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A System of Ordinary and Partial Differential Equations Describing Creep Behaviour of Thin-Walled Shells

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Abstract. The article deals with a system of partial and ordinary differential equations describing creep and damage processes in the material of thin-walled structures. It is shown that if set up in suitable Sobolev spaces, this system may be solved uniquely, locally in time.

Keywords: *Coupled ordinary and partial differential equations, shell models*

AMS subject classification: Primary 35 J 99, 35 Q 72, secondary 73 K 12, 73 K 15

1. Introduction

Metals and alloys exposed to high temperatures over a certain period of time experience irreversible deformations. Such phenomena, called "creep", must be taken into account in analysis and design of thin-walled structures (see [48]). In fact, creep strains cause significant stress redistributions in such structures and may give rise to creep failure, even under moderate loading.

Creep deformations correspond to irreversible changes of material properties, due to nucleation, growth of microcavities, ageing of microstructure or other factors (see [52]). In order to represent such damage effects in a mathematical way, it is usual to consider a nonlinear system of differential equations comprising two kinds of equations. On one hand, there are constitutive equations for the material at hand. They state how the rate of change of the creep tensor depends on stress state, temperature, and some internal state variables. On the other hand, hardening or softening of the respective material is described by an appropriate evolution system for internal state variables. This system may be derived by considering the mechanisms of deformation and of damage evolution acting in a given material (compare [38] for example).

In addition to such a material model, another set of equations, governing kinematics and equilibrium of the respective structure, must be specified. These equations usually

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take the form of a system of partial differential equations, which may be linear or nonlinear depending on the magnitude of the deformations exhibited by the structure (see [49]).

In this paper, we shall perform a mathematical analysis of such a model describing creep-damage processes in thin-walled structures. Corresponding to the indications given above, this model consists of nonlinear ordinary differential equations – governing creep and damage processes in the respective material – and of partial differential equations – governing kinematics and equilibrium of the thin-walled structure under consideration. To simplify our discussion, we reduce the thin-walled structure to a thin, shallow shell. Moreover, we neglect hardening effects and assume creep behaviour to be isotropic, incompressible and independent of the kind of loading involved. Then our model includes just one internal state variable, and it describes creep-damage behaviour only in the isothermal case, under quasistatic loading. It should be remarked, though, that the reduction to a single state variable is not essential for our theory and only serves to diminish the number of equations involved.

As a further simplification, our model does not account for geometrically nonlinear effects of shell deformations. In other words, we assume strains and displacements to be small. In such a case, the total strain tensor ε may be additively decomposed into an elastic part ε^{el} and an irreversible creep part ε^{cr},

$$\varepsilon = \varepsilon^{el} + \varepsilon^{cr}$$

(see [38]). The elastic part of the strains can be calculated from the Hooke's law, that is, the stress tensor σ of the shell is given by

$$\sigma_{ij} = \sum_{k,l=1}^{3} C_{ijkl} \cdot (\varepsilon_{kl} - \varepsilon_{kl}^{cr}) \qquad \text{for } 1 \leq i, j \leq 3 \tag{1.1}$$

where the elastic isotropic material parameter tensor C takes the form

$$C_{ijkl} = \frac{E}{2(1-\nu^2)} \Big((\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk})\,(1-\nu) + 2\nu\,\delta_{ij}\,\delta_{kl} \Big) \tag{1.2}$$

with E denoting Young's modulus, ν Poisson's ratio and δ_{ij} Kronecker's symbol. Note that the tensor C is constant. This, of course, means we consider a homogeneous material, with its elastic behaviour independent of the damage state.

Concerning the relation between strains ε_{ij} and displacements u_i, it is derived from the Kirchhoff-Love hypotheses, under the assumption that both the strains of the middle surface of the shell and the rotational angles of the normal vector of this middle surface are infinitesimal quantities. Then the kinematics of the shell may be characterized by specifying the displacements of the material points of the middle surface. We further assume there is an open bounded set $A \subset \mathbb{R}^2$ such that each point of the middle surface may uniquely be assigned to an element (x_1, x_2) of A. The shape of the shallow shell is described by the principal curvatures $(\kappa_{ij})_{1 \leq i,j \leq 2}$. Then, denoting the thickness of the shell by h, and setting $V = A \times (-\frac{h}{2}, \frac{h}{2})$, we get the following relations between the strains ε_{ij} and displacements u_i:

$$\varepsilon_{ij} = \tfrac{1}{2}(D_i u_j + D_j u_i) + \kappa_{ij}\,u_3 - x_3\,D_i D_j u_3 \tag{1.3}$$

for $x \in V$ and $1 \le i,j \le 2$. The symbols D_i and D_j denote partial derivatives with respect to space variables. Assuming that the shell is loaded by a force $q : A \mapsto \mathbb{R}^3$, the quasistatic equilibrium equations can be put as follows (see [9] and the references given therein):

$$\sum_{i,j,k=1}^{2} C_{ijkl}\, h\Big(D_k D_j u_i(x_1,x_2,t) + \kappa_{ij}(x_1,x_2)\, D_k u_3(x_1,x_2,t) \\ + D_k \kappa_{ij}(x_1,x_2)\, u_3(x_1,x_2,t) Big) \\ = \sum_{i,j,k=1}^{2} C_{ijkl} \int_{-h/2}^{h/2} D_k \varepsilon_{ij}^{cr}(x_1,x_2,x_3,t)\, dx_3 + q_l(x_1,x_2)\, h \tag{1.4}$$

for $1 \le l \le 2$, $(x_1,x_2) \in A$ and $t \in [0,T]$, and

$$\sum_{i,j,k,l=1}^{2} C_{ijkl}\, h\Big(\tfrac{h^2}{12} D_l D_k D_j D_i u_3(x_1,x_2,t) \\ + \kappa_{ij}(x_1,x_2)\, \kappa_{kl}(x_1,x_2)\, u_3(x_1,x_2,t) + \kappa_{kl}(x_1,x_2)\, D_j u_i(x_1,x_2,t)\Big) \\ = \sum_{i,j,k,l=1}^{2} C_{ijkl} \left(\kappa_{ij}(x_1,x_2) \int_{-\frac{h}{2}}^{\frac{h}{2}} \varepsilon_{kl}^{cr}(x_1,x_2,x_3,t)\, dx_3 \right. \\ \left. - \int_{-h/2}^{h/2} D_j D_i \varepsilon_{kl}^{cr}(x_1,x_2,x_3,t)\, x_3 dx_3 \right) + q_3(x_1,x_2)\, h \tag{1.5}$$

for $(x_1,x_2) \in A$ and $t \in [0,T]$. Concerning the inelastic part ε^{cr} of the strain tensor, it is supposed to satisfy the ensuing system of ordinary differential equations

$$\frac{\partial}{\partial t}\varepsilon_{ij}^{cr}(x,t) = A\,\Gamma(u,\varepsilon^{cr})^{n-1}(x,t)\,\Lambda_{ij}(u,\varepsilon^{cr})(x,t)\,(1-d(x,t))^{-n} \tag{1.6}$$

$$\frac{\partial}{\partial t} d(x,t) = B\,\Gamma(u,\varepsilon^{cr})^{m}(x,t)\,(1-d(x,t))^{-\widetilde{m}} \tag{1.7}$$

for $x \in V$ and $1 \le i,j \le 2$. Here u denotes the displacement vector and ε^{cr} the creep strain tensor. The letters $A, B, n, m, \widetilde{m}$ stand for material constants, determined from uniaxial creep tests under stationary loading and constant temperature. The internal state variable $d : V \times [0,T] \to \mathbb{R}$ describes the effect of damage arising in the material. The operators Γ and Λ are defined by

$$\Gamma(u,\varepsilon^{cr}) = \left(\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\cdot\sigma_{22} + 3\,\sigma_{12}^2\right)^{\frac{1}{2}}$$

and

$$\left.\begin{aligned} \Lambda_{11}(u,\varepsilon^{cr}) &= \tfrac{2}{3}\sigma_{11} - \tfrac{1}{3}\sigma_{22} \\ \Lambda_{22}(u,\varepsilon^{cr}) &= \tfrac{2}{3}\sigma_{22} - \tfrac{1}{3}\sigma_{11} \\ \Lambda_{12}(u,\varepsilon^{cr}) &= \Lambda_{21}(u,\varepsilon^{cr}) := \sigma_{12} \end{aligned}\right\}$$

with σ_{ij} introduced via (1.1) and (1.3). The unknowns in system (1.4) - (1.7) are the displacement vector u, the creep strain tensor ε^{cr} and the damage variable d. Equations (1.6), (1.7) represent our material model, and system (1.4), (1.5) describes kinematics and equilibrium of our thin shallow shell.

The relations in (1.6) and (1.7) were proposed by Rabotnov [51]. Hayhurst [32] modified them by introducing a generalized multiaxial stress criterion for damage evolution. This modification implies that different operators should be substituted for Γ in equations (1.6) and (1.7), respectively. The theory we shall develop in the following may easily be adapted to such a situation, provided Γ is replaced by operators which smoothly depend on σ_{ij}. We further note that in our model, the creep strain rate and the damage rate are sensitive only to the von Mises equivalent stress.

