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A General Coincidence Theory for Set-Valued Maps

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Abstract. We present in this paper a coincidence theory for maps with closed graph and maps with continuous selections. Our theory relies on a new fixed point theorem for maps with closed graph. In addition, new minimax theorems and variational inequalities are presented.

Keywords: *Closed maps, continuous selections, acyclic maps, coincidence, minimax, variational inequalities*

AMS subject classification: 46 H 10, 54 C 60, 54 C 65, 54 H 25

1. Introduction

In [17] we established a general nonlinear alternative of Leray-Schauder type for multi-valued condensing maps with closed graphs (i.e. condensing ACG maps). Maps of this type arise naturally when discussing differential and integral inclusions in abstract spaces (in particular, when the dimension of the space is infinite). Our paper will be divided into three main sections. In Section 2 we establish some new fixed point theorems for condensing ACG maps and these results will then be used in Section 3 to establish general coincidence theorems for ACG and CS maps; CS maps will be maps which have a continuous selection (these include the well known $\Phi^\star$ maps [2]). Section 4 presents some applications; in particular, new minimax theorems and variational inequalities are obtained.

For the remainder of this section we describe the maps which we will consider throughout this paper. In this paper 2^E (here E is a Fréchet space) denotes the family of non-empty subsets of E and $CD(E)$ denotes the family of non-empty, closed, acyclic (see [9]) subsets of E. Let X be a Hausdorff topological vector space and Y a Fréchet space.

Definition 1.1. We say $F \in ACG(X,Y)$ if $F: X \to CD(Y)$ has closed graph.

Remark 1.1. In this paper we only consider acyclic maps. However, it is worth remarking that a similar theory could be derived if the acyclic maps are replaced by the approximable [14] maps of Gorniewicz and Granas, or the admissible [15] maps of Gorniewicz, or indeed the admissible [19] maps of Park. Since only minor adjusments are needed in these cases we leave the details to the reader.

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We now state the nonlinear alternative of Leray-Schauder type for ACG maps [17]. For convenience we discuss the case when our space E is a Banach space (the extension to the case when E is a Fréchet space is immediate).

Theorem 1.1. *Let E be a Banach space with U a open, convex subset of E and $x_0 \in U$. Suppose $F \in ACG(\overline{U}, E)$ is a condensing map* (see [12]) *with $F(\overline{U})$ a bounded set in E. Then either*

(A1) *F has a fixed point in $\overline{U}$*

or

(A2) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u) + (1-\lambda)\{x_0\}$.*

Let Z and W be subsets of Hausdorff topological spaces E_1 and E_2, respectively.

Definition 1.2. We say $F \in CS(Z,W)$ if there exists a continuous selection (single-valued) $s : Z \to W$ of F.

CS maps contain many well known maps in the literature (as we will now show). Recall [2] a map $F : Z \to W$ is said to be of *type* $\Phi^\star$, and we write $F \in \Phi^\star(Z,W)$, if W is convex (i.e. a convex subset of a Hausdorff topological vector space), $F(x)$ has convex values for all $x \in Z$, and there exists a selection $B : Z \to W$ of F such that $B(x) \neq \emptyset$ for all $x \in Z$ and the fibres $B^{-1}(y) = \{z : y \in B(z)\}$ are open (in Z) for all $y \in W$.

Remark 1.2. If $A, B : Z \to W$, then B is a selection of A if $B(x) \subseteq A(x)$ for all $x \in Z$.

Example 1.1. If Z is paracompact, W is convex and $F \in \Phi^\star(Z,W)$, then [2] $F \in CS(Z,W)$.

Recently [5, 8] the concept of $\Phi^\star$ map was generalized. We say $F \in DKT(Z,W)$, if W is convex, and there exists a map $B : Z \to W$ with $\text{co}\,(B(x)) \subseteq F(x)$ for all $x \in Z$, $B(x) \neq \emptyset$ for each $x \in Z$ and the fibres $B^{-1}(y)$ are open (in Z) for each $y \in W$.

Example 1.2. If Z is paracompact, W is convex and $F \in DKT(Z,W)$, then [8: Theorem 3.2] (or [5: Theorem 1]) $F \in CS(Z,W)$.

A map $F \in H(Z,W)$ (due to Horvath [6, 7]) if W is a contractible space, and there exists a selection $B : Z \to W$ of F with $B(x) \neq \emptyset$ for all $x \in Z$, $B^{-1}(y)$ is open for all $y \in W$, and for any open set U of Z the set $\cap_{x \in U} F(x)$ is empty or contractible.

