau + \sum_{s<\tau_k<t} Kq\delta e^{\alpha(t-\tau_k)} \|W_n(\tau_k, s)\|.$$

Put

$$u(t) = e^{-\alpha t}\|W_n(t,s)\|, \quad F(t) = Ke^{-\alpha s}, \quad v(t) = Kq\delta, \quad \beta_k = Kq\delta, \quad \alpha_k(t) = 0.$$

From Lemma 1

$$\begin{aligned} \|W(t,s)\| &\leq Ke^{\alpha(t-s)} \prod_{s<\tau_k<t} (1 + Kq\delta)e^{Kq\delta(t-s)} \\ &\leq Ke^{\alpha(t-s)}(1 + Kq\delta)^{p(t-s)+\varepsilon} e^{Kq\delta(t-s)} \\ &= K(1 + Kq\delta)^{\varepsilon} e^{(\alpha + Kq\delta + p\ln(1+Kq\delta))(t-s)} \end{aligned}$$

follows. For $s > t$ we have

$$\begin{aligned}\|W_n(t,s)\| \le Ke^{\alpha(s-t)} \\ &+ \int_t^s Kq\delta e^{\alpha(\tau-t)}\|W_n(\tau,s)\|\,d\tau \\ &+ \sum_{t<\tau_k<s} Kq\delta e^{\alpha(\tau_k-t)}\|W_n(\tau_k,s)\|.\end{aligned}$$

Put

$$u(t) = e^{\alpha t}\|W_n(t,s)\|, \quad F(t) = Ke^{\alpha s}, \quad v(t) = Kq\delta, \quad \beta_k = Kq\delta, \quad \alpha_k(t) = 0.$$

Hence we obtain

$$u(t) \le F(t) + \int_t^s u(\tau)v(\tau) + \sum_{t<\tau_k<s} \beta_k u(\tau - k).$$

Note that for $t < s$ we obtain an inequality of the same form as in the case $t > s$. Then for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$ we get

$$\|W_n(t,s)\| \le K(1+Kq\delta)^{\varepsilon} e^{(\alpha+Kq\delta+p\ln(1+Kq\delta))|t-s|}. \tag{16}$$

From (15) and (16)

$$\begin{aligned}\|Q_{n+1}(t)\| \le \int_{t_0}^{\infty} NK\delta(1+Kq\delta)^{\varepsilon} e^{-\gamma|t-s|}ds \\ + \sum_{k=1}^{\infty} NK\delta(1+Kq\delta)^{\varepsilon} e^{-\gamma|t-\tau_k|}\end{aligned} \tag{17}$$

follows where $\gamma = \Delta - \big(\alpha + Kq\delta + p\ln(1+Kq\delta)\big)$. Thus if we choose δ small enough, then $\alpha < \gamma < \Delta$, and $NK\delta(1+Kq\delta)^{\varepsilon}(\frac{2}{\gamma} + C_\gamma) < q$, where C_γ depends only on τ_k and γ. Based on estimate (17), we obtain

$$\|Q_{n+1}(t)\| \le NK\delta(1+Kq\delta)^{\varepsilon}\Big(\frac{2}{\gamma} + C_\gamma\Big). \tag{18}$$

In view of (18) it follows that $Q_n(t)$ is uniformly bounded.

On the other hand

$$\begin{aligned}Q_{n+1}(t) - Q_n(t) = \int_{t_0}^{\infty} G(t,s)A^{21}(s)\big(W_n(s,t) - W_{n-1}(s,t)\big)\,ds \\ + \sum_{k=1}^{\infty} G(t,\tau_k+0)B_k^{21}\big(W_n(\tau_k,t) - W_{n-1}(\tau_k,t)\big).\end{aligned} \tag{19}$$

