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Global Bifurcation Results for a Semilinear Biharmonic Equation on all of $\mathbb{R}^N$

N. M. Stavrakakis and N. Zographopoulos

Abstract. We prove existence of positive solutions for the semilinear problem

$$(-\Delta)^2 u = \lambda g(x) f(u), \qquad u(x) > 0 \;\; (x \in \mathbb{R}^N), \qquad \lim_{|x|\to+\infty} u(x) = 0$$

under the main hypothesis $N > 4$ and $g \in L^{N/4}(\mathbb{R}^N)$. First, we employ classical spectral analysis for the existence of a simple positive principal eigenvalue for the linearized problem. Next, we prove the existence of a global continuum of positive solutions for the problem above, branching out from the first eigenvalue of the differential equation in the case that $f(u) = u$. This fact is achieved by applying standard local and global bifurcation theory. It was possible to carry out these methods by working between certain equivalent weighted and homogeneous Sobolev spaces.

Keywords: *Biharmonic equations, nonlinear eigenvalue problems, local and global bifurcation theory, maximum principle, indefinite weights*

AMS subject classification: 35 B 32, 35 B 40, 35 J 40, 35 J 65, 35 J 70, 35 H 12

1. Introduction

In this paper we study existence and properties of solutions of the semilinear biharmonic eigenvalue problem

$$(-\Delta)^2 u = \lambda g(x) f(u) \qquad (x \in \mathbb{R}^N) \tag{1.1$_\lambda$}$$

$$u(x) > 0 \;\; (x \in \mathbb{R}^N), \quad \lim_{|x|\to+\infty} u(x) = 0, \tag{1.2}$$

where $\lambda \in \mathbb{R}$ and $N > 4$. The general hypotheses, which will be assumed throughout the paper, are the following ones:

($\mathcal{G}$) g is a smooth function, at least of type $C^{1,\alpha}(\mathbb{R}^N)$, for some $\alpha \in (0,1)$, such that $g \in L^{N/4}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $g(x) \geq 0$.

($\mathcal{F}$) $f : \mathbb{R} \longmapsto [0,\infty)$ is a smooth function such that $f(0) = 0$, $f'(0) > 0$ and $f(u) > 0$ for all $u \neq 0$. Also, $f', f'' \in L^\infty$ and there is $k^* > 0$ such that $|f(s)| \leq k^*|s|$ for all $s \in \mathbb{R}$.

Both authors: Nat. Techn. Univ., Dept. Math., Zografos Campus, 157 80 Athens, Greece
e-mail: nikolas@central.ntua.gr

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The literature for the problem in the bounded domain case is quite complete. For example, among others, we mention the papers of Ph. Clément et al. [6], R. Dalmasso [8], D. E. Edmunds et al. [10] and Y.-G. Gu [16]. We also mention the existence and non-existence results in the papers of E. Mitidieri [20], L. Peletier and R. C. A. M. van der Vorst [23], and P. Pucci and J. Serrin [25] on the subject.

In general, in the unbounded domain case the problem becomes more complicate; among other reasons

(i) compact operators are not expected and

(ii) it is not clear *a priori* in which function spaces solutions of $(1.1)_\lambda$ might lie.

In general, we need spaces which will control the asymptotic behavior of the solutions (and their derivatives) at infinity and such spaces, as we will see in our case, are weighted or/and homogeneous Sobolev spaces.

Recently many authors have studied nonlinear polyharmonic problems in unbounded domains. We refer to more representative of them in the problem: in the radial case the works of Y. Furusho and T. Kusano [13], T. Kusano et al. [17], E. S. Noussair et al. [21], and C. A. Swanson and L. S. Yu [32, 33], in the non-radial sub- (super-) linear case the works of W. Allegretto and L. S. Yu [2], F. Bernis [3], C. A. Swanson [31] as well as the results on the one-dimensional problem by L. Peletier et al. [24]. Also, the fixed point theory is used in several cases as in the paper of T. Kusano et al. [18] (see also the references therein). Maximum principle results for the biharmonic equation in unbounded domains are obtained recently by N. M. Stavrakakis and G. Sweers [30]. Uniqueness questions for the radial case are studied recently in the work of C. A. Swanson [32]. Let also notice that the study in the above mentioned papers [2, 3, 9, 11, 21, 31 - 33] is based in homogeneous Sobolev spaces of type similar to the one used here.

In all these papers the weight function is non-negative. To our knowledge the only works, were the eigenvalue problem for the linear polyharmonic problem with indefinite weight function is discussed, are those published by J. Fleckinger and her co-workers (see, for example, A. Djellit [9], J. Fleckinger and M. L. Lapidus [11] and the references therein). However, their weight function is of a certain fractional type at infinity. Finally, for more general weights of L^P-type, in the semilinear case for the Laplace operator we refer to K. J. Brown and N. M. Stavrakakis [4] and for a quasilinear eigenvalue problem to J. Fleckinger et al. [12].