Example 1.3. If Z is compact, W is contractible and $F \in H(Z,W)$, then [7] $F \in CS(Z,W)$.

A map $F \in D(Z,W)$ (due to Ding [4]) if W is a contractible space, F has the local intersection property (i.e. for each $x \in Z$ we have $F(x) \neq \emptyset$ and there exists an open neighborhood $N(x)$ of x such that $\cap_{z \in N(x)} F(z) \neq \emptyset$), and for any open set U of Z the set $\cap_{x \in U} F(x)$ is empty or contractible.

Example 1.4. If Z is compact, W is contractible and $F \in D(Z,W)$, then [4: pp. 55 - 56] (note $f \circ \phi$ is the continuous selection) implies $F \in CS(Z,W)$.

Remark 1.3. In fact, it is easy to see that Example 1.4 is a consequence of Example 1.3.

Remark 1.4. Many other examples of CS maps could be given here, using for example Michael's selection theorem, the selection theorem of Horvath [8], or indeed more recent selection theorems in the literature.

2. Fixed point theory

In this section we present some fixed point results for ACG maps. These results will then be used in Sections 3 and 4. We will state and prove our results when E is a Banach space (the extension to the case when E is a Fréchet space is immediate).

Theorem 2.1. *Let E be a Banach space with Q a closed, convex subset of E. Suppose $F \in ACG(Q,Q)$ is condensing with $F(Q)$ a bounded set in Q. Then F has a fixed point in Q.*

Proof. Let $x_0 \in Q$. From [9, 11], there exists a compact, convex set X of Q with $x_0 \in X$ and $F : X \to 2^X$. In addition, the values of F are closed and acyclic and also $F|_X$ has closed graph. Now [1: p. 465] implies $F|_X$ is upper semicontinuous. Consequently, $F : X \to CD(X)$ is upper semicontinuous and X is compact. Now [9: Theorem 1] guarantees that F has a fixed point in X ∎

Next we obtain a fixed point theorem for non-selfmaps.

Theorem 2.2. *Let E be a Banach space with Q a closed, convex subset of E and $0 \in Q$. Let $r : E \to Q$ be a continuous retraction with $r(z) \in \partial Q$ for $z \in E \backslash Q$* (see Remark 2.1) *and suppose $G : Q \to 2^E$ is such that $G(Q)$ is a bounded set in E with $G\,r \in ACG(E,E)$ and $G\,r$ a condensing map. In addition, suppose*

$$\left.\begin{array}{l} \text{if } \{(x_j,\lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial Q \times [0,1] \text{ converging to } (x,\lambda) \\ \text{with } x \in \lambda G(x) \text{ and } 0 \le \lambda < 1, \text{ then there exists } j_0 \in \{1,2,....\} \\ \text{with } \{\lambda_j G(x_j)\} \subseteq Q \text{ for each } j \ge j_0 \end{array}\right\} \tag{2.1}$$

holds. Then G has a fixed point in Q.

Remark 2.1. To justify the existence of a continuous retraction $r : E \to Q$ with $r(z) \in \partial Q$ for $z \in E \backslash Q$ see [12].

Remark 2.2. If $G \in ACG(Q,E)$ is a compact map, then $G\,r \in ACG(E,E)$ (also, $G\,r$ is compact so condensing). We must show $G\,r$ has closed graph. Let (x_n) be a sequence in E, (y_n) a sequence in E with $(x_n,y_n) \to (x_0,y_0)$ and $y_n \in G\,r(x_n)$ for every n. Let $z_n = r(x_n), z_0 = r(x_0)$ and note $(z_n,y_n) \to (z_0,y_0)$ with $y_n \in G(z_n)$ for every n. The closedness of G implies $y_0 \in G(z_0)$, i.e. $y_0 \in G\,r(x_0)$. Thus $G\,r \in ACG(E,E)$ [Alternatively, one can notice that since G is a compact map, then $G : Q \to CD(E)$ is upper semicontinuous [1: p. 465] and so $G\,r : E \to CD(E)$ is upper semicontinuous; note we can only apply this argument if G is a compact map].

Remark 2.3. If E is a Hilbert space and $G \in ACG(Q,E)$ is condensing, then $G\,r \in ACG(E,E)$ is condensing if we take r to be the nearest point projection. To see this recall r is non-expansive so $G\,r : E \to CD(E)$ is a condensing map. Of course, this result also holds for certain convex sets in Banach spaces where there is a nearest point retraction that is non-expansive (or more generally 1-set contractive).