Equations (1.4) - (1.7) are supplemented by boundary and initial conditions. For simplicity, displacements and rotation are prescribed everywhere on the boundary ∂A of the shallow shell:

$$\left.\begin{aligned} u(\cdot,t)\,|\,\partial A &= u_0 \\ \frac{\partial u_3(x_1,x_2,t)}{\partial n^{(A)}(x_1,x_2)} &= w_0(x_1,x_2) \end{aligned}\right\} \tag{1.8}$$

for $(x_1,x_2)\in\partial A$ and $t\in[0,T]$, with given functions $u_0:\partial A\mapsto\mathbb{R}^3$ and $w_0:\partial A\mapsto\mathbb{R}$. The symbol $n^{(A)}$ denotes the outward unit normal to A. The initial conditions read as follows:

$$\left.\begin{aligned} \varepsilon^{cr}(x,0) &= \varepsilon_0(x) \\ d(x,0) &= d_0(x) \end{aligned}\right\} \qquad \text{for } x\in V \tag{1.9}$$

where ε_0 and d_0 are given functions.

We refer to [7] for a more thorough discussion on how the preceding model arises in mechanics of solids, and to [46] for comparisons with experiments. In [8], an effective numerical scheme is proposed in order to obtain approximate solutions to equations (1.4) - (1.9). Here we intend to show these equations are well posed in a mathematical sense. In fact, we shall prove that if set up in suitable Sobolev spaces, problem (1.4) - (1.9) may be solved uniquely, locally in time. To this end, we shall assume the domain A has a smooth boundary, and the parameters m and n in (1.6) and (1.7), respectively, verify the relations $n\geq 3$ and $m\geq 2$. The latter assumptions are valid for creep behaviour of metals and alloys under moderate loading and temperature (compare [10], for example).

Our proofs are based on an argument which states that for ε^{cr} given in a suitable class of functions, the solution u to boundary value problem (1.4), (1.5), (1.8) exhibits the property that the functions $\nabla_\varrho u_1(\varrho,t)$, $\nabla_\varrho u_2(\varrho,t)$ and $D^2_\varrho u_3(\varrho,t)$ are bounded in ϱ for each fixed value of t. Since the space variable ϱ is taken from the two-dimensional domain A, and because we seek our solutions in Sobolev spaces, the argument just mentioned is valid due to Sobolev's lemma provided we proceed in one of the following two ways: either equation (1.4) is solved in $W^{s,2}(A)^2$ with some $s>2$ and equation (1.5) in $W^{s,2}(A)$ with $s>3$, or we consider solutions of (1.4), (1.5) in $W^{2,p}(A)^2$ and $W^{3,p}(A)$, respectively, for some $p>2$. We decided for the second alternative because then our theory becomes somewhat less complicated, and our assumptions on the parameters m and n are less restrictive than those which would be necessary in the first case.

Equations (1.6) and (1.7) become singular when the state variable d takes values close to 1. This is the main reason why we can only prove local existence in time of our solutions. The details of our existence result may be found in Theorem 6.1 below.

In the mathematical model given by equations (1.4) - (1.7), internal variables are used in order to describe creep behavior of metals. (These internal variables are, of course, the functions d and ε^{cr}.) Mathematical models involving internal variables and pertaining to bulk materials were considered in [12, 13, 29, 31, 34, 35, 37, 39, 45, 47, 54]. These references essentially deal with constitutive relations which lead to initial-boundary value problems of the type $w_t + C(w) = 0$, where C is a monotone operator. A detailed mathematical theory for constitutive equations of monotone type is given in the monograph [3] by Alber. In the non-monotone case, existence results global in time could be shown by Alber and coworkers in certain special situations (see references [4, 5] dealing with certain constitutive equations in one space dimension, [15 - 19] pertaining to the Bodmer-Partom model, [14] treating constitutive equations of pre-monotone type). A local existence result for Miller's equations is proved in [36]. If the right-hand side of constitutive equations as those in (1.6), (1.7) satisfies a global Lipschitz condition with respect to the internal variables, solutions global in time may be obtained by the arguments presented in [33]. The article [6] gives a presentation – from the point of view of a mathematician – of how constitutive equations with internal variables are derived in continuum mechanics.

We mention that another way of modelling creep behavior of metals consists in introducing integral terms instead of internal variables (see [22 - 25, 59], for example). It should further be indicated that the monograph [40] treats certain systems of ordinary and partial differential equations arising in population dynamics. Concerning the special case of coupled linear partial and linear ordinary differential equations, we refer to [41, 42] for results on well posedness and numerical treatment.

The results on coupled ordinary and partial differential equations established in the preceding references do not cover system (1.4) - (1.7). Similarly, although there is a rich mathematical literature on the theory of thin shells (see the monographs [11, 21, 27, 43] and the references therein), we do not know of any mathematical study pertaining to shell models with internal variables. Thus, in order to solve problem (1.4) - (1.9), a seperate investigation is needed, which will be presented in the present article.

2. Notations and definition of function spaces

If $m, n \in \mathbb{R}$, we shall use the abbreviation $m \vee n$ for the maximum of m and n. For $N \in \mathbb{N}$ and $\varrho \in \mathbb{R}^N$, we put $|\varrho|_1 = |\varrho_1| + \ldots + |\varrho_N|$, whereas $|\varrho|$ denotes the Euclidean norm of ϱ.

Let A be a set. For a function $f : A \mapsto \mathbb{R}$, we put $|f|_0 = \sup\{|f(x)| : x \in A\}$. Assume that $\mathcal{F}$ is a space which contains functions mapping A into $\mathbb{R}$, and take $\sigma \in \mathbb{N}$. Then we define

$$\mathcal{F}^\sigma = \Big\{ F : A \mapsto \mathbb{R}^\sigma \,\Big|\, F_j \in \mathcal{F} \text{ for } 1 \le j \le \sigma \Big\}$$
$$\mathcal{F}^{\sigma\times\sigma} = \Big\{ F : A \mapsto \mathbb{R}^{\sigma\times\sigma} \,\Big|\, F_{ij} \in \mathcal{F} \text{ for } 1 \le i, j \le \sigma \Big\}.$$

Suppose $\mathcal{F}$ is equipped with a norm $\|\cdot\|_{\mathcal{F}}$. Then we define the norm $\|\cdot\|_{\mathcal{F}^\sigma}$ of the space $\mathcal{F}^\sigma$ by

$$\|F\|_{\mathcal{F}^\sigma} = \sum_{j=1}^{\sigma} \|F_i\|_{\mathcal{F}} \qquad (F \in \mathcal{F}^\sigma).$$

The norm $\|\cdot\|_{\mathcal{F}^{\sigma\times\sigma}}$ of $\mathcal{F}^{\sigma\times\sigma}$ is to be understood in an analogous way. Similarly, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are spaces with norm $\|\cdot\|_{\mathcal{B}_1}$ and $\|\cdot\|_{\mathcal{B}_2}$, respectively, we choose the norm

$$\|(v_1, v_2)\|_{\mathcal{B}_1\times\mathcal{B}_2} = \|v_1\|_{\mathcal{B}_1} + \|v_2\|_{\mathcal{B}_2} \qquad (v_1 \in \mathcal{B}_1, v_2 \in \mathcal{B}_2)$$

for the space $\mathcal{B}_1 \times \mathcal{B}_2$. Moreover, for a space $\mathcal{B}$ with norm $\|\cdot\|_{\mathcal{B}}$, for $T \in (0,\infty)$ and $u \in C^0([0,T], \mathcal{B})$, we put

$$\|u\|_{\mathcal{B},\infty} = \sup\big\{\|u(t)\|_{\mathcal{B}} : t \in [0,T]\big\}.$$

Note that if $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space, then $\|\cdot\|_{\mathcal{B},\infty}$ is a norm, and $C^0([0,T],\mathcal{B})$ equipped with this norm is also a Banach space.

Let $N, k \in \mathbb{N}$, $p \in (1,\infty)$ and $B \subset \mathbb{R}^N$ an open set. We write $W^{k,p}(B)$ for the usual Sobolev space of order k and exponent p. The corresponding norm is denoted by $\|\cdot\|_{k,p}$, and the corresponding seminorm by $|\cdot|_{k,p}$, that is,

$$\|u\|_{k,p} = \left(\sum_{\alpha\in\mathbb{N}_0^N, |\alpha|_1\le k} \|D^\alpha u\|_p^p\right)^{\frac{1}{p}} \quad \text{and} \quad |u|_{k,p} = \left(\sum_{\alpha\in\mathbb{N}_0^N, |\alpha|_1= k} \|D^\alpha u\|_p^p\right)^{\frac{1}{p}}$$

for $u \in W^{k,p}(B)$. We define $W_0^{k,p}(\Omega)$ as the closure of $C_0^\infty(B)$ with respect to the norm $\|\cdot\|_{k,p}$. Put $p' = (1-\frac{1}{p})^{-1}$. Let $W_0^{-k,p}(B)$ denote the closure of $L^p(B)$ with respect to the norm

$$\|f\|_{-k,p} = \sup\left\{\left|\int_B f\, v\, dx\right| : \ v \in W_0^{k,p'}(B) \text{ with } \|v\|_{k,p'} = 1\right\} \qquad (f \in L^p(B)).$$

As is well known (see [1: 3.12]), the space $W_0^{-k,p}(B)$ may be identified with the dual space $[W_0^{k,p'}(B)]'$ of $W_0^{k,p'}(B)$ if this dual space is equipped with the usual norm. We write $W^{-k,p}(B)$ for the dual space of $W^{k,p'}(B)$.

