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Existence Theorems for Boundary Value Problems for Strongly Nonlinear Elliptic Systems

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Abstract. Let L be a linear elliptic, a pseudomonotone or a generalized monotone operator (in the sense of F. E. Browder and I. V. Skrypnik), and let F be the nonlinear Nemytskij superposition operator generated by a vector-valued function f. We give two general existence theorems for solutions of boundary value problems for the equation $Lx = Fx$. These theorems are based on a new functional-theoretic approach to the pair (L, F), on the one hand, and on recent results on the operator F, on the other hand. We treat the above mentioned problems in the case of strong non-linearity F, i.e. in the case of lack of compactness of the operator $L - F$. In particular, we do not impose the usual growth conditions on the nonlinear function f; this allows us to treat elliptic systems with rapidly growing coefficients or exponential non-linearities. Concerning solutions, we consider existence in the classical weak sense, in the so-called L_∞-weakened sense in both Sobolev and Sobolev-Orlicz spaces, and in a generalized weak sense in Sobolev-type spaces which are modelled by means of Banach L_∞-modules. Finally, we illustrate the abstract results by some applied problems occuring in nonlinear mechanics.

Keywords: *Strongly nonlinear boundary value problems, existence theorems, solution in the usual weak sense, solution in the L_∞-weakened sense, elliptic operators, pseudomonotone operators, generalized monotone operators, complementary systems, Sobolev spaces, Sobolev-Orlicz spaces, exact embedding theorems*

AMS subject classification: Primary 35 J 65, secondary 46 E 30, 47 H 10, 47 H 17, 47 H 30

0. Introduction

The present paper is concerned with the existence of solutions of strongly nonlinear elliptic boundary value problems, i.e. boundary value problems for elliptic equations and systems of the form

$$Lx = Fx \tag{1}$$

on an open bounded subset Ω of $\mathbb{R}^N$ $(N \geq 2)$, where L is a linear or nonlinear elliptic differential operator of order $2k$ in the divergence form

$$Lx = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha a_a(s, \delta_k x(s)) \tag{2}$$

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with coefficients a_α satisfying conditions (including growth conditions) which guarantee the solvability of the problem $Lx = h$. The Carathéodory function $f(s,u) : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ generating the nonlinear Nemytskij superposition operator F (see, e.g., [3 - 4]) where

$$Fx(s) = f(s, x(s)) \tag{3}$$

satisfies some condition but has otherwise completely inrestricted growth ("strongly nonlinear" growth) with respect to $u \in \mathbb{R}^m$. In other words, the "strong non-linearity" of F means that we have to consider the cases of the operator $L - F$ with "lack of compactness" and of "critical exponents" in the "exact" Sobolev embedding theorems (S. L. Sobolev, S. I. Pokhozaev and N. S. Trudinger and their generalizations). The nonlinear function f is assumed in the present paper to satisfy a general one-sided estimate, which covers all the so-called "sign condition" and "generalized sign condition". We point out that the above one-sided estimate is not sufficient for getting existence results in the non-scalar vector case $m > 1$ (i.e. the case of systems of equations), and so for this case we should suppose some addittional conditions on the growth of f. This one-sided estimate, in particular, "compensates" the lack of both compactness and monotonicity type conditions of $L - F$.

Such problems in the scalar case (i.e. $m = 1$) were first considered by F. E. Browder [9] and later on up to the present time by P. Hess, H. Brezis and F. E. Browder, J. Webb, J.-P. Gossez and others (see, for example, [6 - 8, 10, 15 - 16, 18, 23, 34], and the literature cited therein) via variational and topological methods.

In this paper we shall present two general existence theorems (see Theorem 1 in Section 7 and Theorem 2 in Section 8) following ideas of our dissertation [26] for problem (1). We consider both systems with polynomial growth coefficients and with exponential growth coefficients, as well as, both the existences in the usual weak sense and in the so-called L_∞-weakened sense of H. Brezis and F. E. Browder in Sobolev spaces and of J.-P. Gossez in Sobolev-Orlicz spaces, and in more general spaces as so-called generalized Sobolev spaces (which are modelled by means of Banach function L_∞-modules).

These theorems are based, on the one hand, on a new function-theoretic approach to the pair (L, F), which unifies all the approaches of [7 - 10, 15 - 16], and the approaches [2, 26, 37], and which is applicable not only to linear strongly elliptic operators L or pseudomonotone operators L but also to generalized monotone operators L in the sense of Browder and Skrypnik (see, e.g., [29, 31 - 32]) and, on the other hand, on the recent theories of Banach function L_∞-modules of vector-valued functions and of nonlinear operators acting in such spaces, which are presented in [3 - 4, 25 - 27, 36]. We point out that the structure of such spaces is much more complicated than in the scalar case, and thus the results presented here are not just generalizations from scalar differential equations to systems.

The plan of our paper is the following one. In Section 2 we shall give simplest variants of the two main theorems, namely Theorem 1 in Section 7 and Theorem 2 in Section 8 for the quasilinear elliptic boundary value problem (1) where the nonlinear superposition operator F is a "strongly non-linearity" and L is a operator, which is weakly nonlinear in the sense of M. I. Vishik - Yu. A. Dubinskij (see, e.g., [11]) [$L \in \mathfrak{C}(\mathcal{L})$], pseudomonotone in the sense of H. Brezis [$L \in \mathfrak{C}(BM)$] (in particular, a "variational operator" of J. Leray-J.-L. Lions) (see, e.g., [21]), generalized monotone in the sense of

F. E. Browder - I. V. Skrypnik (see, e.g., [29, 31 - 32]) [$L \in \mathfrak{C}(BS)$], or pseudomonotone with respect to complementary systems in the sense of J.-P. Gossez (see, e.g., [14 - 15, 17]) [$L \in \mathfrak{C}(GM)$].

For these Theorems we shall give the definition of a complementary system

$$(V, V_0; W, W_0)$$

in Section 3, introduce the classes of spaces $\mathfrak{Z}_V(L, F, \mathfrak{F})$ in Section 4 (connected with one-sided estimates and the strong non-linearity of F), $\mathfrak{N}(L, F)$ (with the exact definitions of the classes $\mathfrak{C}(\mathcal{L})$, $\mathfrak{C}(BM)$, $\mathfrak{C}(BS)$, $\mathfrak{C}(GM)$) in Section 5, condition (R1) in Lemma 3 and condition (R2) in Lemma 4 in Section 6, $\mathfrak{Z}_V(L, F, \Gamma)$ and $\mathfrak{N}(L, F, \Gamma)$ and $\mathrm{Tr}(V_0, \Gamma)$ in Section 8, and finally give some applications in nonlinear mechanics in Section 9.

We point out that these theorems can be extended also to variational elliptic inequalities as well as in many papers cited in [17], and we shall treat in detail this in a forthcoming paper.

1. Some terminology and notations

For our analysis in this paper we need the following notions concerning Banach spaces of measurable functions, in particular, the class of Banach L_∞-modules (see [25 - 27, 36]).

Recall [36] that a Banach space X of measurable functions $x : \Omega \to \mathbb{R}^m$ with norm $\|\cdot\|_X$ is called a Banach L_∞-module (Banach function L_∞-module) if $x \in X$ and $\alpha \in L_\infty$ imply that $\alpha x \in X$ and $\|\alpha x\|_X \leq \|\alpha\|_{L_\infty} \|x\|_X$. Here $L_\infty = L_\infty(\Omega, \mathbb{R})$ is the Banach algebra of all essentially bounded measurable scalar functions defined on Ω. In the scalar case $m = 1$, Banach L_∞-modules are just Banach lattices with monotone norm (see, e.g., [5, 22, 33]). In the vector case $m > 1$, the theory of Banach L_∞-modules is more involved and requires tools from Convex and Set-Valued Analysis (see [25 - 26]). A prominent example of a Banach L_∞-module is the Orlicz space $L_M = L_M(\Omega, \mathbb{R}^m)$ with the Luxemburg norm

$$\|x\|_{L_M} = \inf \left\{ \lambda > 0 : \int_\Omega M\Big(s, \frac{x(s)}{\lambda}\Big) ds < \infty \right\}$$

where $M : \Omega \times \mathbb{R}^m \to [0, \infty]$ is a given Young funtion (see, e.g., [5, 24 - 25, 30]). If $M(s, u) = \|u\|^p \quad (1 \leq p < \infty)$, the Orlicz space L_M is just the Lebesgue space L_p. Other important examples of Banach L_∞-modules are the Lorentz space and the Marcinkiewicz space which are modelled by means of L_M (see, e.g., [5, 19]).

Let X be a Banach L_∞-module with the property that there exists a sequence $\{x_i\}_{i \in \mathbb{N}} \subset X$ such that the linear hull of $\{x_i(s)\}_{i \in \mathbb{N}}$ coincides with $\mathbb{R}^m$ for almost all $s \in \Omega$. Then there exists the non-trivial so-called (Köthe) associate space X' of all measurable functions $x' : \Omega \to \mathbb{R}^m$ for which the pairing of measurable functions

$$\langle x, x' \rangle = \int_\Omega (x(s), x'(s))\, ds$$

is finite for every $x \in X$, where $(\cdot,\cdot)$ is the usual scalar product in $\mathbb{R}^m$. The space X' equipped with the natural norm

$$\|x'\|_{X'} = \sup\{\langle x, x'\rangle : \|x\|_X \le 1\}$$

is then also a Banach L_∞-module. X' can be interpreted as closed subspace of all integral linear functionals of the dual space X^*. For example, if $X = L_M$, then X' is the Orlicz space L_{M^*} generated by the associate Young function

$$M^*(s,v) = \sup\Big\{(u,v) - M(s,u) : u \in \mathbb{R}^m\Big\}.$$

We have $(L_p)' = L_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ $(1 \le p \le \infty)$.

The regular part X° of a Banach L_∞-module X is defined as Banach L_∞-module of all measurable function $x \in X$ with absolutely continuous [equi-continuous] norm, i.e.

$$X^\circ = \Big\{x \in X : \lim_{\operatorname{mes}(D)\to 0} \|P_D x\|_X = 0\Big\},$$

where P_D denotes the multiplication operator by the characteristic function χ_D of a measurable set D. If $X = X^\circ$, then X is called a regular space. The space X' coincides with the dual space X^* of X if and only if X is regular. The space L_M is regular if and only if M satisfies the Δ_2-condition (see [24, 26, 30]). The space L_p is regular if $p < \infty$, and is not regular if $p = \infty$.

The embedding $Y \subset X$ of Banach L_∞-modules X and Y is called absolutely bounded if the unit ball of Y is an absolutely bounded subset of X (under other name, it has uniformly absolutely continuous [equi-continuous] norms), i.e.

$$\lim_{\operatorname{mes}(D)\to 0} \sup_{\|x\|_Y \le 1} \|P_D x\|_X = 0.$$

For example, the embedding $L_p \subset L_q$ is absolutely bounded, if $p > q$.

Let a Banach L_∞-module Γ of the associate space X' be total on a Banach L_∞-module X. Then, a subset $\mathfrak{N}$ of X is called a Γ-weakly [$\sigma(X,\Gamma)$-weakly] absolutely bounded set if the condition

$$\lim_{\operatorname{mes}(D)\to 0} \sup_{x \in \mathfrak{N}} \langle P_D x, z\rangle = 0 \qquad (z \in \Gamma)$$

holds. Later on the symbols $\hookrightarrow$ or $\hookrightarrow\!\!\hookrightarrow$ denotes continuous imbedding or compact imbedding of corresponding spaces, respectively. Denote by $S(\Omega,\mathbb{R}^m)$ the metric space of all measurable functions $x : \Omega \to \mathbb{R}^m$, equipped with the metric convergence in measure. For every Banach L_∞-module X we have $X \hookrightarrow S(\Omega,\mathbb{R}^m)$.