Proof of Theorem 2.2. Let

$$H = \{x \in E : x \in G\,r(x)\}.$$

Notice $H \neq \emptyset$ by Theorem 2.1. Also, H is closed. To see this let (x_n) be a sequence in H (i.e. $x_n \in G\,r(x_n)$) with $x_n \to x_0 \in E$. The closedness of $G\,r$ implies $x_0 \in G\,r(x_0)$, i.e. $x_0 \in H$. In fact, H is compact since $H \subseteq G\,r(H)$ and $G\,r$ is condensing. It remains to show $H \cap Q \neq \emptyset$. Suppose $H \cap Q = \emptyset$. Then there exists $\delta > 0$ with dist $(H, Q) > \delta$. Choose $m \in \{1, 2, ...\}$ such that $1 < \delta\, m$. Fix $i \in \{m, m+1, ...\}$. Let

$$U_i = \left\{x \in E : \, d(x, Q) < \frac{1}{i}\right\}$$

where d is the metric associated with the Banach space E. Then $H \cap \overline{U_i} = \emptyset$. Now Theorem 1.1 (applied with $G\,r$ for F and U_i for U) implies (since $H \cap \overline{U_i} = \emptyset$) that there exists $(y_i, \lambda_i) \in \partial U_i \times (0,1)$ with $y_i \in \lambda_i G\,r(y_i)$. Notice in particular since $y_i \in \partial U_i$ that

$$\{\lambda_i G\,r(y_i)\} \not\subseteq Q \quad \text{ for each } i \in \{m, m+1, ...\}. \tag{2.2}$$

Let

$$D = \Big\{x \in E : \, x \in \lambda\, G\,r(x) \text{ for some } \lambda \in [0,1]\Big\}.$$

Now D is closed. To see this let (x_n) be a sequence in D and (λ_n) a sequence in $[0,1]$ with $(x_n, \lambda_n) \to (x_0, \lambda_0)$. Without loss of generality assume $\lambda_n \to \lambda_0 \in (0,1]$. Since $x_n \in D$, there exists $y_n \in G\,r(x_n)$ with $x_n = \lambda_n y_n$. Now $x_n \to x_0$ and $y_n \to \frac{1}{\lambda_0} x_0$. The closedness of $G\,r$ implies $\frac{1}{\lambda_0} x_0 \in G\,r(x_0)$ so $x_0 \in D$ [Alternatively, it is easy to see that $R : \, E \times [0,1] \to CD(E)$, given by $R(x, \lambda) = \lambda\, G\,r(x)$, has closed graph so it is immediate that D is closed]. In fact, D is compact since

$$D \subseteq \overline{\text{co}}\,(G\,r(D) \cup \{0\}).$$

This together with $d(y_j, Q) = \frac{1}{j}$ and $|\lambda_j| \leq 1$ (for $j \in \{m, m+1, ...\}$) implies that we may assume without loss of generality that $\lambda_j \to \lambda^\star$ and $y_j \to y^\star \in \partial Q$. Also, since $y_j \in \lambda_j G\,r(y_j)$ one has (since R given above has closed graph) that $y^\star \in \lambda^\star G\,r(y^\star)$. Now $\lambda^\star \neq 1$ since $H \cap Q = \emptyset$. Thus $0 \leq \lambda^\star < 1$. But in this case (2.1), with $x_j = r(y_j)$ and $x = y^\star = r(y^\star)$, implies that there exists $j_0 \in \{1, 2, ...\}$ with $\{\lambda_j G\,r(y_j)\} \subseteq Q$ for each $j \geq j_0$. This contradicts (2.2). Thus $H \cap Q \neq \emptyset$, i.e. there exists $x \in Q$ with $x \in G\,r(x) = G(x)$ ∎

3. Coincidence theory

Our first result generalizes some well known result in the literature (see [4: Theorem 1] and the references in [4]).

Theorem 3.1. *Let E be a Banach space with Q a closed, convex subset of E and Y any subset of a Hausdorff topological vector space. Suppose $G \in CS(Q,Y)$ (and let $t : Q \to Y$ be a continuous selection of G) and $F \in ACG(Y,Q)$. Let $J = F \circ t : Q \to 2^Q$ be defined by $J(x) = F \circ t(x)$ and suppose J is a condensing map with $J(Q)$ a bounded set in Q. Then G and F^{-1} have a coincidence, i.e. there exists $x_0 \in Q$ with $G(x_0) \cap F^{-1}(x_0) \neq \emptyset$ (i.e. there exists $(x_0, y_0) \in Q \times Y$ with $x_0 \in F(y_0)$ and $y_0 \in G(x_0)$).*

Proof. Notice (see Remark 2.2) that $J \in ACG(Q,Q)$. Apply Theorem 2.1 to deduce that there exists $x_0 \in Q$ with $x_0 \in Ft(x_0)$. Thus $x_0 \in F(y_0)$ where $y_0 = t(x_0) \in G(x_0)$, i.e. $y_0 \in G(x_0) \cap F^{-1}(x_0)$ ∎

We next obtain a generalization of Theorem 3.1 by using Theorem 2.2.