Assume that B is C^2-bounded. Then, for $s \in (0,2)$, we shall use the standard Sobolev spaces $W^{s,p}(\partial B)$ of fractional order s and exponent p.

We further introduce some function spaces which are particular to the theory we shall present in the following. To this end, let A be a bounded open set in $\mathbb{R}^2$, and take

$h \in (0, \infty)$. Abbreviate $V = A \times (-\frac{h}{2}, \frac{h}{2})$. Then we set

$$\mathcal{W} = \Big\{ v \in W_0^{1,2}(A)^3 : v_3 \in W_0^{2,2}(A) \Big\}, \quad \|v\|_{\mathcal{W}} = \|(v_1, v_2)\|_{1,2} + \|v_3\|_{2,2} \quad (v \in \mathcal{W})$$

$$\mathcal{V}_p = \Big\{ v \in W^{2,p}(A)^3 : v_3 \in W^{3,p}(A) \Big\}, \quad \|v\|_{\mathcal{V}_p} = \|(v_1, v_2)\|_{2,p} + \|v_3\|_{3,p} \quad (v \in \mathcal{V}_p)$$

$$\mathcal{V}_p^0 = \Big\{ v \in W^{2,p}(A)^3 \cap W_0^{1,p}(A)^3 : v_3 \in W^{3,p}(A) \cap W_0^{2,p}(A) \Big\}$$

$$\mathcal{X}_p = L^p(A)^2 \times W_0^{-1,p}(A), \qquad \|F\|_{\mathcal{X}_p} = \|(F_1, F_2)\|_p + \|F_3\|_{-1,p} \text{ for } F \in \mathcal{X}_p$$

$$\mathcal{Y}_p = \left\{ \alpha : V \mapsto \mathbb{R} \text{ measurable} \,\middle|\, \begin{array}{l} \alpha(\cdot,\cdot,x_3) \in W^{1,p}(A) \text{ for } x_3 \in (-\frac{h}{2}, \frac{h}{2}) \\ \sup\left\{ \|\alpha(\cdot,\cdot,x_3)\|_{1,p} : x_3 \in (-\frac{h}{2}, \frac{h}{2}) \right\} < \infty \end{array} \right\}$$

$$\|\alpha\|_{\mathcal{Y}_p} = \sup \Big\{ \|\alpha(\cdot,\cdot,x_3)\|_{1,p} : x_3 \in (-\tfrac{h}{2}, \tfrac{h}{2}) \Big\} \text{ for } \alpha \in \mathcal{Y}_p.$$

Note that the mappings $\|\cdot\|_{\mathcal{W}}, \|\cdot\|_{\mathcal{V}_p}, \|\cdot\|_{\mathcal{X}_p}$ and $\|\cdot\|_{\mathcal{Y}_p}$ are norms, and the corresponding spaces are Banach spaces.

3. Auxiliary results

In this section, we give an overview of the tools we shall need. First we mention two Sobolev inequalities which we state here in order to be able to refer to the constants appearing in them.

Theorem 3.1 (some Sobolev inequalities in $\mathbb{R}^2$). *Let $p \in (2, \infty), \Omega \subset \mathbb{R}^2$ open, bounded, with Lipschitz boundary. Then there is a constant $C_1 > 0$ with*

$$\left. \begin{array}{rl} \|u\|_p \le C_1 \|u\|_{1,2} & \quad (u \in W^{1,2}(\Omega)) \\ |u|_0 \le C_1 \|u\|_{1,p} & \quad (u \in W^{1,p}(\Omega)). \end{array} \right\}$$

Proof. See [1: p. 97/98] ∎

For completeness, we state some further inequalities which will turn out to be useful.

Theorem 3.2 (Poincaré inequality). *Let $\nu \in \{1, 2\}, N \in \mathbb{N}, \Omega \subset \mathbb{R}^N$ open and bounded. Then there is a constant $C_2 > 0$ with*

$$\|u\|_{\nu,2} \le C_2 |u|_{\nu,2} \qquad (u \in W_0^{\nu,2}(\Omega)).$$

Proof. See [1: p. 158/159] ∎

Lemma 3.1 (Korn's inequality). *Let $N \in \mathbb{N}, \Omega \subset \mathbb{R}^N$ open and bounded. Then there is a constant $C_3 > 0$ with*

$$\|u\|_{1,2}^2 \le C_3 \sum_{i,j=1}^{N} \|D_i u_j + D_j u_i\|_2^2 \qquad (u \in W_0^{1,2}(\Omega)). \tag{3.1}$$

Proof. As is well known, the lemma follows by integrating by parts on the right-hand side of (3.1) and then applying Theorem 3.2 with $\nu = 1$ [26: p. 1260/1261] ∎

Theorem 3.3 (Minkowski inequality for integrals). *Take $M, N \in \mathbb{N}$ and $p \in (1,\infty)$. Let $X \subset \mathbb{R}^M$ and $Y \subset \mathbb{R}^N$ be measurable sets and $F : X \times Y \mapsto \mathbb{R}$ a measurable function. Then*

$$\left(\int_Y \left(\int_X |F(x,y)|\,dx\right)^p dy\right)^{\frac{1}{p}} \leq \int_X \left(\int_Y |F(x,y)|^p dy\right)^{\frac{1}{p}} dx.$$

Proof. See [53: p. 271] ∎

We shall further need some results on existence and regularity of solutions to the biharmonic equation and to the Lamé system. Concerning the biharmonic equation, we have by [30: Theorem 7.1.2]

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^2$ a bounded domain with C^4-boundary. Let $p \in (1,\infty)$, and define*

$$\mathcal{F} = \mathcal{F}(p,\Omega) : W^{3,p}(\Omega) \cap W_0^{2,p}(\Omega) \mapsto W^{-1,p}(\Omega)$$

by

$$\mathcal{F}(u)(v) = \int_\Omega \nabla\Delta u \cdot \nabla v\, dx$$

for $u \in W^{3,p}(\Omega) \cap W_0^{2,p}(\Omega)$ and $v \in W_0^{1,p'}(\Omega)$. Then the mapping $\mathcal{F}$ is bijective, and there is some constant $C_4 = C_4(p,\Omega) > 0$ with

$$\|u\|_{3,p} \leq C_4 \|\mathcal{F}(u)\|_{-1,p}$$

for $u \in W^{3,p}(\Omega) \cap W_0^{2,p}(\Omega)$.

Concerning the Lamé system, we shall need the following result:

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^2-boundary. Let $\mu, \lambda \in (0,\infty)$ and $p \in (1,\infty)$. Define*

$$\mathcal{G} = \mathcal{G}(p,\Omega,\mu,\lambda) : W^{2,p}(\Omega)^2 \cap W_0^{1,p}(\Omega)^2 \mapsto L^p(\Omega)^2$$

by

$$\mathcal{G}(u) = \mu\,\Delta u + (\mu+\lambda)\,\nabla\,\mathrm{div}\,u$$

for $u \in W^{2,p}(\Omega)^2 \cap W_0^{1,p}(\Omega)^2$. This operator $\mathcal{G}$ is bijective, and there is a constant $C_5 = C_5(p,\Omega,\mu,\lambda) > 0$ with

$$\|u\|_{2,p} \leq C_5 \|\mathcal{G}(u)\|_p$$

for $u \in W^{2,p}(\Omega)^2 \cap W_0^{1,p}(\Omega)^2$.

Proof. This theorem follows from [2] (see [20: p. 296 – 298]) for more details ∎

Finally, we recall some well known facts on trace theorems, repeated here in such a form as will be needed later on.

Theorem 3.6 *Let $N \in \mathbb{N}, p \in (1, \infty), \Omega \subset \mathbb{R}^N$ an open, bounded set with C^3-boundary. Let $n^{(\Omega)}$ denote the outward unit normal to Ω. Take $v_0 \in W^{3-\frac{1}{p},p}(\partial\Omega)$ and $w_0 \in W^{2-\frac{1}{p},p}(\partial\Omega)$. Then there is a function $u_0 \in W^{3,p}(\Omega)$ with*

$$u_0| \, \partial\Omega = v_0 \qquad \text{and} \qquad \sum_{j=1}^{N} D_j u_0| \, \partial\Omega \cdot n_j^{(\Omega)} = w_0$$

where the functions $u_0| \, \partial\Omega$ and $D_j u_0| \, \partial\Omega$ are to be understood in the trace sense ($1 \le j \le N$).