Remark that the theory of the Nemytskij superposition operator F acting in Banach L_∞-modules and in Orlicz spaces is given in [3 - 4, 26] for the vector case $m > 1$, and in [5] for the scalar case $m = 1$.

2. Simplest variants of main existence theorems

In this Section we present the simplest variants of the Main Existence Theorems 1 and 2.

First, we formulate the following simplest variant of Theorem 1 (for operators L in $\mathfrak{C}(\mathcal{L})$, $\mathfrak{C}(BM)$ or $\mathfrak{C}(BS)$) on the existence of solutions for the boundary value problem (1) with $x \in V$ where V with $\mathring{W}_p^k(\Omega) \hookrightarrow V \hookrightarrow W_p^k(\Omega)$ $(1 < p < \infty)$ is a reflexive Banach space of vector functions with values in $\mathbb{R}^m$ and L is a $2k$'s order formal elliptic operator generating the operator $L : V \to V^*$. We denote by Z a "minimal" Banach L_∞-module such that V is continuously (but non-compactly) embedded into Z (the imbedding inclusion $V \hookrightarrow Z$ is defined by means of Sobolev embedding theorems and their various generalizations).

Theorem 1′. *Let X, Y, $\Gamma \subset S(\Omega, \mathbb{R}^m)$ be Banach L_∞-modules, $Z \subset \Gamma$, $Z \subset X$ and $Z \subset Y$, and suppose that following conditions are satisfied:*

- **(P1)** *$L : V \to V^*$ is a bounded operator, either hemicontinuous pseudomonotone in the sense of H. Brezis, or generalized monotone in the sense of F. E. Browder - I. V. Skrypnik.*
- **(P2)** *There exists a Carathéodory function $H : \Omega \times \mathbb{R}^m \to [0, \infty)$ such that $(u, f(s, u)) \le H(s, u)$ and $\langle Lx, x\rangle - H[x] > 0$ $(\|x\|_V \ge r > 0)$ for some $r > 0$.*
- **(P3)** *The embedding $Z \subset Y$ is absolutely bounded, and there exists a Carathéodory function $B : \Omega \times \mathbb{R}^m \to [0, \infty)$ such that $(u, f(s, u)) \le B(s, u)$ and the superposition operator B generated by the function $B(\cdot, \cdot)$ acts from Y into $L(\Omega, \mathbb{R})$.*
- **(P4)** *Either*

(a) $m = 1$ (*the case of scalar equations*) *and $X = \Gamma$ and F acts from Γ into Γ'*

or

(b) $m > 1$ (*the case of systems of equations*), *$Z \subset X$ and $F : X \to \Gamma'$ maps bounded subsets of X into Γ-weakly absolutely bounded subsets of Γ'*

or else

(c) $m > 1$ (*the case of systems of equations*), *the imbedding $Z \subset X$ is absolutely bounded and F acts from X into Γ'.*

Then problem (1) *has at least one solution in $V \subset X$.*

We point out that in the case $L \in \mathfrak{C}(\mathcal{L})$ (in particular, in the case of a linear operator L) we can drop condition (P3). Then we get results containing as particular cases existence theorems obtained in [2] via other methods.

Second, we formulate the following simplest variant of Theorem 2 on the existence of solutions in the Γ-weakened sense for problem (1) where $L \in \mathfrak{C}(BM)$ or $L \in \mathfrak{C}(GM)$). We formulate this variant of Theorem 2 for the case when $L \in \mathfrak{C}(BM), \Gamma = L_\infty$ and $V = \mathring{W}_p^k(\Omega)$ $(1 < p < \infty)$ that corresponds to the Dirichlet problem. Let $L : \mathring{W}_p^k(\Omega) \to W_{p'}^{-k}(\Omega)$ be a $2k$'s order elliptic operator, $p' = \frac{p}{p-1}$. We recall that the boundary value problem

$$Lx = Fx \qquad (x \in \mathring{W}_p^k(\Omega)) \tag{4}$$

is solvable in the L_∞-weakened sense if there exists a function $x_* \in \mathring{W}_p^k(\Omega)$ such that

$$\langle y, Lx_*\rangle_V = \langle y, Fx_*\rangle \qquad \big(y \in L_\infty \cap \mathring{W}_p^k(\Omega)\big) \tag{5}$$
$$\langle x_*, Lx_*\rangle_V = \langle x_*, Fx_*\rangle. \tag{6}$$

Let $X, Y \subset S(\Omega, \mathbb{R}^m)$ be Banach L_∞-modules, $Z \subset X$ and $Z \subset Y$, where $Z = L_{p^*}$ if $kp < N$, $Z = L_\infty$ if $kp > N$, and $Z = L_\phi^\circ$ if $kp = N$ (here $p^* = \frac{Np}{N-kp}$ and $\phi(s,u) = \exp(\|u\|^{\frac{p}{p-1}}) - 1$). By the exact embedding theorems of S. L. Sobolev (see [1]) and of S. I. Pokhozaev - N. S. Trudinger (see [1]) $\mathring{W}_p^k(\Omega)$ is continuously (but non-compactly if $kp \le N$) embedded into Z. Remember that L_ϕ° is the regular part of the Orlicz space L_ϕ (equivalently, it is the closure of the set of continuous functions in L_ϕ).

Theorem 2′. *Suppose that the mentioned above conditions* (P2) - (P4) *for* $\Gamma = L_\infty$ *but without* $Z \subset \Gamma$, *and the following condition*

(P1′) $L : \mathring{W}_p^k(\Omega) \to W_{p'}^{-k}(\Omega)$ *is an operator, which is bounded hemicontinuous and pseudomonotone in the sense of H. Brezis*

are satisfied. Then problem (4) *has at least one solution* x_* *in the* L_∞*-weakened sense* (5) – (6).

We point out that by the scheme for L_∞-weakened solutions in the papers of F. E. Browder, F. E. Browder and H. Brezis, P. Hess, J. Webb, J.-P. Gossez, V. Mustonen, and others mathematicians (see, e.g., [6 - 10, 15 - 16, 18, 34] and the literature cited therein) one can consider only the case when $m = 1$ and $H(s,u) = B(s,u) = b(s)$ with $b \in L(\Omega, \mathbb{R})$ (i.e. when the so-called "sign" condition $(u, f(s,u)) \le b(s)$ is satisfied). We would like to point out that this "sign" condition turns out not to be sufficient for the existence of solutions in the non-scalar vector case $m \ne 1$ (i.e. the case of systems of equations), and so for this case we should suppose some addittional conditions on the growth of f as in the above condition (P4)/(b) and (c).

3. Elliptic differential operator L and complementary system $(\mathbf{V}, \mathbf{V_0}; \mathbf{W}, \mathbf{W_0})$

In this section we give the definition of an elliptic differential operator L in a complementary system $(V, V_0; W, W_0)$ and some facts about the construction of complementary system in applications.

Let Ω be a bounded open subset in the Euclidean space $\mathbb{R}^N$ $(N \ge 2)$, $\alpha = (\alpha_1, \dots, \alpha_N)$ a multi-index with non-negative natural components, $|\alpha| = \sum_j^N \alpha_j$. If $x = x(s) = (x_1(s), \dots, x_m(s))$ is some vector-valued function, then denote

$$D^\alpha x = \frac{\partial^{|\alpha|} x}{\partial s_1^{\alpha_1} \cdots \partial s_N{}^{\alpha_N}}, \qquad \delta_k x = \{D^\alpha x_j\}_{|a| \le k}^{j \le m}, \qquad k(N) = \frac{(N+k)!}{N!\,k!}.$$

Let L be a $2k$'s order elliptic differential operator in the divergence form (2) where $a_\alpha : \Omega \times \mathbb{R}^{mk(N)} \to \mathbb{R}^m$ are Carathéodory functions, i.e. $a_\alpha(\cdot, u)$ is measurable for all u and $a_\alpha(s, \cdot)$ is continuous for almost all s.

Let V and W be two Banach spaces, V of vector-valued functions $x : \Omega \to \mathbb{R}^m$ such that x and its distribution partial derivatives up to order k lie in $S(\Omega, \mathbb{R}^m)$. Suppose that (V, W) is a duality pairing with respect to a bilinear form $\langle \cdot, \cdot \rangle_V$ for which

$$\langle x, y \rangle_V = \langle x, y \rangle \qquad \big(x \in V \subset S(\Omega, \mathbb{R}^m), y \in W \cap S(\Omega, \mathbb{R}^m)\big) \tag{7}$$

holds. Further, let

$$(V, V_0; W, W_0)$$

be a *complementary system* in the sense of T. Donaldson - J.-P. Gossez [13 - 14], i.e. $V_0 \subset V$ and $W_0 \subset W$ are subspaces such that, by means of $\langle \cdot, \cdot \rangle_V$, V_0^* can be identified with (i.e. is linear homeomorphic to) W and W_0^* with V.

Suppose that the formal differential operator (2) defines the operator $L : V \to W$ with domain $D(L) \supset V_0$ (which, in general cases, is not everywhere defined and nonlinear) by

$$\langle y, Lx \rangle_V = L(x, y) \qquad \big(y \in V_0, x \in D(L)\big) \tag{8}$$

and

$$L(x, y) = \int_\Omega \sum_{|\alpha| \le k} \big(a_\alpha(s, \delta_k x(s)), D^\alpha y(s)\big) ds. \tag{9}$$

We point out that in applications for the case of reflexive Banach spaces V we shall use the complementary system $(V, V; V^*, V^*)$. In applications for the non-refexive case we shall use for the Dirichlet problem the system

$$(\mathring{W}^k E, \mathring{W}^k E^\circ; W^{-k} E', W^{-k}(E')^\circ)$$

and for the Neumann problem the system

$$(V, V_0; W, W_0) \qquad \text{with } \ V = W^k E \ \text{ and } \ V_0 = W^k E^\circ$$

(see references in [13 - 14, 17, 19]). Here $E \subset S(\Omega, \mathbb{R}^{mk(N)})$ is a Banach L_∞-module and E° is its regular part (see Section 1), and $W^k E$ is the "generalized" Sobolev-like space (modelled by means of E) of vector-valued functions x with the norm

$$\|x\|_{W^k E} = \|\delta_k x\|_E \tag{10}$$

($\delta_k x$ is the vector of which components are all distribution partial derivatives of x up to order k), and $\mathring{W}^k E$ is the $\sigma(E, (E')^\circ)$-weak closure of $C_\circ^\infty(\Omega, \mathbb{R}^m)$ in $W^k E$. Remember (see, e.g., [13 - 14, 21]) that V with $\mathring{W}_p^k \subset V \subset W_p^k$ (the usual Sobolev spaces) corresponds to boundary value problems for equations having coefficients with polynomial growth, and V with $\mathring{W}^k L_M \subset V \subset W^k L_M$ (the Sobolev-Orlicz spaces) corresponds to boundary value problems for equations having coefficients with rapidly or slowly increasing growths, etc.