It is immediately that the function $V(t) = W_n(t,s) - W_{n-1}(t,s)$ is solution of the system

$$\left.\begin{aligned} \dot V &= \big(A^{11}(t) + A^{12}(t)Q(t)\big)V + A^{12}(t)\big(Q_{n-1}(t) - Q_n(t)\big)W_{n-1}(t,s) \quad && (t \neq \tau_k) \\ \Delta V(\tau_k) &= \big(B_k^{11} + B_k^{12}Q(\tau_k)\big)V(\tau_k) + B_k^{12}\big(Q_{n-1}(t) - Q_n(t)\big)W_{n-1}(t,s) && (k \in \mathbb{N}). \end{aligned}\right\}$$

Then for $t > s$

$$\begin{aligned} V(t) = &\int_s^t W_n(t,\tau)A^{12}(\tau)\big(Q_{n-1}(\tau) - Q_n(\tau)\big)W_(\tau,s)\,d\tau \\ &+ \sum_{s<\tau_k<t} W_n(t,\tau_k)B_k^{12}\big(Q_{n-1}(\tau_k) - Q_n(\tau_k)\big)W_{n-1}(\tau_k,s) \end{aligned}$$

follows. From (16) we obtain

$$\begin{aligned} \|V(t)\| \le &\bigg\{ \int_s^t \big(K(1+Kq\delta)^{\varepsilon}\big)^2 \delta e^{(\alpha + Kq\delta + p\ln(1+Kq\delta))(t-s)} d\tau \\ &\quad + \sum_{s<\tau_k<t} \big(K(1+Kq\delta)^{\varepsilon}\big)^2 \delta e^{(\alpha + Kq\delta + p\ln(1+Kq\delta))(t-s)} \bigg\} \\ &\quad \times \sup_{t\in\mathbb{R}^+} \big\|Q_{n-1}(t) - Q_n(t)\big\| \\ &\le \big(K(1+Kq\delta)^{\varepsilon}\big)^2 \delta\big((1+p)(t-s) + \varepsilon\big) \\ &\quad \times e^{(\alpha + Kq\delta + p\ln(1+Kq\delta))(t-s)} \sup_{t\in\mathbb{R}^+} \big\|Q_{n-1}(t) - Q_n(t)\big\|. \end{aligned}$$

In the case $t < s$ we get

$$\begin{aligned} \|V(t)\| \le &\big(K(1+Kq\delta)^{\varepsilon}\big)^2 \delta\big((1+p)(s-t) + \varepsilon\big) \\ &\times e^{(\alpha + Kq\delta + p\ln(1+Kq\delta))(s-t)} \sup_{t\in\mathbb{R}^+} \big\|Q_{n-1}(t) - Q_n(t)\big\|. \end{aligned}$$

Then for $t \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$

$$\begin{aligned} \|V(t)\| \le &\big(K(1+Kq\delta)^{\varepsilon}\big)^2 \delta\big((1+p)|t-s| + \varepsilon\big) \\ &\times e^{(\alpha + Kq\delta + p\ln(1+Kq\delta))|t-s|} \sup_{t\in\mathbb{R}^+} \big\|Q_{n-1}(t) - Q_n(t)\big\| \end{aligned} \tag{20}$$

is valid. Based on condition (H4) we obtain

$$\left.\begin{aligned} \sum_k^{\infty} e^{-\gamma|t-\tau_k|} < C_\gamma < \infty \\ \sum_k^{\infty} |t-\tau_k| e^{-\gamma|t-\tau_k|} < D_\gamma < \infty \end{aligned}\right\}$$