Furthermore, let $\widetilde{v}_0 \in W^{2-\frac{1}{p},p}(\partial\Omega)$. Then there is a function $\widetilde{u}_0 \in W^{2,p}(\Omega)$ with $\widetilde{u}_0| \, \partial\Omega = \widetilde{v}_0$ in the trace sense.

Proof. See [1: 7.50 - 7.56] ∎

Due to the regularity assumptions required for v_0, w_0 and $\widetilde{v}_0$ in the preceding theorem, we could, of course, impose an additional boundary condition on u_0 and $\widetilde{u}_0$. However, we shall not need this fact.

Theorem 3.7. *Let $N, p, \Omega, n^{(\Omega)}$ be given as in Theorem 3.6. Let $u \in W^{2,p}(\Omega)$. Then $u \in W_0^{2,p}(\Omega)$ if and only if*

$$u| \, \partial\Omega = 0 \qquad \text{and} \qquad \sum_{j=1}^{N} D_j u| \, \partial\Omega \cdot n_j^{(\Omega)} = 0 \quad \text{in the trace sense.}$$

Moreover, take $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if $u| \, \partial\Omega = 0$ in the trace sense.

Proof. See [1: 7.54 and 7.55] ∎

4. System (1.4), (1.5) with a given right-hand side

For the rest of this paper, we shall assume that p is some fixed number from $(2, \infty)$. Moreover, let $A \subset \mathbb{R}^2$ be some fixed bounded domain with C^4-boundary, take $h \in (0, \infty)$, and put $V = A \times (-\frac{h}{2}, \frac{h}{2})$, as in Section 2. With p, A, h chosen in this way, let the spaces $\mathcal{W}, \mathcal{V}_p, \mathcal{V}_p^0, \mathcal{X}_p, \mathcal{Y}_p$ be defined as in Section 2. For $i, j \in \{1, 2\}$, let $\kappa \in C^1(\overline{A})^{2\times 2}$. Assume that $\kappa_{ij} = \kappa_{ji}$ for $1 \le i, j \le 2$.

In this section, we shall study system (1.4), (1.5) under boundary condition (1.8), assuming that ε^{cr} is a given function from $\mathcal{Y}_p^{2\times 2}$. Recall the coefficients C_{ijkl} introduced in (1.2). It follows from this definition

Lemma 4.1. *There is some constant $C_6 > 0$ such that*

$$\sum_{i,j,k,l=1}^{2} C_{ijkl} \, \tau_{ij} \, \tau_{kl} \ge C_6 \sum_{i,j=1}^{2} \tau_{ij}^2$$

for $\tau \in \mathbb{R}^{2\times 2}$ with $\tau_{ij} = \tau_{ji}$.

Next we introduce a family $\{a_\varepsilon\}_{\varepsilon \in [0,1]}$ of bilinear forms, with a_1 corresponding to the variational form of system (1.4), (1.5) considered in [8] (compare Lemma 4.4 below).

Definition 4.1. For $\varepsilon \in [0,1]$ and $v, w \in \mathcal{W}$, put

$$a_\varepsilon(v,w) = \int_A \sum_{i,j,k,l=1}^{2} C_{ijkl}\, h\Big[\big(\tfrac{1}{2}D_i v_j + \tfrac{1}{2}D_j v_i + \sqrt{\varepsilon}\kappa_{ij} v_3\big) \times \big(\tfrac{1}{2}D_k w_l + \tfrac{1}{2}D_l w_k + \sqrt{\varepsilon}\kappa_{kl} w_3\big) + \tfrac{1}{12}h^2 D_i D_j v_3\, D_k D_l w_3\Big]\, d(x_1,x_2).$$

The proof of the ensuing lemma is obvious:

Lemma 4.2. *For any $\varepsilon \in [0,1]$, the mapping a_ε is bilinear. There is a constant $C_7 > 0$ with*

$$|a_\varepsilon(v,w)| \le C_7 \|v\|_{\mathcal{W}}\, \|w\|_{\mathcal{W}}$$

for $v, w \in \mathcal{W}$ and $\varepsilon \in [0,1]$.

In addition, the form a_ε is positive definite:

Lemma 4.3. *There is a constant $C_8 > 0$ with*

$$a_\varepsilon(v,v) \ge C_8 \|v\|_{\mathcal{W}}^2$$

for $v \in \mathcal{W}$ and $\varepsilon \in [0,1]$.

Proof. Let $v \in \mathcal{W}$ and $\varepsilon \in [0,1]$. It readily follows from Lemma 4.1 that $a_\varepsilon(v,v) \ge \frac{1}{12}h^3 C_6 |v_3|_{2,2}$. We may conclude by Poincaré's inequality (Theorem 3.2) that

$$a_\varepsilon(v,v) \ge \mathcal{C}\|v_3\|_{2,2}^2. \tag{4.1}$$

Here and in the rest of this proof, the symbols $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ denote constants which do not depend on ε or v.

On the other hand, once more applying Lemma 4.1, we get

$$\begin{aligned} a_\varepsilon(v,v) &\ge \tfrac{1}{4}C_6 h \sum_{i,j=1}^{2} \|D_i v_j + D_j v_i + 2\sqrt{\varepsilon}\kappa_{ij} v_3\|_2^2 \\ &\ge \tfrac{3}{16}C_6 h \sum_{i,j=1}^{2} \|D_i v_j + D_j v_i\|_2^2 - 3C_6 h \sum_{i,j=1}^{2} |\kappa_{ij}|_0^2\, \|v_3\|_2^2 \end{aligned}$$

where we used the relation

$$(a+b)^2 \ge a^2 + b^2 - 2|ab| \ge \tfrac{3}{4}a^2 - 3b^2 \qquad (a, b \in \mathbb{R})$$

in the last inequality. Thus, referring to Korn's inequality (Lemma 3.1), we have

$$a_\varepsilon(v,v) \ge \mathcal{C}\|(v_1,v_2)\|_{1,2}^2 - \widetilde{\mathcal{C}}\|v_3\|_2^2.$$

Combining this inequality with (4.1) yields

$$\|v\|_{\mathcal{W}} = \|(v_1,v_2)\|_{1,2}^2 + \|v_3\|_{2,2}^2 \le \mathcal{C}\big(a_\varepsilon(v,v) + \|v_3\|_{2,2}^2\big) \le \mathcal{C} a_\varepsilon(v,v)$$

and the lemma is proved ∎

Definition 4.2. For $\varepsilon \in [0,1]$, define the operator $\mathcal{L}_\varepsilon : \mathcal{V}_p^0 \mapsto \mathcal{X}_p$ by setting

$$\mathcal{L}_\varepsilon(v)_l = \sum_{i,j,k=1}^{2} h\, C_{ijkl}\Big(D_k D_j v_i + \sqrt{\varepsilon}\big(\kappa_{ij} D_k v_3 + D_k \kappa_{ij} v_3\big)\Big)$$

for $l \in \{1,2\}$ and $v \in \mathcal{V}_p^0$, and

$$\begin{aligned}\mathcal{L}_\varepsilon(v)_3(\sigma) = \int_A \sum_{i,j,k,l=1}^{2} & h\, C_{ijkl} \\ & \times \Big(\tfrac{1}{12} h^2\, D_k D_j D_i v_3\, D_l \sigma - \varepsilon \kappa_{ij}\kappa_{kl} v_3 \sigma - \sqrt{\varepsilon}\kappa_{kl} D_j v_i \sigma\Big) d(x_1,x_2)\end{aligned}$$

for $v \in \mathcal{V}_p^0$ and $\sigma \in W_0^{1,p'}(A)$. Note that by the definition of the coefficients C_{ijkl}

$$\left.\begin{aligned}\mathcal{L}_0(v)_j &= \mu\Delta v_j + (\mu+\lambda) D_j \operatorname{div}(v_1,v_2) \quad (j \in \{1,2\}) \\ \mathcal{L}_0(v)_3(\sigma) &= \widetilde{\mu} \int_A \nabla\Delta v_3 \nabla\sigma\, dx \quad (\sigma \in W_0^{1,p'}(A))\end{aligned}\right\} \qquad (v \in \mathcal{V}_p^0)$$

with the constants $\mu, \lambda, \widetilde{\mu}$ defined by

$$\mu = \frac{hE}{2(1+\nu)}, \qquad \lambda = \frac{hE\nu}{1-\nu^2}, \qquad \widetilde{\mu} = \frac{h^3 E}{12(1-\nu^2}.$$

Thus, recalling the operator $\mathcal{F} = \mathcal{F}(p,A)$ and $\mathcal{G} = \mathcal{G}(p,A,\mu,\lambda)$ from Theorem 3.4 and 3.5, respectively, we have

$$\left.\begin{aligned}\big(\mathcal{L}_0(v)_l\big)_{1\le l\le 2} &= \mathcal{G}(v_1,v_2) \\ \mathcal{L}_0(v)_3 &= \widetilde{\mu}\,\mathcal{F}(v_3)\end{aligned}\right\} \tag{4.2}$$

for $v \in \mathcal{V}_p^0$.