Theorem 3.2. *Let E be a Banach space, Q a closed, convex subset of E, $0 \in Q$ and Y any subset of a Hausdorff topological vector space. Suppose $G \in CS(Q,Y)$ (and let $t : , Q \to Y$ be a continuous selection of G) and $F : Y \to 2^E$. Let $J = F \circ t : Q \to 2^E$ be defined by $J(x) = F \circ t(x)$ and suppose $J(Q)$ is a bounded set in E. Let $r : E \to Q$ be a continuous retraction with $r(z) \in \partial Q$ for $z \in E \backslash Q$ and suppose $Jr \in ACG(E,E)$ is a condensing map. In addition, suppose*

$$\left.\begin{array}{l} \text{if } \{(x_j,\lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial Q \times [0,1] \text{ converging to } (x,\lambda) \\ \text{with } x \in \lambda J(x) \text{ and } 0 \le \lambda < 1, \text{ then there exists } j_0 \in \{1,2,...\} \\ \text{with } \{\lambda_j J(x_j)\} \subseteq Q \text{ for each } j \ge j_0 \end{array}\right\} \tag{3.1}$$

holds. Then G and F^{-1} have a coincidence.

Proof. Apply Theorem 2.2 and deduce that there exists $x_0 \in Q$ with $x_0 \in Ft(x_0)$ ∎

Remark 3.1. In Theorems 3.1 and 3.2 it is possible to replace ACG maps with AP (see [14] for definition) maps or Ad (see [15]) maps.

Theorem 3.3. *Let E be a Banach space with Q and C closed, convex subsets of E. Suppose $G \in ACG(Q,C)$ and $F \in ACG(C,Q)$. Define the map Ψ by*

$$\Psi(x,y) = F(y) \times G(x) \qquad \text{for } \ (x,y) \in Q \times C$$

and assume $\Psi : Q \times C \to 2^{Q \times C}$ is a condensing map with $\Psi(Q \times C)$ a bounded set in $Q \times C$. Then G and F^{-1} have a coincidence, i.e. there exists $(x_0, y_0) \in Q \times C$ with $y_0 \in G(x_0) \cap F^{-1}(x_0)$.

Proof. Notice we have immediately that $\Psi \in ACG(Q \times C, Q \times C)$. Now apply Theorem 2.1 to deduce that there exists $(x_0, y_0) \in Q \times C$ with $(x_0, y_0) \in \Psi(x_0, y_0)$ ∎

Remark 3.2. We could of course obtain a generalization of Theorem 3.3 if instead of Theorem 2.1 we use Theorem 2.2; we leave the details to the reader.

4. Applications

In this section we first establish some generalized quasi variational inequalities. Then we present new analytic alternatives and minimax inequalities.

Our first three results improve results in [13] (in particular, Theorems 2.5 and 2.6 there can be improved using the ideas in this section).

Theorem 4.1. *Let E be a Banach space with Q a closed, convex subset of E. Suppose the following conditions are satisfied:*

$$f : Q \times Q \to \mathbb{R} \text{ is an upper semicontinuous function} \tag{4.1}$$

$$G : Q \to 2^Q \text{ has compact values} \tag{4.2}$$

$$G : Q \to 2^Q \text{ is an upper semicontinuous map} \tag{4.3}$$

$$\begin{cases} \text{the map } M \text{ (marginal function), defined by} \\ M(x) = \sup_{y \in G(x)} f(x,y) \text{ for } x \in Q, \text{ is lower semicontinuous} \end{cases} \tag{4.4}$$

and

$$\begin{cases} \text{the map } \Phi, \text{ defined by } \Phi(x) = \{y \in G(x) : f(x,y) = M(x)\} \\ \text{for } x \in Q, \text{ is condensing with } \Phi(Q) \text{ a bounded set in } Q; \\ \text{also, } \Phi(x) \text{ is acyclic for each } x \in Q. \end{cases} \tag{4.5}$$

Then there exists $z \in Q$ with

$$z \in G(z) \qquad \text{and} \qquad f(z,z) = M(z)$$

(i.e. there exists $z \in Q$ with $z \in G(z)$ and $f(z,y) \le f(z,z)$ for all $y \in G(z)$).