where D_γ depends only on γ and $\{\tau_k\}$. From (19) and (20) the estimate

$$\begin{aligned}
&\big\|Q_{n+1}(t)-Q_n(t)\big\|\\
&\quad\le\bigg\{\int_{t_0}^{\infty}c\,ds+\sum_{k=1}^{\infty}N\big((1+Kq\delta)^{\varepsilon}\big)^2\delta^2\big((1+p)|t-\tau_k|+\varepsilon\big)e^{-\gamma|t-\tau_k|}\bigg\}\\
&\qquad\times\sup_{t\in\mathbb{R}^+}\big\|Q_n(t)-Q_{n-1}(t)\big\|\\
&\quad\le\bigg\{N\big((1+Kq\delta)^{\varepsilon}\big)^2\delta^2(1+p)\Big(D_\gamma+\frac{3}{\gamma^2}-\frac{1}{\gamma}e^{-\gamma(t-t_0)}-\gamma 1\gamma^2e^{-\gamma(t-t_0)}\Big)\\
&\qquad+\varepsilon\Big(C_\gamma+\frac{2}{\gamma}-\frac{1}{\gamma}e^{-\gamma(t-t_0)}\Big)\bigg\}\sup_{t\in\mathbb{R}^+}\big\|Q_n(t)-Q_{n-1}(t)\big\|
\end{aligned}$$

follows. Then there exists $\delta_0>0$ such that for $\delta\in(0,\delta_0]$ the sequence $\{Q_n(t)\}_{n=1}^{\infty}$ is uniformly convergent to $Q(t)$. The proof of Theorem 1 is complete ∎

Introduce the following conditions:

(H6) There exists a constant $\lambda>0$ such that

$$\begin{aligned}
\sup_{(t,x,y)\in\mathbb{R}^+\times\mathbb{R}^m\times\mathbb{R}^n}\|f(t,x,y)\|\le\lambda, &\qquad \sup_{k\in\mathbb{N},(x,y)\in\mathbb{R}^m\times\mathbb{R}^n}\|I_k(x,y)\|\le\lambda\\
\sup_{(t,x,y)\in\mathbb{R}^+\times\mathbb{R}^m\times\mathbb{R}^n}\|g(t,x,y)\|\le\lambda, &\qquad \sup_{k\in\mathbb{N},(x,y)\in\mathbb{R}^m\times\mathbb{R}^n}\|J_k(x,y)\|\le\lambda.
\end{aligned}$$

(H7) There exists a constant $l>0$ such that

$$\left.\begin{aligned}
\big\|f(t,\overline{x},\overline{y})-f(t,x,y)\big\|&\le l\big(\|\overline{x}-x\|+\|\overline{y}-y\|\big)\\
\big\|g(t,\overline{x},\overline{y})-g(t,x,y)\big\|&\le l\big(\|\overline{x}-x\|+\|\overline{y}-y\|\big)\\
\big\|I_k(\overline{x},\overline{y})-I_k(x,y)\big\|&\le l\big(\|\overline{x}-x\|+\|\overline{y}-y\|\big)\\
\big\|J_k(\overline{x},\overline{y})-J_k(x,y)\big\|&\le l\big(\|\overline{x}-x\|+\|\overline{y}-y\|\big)
\end{aligned}\right\}$$

where $t\in\mathbb{R}^+$, $x,\overline{x}\in\mathbb{R}^m$, $y,\overline{y}\in\mathbb{R}^n$ and $k\in\mathbb{N}$.

Lemma 4. *Let conditions* (H1) - (H7) *be fulfilled, and let the functions*

$$\left.\begin{aligned}
g(t,x,Qx+\eta)-Qf(t,x,Qx+\eta)\\
J_k(x,Qx+\eta)-Q(\tau_k+0)I_k(x,Qx+\eta)
\end{aligned}\right\}$$

where $Q=Q(t)$ is a solution of system (10) *be independent of the variable x. Then for system* (2) *there exists an affinity integral manifold if and only if $\eta=\eta(t)$ is a bounded solution of the system*

$$\left.\begin{aligned}
\dot\eta&=\big(A^{22}(t)-Q(t)A^{12}(t)\big)\eta(t)+H(t,\eta) &&(t\ne\tau_k)\\
\Delta\eta(\tau_k)&=\big(B_k^{22}-Q(\tau_k+0)B_k^{121}\big)\eta(\tau_k)+H(\eta(\tau_k)) &&(k\in\mathbb{N})
\end{aligned}\right\}\tag{21}$$

where

$$\left.\begin{aligned}
H(t,\eta)&=g(t,x,Qx+\eta)-Qf(t,x,Qx+\eta)\\
H_k(\eta)&=J_k(x,Qx+\eta)-Q(\tau_k+0)I_k(x,Qx+\eta).
\end{aligned}\right\}$$