4. Class $\mathfrak{Z}_V(L, F, \mathfrak{F})$

Let $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ be a Carathéodory function generating the superposition operator $F : V \to W$ and having domain $D(F)$. Sections 4 - 8 devote the existence investigation on the boundary value problem (1) where $L : V \to W$ is the operator with domain $D(L)$ which was introduced in Section 3.

We put

$$T_V(r) = \{x \in V : \|x\|_V \le r\} \qquad (0 < r < \infty) \tag{11}$$

and

$$H(r) = \{Lx : x \in T_V(r), Lx = \theta F x \text{ for some } \theta \in \Xi\} \tag{12}$$

where Ξ is the set of all measurable scalar functions such that $0 \le \theta(s) \le 1$ for almost all $s \in \Omega$. We recall that $(Y, \Gamma) \in \mathfrak{Z}_V(L, F, \mathcal{F})$ (here Γ is a total on a Banach L_∞-module Y Banach L_∞-submodule of Y') if the sets $H(r)$ $(0 < r < \infty)$ are Γ-weakly absolutely bounded in Y.

The verification of the inclusion $(Y, \Gamma) \in \mathfrak{Z}_V(L, F, \mathcal{F})$ is connected with many considerable difficulties. We shall describe the main non-trivial scheme for this in the present section.

We shall be interested in the imbedding $V \hookrightarrow\hookrightarrow S(\Omega, \mathbb{R}^m)$ and in constructing a Banach L_∞-module X such that $V \hookrightarrow X$ or $V \hookrightarrow\hookrightarrow X$. The imbedding $V \hookrightarrow X$ can be defined by embedding theorems of Sobolev type, and the imbedding $V \hookrightarrow\hookrightarrow S(\Omega, \mathbb{R}^m)$ follows from theorems of Rellich-Kondrashov type (see, e.g., [1, 19] and the various literature cited therein).

The verification of the imbedding $V \hookrightarrow\hookrightarrow X$ is connected with the following lemma.

Lemma 1. *Let $X_0, X \subset S(\Omega, \mathbb{R}^m)$ be Banach L_∞-modules, $V \hookrightarrow X_0$, $V \hookrightarrow\hookrightarrow S(\Omega, \mathbb{R}^m)$ and $X_0 \subset X$ be an absolutely bounded embedding. Then $V \hookrightarrow\hookrightarrow X$ holds.*

Proof. By assumptions the unit ball of V is pre-compact in $S(\Omega, \mathbb{R}^m)$ and absolutely bounded in the Banach L_∞-module X. Hence, it is pre-compact in X by [27: Theorem 8] ∎

We recall that $(L, F) \in \mathcal{P}_\circ(V, Z)$ (where Z is a Banach L_∞-module and $V \hookrightarrow Z$) if f satisfies the inequalities

$$(\pi u, f(s, u)) \le A_\pi(s, u) \ge 0 \qquad (A_\pi(s, 0) = 0) \tag{13}$$

and if the inequalities

$$\begin{aligned} \|A_\pi x\|_L &\le a_\pi(r) && (\|x\|_Z \le r, \pi \in \mathcal{P}) \\ \langle \pi x, Lx \rangle_V &\ge c_\pi(r) && (\pi \in \mathcal{P}, x \in T_V(r) \cap D(L)) \end{aligned} \tag{14}$$

hold where $a_\pi(r)$ and $c_\pi(r)$ $(r > 0)$ are constants, and $A_\pi : \Omega \times \mathbb{R}^m \to \mathbb{R}$ are superpositionally measurable (for example, Carathéodory) functions generating the superposition operators A_π, and $\mathcal{P} = \{\pi\}$ is some (maybe, empty) family of orthogonal projectors in $\mathbb{R}^m$ which commute with operators L and F. In applications, $\mathcal{P} = \{\emptyset\}$, $\mathcal{P} = \{I\}$ with

I the identity operator or $\mathcal{P} = \{\cdot_j : j = 1, \ldots, m\}$ where $\cdot_j$ is the operator of passing to the j'th component.

We put

$$\Delta(d_\pi) = \left\{ Lx : x \in D(L), \int_\Omega \big|\big(\pi x(s), Lx(s)\big)\big| ds \le d_\pi \right\} \qquad (\pi \in \mathcal{P}). \tag{15}$$

Lemma 2. *Let $(L, F) \in \mathcal{P}_\circ(V, Z)$. Then $H(r) \subset \Delta(d_\pi)$ where $d_\pi = 2a_\pi(cr) - c_\pi(r)$ $(\pi \in \mathcal{P})$ and $c = c(V, Z)$ is the imbedding constant of V into Z.*

Proof. Assume $y \in H(r)$. Then for some $x \in T_V(r) \cap D(L)$ and for some $\theta \in \Xi$, $y = Lx = \theta Fx$. Evidently,

$$\int_\Omega (\pi x, \theta Fx)\, ds = \int_\Omega (\pi x, Lx)\, ds = \langle \pi x, Lx \rangle \ge c_\pi(r).$$

For each measurable set D we have

$$\int_D (\pi x, \theta Fx)\, ds \le \int_D A_\pi x(s)\, ds = \|A_\pi(P_D x)\|_L \le a_\pi(cr).$$

Let $\Omega(\pi)$ denote the set of all $s \in \Omega$ such that $\big(x(s), \theta(s)Fx(s)\big) \ge 0$. Then

$$\begin{aligned} \int_\Omega |(\pi x, Lx)|\, ds = \int_\Omega |(\pi x, \theta Fx)|\, ds &= \left(2\int_{\Omega(\pi)} - \int_\Omega \right) (\pi x, \theta Fx)\, ds \\ &\le d_\pi = 2a_\pi(cr) - c_\pi(r) = d_\pi(r) \end{aligned}$$

and the lemma is proved ∎

Now suppose that f satisfies else the inequality (in this case we write $f \in \mathcal{P}^\circ(V)$)

$$(v, f(s,u)) \le \sum_{\pi \in \mathcal{P}} c_\pi |(\pi u, f(s,u))| + P(s,u) + Q(s,v) + R(s,v,u) \tag{16}$$

where c_π $(\pi \in \mathcal{P})$ are non-negative constants and P, Q, R $(R(s,v,0) = 0)$ are Carathéodory functions such that the corresponding superposition operators P, Q, R map $V, \Gamma, \Gamma \times V$ into $L(\Omega, \mathbb{R})$, respectively, where Γ is a total on Y Banach L_∞-module of Y'.

Proposition 1. *Assume $Y = \Gamma'$, $(L, F) \in \mathcal{P}_\circ(V, Z)$, and $f \in \mathcal{P}^\circ(V)$, and that one of the following conditions is satisfied:*

(i) *$V \hookrightarrow\hookrightarrow X$.*

(ii) *$V \hookrightarrow X$, the operator P is bounded on each ball of X, and the operator $R(z, \cdot)$ is absolutely bounded on each ball of X for every fixed $z \in \Gamma$.*

(iii) *$V \hookrightarrow X$, Γ is a regular Banach L_∞-module, i.e. $\Gamma = \Gamma^\circ$, and the operator $R(z, \cdot)$ is bounded on each ball of X for every fixed $z \in \Gamma$.*

Then $(Y, \Gamma) \in \mathfrak{Z}_V(L, F, \mathcal{F})$.

Proof. From Lemma 2 and (16) the inequality

$$|\langle P_D z, \theta F x\rangle| \le \lambda \sum_{\pi\in\mathcal{P}} c_\pi d_\pi(r) + \lambda\|Px\|_L + \lambda\|P_D Q(\lambda^{-1}z)\|_L + \lambda\|P_D R(\lambda^{-1}z, x)\|_L$$

follows where $0 < \lambda < \infty$, $z \in \Gamma$, $Lx = \theta F x \in H(r)$, $x \in T_V(r)$ and $\theta \in \Xi$. By (17) and the assumptions of Proposition 1 we have

$$\begin{aligned} |\langle P_D z, \theta F x\rangle| \le \lambda \sum_{\pi\in\mathcal{P}} c_\pi d_\pi(r) &+ \lambda \sup_{\|x\|_X \le cr} \|Px\|_L \\ &+ \lambda\|P_D Q(\lambda^{-1}z)\| + \lambda \sup_{\substack{x\in T_V(r)\\ Lx=\theta F x}} \|P_D R(\lambda^{-1}z, x)\|_L \end{aligned} \qquad (0 < \lambda < \infty)$$

where c is the imbedding constant of $V \hookrightarrow X$.

By either $V \hookrightarrow\hookrightarrow X$ or the boundedness of the superposition operator $P : X \to L(\Omega, \mathbb{R})$ it follows that for every fixed $\varepsilon > 0$ we can find a suitable number $\lambda_\varepsilon > 0$ such that

$$|\langle P_D z, \theta F x\rangle| \le \varepsilon + \lambda_\varepsilon\|P_D Q(\lambda_\varepsilon^{-1}z)\| + \lambda_\varepsilon \sup_{\substack{x\in T_V(r)\\ Lx=\theta F x}} \|P_D R(\lambda_\varepsilon^{-1}z, x)\|_L.$$

By the Krasnoselskij type continuity theorem for superposition operators acting in Banach L_∞-modules (see, e.g., [4: Theorem 3]) and of the Vitali-Krasnoselskij compactness criterion (see, e.g. [20] and [27: Theorem 8]) for subsets in $L(\Omega, \mathbb{R})$, in the case when assumption (i) or (ii) holds, the equality

$$\lim_{\mathrm{mes}(D)\to 0} \sup_{\substack{x\in T_V(r)\\ Lx=\theta F x}} |\langle P_D z, \theta F x\rangle| = 0 \tag{18}$$

holds and, in the case when assumption (iii) holds, the inequality

$$\sup_{\substack{x\in T_V(r)\\ Lx=\theta F x}} |\langle z, \theta F x\rangle| < \infty$$

holds which, by regularity of the Banach L_∞-module Γ and the Banach-Steinhaus theorem, is (see [26] and [27: Theorem 8]) equivalent to (18). Equality (18) means that the set $H(r)$ (see (12)) is Γ-weakly absolutely bounded in Y for all $r > 0$, i.e. $(Y, \Gamma) \in \mathfrak{Z}_V(L, F, \mathcal{F})$ ∎

As simple but important specializations of Proposition 1, we obtain the following two corollaries.

Corollary 1. *Suppose $Y = \Gamma'$ and suppose that one of the following conditions is satisfied:*

(i) *$V \hookrightarrow\hookrightarrow X$ and the operator F acts from X into Y.*

(ii) *$V \hookrightarrow X$, and the operator $F : X \to Y$ maps every bounded subset of X into a Γ-weakly absolutely bounded subset of Y.*

(iii) *$V \hookrightarrow X$, Γ is a regular Banach L_∞-module, and the operator $F : X \to Y$ maps every bounded subset of X into a Γ-weakly bounded subset of Y.*

Then $(Y,\Gamma) \in \mathfrak{Z}(L,F,\mathcal{F})$.

Proof. We have

$$(v, f(s,u)) \le R(s,v,u)$$

where $R(s,v,u) = |(v, f(s,u))|$. As assumed, all assumptions of Proposition 1 hold for the case when $\mathcal{P} = \{\emptyset\}$, $P(s,u) = Q(s,v) = 0$. Hence, from Proposition 1, Corollary 1 follows ∎

Corollary 2. *Assume* $Y = \Gamma'$, $m = 1$, $(L,F) \in \mathcal{P}_\circ(V,Z)$ *and* $\mathcal{P} = \{I\}$ *with* I *the identity operator. Suppose, moreover, that the superposition operator* F *acts from* Γ *into* Y. *Then* $(Y,\Gamma) \in \mathfrak{Z}(L,F,\mathcal{F})$ *if* $V \hookrightarrow\hookrightarrow S(\Omega,\mathbb{R}^m)$.