The bilinear form a_ε and the differential operator $\mathcal{L}_\varepsilon$ are related in the following way:

Lemma 4.4. *Let $\varepsilon \in [0,1]$. Then*

$$\int_A \sum_{l=1}^{2} \mathcal{L}_\varepsilon(v)_l\, w_l\, d(x_1,x_2) + \mathcal{L}_\varepsilon(v)_3(w_3) = -a_\varepsilon(v,w)$$

for $v \in \mathcal{V}_p^0$ and $w \in \mathcal{W}$. Since any $F \in \mathcal{X}_p$ may be considered as an element of $\mathcal{W}'$, if $F \in \mathcal{X}_p$ and $v \in \mathcal{V}_p^0$ with $\mathcal{L}_\varepsilon(v) = -F$, then $a_\varepsilon(v,w) = F(w)$ for $w \in \mathcal{W}$.

Proof. This lemma follows by some easy computations ∎

Now we are able to prove the main result of this section:

Theorem 4.1. *For any $F \in \mathcal{X}_p$, there is one and only one function $u \in \mathcal{V}_p^0$ with $\mathcal{L}_1(u) = F$. There is a constant $C_9 > 0$ with*

$$C_9^{-1}\|F\|_{\mathcal{X}_p} \leq \|u\|_{\mathcal{V}_p} \leq C_9\|F\|_{\mathcal{X}_p} \tag{4.3}$$

for F and u as before.

Proof. In the following, we denote by $\mathcal{C}$ any constant which only depends on A, p, h, on the coefficients C_{ijkl}, or on the functions κ_{ij}. For $v \in \mathcal{V}_p^0$ and $\varepsilon \in [0,1]$, we get with Lemmas 4.3 and 4.4

$$\|v\|_{\mathcal{W}}^2 \leq C_8^{-1} a_\varepsilon(v,v) \leq \mathcal{C}\|\mathcal{L}_\varepsilon(v)\|_{\mathcal{X}_p}\big(\|(v_1,v_2)\|_{p'} + \|v_3\|_{1,p'}\big) \leq \mathcal{C}\|\mathcal{L}_\varepsilon(v)\|_{\mathcal{X}_p}\|v\|_{\mathcal{W}}$$

hence $\|v\|_{\mathcal{W}} \leq \mathcal{C}\|\mathcal{L}_\varepsilon(v)\|_{\mathcal{X}_p}$. The latter inequality and Theorem 3.1 imply

$$\|(v_1,v_2)\|_p + \|v_3\|_{1,p} \leq \mathcal{C}\|\mathcal{L}_\varepsilon(v)\|_{\mathcal{X}_p} \tag{4.4}$$

for $v \in \mathcal{V}_p^0$ and $\varepsilon \in [0,1]$. We further observe that

$$\mathcal{L}_\varepsilon(v)_l - \mathcal{L}_0(v)_l = \sum_{i,j,k=1}^{2} h\, C_{ijkl}\sqrt{\varepsilon}\big(\kappa_{ij}D_k v_3 + D_k\kappa_{ij}v_3\big)$$

for $v \in \mathcal{V}_p^0, \varepsilon \in [0,1]$ and $l \in \{1,2\}$. Referring to (4.2) and Theorem 3.5, we conclude for $v \in \mathcal{V}_p^0$ and $\varepsilon \in [0,1]$

$$\|(v_1,v_2)\|_{2,p} \leq C_5\|(\mathcal{L}_0(v)_l)_{1\leq l\leq 2}\|_p \leq C_5\|(\mathcal{L}_\varepsilon(v)_l)_{1\leq l\leq 2}\|_p + \mathcal{C}\|v_3\|_{1,p}$$

with $C_5 = C_5(p, A, \mu, \lambda)$ introduced in Theorem 3.5. It follows with (4.4),

$$\|(v_1,v_2)\|_{2,p} \leq \mathcal{C}\|\mathcal{L}_\varepsilon(v)\|_{\mathcal{X}_p} \tag{4.5}$$

for $v \in \mathcal{V}_p^0$ and $\varepsilon \in [0,1]$. Furthermore, for $v \in \mathcal{V}_p^0, \varepsilon \in [0,1]$ and $\sigma \in W_0^{1,p'}(A)$,

$$\mathcal{L}_\varepsilon(v)_3(\sigma) - \mathcal{L}_0(v)_3(\sigma) = \int_A \sum_{i,j,k,l=1}^{2} -h\, C_{ijkl}\big(\varepsilon\kappa_{ij}\kappa_{kl}v_3 + \sqrt{\varepsilon}\kappa_{kl}D_j v_i\big)\sigma\, d(x_1,x_2).$$

Thus we may conclude from Theorem 3.4, (4.2), (4.4) and (4.5)

$$\begin{aligned}\|v_3\|_{3,p} &\leq \widetilde{\mu}^{-1}C_4\|\mathcal{L}_0(v)_3\|_{-1,p}\\ &\leq \widetilde{\mu}^{-1}C_4\|\mathcal{L}_\varepsilon(v)_3\|_{-1,p} + \mathcal{C}(\|v_3\|_p + \|(v_1,v_2)\|_{1,p})\\ &\leq \mathcal{C}\|\mathcal{L}_\varepsilon(v)\|_{\mathcal{X}_p}\end{aligned} \tag{4.6}$$

for $v \in \mathcal{V}_p^0$ and $\varepsilon \in [0,1]$, with $C_4 = C_4(p, A)$ from Theorem 3.4. Combining (4.5) and (4.6) yields

$$\|v\|_{\mathcal{V}_p} \leq \mathcal{C}\|\mathcal{L}_\varepsilon(v)\|_{\mathcal{X}_p} \tag{4.7}$$

for $v \in \mathcal{V}_p^0$ and $\varepsilon \in [0,1]$. This means in particular the mapping $\mathcal{L}_\varepsilon : \mathcal{V}_p^0 \mapsto \mathcal{X}_p$ is one-to-one, for any $\varepsilon \in [0,1]$.

Let us show that $\mathcal{L}_1$ is onto. To this end, we use a continuity argument with respect to ε. In fact, it is easy to show that, for $v \in \mathcal{V}_p^0$ and $\varepsilon, \varepsilon' \in [0,1]$,

$$\|\mathcal{L}_\varepsilon(v)\|_{\mathcal{X}_p} \leq \mathcal{C} \tag{4.8}$$

$$\|\mathcal{L}_\varepsilon(v) - \mathcal{L}_{\varepsilon'}(v)\|_{\mathcal{X}_p} \leq \mathcal{C}|\sqrt{\varepsilon} - \sqrt{\varepsilon'}|\,\|v\|_{\mathcal{V}_p}. \tag{4.9}$$

By (4.8), the operator $\mathcal{L}_\varepsilon$ is continuous, and by (4.7) it has closed range and is one-to-one, for any $\varepsilon \in [0,1]$. Moreover, we deduce from (4.2) and Theorem 3.4 and 3.5 that $\mathcal{L}_0$ is onto, hence $\mathcal{L}_0$ has index zero. It follows from (4.9) and [44: p. 27/Theorem 3.11] that index$(\mathcal{L}_\varepsilon) = 0$ for any $\varepsilon \in [0,1]$. Note that [44: Theorem 3.11] is valid not only for Fredholm operators as stated in that reference, but also for operators with closed range and finite-dimensional kernel, as is obvious by the (short) proof of [44: Theorem 3.11] and by [44: p. 25/Theorem 3.9]. Thus we get index$(\mathcal{L}_1) = 0$. Referring to (4.7) we conclude that $\mathcal{L}_1$ is one-to-one and onto. Since inequality (4.3) is a consequence of (4.7) and (4.8), the theorem is proved ∎

Now we consider the right-hand side in (1.4) and (1.5).

Definition 4.3. Introduce the mapping $\mathcal{A}_p : \mathcal{Y}_p^{2\times 2} \mapsto \mathcal{X}_p$ by

$$\mathcal{A}_p(\alpha)_l(x_1,x_2) = \sum_{i,j,k=1}^{2} C_{ijkl} \int_{-h/2}^{h/2} D_k \alpha_{ij}(x_1,x_2,x_3)\,dx_3$$

for $l \in \{1,2\}, \alpha \in \mathcal{Y}_p^{2\times 2}, (x_1,x_2) \in A$,

$$\begin{aligned}\mathcal{A}_p(\alpha)_3(\sigma) = \int_A \sum_{i,j,k,l=1}^{2} C_{ijkl}\Big(&- \kappa_{ij}(x_1,x_2)\int_{-h/2}^{h/2} \alpha_{kl}(x_1,x_2,x_3)\,dx_3 \cdot \sigma(x_1,x_2) \\ &- \int_{-h/2}^{h/2} D_j\alpha_{kl}(x_1,x_2,x_3)x_3\,dx_3 \cdot D_i\sigma(x_1,x_2)\Big)d(x_1,x_2)\end{aligned}$$

for $\alpha \in \mathcal{Y}_p^{2\times 2}$ and $\sigma \in W_0^{1,p'}(A)$.