Proof. Now since f is upper semicontinuous and G is upper semicontinuous with compact values, then [1: p. 473] implies M is continuous. In addition, notice [1: p. 44] implies for each $x \in Q$ that $\Phi(x)$ is non-empty and compact. This together with (4.5) implies $\Phi : Q \to CD(Q)$. Next we show that the graph of Φ is closed. Let (x_n, y_n) be a sequence in graph$\,(\Phi)$ with $(x_n, y_n) \to (x,y)$ in $Q \times Q$. Then

$$f(x,y) \ge \limsup f(x_n, y_n) = \limsup M(x_n) = \liminf M(x_n) = M(x). \tag{4.6}$$

In addition, $y_n \in G(x_n)$ together with $x_n \to x, y_n \to y$ and the fact that G is upper semicontinuous implies [20] that $y \in G(x)$. Thus $y \in G(x)$ and $f(x,y) \ge M(x) = \sup_{z \in G(x)} f(x,z)$. Consequently, $f(x,y) = M(x)$ so $(x,y) \in$ graph$\,(\Phi)$. Thus $\Phi : Q \to CD(Q)$ has closed graph and so $\Phi \in ACG(Q,Q)$. Now Theorem 2.1 implies Φ has a fixed point $z \in Q$, i.e. $z \in \Phi(z)$. That is $z \in G(z)$ and $f(z,z) = M(z)$ ∎

Theorem 4.2. *Let E be a Hilbert space with Q a closed, convex subset of E and $0 \in Q$. Suppose the following conditions are satisfied:*

$$f : Q \times E \to \mathbb{R} \text{ is an upper semicontinuous function} \tag{4.7}$$

$$G : Q \to 2^E \textit{ has compact values} \tag{4.8}$$

$$G : Q \to 2^E \textit{ is an upper semicontinuous map} \tag{4.9}$$

$$\begin{cases} \textit{the map } M, \textit{ defined by } M(x) = \sup_{y \in G(x)} f(x,y) \\ \textit{for } x \in Q, \textit{ is lower semicontinuous} \end{cases} \tag{4.10}$$

$$\begin{cases} \textit{the map } \Phi, \textit{ defined by } \Phi(x) = \{y \in G(x) : f(x,y) = M(x)\} \\ \textit{for } x \in Q, \textit{ is condensing with } \Phi(Q) \textit{ a bounded set in } E; \\ \textit{also }, \Phi(x) \textit{ is acyclic for each } x \in Q \end{cases} \tag{4.11}$$

and

$$\begin{cases} \textit{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \textit{ is a sequence in } \partial Q \times [0,1] \textit{ converging to } (x, \lambda) \\ \textit{with } x \in \lambda \Phi(x) \textit{ and } 0 \le \lambda < 1, \textit{ then there exists } j_0 \in \{1, 2, ...\} \\ \textit{with } \{\lambda_j \Phi(x_j)\} \subseteq Q \textit{ for each } j \ge j_0. \end{cases} \tag{4.12}$$

Then there exists $z \in Q$ with

$$z \in G(z) \qquad \textit{and} \qquad f(z,z) = M(z).$$

Proof. As in Theorem 4.1, M is continuous and in fact $\Phi \in ACG(Q,E)$. Let $r : E \to Q$ be the nearest point projection (note r is non-expansive). Then we have immediately that $\Phi r \in ACG(E,E)$ and Φr is a condensing map (see Remark 2.3). Apply Theorem 2.2 to deduce that Φ has a fixed point in Q ∎

Remark 4.1. There is an analogue of Theorem 4.2 in the case when E is a Banach space; we leave the details to the reader.

Our next result replaces sup with inf in Theorem 4.1.