Proof. We set

$$\tilde{f}(s,u) = \sup\big\{|f(s,w)| : |w| \le |u|\big\}.$$

Then (see [4: Lemma 5], noting that $\operatorname{coci} f = [-1,1]\tilde{f}$) $\tilde{f}$ will be also a Carathéodory function and the superposition operator $\tilde{f}$ acts from Γ into Y, too. We have

$$(v, f(s,u)) \le |v|\,|f(s,u)| \le |uf(s,u)| + R(s,v,u)$$

where $R(s,v,u) = |v|\tilde{f}(s,v)$. Hence, condition (1) of Proposition 1 follows in the case when $\mathcal{P} = \{I\}$, $c_I = 1$, $P(s,u) = Q(s,v) = 0$ and X is an arbitrary Banach L_∞-module such that the imbedding $Z \hookrightarrow X$ is absolutely bounded (and then, by Lemma 1, $V \hookrightarrow\hookrightarrow X$). This proves that Corollary 2 follows from Proposition 1/(i) ∎

In the end of Section 4 we remark that the condition $(Y,\Gamma) \in \mathfrak{Z}(L,F,\mathcal{F})$ is analogous to the so-called k_*-compactness and b_*-boundedness of the non-linearity F in the sense of [35]. Via the schemes [35] and [28] one can find other complicated sufficient conditions which guarantee $(Y,\Gamma) \in \mathfrak{Z}_V(L,F,\mathcal{F})$, and, in particular, a vector analog of Corollary 2 by using $\mathcal{P} = \{\cdot_j : j = 1,\dots,m\}$ where $\cdot_j$ is the operator of passing to the j'th component.

5. Classes $\mathfrak{N}(L,F)$, $\mathfrak{C}(\mathcal{L})$, $\mathfrak{C}(BM)$, $\mathfrak{C}(BS)$ and $\mathfrak{C}(GM)$

We recall that $(V,V_0;W,W_0) \in \mathfrak{N}(L,F)$ if for any sequences $\{x_n\} \subset D(L)$ and $\{\theta_n\} \subset \Xi$ such that

$Lx_n = \theta_n F x_n$
x_n converges to x in the metric of $S(\Omega,\mathbb{R}^m)$
x_n converges in the weak topology $\sigma(V,W_0)$,
θ_n converges to 1 in $S(\Omega,\mathbb{R}^m)$,
Lx_n converges to κ in the weak topology $\sigma(W,V_0)$,

$Lx = \kappa = Fx$ follows. Remember that, as before, Ξ is the family of all measurable scalar functions θ such that $0 \le \theta(s) \le 1$ for a.a. $s \in \Omega$.

The verification of the inclusion $(V,V_0;W,W_0) \in \mathfrak{N}(L,F)$ is connected with the following well-known concrete classes of operators L and with the inclusion $(Y,\Gamma) \in \mathfrak{Z}_V(L,F,\mathcal{F})$. In this paper we use the classes $\mathfrak{C}(\mathcal{L})$, $\mathfrak{C}(BM)$, $\mathfrak{C}(BS)$ and $\mathfrak{C}(GM)$ of L which will be defined below.

We recall (see Yu. A. Dubinskij [11], and also [31 - 32]) that $L \in \mathfrak{C}(\mathcal{L})$ if $V = V_0$ is a reflexive Banach space, $W = W_0 = V^*$, and the operator $L : V \to W \ \ (D(L) = V)$ is bounded and weakly continuous (i.e. Lx_n weakly converges to Lx if a generalized sequence x_n converges weakly to x; in the case of a separable Banach space V it is sufficient to consider a usual sequence x_n).

We recall (see [21] and Brezis's papers cited therein) that $L \in \mathfrak{C}(BM)$ if $V = V_0$ is a reflexive Banach space, $W = W_0 = V^*$, the operator $L : V \to W \ \ (D(L) = V)$ is bounded, weakly continuous on every finite-dimensional subspace of V and of the type (M), i.e. has the following property (M): for every sequence $\{x_n\}$ such that $x_n \to x$ weakly, $Lx_n \to \kappa$ weakly and $\limsup_{n\to\infty}\langle x_n, Lx_n\rangle_V \leq \langle x, \kappa\rangle_V$ it follows that $\kappa = Lx$.

We recall (see, e.g., [29, 31 - 32, 38]) that $L \in \mathfrak{C}(BS)$ if $V = V_0$ is a reflexive Banach space, $W = W_0 = V^*$, $L : V \to W \ \ (D(L) = V)$ is a bounded operator, which is demicontinuous (i.e. which maps every convergent in norm sequence into a weakly convergent sequence) and it has the Browder - Skrypnik property (so-called condition $(\mathrm{S})_+$): for any sequence $\{x_n\}$ such that $x_n \to x$ weakly and $\lim_n\langle x_n - x, Lx_n\rangle_V \leq 0$ it follows that $x_n \to x$ in norm.

We recall (see J.-P. Gossez [13 - 14], and also [17]) that $L \in \mathfrak{C}(GM)$ if

1) $D(L) \supset V_0$ and L is continuous on every finite-dimensional subspace of V as an operator acting from V into $(W, \sigma(W, V_0))$

2) L is sequentially pseudomonotone, i.e. for any sequence $\{x_n\} \subset D(L)$ such that $x_n \to x$ in the weak topology $\sigma(V, W_0)$, $Lx_n \to \kappa$ in the weak topology $\sigma(W, V_0)$ and $\limsup_{n\to\infty}\langle x_n, Lx_n\rangle_V \leq \langle x, \kappa\rangle_V$ it follows that $x \in D(L)$, $\kappa = Lx$ and $\langle x_n, Lx_n\rangle_V$ converges to $\langle x, \kappa\rangle_V$.

We recall that $V \in \mathfrak{N}(F)$ if for any sequences $\{x_n\}$ and $\{\theta_n\} \subset \Xi$ such that $x_n \to x$ in the weak topology $\sigma(V, W_0)$ and in $S(\Omega, \mathbb{R}^m)$, $\theta_n \to 1$ in $S(\Omega, \mathbb{R})$ and $|\langle x_n, \theta_n F x_n\rangle| < \infty$, the inequality

$$\limsup_{n\to\infty}\langle x_n, \theta_n F x_n\rangle \leq \langle x, Fx\rangle < \infty \tag{19}$$

follows.

Proposition 2. *Suppose* $(u, f(s,u)) \leq B(s,u) \geq 0$, *with* $B : \Omega \times \mathbb{R}^m \to \mathbb{R}$ *a Carathéodory function generating the superposition operator* $B : Z_* \to L(\Omega, \mathbb{R})$ *where* $V \hookrightarrow\hookrightarrow Z_*$ *and* $Z_* \subset S(\Omega, \mathbb{R}^m)$ *is a Banach* L_∞*-module. Then* $V \in \mathfrak{N}(F)$.

Proof. Assume that $\{x_n\}$ and $\{\theta_n\}$ are sequences of functions in the definition of $V \in \mathfrak{N}(F)$. Then from $V \hookrightarrow\hookrightarrow Z_*$, it follows that $x_n \to x$ in the norm of Z_*. By the Krasnoselskij-type continuity theorem (see, e.g., [4: Theorem 3]), the superposition operator $B : Z_* \to L(\Omega, \mathbb{R})$ is continuous, and consequently, $Bx_n \to Bx$ in $L(\Omega, \mathbb{R})$. By the Nemytskij theorem (see, e.g., in [20]), $\theta_n F x_n \to Fx$ in $S(\Omega, \mathbb{R}^m)$.

Further, by $Bx_n(s) - (x_n(s), \theta_n F x_n(s)) \geq 0$, from the Fatou lemma the inequality

$$\liminf_{n\to\infty}\int_\Omega \big(Bx_n(s) - \big(x_n(s), \theta_n F x_n(s)\big)\big)\,ds \geq \int_\Omega \big(Bx - (x, Fx)\big)\,ds \tag{20}$$

follows. This inequality implies that

$$\begin{aligned} 0 &\le \int_\Omega \big(Bx - (x, Fx)\big)\, ds \\ &\le \liminf_{n\to\infty} \int_\Omega Bx_n(s)\, ds + \liminf_{n\to\infty} \int_\Omega \big(- (x_n, \theta_n F x_n)\big) ds \\ &= \int_\Omega Bx(s)\, ds - \limsup_{n\to\infty} \int_\Omega (x_n, \theta_n F x_n)\, ds. \end{aligned}$$

Consequently,

$$\limsup_{n\to\infty} \int_\Omega (x_n, \theta_n F x_n)\, ds \le \int_\Omega (x, Fx)\, ds \le \int_\Omega Bx(s)\, ds < \infty$$

which proves (19) ∎

Proposition 3. *Assume $\Gamma' = Y, V_0 \hookrightarrow \Gamma$ and that one of the following conditions is satisfied:*

(i) $L \in \mathfrak{C}(\mathcal{L})$, $(Y, \Gamma) \in \mathfrak{Z}_V(L, F, \mathcal{F})$.

(ii) $L \in \mathfrak{C}(BM)$, $(Y, \Gamma) \in \mathfrak{Z}_V(L, F, \mathcal{F})$, $V \in \mathfrak{N}(F)$.

(iii) $L \in \mathfrak{C}(BS)$, $(Y, \Gamma) \in \mathfrak{Z}_V(L, F, \mathcal{F})$, $V \in \mathfrak{N}(F)$.

(iv) $L \in \mathfrak{C}(GM)$, $(Y, \Gamma) \in \mathfrak{Z}_V(L, F, \mathcal{F})$, $V \in \mathfrak{N}(F)$.

Then $(V, V_0; W, W_0) \in \mathfrak{N}(L, F)$.

Proof. Let x_n, θ_n, κ, x be functions in the definition of $\mathfrak{N}(L, F)$. First, we consider the case when condition (i) is satisfied. Then, by $L \in \mathfrak{C}(\mathcal{L})$ we have at once the equality $Lx = \kappa$. It remains to prove $\kappa = Fx$. Since $\theta_n \to 1$ in $S(\Omega, \mathbb{R})$ and $x_n \to x$ in $S(\Omega, \mathbb{R}^m)$, then by the Nemytskij theorem (see, e.g., in [5, 20]) $\theta_n F x_n \to Fx$ in $S(\Omega, \mathbb{R}^m)$. Since $(Y, \Gamma) \in \mathfrak{Z}(L, F, \mathcal{F})$ and the set $\{x_n\}$ is bounded in V, the set $\{\theta_n F x_n\}$ is Γ-weakly absolutely bounded in Y. By the Vitali-Krasnoselskij convergence theorem (see, e.g., [20] and [27: Theorem 8]) in $L(\Omega, \mathbb{R})$,

$$\langle z, \theta_n F x_n \rangle \to \langle z, Fx \rangle \qquad (z \in \Gamma) \tag{21}$$

as $n \to \infty$ and therefore $\theta_n F x_n \to Fx$ Γ-weakly in Y. Since $V_0 \hookrightarrow \Gamma$ and $\Gamma' = Y$, the restriction of every functional $\langle \cdot, y \rangle$ on V_0 $(y \in Y)$ is a element of $V_0^* = W$. By the definition of a complementary system $(V, V_0; W, W_0)$ (see Section 3) it follows that $\langle z, \theta_n F x_n \rangle = \langle z, \theta_n F x_n \rangle_V$ and $\langle z, Fx \rangle = \langle z, Fx \rangle_V$ $(z \in V_0)$. From (21) it follows

$$\langle z, \theta_n F x_n \rangle_V \to \langle z, Fx \rangle_V \qquad (z \in V_0), \tag{22}$$

as $n \to \infty$, i.e. $\theta_n F x_n = Lx_n \to Fx$ in the weak topology $\sigma(W, V_0)$. Hence, $Fx = \kappa$.