Lemma 4.5. *The mapping $\mathcal{A}_p$ is well defined, that is, $\mathcal{A}_p(\alpha) \in \mathcal{X}_p$ for $\alpha \in \mathcal{Y}_p^{2\times 2}$, and there is a constant $C_{10} > 0$ such that*

$$\|\mathcal{A}_p(\alpha)\|_{\mathcal{X}_p} \leq C_{10}\|\alpha\|_{\mathcal{Y}_p^{2\times 2}}$$

for $\alpha \in \mathcal{Y}_p^{2\times 2}$.

Proof. For $\alpha \in \mathcal{Y}_p^{2\times 2},\ j,k,l \in \{1,2\}$ and $\nu \in \{0,1\}$, we get by Theorem 3.3

$$\left(\int_A \left|\int_{-h/2}^{h/2} \alpha_{kl}(x)\,dx_3\right|^p d(x_1,x_2)\right)^{\frac{1}{p}} \leq \int_{-h/2}^{h/2} \|\alpha_{kl}(\cdot,\cdot,x_3)\|_p dx_3 \leq h\|\alpha_{kl}\|_{\mathcal{Y}_p}$$

and

$$\left(\int_A \left|\int_{-h/2}^{+h/2} D_j\alpha_{kl}(x)x_3^\nu dx_3\right|^p d(x_1,x_2)\right)^{\frac{1}{p}} \leq h^\nu \int_{-h/2}^{+h/2} \|D_j\alpha_{kl}(\cdot,\cdot,x_3)\|_p dx_3 \leq h^{1+\nu}\|\alpha_{kl}\|_{\mathcal{Y}_p}.$$

The lemma follows from these inequalities, after some easy computations ∎

Definition 4.4. Define the operator $\mathcal{L}$ in the same way as $\mathcal{L}_1$, but with the domain of the former operator enlarged from $\mathcal{V}_p^0$ to $\mathcal{V}_p$.

Corollary 4.1. *For $u_0 \in \mathcal{V}_p, \alpha \in \mathcal{Y}_p^{2\times 2}$ and $F \in \mathcal{X}_p$, there is one and only one function $v = v(u_0, \alpha, F) \in \mathcal{V}_p^0$ with*

$$\mathcal{L}_1(v) = -\mathcal{L}(u_0) + \mathcal{A}_p(\alpha) + F. \tag{4.10}$$

There is a constant $C_{11} > 0$ such that

$$\|v(u_0, \alpha, F)\|_{\mathcal{V}_p} \leq C_{11}\big(\|u_0\|_{\mathcal{V}_p} + \|\alpha\|_{\mathcal{Y}_p^{2\times 2}} + \|F\|_{\mathcal{X}_p}\big)$$

for $u_0 \in \mathcal{V}_p, \alpha \in \mathcal{Y}_p^{2\times 2}$ and $F \in \mathcal{X}_p$.

Proof. Combine Theorem 4.1 and Lemma 4.5 ∎

In view of system (1.4) - (1.7) which we ultimately want to solve, we state Corollary 4.1 for the case the right-hand side in (4.10) depends on time.

Corollary 4.2. *For $u_0 \in \mathcal{V}_p, T \in (0, \infty), q \in C^0([0,T], \mathcal{X}_p), g \in C^0([0,T], \mathcal{Y}_p^{2\times 2})$ there is one and only one mapping $U = U(u_0, g, q) \in C^0([0,T], \mathcal{V}_p^0)$ such that*

$$\mathcal{L}_1(U(t)) = -\mathcal{L}(u_0) + \mathcal{A}_p(g(t)) + q(t)$$

for $t \in [0,T]$. Moreover,

$$\|U(u_0, g, q)\|_{\mathcal{V}_p,\infty} \leq C_{11}\big(\|u_0\|_{\mathcal{V}_p} + \|g\|_{\mathcal{Y}_p^{2\times 2},\infty} + \|q\|_{\mathcal{X}_p,\infty}\big)$$

$$\|U(u_0, g, q) - U(u_0, g', q')\|_{\mathcal{V}_p,\infty} \leq C_{11}\big(\|g - g'\|_{\mathcal{Y}_p^{2\times 2},\infty} + \|q - q'\|_{\mathcal{X}_p,\infty}\big)$$

hold for $u_0 \in \mathcal{V}_p$, $g, g' \in C^0([0,T], \mathcal{Y}_p^{2\times 2})$ and $q, q' \in C^0([0,T], \mathcal{X}_p)$, with C_{11} from Corollary 4.1.

5. An estimate of the solution to system (1.6), (1.7) when the function u is given

In this section, we consider system (1.6), (1.7) of nonlinear ordinary differential equations in suitable function spaces, under the assumption that the function u is given in $C^0([0,T], \mathcal{V}_p)$. We begin by defining the right-hand side in (1.6), (1.7) in a more formal way than in Section 1.

Definition 5.1. Define

$$\sigma(1,1)=1,\quad \sigma(1,2)=2,\quad \sigma(2,1)=3,\quad \sigma(2,2)=4$$

$$F_{kl}(z,x)=\tfrac{1}{2}z_1+\tfrac{1}{2}z_2+\kappa_{kl}(x_1,x_2)z_3-x_3z_4-z_5 \quad (z\in\mathbb{R}^5, x\in V, 1\le k,l\le 2)$$

$$G_{ij}(\varrho,x)=\sum_{k,l=1}^{2}C_{ijkl}F_{kl}\big(\varrho_{\sigma(k,l)},\varrho_{\sigma(l,k)},\varrho_5,\varrho_{\sigma(k,l)+5},\varrho_{\sigma(k,l)+9},x\big)$$
$$(\varrho\in\mathbb{R}^{13}, x\in V, i,j\in\{1,2\})$$

$$P(z)=(z_1^2+z_2^2-z_1z_2+3z_3^2)^{\frac{1}{2}} \quad (z\in\mathbb{R}^3)$$

$$Q_{11}(z)=\tfrac{2}{3}z_1-\tfrac{1}{3}z_2,\quad Q_{22}(z)=-\tfrac{1}{3}z_1+\tfrac{2}{3}z_2,\quad Q_{12}(z)=z_3 \quad (z\in\mathbb{R}^3)$$

$$\Gamma(\varrho,x)=P\big(G_{11}(\varrho,x),G_{22}(\varrho,x),G_{12}(\varrho,x)\big) \quad (\varrho\in\mathbb{R}^{13}, x\in V)$$

$$\Lambda_{ij}(\varrho,x)=Q_{ij}\big(G_{11}(\varrho,x),G_{22}(\varrho,x),G_{12}(\varrho,x)\big) \quad (\varrho\in\mathbb{R}^{13}, x\in V, 1\le i,j\le 2).$$

Let $\widetilde{A},\widetilde{B}\in\mathbb{R}, n\in[3,\infty), m\in[2,\infty)$ and $\widetilde{m}\in(0,\infty)$ be fixed. Then put

$$\left.\begin{aligned} R_{ij}(\varrho,\delta,x)&=\widetilde{A}\,\Gamma^{n-1}(\varrho,x)\,\Lambda_{ij}(\varrho,x)\,(1-\delta)^{-n}\\ S(\varrho,\delta,x)&=\widetilde{B}\,\Gamma^{m}(\varrho,x)\,(1-\delta)^{-\widetilde{m}}\end{aligned}\right\}$$

for $\varrho\in\mathbb{R}^{13},\delta\in(-\infty,1), x\in V$ and $1\le i,j\le 2$.

With these notations, system (1.6), (1.7) of differential equations may be rewritten in the form

$$\left.\begin{aligned} \frac{\partial}{\partial t}\varepsilon^{cr}(x,t)&=R\big(\mathcal{B}(u,\varepsilon^{cr},d)(x,t)\big)\\ \frac{\partial}{\partial t}d(x,t)&=S\big(\mathcal{B}(u,\varepsilon^{cr},d)(x,t)\big)\end{aligned}\right\} \tag{5.1}$$

with $\mathcal{B}(u,\varepsilon^{cr},d)(x,t)$ defined by

$$\begin{aligned}\mathcal{B}(v,g,\delta)(x,t)=\Big(&D_1v_1(\overline{x},t),D_1v_2(\overline{x},t),D_2v_1(\overline{x},t),D_2v_2(\overline{x},t),v_3(\overline{x},t),\\ &D_1D_1v_3(\overline{x},t),D_1D_2v_3(\overline{x},t),D_2D_1v_3(\overline{x},t),D_2D_2v_3(\overline{x},t),\\ &g_{11}(x,t),g_{12}(x,t),g_{21}(x,t),g_{22}(x,t),\delta(x,t),x\Big)\end{aligned} \tag{5.2}$$

for $x\in V,\overline{x}:=(x_1,x_2), t\in[0,T]$, with some $T\in\mathbb{R}$, and for functions

$$\left.\begin{aligned} v&:\ A\times[0,T]\mapsto\mathbb{R}^3\\ g&:\ V\times[0,T]\mapsto\mathbb{R}^{2\times 2}\\ \delta&:\ V\times[0,T]\mapsto\mathbb{R}\end{aligned}\right\}$$

with $v(\cdot,t)\in\mathcal{W}$ for $t\in[0,T]$.