To be able to carry out our study and especially to apply the bifurcation methods, we introduce certain equivalent weighted and homogeneous Sobolev spaces. This is done in Section 2. We construct two function spaces which will form the base to develop our theory for both the linear and semilinear problem. These spaces are, on the one hand, the Hilbert space $\mathcal{V}_2$, i.e. the closure of the $C_0^\infty(\mathbb{R}^N)$-functions with respect to the norm

$$|||u|||_2 = \left\{ \int_{\mathbb{R}^N} |\Delta u|^2 dx - \frac{\alpha}{2} \int_{\mathbb{R}^N} g|u|^2 dx \right\}$$

for an appropriate positive constant α to be chosen later and, on the other hand, the standard "energy" space $\mathcal{D}^{2,2}$, i.e. the closure of the $C_0^\infty(\mathbb{R})$-functions with respect to the norm

$$||u||^2_{\mathcal{D}^{2,2}} = \int_{\mathbb{R}} |\Delta u|^2 dx.$$

Also, we briefly state results, to be used later, concerning the existence, regularity and the asymptotic behavior of the first positive principal eigenvalue λ_1 of the linearized problem $(2.1)_\lambda$.

In Section 3, we study the existence of a local continuum of positive solutions branching out from the first eigenvalue of the semilinear biharmonic problem $(1.1)_\lambda$-(1.2).

Finally, we get the global character of this continuum in Section 4. This fact is achieved by applying standard global bifurcation theory in the homogeneous Sobolev space $\mathcal{D}^{2,2}$.

Notation: We denote by $B_r(z)$ the open ball with center $z \in \mathbb{R}^N$ and radius r. For simplicity we use the symbol $||\cdot||_p$ for the norm $||\cdot||_{L^p(\mathbb{R})}$ and L^p for the space $L^p(\mathbb{R}^N)$ $(1 \leq p \leq \infty)$. The end of proofs is marked by the sign ∎

2. Space setting - the linearized problem

In this section we shall discuss the existence of non-zero simple principal eigenvalues for the linearized biharmonic problem

$$(-\Delta)^2 u = \lambda g(x) u \qquad (x \in \mathbb{R}^N) \tag{2.1$_\lambda$}$$

$$u(x) > 0 \quad (x \in \mathbb{R}^N), \quad \lim_{|x|\to+\infty} u(x) = 0. \tag{2.2}$$

To simplify notation but without loss of generality we shall assume that $f'(0) = 1$, so that equation $(2.1)_\lambda$ becomes the linearization of equation $(1.1)_\lambda$. The proofs of all results presented in this section are given in detail by L. Peletier and N. M. Stavrakakis in [22]. Also, the main results where announced in [28].

The natural space setting for the eigenfunctions of this problem, as we show next, will be the space $\mathcal{D}^{2,2}(\mathbb{R}^N)$, i.e. the closure of the $C_0^\infty(\mathbb{R}^N)$-functions with respect to the norm

$$||u||_{\mathcal{D}^{2,2}} = \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx \right)^{1/2}.$$

It can be shown (see, for example, in C. A. Swanson [30]) that

$$\mathcal{D}^{2,2} = \left\{ u \in L^{\frac{2N}{N-4}}(\mathbb{R}^N) : |\nabla^2 u| \in L^2(\mathbb{R}^N) \right\}$$

and that there exists $K > 0$ such that for all $u \in \mathcal{D}^{2,2}$

$$||u||_{\frac{2N}{N-4}} \leq K \, ||u||_{\mathcal{D}^{2,2}}.$$

So $\mathcal{D}^{2,2}(\mathbb{R}^N)$ is a reflexive Banach space.

Our approach is based on the following inequality of generalized Poincaré type.

Lemma 2.1. *Suppose $g \in L^{N/4}(\mathbb{R}^N)$. Then there exists $\alpha = 1/K||g||_{N/4}^{1/2} > 0$ such that*

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx \geq \alpha \int_{\mathbb{R}^N} |g|\, |u|^2 dx,$$

for all $u \in C_0^\infty(\mathbb{R}^N)$.

Thus if $g \in L^{N/4}(\mathbb{R}^N)$ and $\alpha > 0$ is as in Lemma 2.1, we can define an inner product on $C_0^\infty(\mathbb{R}^N)$ by

$$\langle u, v\rangle_2 = \int_{\mathbb{R}^N} \Delta u \Delta v\, dx - \frac{\alpha}{2} \int_{\mathbb{R}^N} guv\, dx.$$

Next we define $\mathcal{V}_2$ to be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the above product. The space $\mathcal{V}_2$ depends on the function g; it is natural to expect that $\mathcal{V}_2$ grows as $|g|$ becomes smaller. However, under condition $(\mathcal{G})$ we prove that $\mathcal{V}_2$ is independent of this function. In fact, the space $\mathcal{V}_2$ is characterized by the following

Lemma 2.2. *Suppose $g \in L^{N/4}(\mathbb{R}^N)$. Then $\mathcal{V}_2 = \mathcal{D}^{2,2}$.*

Thus we may henceforth suppose that $|||\cdot|||_2$, the norm in $\mathcal{V}_2$, coincides with the norm in $\mathcal{D}^{2,2}$ and that the inner product in $\mathcal{V}_2$ is given by

$$\langle u, v\rangle = \int_{\mathbb{R}^N} \Delta u \Delta v\, dx.$$

Proceeding as for example in [4], we define a bilinear form by

$$\beta(u, v) = \int_{\mathbb{R}^N} guv\, dx \qquad (u, v \in \mathcal{V}_2).$$

By the fact that $\mathcal{V}_2 \subseteq L^{\frac{2N}{N-4}}(\mathbb{R}^N)$ we obtain that β is *bounded* in $\mathcal{V}_2$. Hence by the Riesz Representation Theorem we can define a bounded linear operator M such that

$$\beta(u, v) = \langle Mu, v\rangle \qquad (u, v \in \mathcal{V}_2).$$

It is standard to check the following result.