We are going to consider the case when condition (ii) is satisfied. As in the preceding case one can prove that $\theta_n F x_n = Lx_n \to Fx$ in the weak topology $\sigma(W, V_0)$ and $Fx = \kappa$. It remains to prove that $Lx = \kappa$. By $V \in \mathfrak{N}(F)$ we have (19). Then

$$\limsup_{n\to\infty} \langle x_n, Lx_n \rangle_V = \limsup_{n\to\infty} \langle x_n, \theta_n F x_n \rangle \le \langle x, Fx \rangle = \langle x, \kappa \rangle_V.$$

By property (M) for $L \in \mathfrak{C}(BM)$, $Lx = \kappa$ follows. Thus, $Lx = \kappa = Fx$.

Now we consider the case when condition (iii) is satisfied. As in the first case one can prove that $\theta_n F x_n = L x_n \to Fx$ in the weak topology $\sigma(W, V_0)$ and $\kappa = Fx$. It remains to prove that $Lx = \kappa$. By $V \in \mathfrak{N}(F)$ we obtain (19). Hence, since $\langle x_n - x, Lx_n\rangle_V = \langle x_n, \theta_n F x_n\rangle_V - \langle x, \theta_n F x_n\rangle_V$, $x \in V = V_0 \hookrightarrow \Gamma$ and $Y = \Gamma'$, it follows that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle x_n - x, Lx_n\rangle_V \\
&\qquad = \limsup_{n\to\infty} \langle x_n, \theta_n F x_n\rangle + \liminf_{n\to\infty} (-\langle x, \theta_n F x_n\rangle) \\
&\qquad = \limsup_{n\to\infty} \langle x_n, \theta_n F x_n\rangle - \langle x, Fx\rangle \\
&\qquad \le 0.
\end{aligned}$$

Consequently, by the Browder-Skrypnik property of $L \in \mathfrak{C}(BS)$, $x_n \to x$ in the norm of V. By the demicontinuity of $L \in \mathfrak{C}(BS)$, $Lx_n \to Lx$ weakly, and consequently, $Lx = \kappa$. Thus, $Fx = \kappa = Lx$.

We consider the case when condition (iv) is satisfied. As in the case (i) one can prove that $\theta_n F x_n = L x_n \to Fx = \kappa$ in the weak topology $\sigma(W, V_0)$, and by $V \in \mathfrak{N}(F)$ we obtain (19). Then

$$\langle z, Lx_n\rangle_V = \langle z, \theta_n F x_n\rangle_V \to \langle z, \kappa\rangle_V = \langle z, Fx\rangle \qquad (z \in V_0). \tag{23}$$

as $n \to \infty$. Further, we have

$$\limsup_{n\to\infty} \langle x_n, Lx_n\rangle_V = \limsup_{n\to\infty} \langle x_n, \theta_n F x_n\rangle \le \langle x, Fx\rangle = \langle x, \kappa\rangle_V.$$

By $L \in \mathfrak{C}(GM)$ it follows that $x \in D(L)$ and $Lx = \kappa$. Thus, $Fx = k = Lx$ ∎

6. Auxiliary lemmas

We need else the following technical notions [25 - 26] in the theory of Banach L_∞-modules for some further proofs in this paper.

Let $B : \Omega \to 2^{\mathbb{R}^m}$ be a multifunction, for which there exists a sequence of measurable functions $\{x_i\}_{i\in\mathbb{N}}$ such that the closure hull of $\{x_i(s)\}_{i\in\mathbb{N}}$ coincides with $B(s)$ for a.a. $s \in \Omega$ and $B(s)$ is a symmetric absolutely convex compact subset of $\mathbb{R}^m$. Then B is called (see [25 - 26]) an *infra-semi-unit or m-unit* of X if $x \in X$ for every so-called *measurable selector* x of B, i.e. measurable function x such that $x(s) \in B(s)$ for a.a. $s \in \Omega$. In the scalar case $m = 1$, any 1-unit B has the form $B(s) = [-b(s), +b(s)]$ where b is a non-negative element in X. A subset $\mathfrak{N} \subset X$ is called (see [25 - 26]) *U-bounded*, if all elements of $\mathfrak{N}$ are measurable selectors for some fixed m-unit B of X. In the scalar case $m = 1$, this notion is equivalent to the notion of order-boundedness of $\mathfrak{N}$, i.e. to the existence of a non-negative function $b \in X$ such that $|x(s)| \le b(s)$ for a.a. $s \in \Omega$ and for all $x \in \mathfrak{N}$.

An analog of the space $\mathbb{L}_\infty$ is the Banach L_∞-module M_B of measurable functions x with the norm

$$\|x\|_{M_B} = \inf\big\{\lambda > 0 : x(s) \in \lambda B(s) \ \text{ a.e.}\big\} < +\infty.$$

A subset $\mathfrak{N}$ is U-bounded if and only if $\mathfrak{N}$ is bounded in the norm of M_B for some m-unit of X. The *associtate space* $(M_B)'$ (denoted by L_{B°) as an analog of the Lebesgue space L_1 is a regular space [26]. For example, in the scalar case $m = 1$, when $B(s) = [-b(s), +b(s)]$, L_{B° is just the Lebesgue space $L_1(b)$ with weight b.

Recall that an operator $F : X \to Y$ is called *U-bounded* on the whole space X, if $F(X)$ is U-bounded in Y.

Lemma 3. *Let $V \hookrightarrow\hookrightarrow E$, $E \subset S(\Omega, \mathbb{R}^m)$ be a Banach L_∞-module, $L \in \mathfrak{C}$, where $\mathfrak{C}$ is one of the classes $\mathfrak{C}(\mathcal{L})$, $\mathfrak{C}(BM)$ or $\mathfrak{C}(BS)$, and the superposition operator $F : E \to E'$ be U-bounded on whole space E. Suppose that the following condition is satisfied:*

(R1) *There exists a superpositionally measurable function $H(\cdot,\cdot) : \Omega \times \mathbb{R}^m \to [0, \infty)$ such that $(u, f(s, u)) \le H(s, u)$, the superposition operator H generated by $H(\cdot,\cdot)$ acts from V into $L(\Omega, \mathbb{R})$, and $\langle x, Lx\rangle_V - Hx > 0$ $\ (\|x\| \ge r)$ for some $r > 0$.*

Then the equation $Lx = Fx$ has at least one solution $x^* \in T_V(r)$ for which $\|Lx^*\|_W \le c$ where $c = \sup\{\|Lz\|_W : \|z\|_V \le r\}$.

Proof. Remember that $E \hookrightarrow S(\Omega, \mathbb{R}^m)$ by [27: Theorem 4]. First, we prove that $L - F \in \mathfrak{C}$. Since $F : E \to E'$ is U-bounded on whole E, there exists an m-unit B of E' such that $F(X) \subset M_B \hookrightarrow E'$, $E \hookrightarrow (M_B)' = L_{B^\circ}$, and $F : E \to M_B$ is a bounded operator on whole E (see [27: Theorem 4] and [25: Theorem 5]). In the scalar case $m = 1$, this 1-unit B has the form $B(s) = [-b(s), +b(s)]$ where b is a non-negative element in E' with $\operatorname{supp} b = \Omega$. Let $\|x_n - x\|_E \to 0$ as $n \to \infty$. Then x_n converges in the norm of L_{B° and, consequently, it has uniformly absolutely continuous norms as L_{B° is regular [26] by the Vitali-Krasnoselskij type convergence theorem (see, e.g., [20] and [27: Theorem 8]). Hence, for $z \in L_{B^\circ} \supset E$ we get

$$\lim_{\operatorname{mes}(D)\to 0} \sup_n \int_D |(z, Fx_n)|\, ds \le \lim_{\operatorname{mes}(D)\to 0} \|P_D z\|_{L_{B^\circ}} \sup_n \|Fx_n\|_{M_B} = 0.$$

By the Nemytskij theorem (see, e.g., [5, 20]) $Fx_n \to Fx$ in the metric of $S(\Omega, \mathbb{R}^m)$, and hence it follows from the Vitali-Krasnoselskij convergence theorem in $L(\Omega, \mathbb{R})$ (see, e.g., [20] and [27: Theorem 8]) that $\langle Fx_n, z\rangle \to \langle Fx, z\rangle$ as $n \to \infty$, for every $z \in E$. Therefore, Fx_n converges in the weak topology $\sigma(E', E)$. This proves that $f : E \to (E', \sigma(E', E))$ is continuous.

We consider the case $\mathfrak{C} = \mathfrak{C}(\mathcal{L})$. Since $V \hookrightarrow\hookrightarrow E$ and $F : E \to E'$ is U-bounded, boundedness of $F : V = V_0 \to W = W_0$ follows and consequently, boundedness of $L - F : V \to W$, since $L \in \mathfrak{C}(\mathcal{L})$ is bounded. Further, since $V = V_0 \hookrightarrow\hookrightarrow E$ and V is a reflexive Banach space, from weak convergence in V strong convergence in E follows. Then the proved continuity of $F : E \to (E', \sigma(E', E))$ implies weak continuity of $F : V \to W$, and consequently $L - F : V \to W$ is weakly continuous, since $L \in \mathfrak{C}(\mathcal{L})$ is weakly continuous. Thus, $L - F \in \mathfrak{C}(\mathcal{L})$.