Proof. Lemma 4 can be proved by straightforward calculations and thus its proof is omitted ∎

Theorem 2. *Let conditions* (H1) - (H7) *be fulfilled. Further, assume that the following is true:*

1. *There exists a constant* $\delta > 0$ *such that* $\sup_{t\in\mathbb{R}^+}\|A^{ij}(t)\| \le \delta$ *and* $\sup_{k\in\mathbb{N}}\|B_k^{ij}\| \le \delta$ *where* $i = 1, 2$ *and* $j = 3 - i$.

2. *The functions*

$$\left.\begin{array}{r} g(t,x,Qx+\eta) - Qf(t,x,Qx+\eta) \\ J_k(x,Qx+\eta) - Q(\tau_k+0)I_k(x,Qx+\eta) \end{array}\right\}$$

where $Q = Q(t)$ *is a solution of system* (10) *are independend of the variable* x.

3. *There exists constants* $\mu > 0$ *and* $L > 0$ *such that*

$$\left.\begin{array}{r} \sup\limits_{t\in R^+}\|H(t,0)\| \\ \sup\limits_{k\in\mathbb{N}}\|H_k(0)\| \end{array}\right\} \le \mu \qquad \text{and} \qquad \left.\begin{array}{r} \|H(t,\overline{\eta}) - H(t,\eta)\| \\ \|H_k(\overline{\eta}) - H_k(\eta)\| \end{array}\right\} \le L\|\overline{\eta} - \eta\|$$

where $t \in \mathbb{R}^+$, $\eta, \overline{\eta} \in \mathbb{R}^m$ *and* $k \in \mathbb{N}$.

Then there exist positive constants μ_0, δ_1 *with* $\delta_1 < \delta_0$ *and* L_0 *such that for* $\mu \in (0, \mu_0]$, $L \in (0, L_0]$ *and* $\delta \in (0, \delta_1]$ *an affinity integral manifold for system* (2) *exists.*

Proof. The parameter function $\varphi(t,x) = Q(t)x + \eta(t)$ we shall obtain by the method of consistent approach. Set $\eta_0(t) = 0$ and

$$\begin{aligned} \eta_{n+1}(t) = &\int_{t_0}^{\infty} G(t,s)\Big(H(s,\eta_n(s)) - Q(s)A^{12}(s)\eta_n(s)\Big)ds \\ &+ \sum_{k=1}^{\infty} G(t,\tau_k+0)\Big(H_k(\eta_n(\tau_k)) - Q(\tau_k+0)B_k^{12}\eta_n(\tau_k)\Big) \end{aligned} \tag{22}$$

for $n \in \mathbb{N}$. From Theorem 1 it follows that there exists function $Q = Q(t)$ which is solution of system (10). From (22) for $n = 0$ we obtain

$$\begin{aligned} \|\eta_1(t)\| &\le \int_{t_0}^{\infty} \|G(t,s)\|\Big(\|H(s,0)\|\,ds + \sum_{k=1}^{\infty}\|G(t,\tau_k+0)\|\,\|H_k(0)\|\Big) \\ &\le N\mu\Big(\frac{2}{\Delta} + C_\Delta\Big). \end{aligned}$$

On the other hand, if $\|\eta_n(t)\| \le \sigma$ for some $\sigma > 0$, then

$$\begin{aligned} \|\eta_{n+1}\| \le &\int_{t_0}^{\infty} \|G(t,s)\|\Big(\|H(s,\eta_n(s))\| + \|Q(s)\|\,\|A^{12}(s)\|\,\|\eta_n(s)\|\Big)ds \\ &+ \sum_{k=1}^{\infty}\|G(t,\tau_k+0)\|\Big(\|H_k(\eta_n(\tau_k))\| + \|Q(\tau_k+0)\|\,\|B_k^{12}\|\,\|\eta_n(\tau_k)\|\Big) \\ \le &\, N(L\sigma + \mu + q\delta\sigma)\Big(\frac{2}{\Delta} + C_\Delta\Big) \end{aligned} \tag{23}$$