Lemma 2.3. *Suppose $g \in L^{N/4}(\mathbb{R}^N)$. Then the operator M is selfadjoint and compact.*

By means of classical spectral methods we have proved the existence, positivity and principality of the first eigenvalue. These results are given by the following

Theorem 2.4. *Equation $(2.1)_\lambda$ admits a positive first eigenvalue λ_1 given by*

$$\lambda_1 = \inf_{\langle Mu, u\rangle = 1} ||u||^2_{\mathcal{D}^{2,2}}.$$

The associated eigenfunction ϕ belongs to $\mathcal{D}^{2,2}(\mathbb{R}^N)$.

For the weak formulation of the problem the following result is necessary.

Theorem 2.5. *Suppose that g satisfies hypothesis $(\mathcal{G})$ and $u \in \mathcal{D}^{2,2}$. Then there exists a sequence $\{R_n\} \subset \mathbb{R}$ with $R_n \to \infty$ as $n \to +\infty$ such that*

$$\lim_{n\to\infty} \int_{\partial B_{R_n}} \nabla u \frac{\partial \nabla u}{\partial n}\, dS = 0 = \lim_{n\to\infty} \int_{\partial B_{R_n}} u \frac{\partial \Delta u}{\partial n}\, dS = 0.$$

Having in mind the application of bifurcation theory to the study of problem $(2.1)_\lambda$, information concerning the dimension of the eigenspace associated to the principal eigenvalues of the linearized biharmonic problem $(2.1)_\lambda$ are of basic importance. The main results in this direction, needed in the rest of the paper, can be stated as follows.

Theorem 2.6. *Let g satisfy hypothesis $(\mathcal{G})$. Then we have:*

(i) *The eigenspace corresponding to the principal eigenvalue λ_1 is of dimension* 1.

(ii) *λ_1 is the only eigenvalue of* $(2.1)_\lambda$ *to which a positive eigenfunction corresponds.*

The proof of this theorem is long and technical. We again refer to [21] for the detailed proof.

Remark 2.7. The algebraic and the geometric multiplicities of the eigenvalues of the problem under discussion are equal since the operator M is compact and selfadjoint (see Lemma 2.3.) (see E. Zeidler [34]).

3. Local bifurcation results

In this section we shall obtain results on the existence of solutions for the nonlinear problem $(1.1)_\lambda$-(1.2), close to $(\lambda_1, 0)$, by considering local bifurcation of solutions from the zero solution. First, we state a general asymptotic and regularity result for the solutions of this problem. Using Agmon's theorem (see [1: Theorem 6.1]) and Serrin's estimates from [27] (see also [15: Theorems 8.17 and 9.19]) we can prove the L^σ-character and the $C^{4,\alpha}_{loc}$-regularity as well as the asymptotic properties of the solutions of $(1.1)_\lambda$. For these technics we refer also to S. Luckhaus [19]. The main results of this and the next section were announced in the paper [29].

Lemma 3.1. *Suppose that $u \in \mathcal{D}^{2,2}$ is a solution of equation* $(1.1)_\lambda$. *Then:*

(i) *u is a classical solution, i.e. $u \in C^{4,\alpha}_{loc}(\mathbb{R}^N)$. Moreover, for any $x_0 \in \mathbb{R}^N$ and $p > 1$ we have*

$$||u||_{W^{4,p}(B_1(x_0))} \le C||u||_{L^{\frac{2N}{N-4}}(B_2(x_0))}, \tag{3.1}$$

where $C = C(x_0, \lambda, N, m, ||g||_\infty, |||u|||_2)$.

(ii) *$D^\beta u(x)$ decay uniformly to zero as $|x| \to +\infty$, for all $|\beta| \le 3$.*

Proof. For the detailed proof we refer to [2: Lemma 5] ∎

To apply local bifurcation theory we introduce the nonlinear operator $P : \mathbb{R} \times \mathcal{V}_2 \to \mathcal{V}_2$ through the relation

$$\langle P(\lambda, u), \phi\rangle = \int_{\mathbb{R}^N} \Delta u \Delta \phi \, dx - \lambda \int_{\mathbb{R}^N} g f(u)\phi\, dx \qquad (3.2),$$

for all $\phi \in \mathcal{V}_2$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{D}^{2,2}$.

Lemma 3.2. *The operator P is well defined by* (3.2).

Proof. For fixed $u \in \mathcal{D}^{2,2}$ we define the functional

$$F(\phi) = \int_{\mathbb{R}^N} \Delta u \Delta \phi \, dx - \lambda \int_{\mathbb{R}^N} g f(u) \phi \, dx,$$

for $\phi \in \mathcal{D}^{2,2}$. Since f satisfies hypothesis $(\mathcal{F})$, then $f \in L^{\frac{2N}{N-4}}(\mathbb{R}^N)$ and so for some constant K_1

$$\begin{aligned} |F(\phi)| &\le ||\Delta u||_2 \, ||\Delta \phi||_2 + |\lambda| \, ||g||_{N/4} \, ||f(u)||_{\frac{2N}{N-4}} \, ||\phi||_{\frac{2N}{N-4}} \\ &\le K_1 \big(||\Delta u||_2 + |\lambda| \, ||g||_{N/4} \, ||f(u)||_{\frac{2N}{N-4}} \big) ||\phi||_{\mathcal{V}_2}. \end{aligned}$$