Next we consider the case $L \in \mathfrak{C} = \mathfrak{C}(BM)$. As in the first case we can prove that $L - F : V = V_0 \to W = W_0$ is bounded and $F : V \to W$ is weakly continuous, in particular, $F : V \to W$ is weakly continuous on every finite-dimensional subspace of V. Since $L \in \mathfrak{C}(BM)$ is weakly continuous on every finite-dimensional subspace of V, $L - F$ has the same continuity property. It remains to prove that L is an operator of type (M). For this suppose that $x_n \to x$ weakly in V, $(L-F)x_n \to \kappa$ weakly in W and $\limsup_{n\to\infty} \langle x_n, (L-F)x_n\rangle_V \le \langle x, \kappa\rangle_V$. By $V = V_0 \hookrightarrow\hookrightarrow E$, $x_n \to x$ in the norm of E and in the metric of $S(\Omega, \mathbb{R}^m)$. Consequently, $(x_n, Fx_n) \to (x, Fx)$ in $S(\Omega, \mathbb{R})$. Further, $Fx_n \in M_B$ and $x_n \to x$ in L_{B°. We have

$$\begin{aligned}
\lim_{\operatorname{mes}(D)\to 0} \sup_n \int_D |(x_n, Fx_n)|\, ds &\le \lim_{\operatorname{mes}(D)\to 0} \sup_n \|Fx_n\|_{M_B} \|P_D x_n\|_{L_{B^\circ}} \\
&\le c \lim_{\operatorname{mes}(D)\to 0} \sup_n \|P_D x_n\|_{L_{B^\circ}} \\
&= 0
\end{aligned}$$

where $c = \sup_n \|Fx_n\|_{M_B} < \infty$, since $F : E \to M_B$ is bounded and L_{B° is regular (see [26]). Hence it follows by the Vitali-Krasnoselskij theorem in $L(\Omega, \mathbb{R})$ (see, e.g., [20] or [27: Theorem 8]) that $(x_n, Fx_n) \to (x, Fx)$ in $L(\Omega, \mathbb{R})$ and, in particular,

$$\lim_{n\to\infty} \langle x_n, Fx_n\rangle = \langle x, Fx\rangle = \langle x, Fx\rangle_V. \tag{24}$$

We have that $Fx_n \to Fx$ weakly in W and, consequently, $Lx_n \to \kappa + Fx$ weakly. Thus,

$$\begin{aligned}
\limsup_{n\to\infty} \langle x_n, Lx_n\rangle_V &\le \limsup_{n\to\infty} \langle x_n, (L-F)x_n\rangle_V + \limsup_{n\to\infty} \langle x_n, Fx_n\rangle \\
&\le \langle x, \kappa\rangle_V + \langle x, Fx\rangle_V \\
&= \langle x, \kappa + Fx\rangle_V.
\end{aligned}$$

Consequently, $Lx = \kappa + Fx$, since $L \in \mathfrak{C}(BM)$ is an operator of type (M). This completes the proof of $L - F \in \mathfrak{C}(BM)$.

At last we consider now the case that $L \in \mathfrak{C}(BS)$. As in the preceding case one can prove that $F : V = V_0 \to W = W_0$ is bounded and weakly continuous, in particular, demicontinuous. Since $L \in \mathfrak{C}(BS)$ is bounded and demicontinuous, $L - F$ has also these properties. It remains to prove the Browder-Skrypnik property of $L - F \in \mathfrak{C}(BS)$. For this suppose that $x_n \to x$ weakly and $\lim_{n\to\infty} \langle x_n - x, (L-F)x_n\rangle_V \le 0$. Analogously as in the proof of (24) one can prove that

$$\lim_{n\to\infty} \langle x_n - x, Fx_n\rangle = 0. \tag{25}$$

Hence it follows that

$$\begin{aligned}
\lim_{n\to\infty} \langle x_n - x, Lx_n\rangle_V &= \lim_{n\to\infty} \langle x_n - x, (L-F)x_n\rangle_V + \lim_{n\to\infty} \langle x_n - x, Fx_n\rangle_V \\
&\le 0 + 0 \\
&= 0.
\end{aligned}$$

By the Browder-Skrypnik property of $L \in \mathfrak{C}(BS)$ it follows that $x_n \to x$ in the norm of V. This proves that $L-F$ has the Browder-Skrypnik property and that $L-F \in \mathfrak{C}(BS)$.

Thus we proved that $L - F \in \mathfrak{C}$. By condition (R1) of Lemma we have

$$(u, f(s,u)) \le H(s,u) \ge 0$$

and

$$\langle x, (L-F)x\rangle_V = \langle x, Lx\rangle_V - \langle x, Fx\rangle \ge \langle x, Lx\rangle_V - Hx > 0 \qquad (\|x\|_V \ge r > 0).$$

Then by well-known existence theorems for operators of the class $\mathfrak{C}$ (see, e.g., [11, 21, 29, 31 - 32, 38]) it follows that $Lx = Fx$ has at least one solution $x^* \in T_V(r)$ ∎

Lemma 4. *Let $V \hookrightarrow\hookrightarrow E$, $E \subset S(\Omega, \mathbb{R}^m)$ be a Banach L_∞-module, $L \in \mathfrak{C}(GM)$ and the superposition operator $F : E \to E'$ be U-bounded on whole space E. Suppose that the following condition is satisfied:*

- **(R2)** *There exist a function $y \in V_0$, a superpositionally measurable function $H(\cdot,\cdot) : \Omega \times \mathbb{R}^m \to [0,\infty)$ and a number $r > 0$ such that $(u - y(s), f(s,u)) \le H(s,u)$, the superposition operator $H : V \to L(\Omega,\mathbb{R})$ generated by $H(\cdot,\cdot)$ is bounded on bounded sets, $\langle x-y, Lx\rangle_V - Hx > 0$ $(\|x\|_V \ge r > 0)$, and $\|Lx\|_W \le c(c_1,c_2) < \infty$ where $\|x\| \le c_1 < \infty$ and $\langle x - y, Lx\rangle_V \le c_2 < \infty$.*

Then the equation $Lx = Fx$ has at least one solution x^* for which $\|Lx^*\|_W \le c'$ where $c' = c'\big(r, \sup_{\|z\|_V \le r} \|Hz\|_L\big) < \infty$.

Proof. Analogously as in the proof of Lemma 3 in the case $L \in \mathfrak{C}(BM)$ one can prove that $L - F \in \mathfrak{C}(GM)$ and that F is bounded in W on every bounded in V subset. Further, we prove that $L - F$ satisfies also condition (R2) for same y and $H(s,u)$ and for some function $c'(c_1,c_2)$. First we have

$$\begin{aligned} \big(u - y(s), f(s,u)\big) &\le H(s,u) \ge 0 \\ \langle x-y, (L-F)x\rangle_V &= \langle x-y, Lx\rangle_V - \langle x-y, Fx\rangle_V \\ &\ge \langle x-y, Lx\rangle_V - Hx \\ &> 0 \quad (\|x\|_V \ge r > 0). \end{aligned} \tag{26}$$

Now let $\|x\|_V \le c_1 < \infty$ and $\langle x-y, (L-F)x\rangle_V \le c_2 < \infty$. Then $x \in D(L)$ (since $Lx = (L-F)x + Fx$) and

$$\begin{aligned} \langle x-y, Lx\rangle_V &= \langle x-y, (L-F)x\rangle_V + \langle x-y, Fx\rangle \\ &\le c_2 + (\|x\|_V + \|y\|_V) \sup_{\|z\|_V \le c_1} \|fz\|_W \\ &= c_2' \\ &< \infty. \end{aligned}$$

Consequently, by condition (R2) for L we have $\|Lx\|_W \le c(c_1,c_2') < \infty$ where $\|Fx\|_W \le d(c_1) = \sup_{\|z\|_V \le c_1} \|fz\|_W < \infty$. Hence

$$\|(L-F)x\|_W \le c'(c_1,c_2) = c(c_1,c_2') + d(c_1) \tag{27}$$

follows where $\|x\|_V \le c_1 < \infty$ and $\langle x-y, (L-F)x\rangle_V \le c_2 < \infty$. By $(L-F) \in \mathfrak{C}(GM)$ and (26) - (27), from known existence theorems (see, e.g., [13, 17]) it follows that the equation $Lx = Fx$ has at least one solution $x^* \in T_V(r)$. The estimation of the norm $\|Lx^*\|_W$ follows from condition (R2) and the relations $\langle x^*-y, Lx^*\rangle = \langle x^*-y, Fx^*\rangle_V \le \|Hx^*\|_L$ ∎

7. Main existence theorem in the usual weak sense

We state now the general result on the existence of solutions in V for the problem $Lx = Fx$ via the abstract notion $(V, V_0; W, W_0) \in \mathfrak{N}(L, F)$. We draw attention of the reader to the fact that verifiable sufficient conditions for $(V, V_0; W, W_0) \in \mathfrak{N}(L, F)$ were given before in Proposition 3 (Section 5) together with Proposition 2 (Section 5) and Proposition 1 with Corollaries 1 - 2 (Section 4).

Theorem 1. *Let $(V, V_0; W, W_0)$ be a complementary system in the sense of Section 3, and let $E, Y, \Gamma \subset S(\Omega, \mathbb{R}^m)$ be Banach L_∞-modules such that $V \hookrightarrow\hookrightarrow E, \Gamma' = Y$ and $V_0 \hookrightarrow \Gamma$. Further, suppose that one of the following conditions is satisfied:*

(i) $(V, V_0; W, W_0) \in \mathfrak{N}(L, F)$*, and condition* (R1) *in Lemma* 3 *for $L \in \mathfrak{C}$ is satisfied where $\mathfrak{C}$ is one of the classes $\mathfrak{C}(\mathcal{L})$, $\mathfrak{C}(BS)$, $\mathfrak{C}(BM)$.*

(ii) $(V, V_0; W, W_0) \in \mathfrak{N}(L, F)$*, and condition* (R2) *in Lemma* 4 *for $L \in \mathfrak{C}(GM)$ is satisfied.*

Then the equation $Lx = Fx$ has at least one solution in V.

Proof. Fix an m-unit B of E' such that the linear hull of $B(s)$ coincides with $\mathbb{R}^m$ for a.a. $s \in \Omega$ (see Sections 1 and 6, and [25: Theorem 5]; in the scalar case $m = 1$, this 1-unit B has the form $B(s) = [-b(s), +b(s)]$ where b is a non-negative element in E' with $\operatorname{supp} b = \Omega$). We define the Carathéodory function $\varphi_B(s, u) = \inf\{\lambda > 0 : u \in \lambda B(s)\}$, and then $\varphi_B(s, \cdot)$ is a norm in $\mathbb{R}^m$ for a.a. $s \in \Omega$. Put

$$f_n(s, u) = \min\bigl\{1, n\bigl(\varphi_B(s, f(s, u))\bigr)^{-1}\bigr\} f(s, u). \tag{28}$$

Evidently, for each $n \in \mathbb{N}$, f_n [the so-called *relaxation of* f] is a Carathéodory function and it defines the superposition operator $F_n : E \to E'$, which is U-bounded on whole space E, since $F_n(X)$ is contained in the ball of M_B with radius n. Further, for the sequence of scalar functions

$$\theta_n(s, u) = \min\bigl\{1, n\bigl(\varphi_B(s, f(s, u))\bigr)^{-1}\bigr\}$$

we have $\theta_n \to 1$ a.e. and, therefore, $f_n \to f$ a.e. In the end, for each function f_n condition (R1) in Lemma 3 or condition (R2) in Lemma 4 holds. By Lemmas 3 and 4 the equation $Lx = F_n x_n$ has at least one solution $x_n \in T_V(r)$ for which $\|Lx_n\|_W \le c < \infty$ for some numbers $r, c > 0$ not depending on n. Obviously, $Lx_n = \theta_n F x_n$ where $\theta_n(\cdot) = \theta_n(s, x_n(\cdot)) \in \Xi$ and $\theta \to 1$ in $S(\Omega, \mathbb{R})$.