Theorem 4.3. *Let E be a Banach space with Q a closed, convex subset of E. Suppose the following conditions are satisfied:*

$$f : Q \times Q \to \mathbb{R} \textit{ is a continuous function} \tag{4.13}$$

$$G : Q \to 2^Q \textit{ has compact values} \tag{4.14}$$

$$G : Q \to 2^Q \textit{ is a continuous map} \tag{4.15}$$

and

$$\begin{cases} \textit{the map } \Psi, \textit{ defined by } \Psi(x) = \{y \in G(x) : f(x,y) = N(x)\} \textit{ for } x \in Q \\ (\textit{here } N(x) = \inf\limits_{z \in G(x)} f(x,z)), \textit{ is condensing with } \Psi(Q) \textit{ a bounded set in } Q; \\ \textit{also}, \Psi(x) \textit{ is acyclic for each } x \in Q. \end{cases} \tag{4.16}$$

Then there exists $w \in Q$ with

$$w \in G(w) \qquad \textit{and} \qquad f(w,w) = N(w)$$

(i.e. there exists $w \in Q$ with $w \in G(w)$ and $f(w,y) \ge f(w,w)$ for all $y \in G(w)$).

Proof. Now [1: pp. 472 - 473] imply N is continuous. As in Theorem 4.1 it is easy to check that $\Psi : Q \to CD(Q)$ and in fact $\Psi \in ACG(Q,Q)$ (note in (4.6) we have equality now since f is continuous). Now Theorem 2.1 implies that there exists $w \in Q$ with $w \in \Psi(w)$ ∎

Remark 4.2. We could also obtain a generalization of Theorem 4.3 if we use Theorem 2.2 instead of Theorem 2.1; we leave the details to the reader.

Remark 4.3. The results in [3, 16] could also be easily improved using the ideas above.

Next we establish some new analytic alternatives. These improve results in the literature [2, 10, 14, 15].

Theorem 4.4. *Let E be a Banach space with Q a closed, convex subset of E and Y any subset of a Hausdorff topological vector space. Suppose $F \in ACG(Y, Q)$ and $f : Q \times Y \to \mathbb{R}$. Fix $\alpha \in \mathbb{R}$ and let*

$$G(x) = \{y \in Y : f(x,y) > \alpha\} \quad \textit{for } x \in Q.$$

Suppose the following condition is satisfied:

$$\begin{cases} \textit{if } G(x) \neq \emptyset \textit{ for every } x \in Q, \ \textit{then } G \in CS(Q,Y) \\ (\textit{and let } t : Q \to Y \textit{ be a continuous selection of } G) \\ \textit{and } J = F \circ t : Q \to 2^Q \ (\textit{defined by } J(x) = F \circ t(x)) \\ \textit{is a condensing map with } J(Q) \textit{ a bounded set in } Q. \end{cases} \tag{4.17}$$

Then either

(A1) *there exists $z_0 \in Q$ with $f(z_0, y) \leq \alpha$ for all $y \in Y$ (i.e. there exists $z_0 \in Q$ with $G(z_0) = \emptyset$)*

or

(A2) *there exists $(x_0, y_0) \in Q \times Y$ with $x_0 \in F(y_0)$ and $f(x_0, y_0) > \alpha$*

occurs.

Remark 4.4. For $\Phi^\star$ maps (and a similar comment applies for DKT and H maps) it is possible to replace "if $G(x) \neq \emptyset$ for every $x \in Q$, then $G \in \Phi^\star(Q,Y)$" in (4.17) by "if $B(x) \neq \emptyset$ for every $x \in Q$, then $G \in \Phi^\star(Q,Y)$" (here B is the selection of G as described in the definition of $\Phi^\star$ maps). Of course, statement (A1) would now become "there exists $z_0 \in Q$ with $B(z_0) = \emptyset$". A similar remark will apply for Theorem 4.5.

Remark 4.5. Conditions so that "if $G(x) \neq \emptyset$ for every $x \in Q$, then $G \in \Phi^\star(Q,Y)$" may be found in [15] (conditions of this type are standard in the literature, see [2, 10]).

Proof of Theorem 4.4. Either $G(x) \neq \emptyset$ for every $x \in Q$ or not. If $G(x) \neq \emptyset$ for every $x \in Q$, then $G \in CS(Q,Y)$ and Theorem 3.1 implies that there exists $(x_0, y_0) \in Q \times Y$ with $x_0 \in F(y_0)$ and $y_0 \in G(x_0)$ (i.e. $x_0 \in F(y_0)$ and $f(x_0, y_0) > \alpha$) so statement $(A2)$ occurs. If $G(x) \neq \emptyset$ for every $x \in Q$ does not hold, then there exists $z_0 \in Q$ with $G(z_0) = \emptyset$, i.e. there exists $z_0 \in Q$ with $f(z_0, y) \leq \alpha$ for all $y \in Y$ so statement $(A1)$ occurs ∎