So F is a bounded linear functional. Hence by the Riesz Representation Theorem we may define P as in (3.2) ∎

Lemma 3.3. *The operator P defined by* (3.2) *is continuous and for* $N = 5, 6, \ldots, 12$ *Fréchet differentiable with continuous Fréchet derivatives given by*

$$\begin{aligned} \langle P_u(\lambda, u)\phi, \psi \rangle &= \int_{\mathbb{R}^N} \Delta \phi \Delta \psi \, dx - \lambda \int_{\mathbb{R}^N} g f'(u) \phi \psi \, dx, \\ \langle P_\lambda(\lambda, u), \phi \rangle &= - \int_{\mathbb{R}^N} g f(u) \phi \, dx, \\ \langle P_{\lambda u}(\lambda, u)\phi, \psi \rangle &= - \int_{\mathbb{R}^N} g f'(u) \phi \psi \, dx, \end{aligned}$$

for all $\phi, \psi \in \mathcal{D}^{2,2}$.

Proof. For completeness we juct sketch the proof.

(i) To prove the continuity of P, we have

$$\begin{aligned} &\big\langle P(\lambda, u) - P(\mu, w), v \big\rangle \\ &\quad = \big\langle P(\lambda, u) - P(\lambda, w), v \big\rangle + \big\langle P(\lambda, w) - P(\mu, w), v \big\rangle \\ &\quad = \left| \int_{\mathbb{R}^N} (\Delta u - \Delta w) \Delta v \, dx \right| \\ &\qquad + \lambda \left| \int_{\mathbb{R}^N} g(f(u) - f(w)) v \, dx \right| + |\lambda - \mu| \left| \int_{\mathbb{R}^N} g f(w) v \, dx \right| \\ &\quad \le \Big\{ 1 + \lambda \sup_{\tau \in (0,1)} ||g||_{N/4} \, ||f'(u + \tau w)||_\infty \Big\} ||u - w||_{\mathcal{D}^{2,2}} ||v||_{\mathcal{D}^{2,2}} \\ &\qquad + c|\lambda - \mu| \, ||g||_{N/4} ||w||_{\mathcal{D}^{2,2}} ||v||_{\mathcal{D}^{2,2}}. \end{aligned}$$

So P is continuous at any $(\lambda, u) \in \mathbb{R} \times \mathcal{D}^{2,2}$.

(ii) Let $u, h \in \mathcal{D}^{2,2}$ and

$$P(\lambda, u, h) = P(\lambda, u + h) - P(\lambda, u) - P_u(\lambda, u)h.$$

Then we have

$$\begin{aligned}
&\big\|P(\lambda, u+h) - P(\lambda, u) - P_u(\lambda, u)h\big\|^2_{\mathcal{D}^{2,2}} \\
&\quad \le |\lambda| \left| \int_{\mathbb{R}^N} g\big(f(u+h) - f(u) - f'(u)h\big)\, P(\lambda, u, h)dx \right| \\
&\quad \le |\lambda| \sup_{t\in(0,1)} ||f''(u+th)||_\infty\, ||g||_q\, ||h||_p^2\, ||P(\lambda, u, h)||_p,
\end{aligned}$$

where $p = \frac{2N}{N-4}$ and $\frac{1}{q} + \frac{3}{p} = 1$, i.e. $q = \frac{2N}{12-N}$. Hence for $N = 5, 6, \ldots, 12$ we get that $q \ge \frac{N}{4}$ and so $g \in L^q(\mathbb{R}^N)$. Hence we have

$$\begin{aligned}
&\big\|P(\lambda, u+h) - P(\lambda, u) - P_u(\lambda, u)h\big\|_{\mathcal{D}^{2,2}} \\
&\quad \le |\lambda| \sup_{t\in(0,1)} ||f''(u+th)||_\infty\, ||g||_q\, ||h||^2_{\mathcal{D}^{2,2}} \\
&\quad = o(||h||_{\mathcal{D}^{2,2}}).
\end{aligned}$$

So P is Fréchet differentiable in $\mathbb{R} \times \mathcal{D}^{2,2}$.

(iii) Let $u, v, \phi \in \mathcal{D}^{2,2}$. Then we have

$$\begin{aligned}
&\big|\big\langle P_u(\lambda, u)h - P_u(\mu, v)h, \phi\big\rangle\big| \\
&\quad \le \Big|\big\langle P_u(\lambda, u)h - P_u(\lambda, v)h, \phi\big\rangle + \big\langle P_u(\lambda, v)h - P_u(\mu, v)h, \phi\big\rangle\Big| \\
&\quad \le |\lambda| \left| \int_{\mathbb{R}^N} g(f'(u) - f'(v))h\phi\, dx \right| + |\lambda - \mu| \left| \int_{\mathbb{R}^N} gf'(v)\phi\, dx \right| \\
&\quad \le |\lambda| \sup_{t\in(0,1)} ||f''(tu + (1-t)v)||_\infty\, ||g||_q\, ||u-q||_{\mathcal{D}^{2,2}}\, ||h||_{\mathcal{D}^{2,2}}\, ||\phi||_{\mathcal{D}^{2,2}} \\
&\qquad + |\lambda - \mu|\, ||g||_{N/4}\, ||f'(v)||_\infty\, ||h||_p\, ||\phi||_p,
\end{aligned}$$

where $p = \frac{2N}{N-4}$ and $q = \frac{2N}{12-N}$ for $N = 5, 6, \ldots, 12$. Hence we have that P_u is continuous in $\mathbb{R} \times \mathcal{D}^{2,2}$ ∎

Next we have the following local result.