By $V \hookrightarrow\hookrightarrow E \hookrightarrow S(\Omega, \mathbb{R}^m)$ (from our assumptions and [27: Theorem 4]) and by the Alaoglu-Bourbaki theorem [12] hence the existence of a subsequence $\{n_k\}$ follows such that, for some $x^* \in T_V(r)$, $x_{n_k} \to x^*$ in the weak topology $\sigma(V, W_0)$, in norm of E and in the metric of $S(\Omega, \mathbb{R}^m)$, and, for some $\kappa \in T_W(r)$, $Lx_{n_k} \to \kappa$ in the weak topology $\sigma(W, V_0)$. By $(V, V_0; W, W_0) \in \mathfrak{N}(L, F)$ hence it follows that $Lx^* = \kappa = Fx^*$, i.e. the equation $Lx = Fx$ has a solution in V ∎

8. Main existence theorem in the Γ-weakened sense

8.1 Solvability in the Γ-weakened sense. Let $(V, V_0; W, W_0)$ be a complementary system in the sense of Section 3. Recall that the equation $Lx = Fx$ is solvable in the Γ-weakened sense (here Γ is some fixed Banach L_∞-module in $S(\Omega, \mathbb{R}^m)$) if there exists x_* such that

$$\langle y, Lx_* \rangle_V = \langle y, Fx_* \rangle \quad (y \in V \cap \Gamma) \tag{29}$$

$$\langle x_*, Lx_* \rangle_V = \langle x_*, Fx_* \rangle. \tag{30}$$

Remark that the solvability in the L_∞-weakened sense is considered, e.g., in [6 - 10, 15 - 16, 18, 34] (see also the literature cited therein). In this section we get the general Theorem 2 on the existence in the Γ-weakened sense by means of the method presented in Sections 4 - 7 that refines results of these mathematicians for the case when $\Gamma = L_\infty$ and $m = 1$. From this Theorem it follows that the so-called "sign condition" in the above mentioned articles is not sufficient for the solvability of the equation $Lx = Fx$ in the vector case $m \neq 1$.

We state now the general result on the existence of solutions in the Γ-weakened sense in V for the problem $Lx = Fx$ via the following abstract notion $(V, V_0; W, W_0) \in \mathfrak{N}(L, F, \Gamma)$. We draw attention of the reader to the fact that verifiable sufficient conditions for the inclusion $(V, V_0; W, W_0) \in \mathfrak{N}(L, F, \Gamma)$ will be given in Proposition 5 (Subsection 8.2) together with Proposition 4 (Subsection 8.2), the [mentioned in the first part of Subsection 8.2] sufficient conditions for the inclusion $(V, V_0; W, W_0) \in \mathrm{Tr}(V_0, \Gamma)$, and Proposition 1 and its Corollaries 1 - 2 (Section 4).

Let $\mathfrak{C}$ be a fixed class from $\mathfrak{C}(BM)$ or $\mathfrak{C}(GM)$ (see Section 5). Following Section 6, we recall that $(V, V_0; W, W_0) \in \mathfrak{N}(L, F, \Gamma)$ if for any sequences $\{x_n\} \subset D(L)$ and $\{\theta_n\} \subset \Xi$ such that

$Lx_n = \theta_n F x_n$
$x_n \to x_*$ in the metric of $S(\Omega, \mathbb{R}^m)$
$x_n \to x^*$ in the weak topology $\sigma(V, W_0)$
$\theta_n \to 1$ in $S(\Omega, \mathbb{R})$
$Lx_n \to \kappa$ in the weak topology $\sigma(W, V_0)$

it follows that $Lx_* = \kappa$ and relations (29) - (30) hold. Remember that, as before, Ξ is the set of all measurable scalar functions θ such that $0 \leq \theta(s) \leq 1$.

Theorem 2. *Let $(V, V_0; W, W_0)$ be a complementary system in the sense of Section 3, and $E, Y, \Gamma \subset S(\Omega, \mathbb{R}^m)$ be Banach L_∞-modules such that $V \hookrightarrow\hookrightarrow E$ and $\Gamma' = Y$ is regular (in particular, $\Gamma = \mathbb{L}_\infty$). Further, suppose that one of the following conditions is satisfied:*

(i) *$(V, V_0; W, W_0) \in \mathfrak{N}(L, F, \Gamma)$ and condition* (R1) *in Lemma* 3 *is satisfied for $L \in \mathfrak{C}(BM)$.*

(ii) *$(V, V_0; W, W_0) \in \mathfrak{N}(L, F, \Gamma)$ and condition* (R2) *in Lemma* 4 *is satisfied for $L \in \mathfrak{C}(GM)$.*

Then the equation $Lx = Fx$ has at least one solution $x_* \in V$ in the Γ-weakened sense.

Proof. Let f_n be the functions defined in (28) and θ_n be the functions in the proof of Theorem 1. As proved there, each equation $Lx = F_n x$ has at least one solution x_n such that $x_n \in T_V(r)$ and $\|Lx_n\|_W \leq c < \infty$ for some $r, c > 0$ not depending on n. Obviously, $Lx_n = \theta_n F x_n$ where $\theta_n(\cdot) = \theta_n(\cdot, x_n(\cdot)) \to 1$ in $S(\Omega, \mathbb{R})$. By $V \hookrightarrow\hookrightarrow E \hookrightarrow S(\Omega, \mathbb{R}^m)$ (from our assumptions and [27: Theorem 4]) and the Alaoglu-Bourbaki theorem [12] hence there follows the existence of a subsequence $\{n_k\}$ such that, for some $x^* \in T_V(r)$, $x_{n_k} \to x_*$ in the weak topology $\sigma(V, W_0)$, in the norm of E and in the metric of $S(\Omega, \mathbb{R}^m)$, and, for some $\kappa \in T_W(c)$, $Lx_{n_k} \to \kappa$ in the weak topology $\sigma(W, V_0)$. By $(V, V_0; W, W_0) \in \mathfrak{N}(L, F, \Gamma)$ hence $Lx_* = \kappa$ and relations (29) - (30) follow ∎

8.2. Verifiable sufficient conditions for $(V, V_0; W, W_0) \in \mathfrak{N}(L, F, \Gamma)$. The verification of the inclusion $(V, V_0; W, W_0) \in \mathfrak{N}(L, F, \Gamma)$ is connected with the inclusion $(Y, \Gamma) \in \mathfrak{Z}_V(L, F, \mathfrak{F})$ and with "good" properties of the spaces V and V_0. We recall that $(V, V_0; W, W_0) \in \mathrm{Tr}(V_0, \Gamma)$ if the following two conditions are satisfied:

- **(T1)** (*Brezis-Browder-Gossez condition*). From that $\kappa \in W$, $z \in \Gamma'$, $\langle y, \kappa\rangle_V = \langle y, z\rangle$ $(y \in V_0 \cap \Gamma)$, $x \in V$, $(x(s), z(s)) \leq h(s)$ for a.a. $s \in \Omega$ and $h \in L(\Omega)$ it follows that $\langle x, \kappa\rangle_V = \langle x, z\rangle$.
- **(T2)** For each $y \in V \cap \Gamma$ there exists $y_j \in V_0 \cap \Gamma$ such that $y_j \to y$ in the weak topology $\sigma(V, W)$ and in the metric of $S(\Omega, \mathbb{R}^m)$, and $\sup_{j \in \mathbb{N}} \|y_j\|_\Gamma < \infty$.

The verification of the inclusion $(V, V_0; W, W_0) \in \mathrm{Tr}(V_0, \Gamma)$ in the non-trivial case $V_0 \cap \Gamma \neq V_0$ is connected with many considerable difficulties. At the present from results of H. Brezis and F.E. Browder [7 - 8], J.-P. Gossez [15 - 16], A. Benkirane and J.-P. Gossez [6] it follows that $(V, V_0; W, W_0) \in \mathrm{Tr}(V_0, \Gamma)$ if one of the following conditions is satisfied:

(i) $V = V_0 = \mathring{W}_p^k(\Omega)$ and $\Gamma = L_\infty(\Omega)$ (see [7 - 8]).

(ii) $V = V_0 = W_p^k(\Omega)$, $\Gamma = L_\infty(\Omega)$ and Ω satisfies the cone condition and the segment condition (see [7 - 8]).

(iii) $V = \mathring{W}^1 L_M(\Omega)$, $V_0 = \mathring{W}^1 L_M^\circ(\Omega)$ and $\Gamma = L_\infty(\Omega)$ where Ω satisfies the segment condition (see [15 - 16]).

(iv) $V = W^1 L_M(\Omega)$, $V_0 = W^1 L_M^\circ(\Omega)$ and $\Gamma = L_\infty(\Omega)$ where Ω satisfies the cone condition and the segment condition (see [15 - 16]).

(v) $V = \mathring{W}^k L_M(\Omega)$, $V_0 = \mathring{W}^k L_M^\circ(\Omega)$ and $\Gamma = L_\infty(\Omega)$ where $k \geq 2$ and Ω satisfies the segment condition, and both Young functions M and M^* satisfy the Δ_2-condition (see [6]).

As an analog to the inclusion $V \in \mathfrak{N}(F)$ in Section 6 we recall that $V \in \mathfrak{N}'(F)$ if for any sequences $\{x_n\}$ and $\{\theta_n\}$ such that $x_n \to x$ in the weak topology $\sigma(V, W_0)$ and in the metric of $S(\Omega, \mathbb{R}^m)$, $\theta_n \to 1$ in $S(\Omega, \mathbb{R})$ and $|\langle x_n, \theta_n F x_n\rangle| < \infty$ it follows that

$$\limsup_{n \to \infty} \langle x_n, \theta_n F x_n\rangle \leq \langle x, Fx\rangle < \infty$$

and $(x(s), Fx(s)) \leq h(s)$ holds for a.a. $s \in \Omega$ where h is some function of $L(\Omega, \mathbb{R})$.

Proposition 4. *Let* $(u, f(s,u)) \leq B(s,u) \geq 0$ *with* $B(\cdot,\cdot)$ *a Carathéodory function generating the superposition operator* $B : Z_* \to L(\Omega, \mathbb{R})$ *where* $V \hookrightarrow\hookrightarrow Z_*$ *and* $Z_* \subset S(\Omega, \mathbb{R}^m)$ *is a Banach* L_∞*-module. Then* $V \in \mathfrak{N}'(F)$.

Proof. By Proposition 2 we have $V \in \mathfrak{N}(F)$. Since $x \in V \hookrightarrow Z_*$, we have $(x(s), Fx(s)) \leq h(s)$, $h(s) = Bx(s) \geq 0$ and $h \in L(\Omega, \mathbb{R})$. This completes the proof of $V \in \mathfrak{N}'(F)$ ∎

Proposition 5. *Let* $\Gamma' = Y$ *be a regular Banach* L_∞*-module (in particular,* $\Gamma = \mathbb{L}_\infty$*),* $(Y, \Gamma) \in \mathfrak{Z}_V(L, F, \mathcal{F})$*,* $V \in \mathfrak{N}'(F)$*,* $(V, V_0; W, W_0) \in \mathrm{Tr}(V_0, \Gamma)$*, and* $L \in \mathfrak{C}$ *where* $\mathfrak{C}$ *is one of the classes* $\mathfrak{C}(BM)$ *or* $\mathfrak{C}(GM)$*. Then* $(V, V_0; W, W_0) \in \mathfrak{N}(L, F, \Gamma)$.