and

$$\begin{aligned}\big\|\eta_{n+1}-\eta_n(t)\big\| \le \int_{t_0}^{\infty} &\|G(t,s)\|\Big(\big\|H(s,\eta_n(s))-H(s,\eta_{n-1}(s))\big\| \\ &+\|Q(s)\|\,\|A^{12}(s)\|\,\big\|\eta_n(s)-\eta_{n-1}(s)\big\|\Big)ds \\ &+\sum_{k=1}^{\infty}\|G(t,\tau_k+0)\|\Big(\big\|H_k(\eta_n(\tau_k))-H_k(\eta_{n-1}(\tau_k))\big\| \\ &+\big\|Q(\tau_k+0)\big\|\,\big\|B_k^{12}\big\|\,\big\|\eta_n(\tau_k)-\eta_{n-1}(\tau_k)\big\|\Big) \\ \le N(L+q\delta)&\Big(\frac{2}{\Delta}+C_\Delta\Big)\sup_{t\in\mathbb{R}^+}\big\|\eta_n(t)-\eta_{n-1}(t)\big\|.\end{aligned} \tag{24}$$

Based on estimates (23) and (24) we obtain that there exist positive constants μ_0, δ_1 with $\delta_1 < \delta_0$ and L_0 such that for $\mu \in (0, \mu_0]$, $L \in (0, L_0]$ and $\delta \in (0, \delta_1]$ the sequence $\{\eta_n(t)\}_{n=0}^{\infty}$ converges to $\eta(t)$. Then the proof of Theorem 2 follows from Lemma 4 ∎

Example 1. Consider the system of impulsive differential equations

$$\left.\begin{aligned} \dot{x} &= x && (t \neq \tau_k) \\ \Delta x(\tau_k) &= a_k x(\tau_k) + b_k y(\tau_k) && (k \in \mathbb{N}) \\ \dot{y} &= 2y - \sin tx && (t \neq \tau_k) \\ \Delta y(\tau_k) &= \big(a_k + \tfrac{1}{2}(-1)^k b_k\big)\, y(\tau_k) && (k \in \mathbb{N}) \end{aligned}\right\} \tag{25}$$

where $t \in \mathbb{R}^+$, $x, y \in \mathbb{R}$, $\tau_k = k\pi$ $(k \in \mathbb{N}$, $\{a_k\}$ and $\{b_k\}$ are real bounded sequences. The function $Q(t) = \frac{1}{2}(\sin t + \cos t)$ is solution of a system of the form (10) and the conditions of Theorem 1 are fulfilled. Then for system (25) there exists an integral manifold with parameter function $\varphi(t, x) = \frac{1}{2}(\sin t + \cos t)$.

Example 2. Consider the system

$$\left.\begin{aligned} \dot{x} &= x + xy + 2 && (t \neq \tau_k) \\ \Delta x(\tau_k) &= a_k y(\tau_k) && (k \in \mathbb{N}) \\ \dot{y} &= \cos tx + y + xy \sin t && (t \neq \tau_k) \\ \Delta y(\tau_k) &= (-1)^k a_k y(\tau_k) && (k \in \mathbb{N}) \end{aligned}\right\} \tag{26}$$

where $t \in \mathbb{R}^+$, $x, y \in \mathbb{R}$, $\{a_k\}_{k\in\mathbb{N}}$ is a bounded real sequence and $\tau_k = \frac{2k+1}{2}\pi$. The functions $Q(t) = \sin t$ and $\eta(t) = \sin t + \cos t$ are solutions of system (10) and (21), respectively, and the conditions of Theorem 2 are fulfilled. Then for system (26) there exists an integral manifold with parameter function $\varphi(t, x) = \sin tx + \sin t + \cos t$.

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