Theorem 3.4 (Local Bifurcation). *There exists $\varepsilon_0 > 0$ and continuous functions $\eta : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ and $\psi : (-\varepsilon_0, \varepsilon_0) \to [\phi]^\perp$ such that $\eta(0) = \lambda_1$, $\psi(0) = 0$ and every non-trivial solution of $P(\lambda, u) = 0$ in a small neighbourhood of $(\lambda_1, 0)$ is of the form $(\lambda_\varepsilon, u_\varepsilon) = (\eta(\varepsilon), \varepsilon\phi + \varepsilon\psi(\varepsilon))$.*

Proof. We shall prove that the operator P satisfies all the hypotheses of the local bifurcation theorem in M. Crandall and P. Rabinowitz [7].

(i) The operator $P_u(\lambda_1, 0)$ is linear, compact, selfadjoint and $P_u(\lambda_1, 0)\phi = 0$ if and only if $\phi \in \mathcal{V}_2$ is a solution of equation $(2.1)_{\lambda_1}$. Therefore $N(P_u(\lambda_1, 0)) = [\phi]$ where ϕ is the principal eigenfunction of $(2.1)_{\lambda_1}$. So $\psi \in R(P_u(\lambda_1, 0))$ if and only if there exists $w \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ and $\langle\psi, \phi\rangle = \langle P_u(\lambda_1, 0)w, \phi\rangle$. But selfadjointness of $P_u(\lambda_1, 0)$ implies that

$$\langle P_u(\lambda_1, 0)w, \phi\rangle = \langle w, P_u(\lambda_1, 0)\phi\rangle = \langle w, 0\rangle = 0.$$

Hence $R(P_u(\lambda_1,0)) = [\phi]^\perp$.

(ii) Let $w \in N(P_u(\lambda_1,0)) \cap R(P_u(\lambda_1,0))$. Then $P_u(\lambda_1,0)w = 0$ and there exists $\psi \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ such that $\langle P_u(\lambda_1,0)\psi, w\rangle = \langle w,w\rangle$. Again selfadjointness implies that $0 = \langle \psi, P_u(\lambda_1,0)w\rangle = \langle w,w\rangle$. So

$$\langle w,w\rangle = ||w||_{\mathcal{D}^{2,2}} = \left(\int_{\mathbb{R}^N} |\Delta w|^2\,dx\right)^{1/2} = 0.$$

Then using Lemma 3.1/(ii) we obtain that $w \equiv 0$ in $\mathbb{R}^N$, i.e.

$$N(P_u(\lambda_1,0)) \cap R(P_u(\lambda_1,0)) = \{0\}.$$

Also, it is easy to see that

$$N(P_u(\lambda_1,0)) \oplus R(P_u(\lambda_1,0)) = \mathcal{D}^{2,2}(\mathbb{R}^N).$$

(iii) Finally, we have that $P_{\lambda u}(\lambda_1,0)\phi \notin R(P_u(\lambda_1,0))$ (transversality condition) since

$$\langle P_{\lambda u}(\lambda_1,0)\phi,\phi\rangle = -\int_{\mathbb{R}^N} g\phi^2 dx < 0$$

and the proof is completed ∎

The last statement of this section describes the sign of solutions of equation $(1.1)_\lambda$ close to the bifurcation point.

Theorem 3.5. *Let $(\lambda_\varepsilon, u_\varepsilon)$ be solutions of equation $(1.1)_\lambda$ given by Theorem 3.4, $\varepsilon > 0$. Then there exists $\varepsilon_0 > 0$ such that $u_\varepsilon > 0$ in $\mathbb{R}^N$ whenever $0 < \varepsilon \le \varepsilon_0$.*

Proof. From the above theorem we have that $u_\varepsilon = \varepsilon\phi + \varepsilon\psi(\varepsilon)$ satisfies $(1.1)_\lambda$ where $\lambda = \eta(\varepsilon)$. We have

$$(-\Delta)^2\psi(\varepsilon) = \eta(\varepsilon) g \frac{f(\varepsilon\phi + \varepsilon\psi(\varepsilon))}{\varepsilon} - \lambda_1 g\phi,$$

which implies that

$$(-\Delta)^2\psi(\varepsilon) - \rho(x)\psi(\varepsilon) = (\eta(\varepsilon) - \lambda_1)g\phi + \frac{1}{2}\eta(\varepsilon) g\phi f''(\xi(\varepsilon,x))u_\varepsilon,$$

where $\rho(x) = \eta(\varepsilon) g\big[1 + \frac{1}{2} f''(\xi(\varepsilon,x))u_\varepsilon\big]$ and $\xi(\varepsilon,x)$ lies between 0 and $u_\varepsilon(x)$. Whenever $0 < \varepsilon \le \varepsilon_0$, it is easy to prove that, for some positive constant C, $|u_\varepsilon(x)| \le C$ for all $x \in \mathbb{R}^N$, and so ρ is uniformly bounded (i.e. $||\rho||_\infty < +\infty$). Moreover,

$$||g\phi||_{\frac{2N}{N-4}} \le ||g||_\infty\, ||\phi||_{\frac{2N}{N-4}} \le ||g||_\infty\, ||\phi||_{\mathcal{V}_2}$$

and

$$||g\phi u_\varepsilon||_{\frac{2N}{N-4}} \le ||g||_\infty\, ||\phi||_\infty\, ||u_\varepsilon||_{\frac{2N}{N-4}} \le ||g||_\infty\, ||\phi||_\infty\, ||u_\varepsilon||_{\mathcal{V}_2}.$$