Proof. Let $x_n, \theta_n, \kappa, x_*$ be functions in the definition of $(V, V_0; W, W_0) \in \mathfrak{N}(L, F, \Gamma)$. Then by $(Y, \Gamma) \in \mathfrak{Z}_V(L, F, \mathcal{F})$ and by the results of the proof of Proposition 2 (see (21)) we have $\lim_{n\to\infty}\langle y, \theta_n F x_n\rangle = \langle y, Fx_*\rangle$ $(y \in \Gamma)$ and, consequently,

$$\langle y, \kappa\rangle_V = \lim_{n\to\infty}\langle y, Lx_n\rangle_V = \lim_{n\to\infty}\langle y, \theta_n F x_n\rangle = \langle y, Fx_*\rangle \qquad (y \in V_0 \cap \Gamma). \tag{31}$$

By $V \in \mathfrak{N}'(F)$ we obtain

$$\limsup_{n\to\infty}\langle x_n, Lx_n\rangle_V = \limsup_{n\to\infty}\langle x_n, \theta_n F x_n\rangle \leq \langle x_*, Fx_*\rangle < \infty \tag{32}$$

and $(x_*(s), Fx_*(s)) \leq h(s)$ for some $h \in L(\Omega, \mathbb{R})$. By (31) and condition (T1) in the definition of $(V, V_0; W, W_0) \in \mathrm{Tr}(V_0, \Gamma)$ hence

$$\langle x_*, \kappa\rangle_V = \langle x_*, Fx_*\rangle \tag*{$(33)_a$}$$

follows. Consequently, by (32) we have

$$\limsup_{n\to\infty}\langle x_n, Lx_n\rangle_V \leq \langle x_*, \kappa\rangle_V. \tag*{$(33)_b$}$$

Since $\mathfrak{C} = \mathfrak{C}(BM)$ or $\mathfrak{C} = \mathfrak{C}(GM)$, $Lx_* = \kappa$ follows.

Now by condition (T2) in the definition of $(V, V_0; W, W_0) \in \mathrm{Tr}(V_0, \Gamma)$ there exists for any $y \in V \cap \Gamma$ a sequence $\{y_j\} \subset V_0 \cap \Gamma$ such that $y_j \to y$ in the weak topology $\sigma(V, W)$ and in the metric of $S(\Omega, \mathbb{R}^m)$, and $\sup_j \|y_j\| < \infty$. Then, by the regularity of $\Gamma' = Y$ we have

$$\lim_{\mathrm{mes}\,(D)\to 0}\sup_j \int_D \big|\big(y_j(s), Fx_*(s)\big)\big| ds \leq \sup_j \|y_j\|_\Gamma \lim_{\mathrm{mes}\,(D)\to 0}\|P_D F x_*\|_Y = 0.$$

Consequently, by the Vitali-Krasnoselskij convergence theorem (see, e.g., [20] or [27: Theorem 8]), $\big(y_j(\cdot), Fx_*(\cdot)\big) \to \big(y(\cdot), Fx_*(\cdot)\big)$ in $L(\Omega, \mathbb{R})$ and, in particular, $\lim_{j\to\infty}\langle y_j, Fx_*\rangle = \langle y, Fx_*\rangle$. From (31) hence it follows that

$$\langle y, \kappa\rangle_V = \lim_{j\to\infty}\langle y_j, \kappa\rangle_V = \lim_{j\to\infty}\langle y_j, Fx_*\rangle = \langle y, Fx_*\rangle \qquad (y \in V \cap \Gamma). \tag{34}$$

From the proved relations $Lx_* = \kappa$ and (33) - (34) it follows that $x_* \in D(L)$ and relations (29) - (30) hold ∎

9. Applications to boundary value problems in nonlinear mechanics

In this Section by studying two concrete examples we explain the scheme of applications of Theorems 1 and 2 to some "perturbed" applied boundary value problems in Nonlinear Mechanics (mechanics of visco-plastic media, mechanics of elastic media, etc.) of the type $Lx = Fx$ where $x(s)$ is the "state" of the system, and the perturbation term $Fx(s) = f[s, x(s)]$ is the "force" acting on the system and depending on the "state".

First we consider the "perturbed" boundary value problem finding the elastic-plastic torsion of strengthened bars

$$\left.\begin{aligned} \frac{\partial}{\partial s_1}\left[H(T^2(x))\frac{\partial x}{\partial s_1}\right] + \frac{\partial}{\partial s_2}\left[H(T^2(x))\frac{\partial x}{\partial s_2}\right] + f\big(s_1, s_2, x(s_1, s_2)\big) = 0 \\ x_{|\partial\Omega} = 0 \end{aligned}\right\} \tag{35}$$

where Ω is a bounded domain of $\mathbb{R}^2$, $T^2(x) = \left(\frac{\partial x}{\partial s_1}\right)^2 + \left(\frac{\partial x}{\partial s_2}\right)^2$, and $H(\alpha)$ characterizes material properties of a bar. Following [31], suppose that $H(\alpha)$ is continuous for $\alpha > 0$, $a_0 + a_1\alpha^{\frac{p}{2}-1} \le H(\alpha) \le A_0 + A_1\alpha^{\frac{p}{2}-1}$ $(\alpha \ge 0, p > 1, a_0 \ge 0, a_1 > 0)$ and $A_0 = 0$ for $1 < p < 2$. Then by [31] the system (35) is equivalent to the operator equation $Lx = Fx$ in $\mathring{W}_p^1(\Omega)$ where $L : \mathring{W}_p^1(\Omega) \to W_{p'}^{-1}(\Omega)$ is an operator of the class $\mathfrak{C}(BS)$ and

$$\langle Lx, x\rangle = \iint_\Omega H(T^2(x))T^2(x)\, ds dt \to +\infty \qquad (\|x\|_{\mathring{W}_p^1(\Omega)} \to +\infty).$$

From Theorem 1 and embedding theorems of S. I. Sobolev and S. I. Pokhozaev - N. S. Trudinger mentioned in Section 2, we have the following

Corollary 3. *Let $Z = \mathbb{L}_{\frac{2p}{2-p}}$ for $p < 2$, $Z = L_\infty$ for $p > 2$, $Z = L_\varphi^\circ$ for $p = 2$ where $\varphi(s, u) = \exp u^2 - 1$. Suppose the following:*

- **(i)** *$(u, f(s, u)) \le b(s)$ for a.a. $s \in \Omega$ where $b \in L_1(\Omega, \mathbb{R})$.*
- **(ii)** *$Z = \Gamma$, and the superposition operator f acts from Γ into Γ'.*

Then the boundary value problem (35) has at least one solution x of $\mathring{W}_p^1(\Omega, \mathbb{R})$.

Now we are going to consider the "perturbed" three-dimensional problem finding the elastic-plastic equilibrium of bodies in Plastic Deformation Theory, i.e.

$$\left.\begin{aligned} \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial s_j} + \rho f_j(s, x(s)) = 0 \qquad (i = 1, 2, 3) \\ x_{|\partial\Omega} = 0 \end{aligned}\right\} \tag{36}$$

where $\Omega \subset \mathbb{R}^3$ is a considered body, ρ is its density, $x = (x_1, x_2, x_3)$ is the displacement vector, $f(s, x(s)) = (f_j(s, x(s))$ $(i = 1, 2, 3)$ is the "perturbed" mass force depending on a displacement vector, σ_{ij} are the components of the stress tensor defined by means of the deformation tensor ε_{ij} as $\sigma_{ij}(x) = \frac{\varepsilon(x)}{2k}\delta_{ij} + 2g(\Gamma(x))\{\varepsilon_{ij}(x) - \varepsilon(x)\delta_{ij}\}$ where k is the volume compression coefficient, δ_{ij} are the Kronecker symbols,

$$\varepsilon_{ij}(x) = \varepsilon_{ji}(x) = \frac{1}{2}\Big(\frac{\partial x_i}{\partial s_j} + \frac{\partial x_j}{\partial s_i}\Big), \quad \Gamma(x) = \Big(2\sum_{i,j} e_{i,j}(x)\Big)^{\frac{1}{2}}$$

$$\varepsilon(x) = \sum_{i,j}\varepsilon_{ij}\delta_{ij} = \sum_i \varepsilon_{ii}(x), \quad e_{ij}(x) = \varepsilon_{ij}(x) - \frac{1}{3}\varepsilon(x)\delta_{ij}$$

g is some function (so-called "plastic module") characterizing properties of a given material. Following [31], we suppose that g is continuous on $\alpha > 0$, and that $a_0 + a_1\alpha^{p-2} \le g(\alpha) \le A_0 + A_1\alpha^{p-2}$ $(\alpha \ge 0, p > 1, a_0 \ge 0, a_1 > 0)$ and $A_0 = 0$ for $1 < p < 2$. Then by [31] the system (36) is equivalent to the operator equation $Lx = \rho Fx$ in $\mathring{W}^1_p(\Omega)$ where $L : \mathring{W}^1_p(\Omega) \to W^{-1}_{p'}(\Omega)$ is an operator of the class $\mathfrak{C}(BS)$ and

$$\langle Lx, x\rangle \ge \int_\Omega \left\{\frac{\varepsilon^2(x)}{2k} + g(\Gamma(x))\Gamma^2(x)\right\} ds \to +\infty \qquad (\|x\|_{\mathring{W}^1_p(\Omega)} \to +\infty).$$

From Theorem 1 and embedding theorems of S. I. Sobolev and S. I. Pokhozaev - N. S. Trudinger mentioned in Section 2, it follows

Corollary 4. *Let $Z = \mathbb{L}_{\frac{3p}{3-p}}$ for $p < 3$, $Z = L_\infty$ for $p > 3$, $Z = \mathbb{L}^\circ_\varphi$ for $p = 3$ where $\varphi(s,u) = \exp(\|u\|^{\frac{3}{2}}) - 1$. Suppose that condition* (i) *in Corollary* 3 *is satisfied and that for some Banach L_∞-modules $X, \Gamma \subset S(\Omega, \mathbb{R}^3)$ the following condition is satisfied:*

(iii) *Either*

(a) *$Z \subset X$ and $Z \subset \Gamma$ with Γ regular, and the superposition operator ρF acts from X into Γ'*

or

(b) *$Z \subset X$ and $Z \subset \Gamma$, and $\rho F : X \to \Gamma'$ maps every bounded subset into a Γ-weakly absolutely bounded set*

or else

(c) *$Z \subset \Gamma$, the embedding $Z \subset X$ is absolutely bounded and ρF acts from X into Γ'.*

Then the boundary value problem (36) *has at least one solution x_* of $\mathring{W}^1_p(\Omega, \mathbb{R}^3)$.*

We remark that from the conditions of Corollaries 3 or 4 it follows that $L - F$ is bounded and demicontinuous, but it does not follow that $L - F$ has the Browder-Skrypnik property, and therefore, generally speaking, $(L - F) \notin \mathfrak{C}(BS)$, and so the general existence theorem of I. V. Skrypnik [31] is not applicable.

In conclusion of this section we remark that Corollaries 3 and 4 remain true if we replace in (35) - (36) $y_0 + f(\cdot,\cdot)$ by $f(\cdot,\cdot)$ where y_0 is a fixed element of $W^{-1}_{p'}(\Omega)$), since we can apply Theorem 1 for $(Lx - y_0) = Fx$ instead of $Lx = Fx$.

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Let us to inform that at the moment of acceptance of the paper the author succeeded in establishing some particular analogs of Theorems 1 - 2 for the case of discontinuity in u (or set-valuedness) of the nonlinear function $f(s,u)$; here for the construction of the "appropriate" Carathéodory relaxations f_n [cf. (28)] we must use the approximate CM-selection theorems obtained by H. T. Nguyễn, M. Juniewicz and J. Ziemińska in December 1996. The author got also some new existence theorems for equation (1) in the resonance case of the strongly nonlinear pair (L, F) via the modification of the topologic-approximate methods of the present paper; some different existence theorems in such a case were obtained via variational methods by A. Anane and J.-P. Gossez (1990) etc. All above results will be presented in forthcoming papers.

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