We choose some number $\beta\in(0,\frac{1}{2})$ which will be kept fixed for the rest of this paper. The following estimates will be basic to our arguments:

Lemma 5.1. *There is a constant $C_{12} > 0$ such that, for $i,j \in \{1,2\}, \varrho, \varrho' \in \mathbb{R}^{13}, \theta, \theta' \in (-\infty, 1-\frac{\beta}{2}], x \in V$ and $\nu \in \{0,1,...,16\}$,*

$$\left|D_\nu R_{ij}(\varrho,\theta,x)\right| + \left|D_\nu S(\varrho,\theta,x)\right| \le C_{12}(1+|\varrho|_1)^{m\vee n}$$

and

$$\begin{aligned}&\left|D_\nu R_{ij}(\varrho,\theta,x) - D_\nu R_{ij}(\varrho',\theta',x)\right| + \left|D_\nu S(\varrho,\theta,x) - D_\nu S(\varrho',\theta',x)\right| \\ &\qquad \le C_{12}(1+|\varrho|_1+|\varrho'|_1)^{m\vee n}\big(|\varrho-\varrho'|_1 + |\theta-\theta'|\big)\end{aligned}$$

where $D_0 R_{ij} = R_{ij}$ and $D_0 S = S$.

For brevity, we wrote $D_1, ..., D_{13}$ for derivatives with respect to $\varrho_1, ..., \varrho_{13}$, and D_{14}, D_{15}, D_{16} for derivatives with respect to θ, x_1, x_2.

Proof of Lemma 5.1. For $z \in \mathbb{R}^3$,

$$P(z) = \Big(\frac{z_1^2}{2} + \frac{z_2^2}{2} + \frac{(z_1-z_2)^2}{2} + 3z_3^2\Big)^{\frac{1}{2}} \ge \frac{|z|}{\sqrt{2}}.$$

Thus, for $q \in [2,\infty)$ there is a constant $\mathcal{C}(q) > 0$ with

$$\begin{gathered}|P^q(z)| \le \mathcal{C}(q)|z|_1^q \\ |D_\nu(P^q)(z)| \le \mathcal{C}(q)|z|_1^{q-1} \\ |D_\nu(P^q)(z) - D_\nu(P^q)(z')| \le \mathcal{C}(q)(|z|_1+|z'|_1)^{q-2}|z-z'|_1\end{gathered}$$

for $z, z' \in \mathbb{R}^3$ and $\nu \in \{1,2,3\}$. The lemma may be deduced from these observations by some easy but tedious computations, which we omit here ∎

Lemma 5.2. *There is a constant $C_{13} > 0$ with the properties to follow: Take $M, T \in (0,\infty)$, $v^{(1)}, v^{(2)} \in C^0([0,T],\mathcal{V}_p)$ with $\|v^{(\tau)}\|_{\mathcal{V}_p,\infty} \le M$ for $\tau \in \{1,2\}$. Moreover, take $\varrho^{(1)}, \varrho^{(2)} \in C^0([0,T],\mathcal{Y}_p^{2\times2})$, $\delta^{(1)}, \delta^{(2)} \in C^0([0,T],\mathcal{Y}_p)$ with*

$$\left.\begin{aligned}\delta^{(\tau)}(x,t) \le 1 - \frac{\beta}{2} \qquad & (x\in V) \\ \|\varrho^{(\tau)}(t) - \varrho^{(\tau)}(0)\|_{\mathcal{Y}_p^{2\times2}} + \|\delta^{(\tau)}(t) - \delta^{(\tau)}(0)\|_{\mathcal{Y}_p} \le 1 \qquad & (\tau\in\{1,2\})\end{aligned}\right\}$$

for $t \in [0,T]$. Abbreviate $\mathcal{B}^{(\tau)} = \mathcal{B}(v^{(\tau)}, \varrho^{(\tau)}, \delta^{(\tau)})$ for $\tau \in \{1,2\}$ (see (5.2)). Then

$$(R\circ\mathcal{B}^{(\tau)})(t) \in \mathcal{Y}_p^{2\times2} \qquad and \qquad (S\circ\mathcal{B}^{(\tau)})(t) \in \mathcal{Y}_p$$

for $t \in [0,T]$ and $\tau \in \{1,2\}$,

$$\begin{aligned}&\|R\circ\mathcal{B}^{(\tau)}\|_{\mathcal{Y}_p^{2\times2},\infty} + \|S\circ\mathcal{B}^{(\tau)}\|_{\mathcal{Y}_p,\infty} \\ &\qquad \le C_{13}\big(M + \|\varrho^{(\tau)}(0)\|_{\mathcal{Y}_p^{2\times2}} + \|\delta^{(\tau)}(0)\|_{\mathcal{Y}_p} + 1\big)^{(m\vee n)+1}\end{aligned}$$

for $\tau \in \{1,2\}$, *and*

$$
\begin{aligned}
&\|R\circ\mathcal{B}^{(1)} - R\circ\mathcal{B}^{(2)}\|_{\mathcal{Y}_p^{2\times 2},\infty} + \|S\circ\mathcal{B}^{(1)} - S\circ\mathcal{B}^{(2)}\|_{\mathcal{Y}_p,\infty}\\
&\quad\le C_{13}\left(M + \sum_{\tau=1}^{2}\left(\|\varrho^{(\tau)}(0)\|_{\mathcal{Y}_p^{2\times 2}} + \|\delta^{(\tau)}(0)\|_{\mathcal{Y}_p}\right) + 1\right)^{(m\vee n)+1}\\
&\qquad\times\left(\|v^{(1)} - v^{(2)}\|_{\mathcal{V}_p,\infty} + \|\varrho^{(1)} - \varrho^{(2)}\|_{\mathcal{Y}_p^{2\times 2},\infty} + \|\delta^{(1)} - \delta^{(2)}\|_{\mathcal{Y}_p,\infty}\right).
\end{aligned}
$$

Proof. Lemma 5.2 follows from Lemma 5.1 and Theorem 3.1. To give an example of the arguments involved, consider the term

$$
\mathcal{K}(x,s) = \left|D_\nu(R_{ij}\circ\mathcal{B}^{(1)})(x,s) - D_\nu(R_{ij}\circ\mathcal{B}^{(2)}(x,s)\right|
$$

with $x\in V, s\in[0,T]$ and $\nu, i, j\in\{1,2\}$. Then

$$
\begin{aligned}
\mathcal{K}(x,s) &\le \sum_{r=1}^{14}\left|D_r R_{ij}(\mathcal{B}^{(1)}(x,s)) - D_r R_{ij}(\mathcal{B}^{(2)}(x,s))\right|\left|D_\nu\mathcal{B}_r^{(1)}(x,s)\right|\\
&\quad+\sum_{r=1}^{14}\left|D_r R_{ij}(\mathcal{B}^{(2)}(x,s))\right|\left|D_\nu\mathcal{B}_r^{(1)}(x,s) - D_\nu\mathcal{B}_r^{(2)}(x,s)\right|\\
&\quad+\left|D_{14+\nu}R_{ij}(\mathcal{B}^{(1)}(x,s)) - D_{14+\nu}R_{ij}(\mathcal{B}^{(2)}(x,s))\right|.
\end{aligned}
\tag{5.3}
$$

But we have by Lemma 5.1 and Theorem 3.1

$$
\begin{aligned}
&\sum_{r=1}^{16}\left|D_r R_{ij}\big(\mathcal{B}^{(1)}(x,s)\big) - D_r R_{ij}\big(\mathcal{B}^{(2)}(x,s)\big)\right|\\
&\quad\le \mathcal{C}\left(\sum_{\tau=1}^{2}\sum_{r=1}^{13}|\mathcal{B}_r^{(\tau)}(x,s)| + 1\right)^{m\vee n}\sum_{r=1}^{14}|\mathcal{B}_r^{(1)}(x,s) - \mathcal{B}_r^{(2)}(x,s)|\\
&\quad\le \mathcal{C}\left[\sum_{\tau=1}^{2}\left(\sum_{i,j=1}^{2}\left(\|D_i v_j^{(\tau)}(s)\|_{1,p} + \|D_iD_j v_3^{(\tau)}(s)\|_{1,p} + \|\varrho_{ij}^{(\tau)}(\cdot,x_3,s)\|_{1,p}\right)\right.\right.\\
&\qquad\left.\left.+\|v_3^{(\tau)}(s)\|_{1,p}\right) + 1\right]^{m\vee n}\\
&\qquad\times\left(\sum_{i,j=1}^{2}\left(\|D_i v_j^{(1)}(s) - D_i v_j^{(2)}(s)\|_{1,p} + \|D_iD_j v_3^{(1)}(s) - D_iD_j v_3^{(2)}(s)\|_{1,p}\right.\right.\\
&\qquad\left.+\|\varrho_{ij}^{(1)}(\cdot,x_3,s) - \varrho_{ij}^{(2)}(\cdot,x_3,s)\|_{1,p}\right)\\
&\qquad\left.+\|v_3^{(1)}(s) - v_3^{(2)}(s)\|_{1,p} + \|\delta^{(1)}(s) - \delta^{(2)}(s)\|_{1,p}\right)\\
&\quad\le \mathcal{C}\left[\sum_{\tau=1}^{2}\left(\|v^{(\tau)}\|_{\mathcal{V}_p,\infty} + \|\varrho^{(\tau)}\|_{\mathcal{Y}_p^{2\times 2},\infty}\right) + 1\right]^{m\vee n}
\end{aligned}
\tag{5.4}
$$

$$\times\Big(\|v^{(1)}-v^{(2)}\|_{\mathcal{V}_p,\infty}+\|\varrho^{(1)}-\varrho^{(2)}\|_{\mathcal{Y}_p^{2\times 2},\infty}+\|\delta^{(1)}-\delta^{(2)}\|_{\mathcal{Y}_p,\infty}\Big).$$