If $N = 5, 6, 7$ and $\frac{s}{2} = \frac{2N}{N-4}$, then $s > N$. Hence by Lemma 3.1 with $p = \frac{s}{2} = \frac{2N}{N-4}$, there exists a constant $K > 0$ (independent of ε) such that

$$\sup_{|x|\le R_0} |\psi(\varepsilon)(x)| \le K||g||_\infty \big\{(\eta(\varepsilon) - \lambda_1)||\phi||_{\mathcal{V}_2} + ||\phi||_\infty \, ||u_\varepsilon||_{\mathcal{V}_2}\big\}.$$

Since $\phi(x) > 0$ for all x in the compact set $B_{R_0} = \{x \in \mathbb{R}^N : |x| \le R_0\}$, it follows that there exists $\varepsilon_1 > 0$ such that $\phi(x) + \psi(\varepsilon)(x) > 0$ for all $x \le R_0$ provided that $0 < \varepsilon < \varepsilon_1$. Suppose $0 < \varepsilon < \varepsilon_1$ and that $u_\varepsilon(x_0) < 0$ for some $x_0 \in \mathbb{R}^N$. We put equation $(1.1)_\lambda$ in form of the system

$$\left.\begin{aligned} -\Delta u_\varepsilon &= v_\varepsilon, \\ -\Delta v_\varepsilon &= \lambda g(x) f(u_\varepsilon) \end{aligned}\right\} \tag{3.3}$$

where we know by Lemma 3.1 that $\lim_{|x|\to+\infty} u_\varepsilon(x) = \lim_{|x|\to+\infty} v_\varepsilon(x) = 0$. Since $g(x)f(u) \ge 0$ in $\mathbb{R}^N \times \mathbb{R}^+$, for any $x_0 \in \mathbb{R}^N$ we may apply the maximum principle on the ball $B_r(x_0)$, for r large enough to obtain $v_\varepsilon(x_0) \ge \inf_{|x|=r} v_\varepsilon(x)$. Letting $r \to +\infty$ we get that $-\Delta u \ge 0$ so $u_\varepsilon \ge 0$. Therefore applying the maximum principle again (see [15: Theorem 8.18]) on the non-trivial solution u_ε we get that $u_\varepsilon > 0$.

Since $\lim_{|x|\to\infty} u_\varepsilon(x) = 0$, it follows that there must exist x_1, $|x_1| > R$, such that u_ε attains a negative minimum at x_1. But then

$$-\Delta u_\varepsilon(x_1) = \lambda g(x_1) f(u_\varepsilon(x_1)) > 0,$$

which is impossible. Hence $u_\varepsilon(x) > 0$ for all $x \in \mathbb{R}^N$ whenever $0 < \varepsilon < \varepsilon_1$ ∎

4. Global continuation of the branch

To discuss the global nature of the continuum of solutions bifurcating from $(\lambda_1, 0)$, we write the operator P in the form $P(\lambda, u) = u - \lambda R(u)$, where

$$\langle R(u), \phi\rangle = \int_{\mathbb{R}^N} g(x) f(u(x))\phi(x)\, dx \qquad (\phi \in \mathcal{D}^{2,2}).$$

Also, we assume that R satisfies the relation

$$R(u) = Mu + \mathcal{H}(u),$$

where M denotes the same linear operator as in Section 2, i.e.

$$\langle Mu, v\rangle = \int_{\mathbb{R}^N} guv\, dx \qquad (u, v \in \mathcal{D}^{2,2})$$

and $\mathcal{H}(u) = O(|||u|||_2^\beta)$ as $|||u|||_2 \to 0$, for some $\beta > 1$.

It is shown in Section 2 that λ_1 is an eigenvalue of L and by Theorem 2.6 the eigenspace associated with λ_1 has algebraic and geometric (see Remark 2.7) multiplicity 1. Also, as f is a Lipschitz function, it can be proved by modifying slightly the proof of Lemma 2.3 that R is a compact operator (see in [22]). Thus we can apply the classical global bifurcation theorem (see P. Rabinowitz [26]) to obtain the following

Theoprem 4.1. *There exists a continuum $\mathcal{C}$ of non-zero solutions of problem $(1.1)_\lambda - (1.2)$ bifurcating from $(\lambda_1, 0)$, which is either*

(i) *unbounded*

or

(ii) *contains a point $(\lambda, 0)$, where $\lambda \neq \lambda_1$ is an eigenvalue of $(2.1)_\lambda$.*

Moreover, $\mathcal{C}$ has a connected subset $\mathcal{C}^+ \subset \mathcal{C} - \{(\eta(\varepsilon), u_\varepsilon) : -\varepsilon_0 \leq \varepsilon \leq 0\}$ for some $\varepsilon_0 > 0$ such that $\mathcal{C}^+$ also satisfies one of the above alternatives. Close to the bifurcation point $(\lambda_1, 0)$, $\mathcal{C}^+$ consists of the curve $\varepsilon \to (\eta(\varepsilon), u_\varepsilon)$ $(0 < \varepsilon \leq \varepsilon_0)$.