Theorem 4.5. *Let E be a Hilbert space, Q a closed, convex subset of E, $0 \in Q$ and Y any subset of a Hausdorff topological vector space. Suppose $F \in ACG(Y, E)$ and $f : Q \times Y \to \mathbb{R}$. Fix $\alpha \in \mathbb{R}$ and let*

$$G(x) = \{y \in Y : f(x,y) > \alpha\} \quad \textit{for } x \in Q.$$

Suppose the following condition is satisfied:

$$\begin{cases} \textit{if } G(x) \neq \emptyset \textit{ for every } x \in Q, \textit{ then } G \in CS(Q,Y) \\ (\textit{and let } t: Q \to Y \textit{ be a continuous selection of } G) \\ \textit{with } J = F \circ t: Q \to 2^E \ (\textit{defined by } J(x) = F \circ t(x)) \\ \textit{a condensing map }, J(Q) \textit{ a bounded set in } E, \\ \textit{and if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \textit{ is a sequence in } \partial Q \times [0,1] \textit{ converging to } (x,\lambda) \\ \textit{with } x \in \lambda J(x) \textit{ and } 0 \leq \lambda < 1, \textit{ then there exists } j_0 \in \{1,2,....\} \\ \textit{with } \{\lambda_j J(x_j)\} \subseteq Q \textit{ for each } j \geq j_0. \end{cases} \tag{4.18}$$

Then either

(A1) *there exists* $z_0 \in Q$ *with* $f(z_0, y) \leq \alpha$ *for all* $y \in Y$ *(i.e. there exists* $z_0 \in Q$ *with* $G(z_0) = \emptyset$*)*

or

(A2) *there exists* $(x_0, y_0) \in Q \times Y$ *with* $x_0 \in F(y_0)$ *and* $f(x_0, y_0) > \alpha$

occurs.

Proof. If $G(x) \neq \emptyset$ for every $x \in Q$ does not hold, then as in Theorem 4.4 statement $(A1)$ occurs. It remains to consider the case $G(x) \neq \emptyset$ for every $x \in Q$. Then $G \in CS(Q, Y)$. Let $r: E \to Q$ be the nearest point projection. We have immediately that $Jr \in ACG(E, E)$ and Jr is a condensing map. Now Theorem 3.2 implies that there exists $(x_0, y_0) \in Q \times Y$ with $x_0 \in F(y_0)$ and $y_0 \in G(x_0)$, i.e. statement $(A2)$ occurs ∎

Theorem 4.6. *Let E be a Banach space with Q and C closed, convex subsets of E. Let $f, g: Q \times C \to \mathbb{R}$ with*

$$g(x,y) \leq f(x,y) \quad \textit{for all } (x,y) \in Q \times C. \tag{4.19}$$

Fix $\alpha \in \mathbb{R}$ and let

$$G(x) = \{y \in C : g(x,y) > \alpha\} \quad \textit{for } x \in Q$$

and

$$F(y) = \{x \in Q : f(x,y) < \alpha\} \quad \textit{for } y \in C.$$

Suppose the following condition is satisfied:

$$\begin{cases} \textit{if } G(x) \neq \emptyset \textit{ for every } x \in Q \textit{ and } F(y) \neq \emptyset \textit{ for every } y \in C, \\ \textit{then } G \in ACG(Q,C) \textit{ and } F \in ACG(C,Q) \textit{ with } \Psi: Q \times C \to 2^{Q \times C} \\ (\textit{defined by } \Psi(x,y) = F(y) \times G(x)) \textit{ a condensing map} \\ \textit{and } \Psi(Q \times C) \textit{ is a bounded set in } Q \times C. \end{cases} \tag{4.20}$$

Then either

(A1) *there exists* $z_0 \in Q$ *with* $g(z_0, y) \leq \alpha$ *for all* $y \in C$

or

(A2) *there exists* $w_0 \in C$ *with* $f(x, w_0) \geq \alpha$ *for all* $x \in Q$
occurs.

Remark 4.6. Conditions so that "if $G(x) \neq \emptyset$ for every $x \in Q$, then $G \in ACG(Q, C)$" may be found in [14].

Proof of Theorem 4.6. There are three cases to consider.

Case (i): $G(x) \neq \emptyset$ for every $x \in Q$ and $F(y) \neq \emptyset$ for every $y \in C$. In this case Theorem 3.3 implies that there exists $(x_0, y_0) \in Q \times C$ with $y_0 \in G(x_0)$ and $x_0 \in F(y_0)$, i.e. $f(x_0, y_0) < \alpha < g(x_0, y_0)$. This contradicts (4.19).