The following lemmas describe the nature of solutions lying on $\mathcal{C}^+$.

Lemma 4.2. *There exists $\lambda_* > 0$ such that $\lambda > \lambda_*$ whenever $(\lambda, u) \in \mathcal{C}^+$.*

Proof. Suppose $u \in \mathcal{V}_2$ is a solution of problem $(1.1)_\lambda - (1.2)$. Multiplying equation $(1.1)_\lambda$ by u, integrating over B_R, letting $R \to \infty$ and using Theorem 2.5 we obtain

$$\begin{aligned} |||u|||_2^2 &= \int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &= \lambda \int_{\mathbb{R}^N} g f(u) u \, dx \\ &\leq \lambda k^* ||g||_{N/4} \, ||u||^2_{\frac{2N}{N-4}} \qquad \text{(where } |\mathrm{f(u)}| \leq \mathrm{k^* u \text{ for all } u)} \\ &\leq \lambda K_1 ||g||_{N/4} \, |||u|||_2^2, \end{aligned}$$

where K_1 is a constant and the result follows. Therefore we may get that $\lambda_* = 1/K_1 ||g||_{N/4} > 0$ ∎

In order to proceed further we must investigate the L^∞-closeness of solutions which are close in $\mathbb{R} \times \mathcal{D}^{2,2}$; since $\mathcal{D}^{2,2}$ does not embed into $L^\infty(\mathbb{R}^N)$, this is not immediately obvious. Actually, we can prove

Lemma 4.3. *Let $N < 8$. Then there exist constants $K_1 > 0$ and $K_2 > 0$ such that*

$$|u_\lambda(x) - u_\mu(x)| \leq K_1 |\lambda - \mu| + K_2 ||u_\lambda - u_\mu||_{\mathcal{D}^{2,2}}, \qquad \textit{for all } \ x \in \mathbb{R}^N,$$

whenever μ is close to λ and $u_\lambda, u_\mu \in \mathcal{D}^{2,2}$ are solutions of equations $(1.1)_\lambda$ and $(1.1)_\mu$, respectively.

Proof. Indeed, it is easy to see that

$$(-\Delta)^2 (u_\lambda - u_\mu) = g\{\lambda f(u_\lambda) - \mu f(u_\mu)\}.$$

Hence by Lemma 3.1 there exists $C > 0$ such that

$$\begin{aligned} &|u_\lambda(x) - u_\mu(x)| \\ &\quad \leq \sup_{y \in B_1(x)} |u_\lambda(y) - u_\mu(y)| \\ &\quad \leq C \Big\{ ||u_\lambda - u_\mu||_{L_p(B_2(x))} + ||g\,[\lambda f(u_\lambda) - \mu f(u_\mu)]||_p \Big\} \\ &\qquad \left(\text{where } \mathrm{p} = \tfrac{2\mathrm{N}}{\mathrm{N}-4} \text{ and by Lemma 3.1 } 2\mathrm{p} = \tfrac{4\mathrm{N}}{\mathrm{N}-4} > \mathrm{N}, \text{ so } \mathrm{N} < 8\right) \\ &\quad \leq C_1 |||u_\lambda - u_\mu|||_2 + C_2 ||g||_\infty \Big\{ ||(\lambda - \mu) f(u_\lambda)||_p + ||\mu[f(u_\lambda) - f(u_\mu)]||_p \Big\} \\ &\quad \leq C_1 |||u_\lambda - u_\mu|||_2 + C_2 |\lambda - \mu| \, ||g||_\infty \, |||u_\lambda|||_2 + C_3 |\mu| \, ||g||_\infty \, |||u_\lambda - u_\mu|||_2, \end{aligned}$$

where C_1, C_2 and C_3 are constants and the result follows ∎

Theorem 4.4. *$\mathcal{C}^+$ contains no points of the form $(\lambda, 0)$, where $\lambda \neq \lambda_1$.*

Proof. Suppose that there exists $(\lambda, u) \in \mathcal{C}^+$ such that $u(x_0) < 0$ for some $x_0 \in \mathbb{R}^N$. By Theorem 3.5, $u(x) > 0$ for all $x \in \mathbb{R}^N$ whenever $(\lambda, u) \in \mathcal{C}^+$ is close to $(\lambda_1, 0)$. Moreover, by Lemma 4.3, points in $\mathcal{C}^+$ which are close in $\mathbb{R} \times \mathcal{V}$ must also be close in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$. Hence there must exist $(\lambda_0, u_0) \in \mathcal{C}^+$ such that $u_0(x) \geq 0$ for all $x \in \mathbb{R}^N$ but $u_0(x_0) = 0$ for some $x_0 \in \mathbb{R}^N$, and in any neighbourhood of (λ_0, u_0) we can find a point $(\hat{\lambda}, \hat{u}) \in \mathcal{C}^+$ with $\hat{u}(x) < 0$ for some $x \in \mathbb{R}^N$. Let B denote any open ball containing x_0. Then

$$\left.\begin{array}{rl} -\Delta u_0(x) - \lambda g(x)\dfrac{f(u_0(x))}{u_0(x)}u_0(x) = 0 & \text{on } B \\ u_0(x) \geq 0 & \text{on } \partial B. \end{array}\right\}$$

It follows from the Serrin Maximum principle (see [14]) that $u_0 \equiv 0$ on B. Hence $u_0 \equiv 0$ on $\mathbb{R}^N$.

Thus we can construct a sequence $\{(\lambda_n, u_n)\} \subseteq \mathcal{C}^+$ such that $u_n(x) > 0$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$, $u_n \to 0$ in $\mathcal{V}_\in$ and $\lambda_n \to \lambda_0$. Let $v_n = \frac{u_n}{|||u_n|||_2}$. Since $u_n = \lambda_n M(u_n) + \lambda_n \mathcal{H}(u_n)$ we have

$$v_n = \lambda_n M(v_n) + \lambda_n \frac{\mathcal{H}(u_n)}{||u_n||_{\mathcal{V}}}.$$

Since M is compact, there exists a subsequence of $\{v_n\}$ (which we again denote by $\{v_n\}$) such that $\{M(v_n)\}$ is convergent. Since $\lim_{n\to\infty} \frac{\mathcal{H}(u_n)}{|||u_n|||_2} = 0$, $\{v_n\}$ is convergent to v_0, say, and $v_0 = \lambda_0 M(v_0)$. Since $v_n \geq 0$ for all $n \in \mathbb{N}$, $v_0 \geq 0$. Since by Theorem 2.6 λ_1 is the only positive eigenvalue corresponding to a positive eigenfunction, it follows that $\lambda_1 = \lambda_0$. Thus $(\lambda_0, u_0) = (\lambda_1, 0)$ and this contradicts the fact that every neighbourhood of (λ_0, u_0) must contain a solution $(\hat{\lambda}, \hat{u}) \in \mathcal{C}^+$ with $\hat{u}(x) < 0$, for some $x \in \mathbb{R}^N$. Hence $u(x) > 0$ for all $x \in \mathbb{R}^N$ whenever $(\lambda, u) \in \mathcal{C}^+$ ∎

Corollary 4.5. *$\mathcal{C}^+$ must connect $(\lambda_1, 0)$ to ∞ in $\mathbb{R} \times \mathcal{D}^{2,2}$.*

The next theorem shows that $\mathcal{C}^+$ cannot become unbounded at a finite value of λ; in order to obtain this result we must impose some further restrictions on f and g.

Theorem 4.6. *Suppose that, for some $\gamma \in (0,1)$, $f(s) \sim |s|^\gamma$ at infinity and $g \in L^p(\mathbb{R}^N)$, where $p < \frac{2N}{2N-(\gamma+1)(N-4)}$. Then there exists a continuous function $\mathcal{K} : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|||u|||_2 \leq \mathcal{K}(\lambda)$ whenever $(\lambda, u) \in \mathcal{C}^+$.*

Proof. As in Lemma 4.2 we obtain that if u satisfies problem $(1.1)_\lambda - (1.2)$, then

$$|||u|||_2^2 = \lambda \int_{\mathbb{R}^N} gf(u)u\,dx \leq \lambda K \int_{\mathbb{R}^N} |g|u^{1+\gamma}dx \leq \lambda K ||g||_p^{1/p} ||u||_{(1+\gamma)q}^{1/q}$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. We set $(1+\gamma)q = \frac{2N}{N-4}$. Then we get $\frac{1}{q} = \frac{(1+\gamma)(N-4)}{2N} = \beta < 2$. We obtain $||u||_2^2 \leq \lambda K ||g||_p^{1/p} ||u||_2^\beta$. So $||u||_2 \leq \lambda K ||g||_p^{\frac{1}{2-\beta}} \equiv \mathcal{K}(\lambda)$ and the proof is complete ∎

As an immediate consequence of the previous results we can give the following complete description of the continuum $\mathcal{C}^+$.

Theorem 4.7 (Global Bifurcation). *Suppose that* $N = 5, 6, 7,$ *for some* $\gamma \in (0,1)$, $Sf(s) \sim |s|^\gamma$ *at infinity and* $g \geq 0$, $g \in L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, *where* $1 < p < \frac{2N}{2N-(\gamma+1)(N-4)}$. *Then there exists a continuum* $\mathcal{C}^+ \subseteq \mathbb{R} \times \mathcal{D}^{2,2}$ *of solutions of problem* $(1.1)_\lambda - (1.2)$ *bifurcating from the zero solution at* $(\lambda_1, 0)$ *such that:*

(i) *If* $(\lambda, u) \in \mathcal{C}^+$, *then* $\lambda > \lambda_* > 0$, $u \in L^\infty$ *and* $u(x) > 0$ *for all* $x \in \mathbb{R}^N$.

(ii) $\{\lambda : (\lambda, u) \in \mathcal{C}^+$ *for some* $u \in \mathcal{D}^{2,2}\} \supseteq (\lambda_1, \infty]$.

In particular, problem $(1.1)_\lambda - (1.2)$ *has a non-trivial solution* $u \in \mathcal{D}^{2,2}$ *such that* $u(x) > 0$ *for all* $x \in \mathbb{R}^N$ *whenever* $\lambda > \lambda_1$.

Remark 4.7. The restriction on the dimension seems to be relared, via the Sobolev embeddings, to the certain technique used. For example, in the case of the Laplacian problem the authors of the work [4] have proved that the restriction on the dimension appearing there, based on the bifurcation technique, may be removed by an other method (super- and sub-solutions) as it is proved in the work [5]. However, in the work [5] we got a general existence result, instead of the more geometric inside in the problem we gained by applying bifurcation methods. We guess that the dimension restriction is the price we have to pay for the geometric clarity of the solution branches.

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