Case (ii): $G(x) \neq \emptyset$ for every $x \in Q$ does not hold. Then there exists $z_0 \in Q$ with $G(z_0) = \emptyset$, i.e. statement $(A1)$ occurs.

Case (iii): $F(y) \neq \emptyset$ for every $y \in C$ does not hold. Then there exists $w_0 \in C$ with $F(w_0) = \emptyset$, i.e. statement $(A2)$ occurs ∎

Finally, in this section we obtain two new minimax theorems.

Theorem 4.7. *Let E be a Banach space with Q a closed, convex subset of E and Y any subset of a Hausdorff topological vector space. Suppose $F \in ACG(Y, Q)$ and $f : Q \times Y \to \mathbb{R}$. Define for each $\alpha \in \mathbb{R}$*

$$G_\alpha(x) = \{y \in Y : f(x, y) > \alpha\} \qquad \text{for } x \in Q.$$

Suppose the following condition is satisfied:

$$\begin{cases} \text{for any } \alpha \in \mathbb{R}, \text{ if } G_\alpha(x) \neq \emptyset \text{ for every } x \in Q, \text{ then } G_\alpha \in CS(Q, Y) \\ (\text{and let } t_\alpha : Q \to Y \text{ be a continuous selection of } G_\alpha) \\ \text{and } J_\alpha = F \circ t_\alpha : Q \to 2^Q \ (\text{defined by } J_\alpha(x) = F \circ t_\alpha(x)) \\ \text{is a condensing map with } J_\alpha(Q) \text{ a bounded set in } Q. \end{cases} \tag{4.21}$$

Then

$$\inf_{x \in Q} \sup_{y \in Y} f(x, y) \leq \sup \big\{ f(x, y) : x \in Q, y \in Y, x \in F(y) \big\}. \tag{4.22}$$

Remark 4.7. Other results could be obtained if we use Remark 4.4.

Proof of Theorem 4.7. Let

$$\alpha = \sup \big\{ f(x, y) : x \in Q, y \in Y, x \in F(y) \big\}.$$

The case $\alpha = \infty$ is trivial, so from now on we assume $\alpha < \infty$. Apply Theorem 4.4. Notice statement $(A2)$ cannot occur (see the definition of α). Then there exists $z_0 \in Q$ with $f(z_0, y) \leq \alpha$ for all $y \in Y$. That is $\sup_{y \in Y} f(z_0, y) \leq \alpha$ so (4.22) follows ∎

Theorem 4.8. *Let E be a Hilbert space, Q a closed, convex subset of E, $0 \in Q$ and Y any subset of a Hausdorff topological vector space. Suppose $F \in ACG(Y, E)$ and $f : Q \times Y \to \mathbb{R}$. Define for each $\alpha \in \mathbb{R}$,*

$$G_\alpha(x) = \{y \in Y : f(x, y) > \alpha\} \qquad \text{for } x \in Q.$$

Suppose the following condition is satisfied:

$$\begin{cases} \textit{for any } \alpha \in \mathbb{R}, \ \textit{if } G_\alpha(x) \neq \emptyset \textit{ for every } x \in Q, \ \textit{then } G_\alpha \in CS(Q,Y) \\ (\textit{and let } t_\alpha : Q \to Y \textit{ be a continuous selection of } G_\alpha) \\ \textit{with } J_\alpha = F \circ t_\alpha : Q \to 2^E (\textit{defined by } J_\alpha(x) = F \circ t_\alpha(x)) \\ \textit{a condensing map }, J_\alpha(Q) \textit{ a bounded set in } E, \\ \textit{and if } \{(x_j,\lambda_j)\}_{j=1}^\infty \textit{ is a sequence in } \partial Q \times [0,1] \textit{ converging to } (x,\lambda) \\ \textit{with } x \in \lambda J_\alpha(x) \textit{ and } 0 \le \lambda < 1, \ \textit{then there exists } j_0 \in \{1,2,....\} \\ \textit{with } \{\lambda_j J_\alpha(x_j)\} \subseteq Q \textit{ for each } j \ge j_0. \end{cases} \tag{4.23}$$

Then

$$\inf_{x\in Q} \sup_{y\in Y} f(x,y) \le \sup\big\{f(x,y) : x \in Q, y \in Y, x \in F(y)\big\}.$$

Proof. Let

$$\alpha = \sup\big\{f(x,y) : x \in Q, y \in Y, x \in F(y)\big\}$$

and assume without loss of generality that $\alpha < \infty$. Apply Theorem 4.5 so there exists $z_0 \in Q$ with $f(z_0,y) \le \alpha$ for all $y \in Y$ ∎

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