Here and in the following, the letter $\mathcal{C}$ denotes constants which do not depend on $x, s, i, j, v^{(1)}, v^{(2)}, \varrho^{(1)}, \varrho^{(2)}, \delta^{(1)}$ or $\delta^{(2)}$. We get in a similar way

$$\sum_{r=1}^{14}\big|D_r R_{ij}(\mathcal{B}^{(2)}(x,s))\big|\leq \mathcal{C}\big(\|v^{(2)}\|_{\mathcal{V}_p,\infty}+\|\varrho^{(2)}\|_{\mathcal{Y}_p^{2\times 2},\infty}+1\big)^{m\vee n}. \tag{5.5}$$

Obviously,

$$\begin{aligned}\left(\int_A\left(\sum_{r=1}^{14}|D_\nu\mathcal{B}_r^{(1)}(x,s)|\right)^p d(x_1,x_2)\right)^{\frac{1}{p}}&\\ \leq \mathcal{C}\Big(\|v^{(1)}(s)\|_{\mathcal{V}_p,\infty}+\|\varrho^{(1)}\|_{\mathcal{Y}_p^{2\times 2},\infty}+\|\delta^{(1)}\|_{\mathcal{Y}_p,\infty}\Big)&\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}\left(\int_A\left(\sum_{r=1}^{14}\big|D_\nu\mathcal{B}_r^{(1)}(x,s)-D_\nu\mathcal{B}_r^{(2)}(x,s)\big|\right)^p d(x_1,x_2)\right)^{\frac{1}{p}}&\\ \leq \mathcal{C}\Big(\|v^{(1)}(s)-v^{(2)}(s)\|_{\mathcal{V}_p,\infty}+\|\varrho^{(1)}-\varrho^{(2)}\|_{\mathcal{Y}_p^{2\times 2},\infty}+\|\delta^{(1)}-\delta^{(2)}\|_{\mathcal{Y}_p,\infty}\Big).&\end{aligned} \tag{5.7}$$

Combining (5.3) - (5.7) yields

$$\begin{aligned}&\left(\int_A|\mathcal{K}(x,s)|^p d(x_1,x_2)\right)^{\frac{1}{p}}\\ &\leq \mathcal{C}\left(\sum_{\tau=1}^{2}\Big(\|v^{(\tau)}\|_{\mathcal{V}_p,\infty}+\|\varrho^{(\tau)}\|_{\mathcal{Y}_p^{2\times 2},\infty}+\|\delta^{(\tau)}\|_{\mathcal{Y}_p,\infty}\Big)+1\right)^{(m\vee n)+1}\\ &\qquad\times\Big(\|v^{(1)}-v^{(2)}\|_{\mathcal{V}_p,\infty}+\|\varrho^{(1)}-\varrho^{(2)}\|_{\mathcal{Y}_p^{2\times 2},\infty}+\|\delta^{(1)}-\delta^{(2)}\|_{\mathcal{Y}_p,\infty}\Big).\end{aligned}$$

Similar arguments may be used in order to estimate the expressions

$$\left.\begin{aligned}&\big|R_{ij}\circ\mathcal{B}^{(\tau)}(x,s)\big|\\ &\big|D_\nu(R_{ij}\circ\mathcal{B}^{(\tau)})(x,s)\big|\\ &\big|R_{ij}\circ\mathcal{B}^{(1)}(x,s)-R_{ij}\circ\mathcal{B}^{(2)}(x,s)\big|.\end{aligned}\right\}$$

No additional difficulties arise if R_{ij} is replaced by S ∎

By means of the preceding lemma, we may solve system (5.1) if the function u is taken from $C^0([0,T],\mathcal{V}_p)$. In fact, the following statement holds.

Theorem 5.1. *Let $M\in(0,\infty), \varepsilon_0\in\mathcal{Y}_p^{2\times 2}$ and $d_0\in\mathcal{Y}_p$ with $d_0(x)\leq 1-\beta$ for $x\in V$. Put*

$$T_0=\Big[C_{13}\big(M+2\|\varepsilon_0\|_{\mathcal{Y}_p^{2\times 2}}+2\|d_0\|_{\mathcal{Y}_p}+1\big)^{(m\vee n)+1}2\frac{1+C_1}{\beta}\Big]^{-1}$$

with C_1 from Theorem 3.1 and C_{13} from Lemma 5.2. Let $T' \in (0, T_0]$ and $v \in C^0([0,T'], \mathcal{V}_p)$ with $\|v\|_{\mathcal{V}_p,\infty} \le M$. Then there is a uniquely determined mapping

$$(g, \delta) = \big(g(v, \varepsilon_0, d_0), \delta(v, \varepsilon_0, d_0)\big) \in C^0([0,T'], \mathcal{Y}_p^{2\times 2} \times \mathcal{Y}_p)$$

with

$$\|g(t) - \varepsilon_0\|_{\mathcal{Y}_p^{2\times 2}} + \|\delta(t) - d_0\|_{\mathcal{Y}_p} \le \frac{\beta}{2(1+C_1)} \le \frac{1}{2} \tag{5.8}$$

and

$$(g,\delta)(t) = (\varepsilon_0, d_0) + \int_0^t \big(R \circ \mathcal{B}(v,g,\delta),\ S \circ \mathcal{B}(v,g,\delta)\big)(s)\, ds \tag{5.9}$$

for $t \in [0,T']$, where $\mathcal{B}(v,g,\delta)$ is defined in (5.2), the integral in (5.9) is to be understood as a Bochner integral in $\mathcal{Y}_p^{2\times 2} \times \mathcal{Y}_p$, and relation (5.8) means in particular

$$\delta(x,t) \le 1 - \frac{\beta}{2} \tag{5.10}$$

for $x \in V$ and $t \in [0,T']$. In addition,

$$(g,\delta) \in C^1([0,T'], \mathcal{Y}_p^{2\times 2} \times \mathcal{Y}_p) \tag{5.11}$$

$$(g,\delta)'(t) = \big(R \circ \mathcal{B}(v,g,\delta), S \circ \mathcal{B}(v,g,\delta)\big)(t) \quad (t \in [0,T']). \tag{5.12}$$

Moreover, g and δ considered as functions on $V \times [0,T']$ are partial differentiable with respect to $t \in [0,T']$, and

$$\left.\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= R\big(\mathcal{B}(v,g,\delta)(x,t)\big) \\ \frac{\partial}{\partial t} \delta(x,t) &= S\big(\mathcal{B}(v,g,\delta)(x,t)\big) \end{aligned}\right\} \tag{5.13}$$

for $x \in V$ and $t \in [0,T']$. Furthermore,

$$\left.\begin{aligned} g(0) &= \varepsilon_0 \\ \delta(0) &= d_0 \end{aligned}\right\}. \tag{5.14}$$

Proof. We adapt the standard proof for existence of solutions to ordinary differential equations in Banach spaces (see [28: Section 10.4]). To this end, we set

$$\mathcal{M} = \left\{ (\sigma,\kappa) \in C^0([0,T'], \mathcal{Y}_p^{2\times 2} \times \mathcal{Y}_p) \,\middle|\, \begin{array}{l} (\sigma,\kappa)(0) = (\varepsilon_0, d_0) \ \text{ and, for all } \ t \in [0,T'], \\ \|\sigma(t) - \varepsilon_0\|_{\mathcal{Y}_p^{2\times 2}} + \|\kappa(t) - d_0\|_{\mathcal{Y}_p} \le \dfrac{\beta}{2(1+C_1)} \end{array} \right\}.$$

For $(\sigma,\kappa) \in \mathcal{M}$, $x \in V$ and $t \in [0,T']$, we obtain by applying Theorem 3.1

$$\begin{aligned} \kappa(x,t) &\le d_0(x) + |\kappa(x,t) - d_0(x)| \\ &\le 1 - \beta + C_1 \|\kappa(\cdot, x_3, t) - d_0(\cdot, x_3)\|_{1,p} \\ &\le 1 - \beta + C_1 \|\kappa(t) - d_0\|_{\mathcal{Y}_p} \\ &\le 1 - \frac{\beta}{2}. \end{aligned} \tag{5.15}$$

Obviously,

$$\|\sigma(t)-\varepsilon_0\|_{\mathcal{Y}_p^{2\times 2}}+\|\kappa(t)-d_0\|_{\mathcal{Y}_p}\le 1 \tag{5.16}$$

for $(\sigma,\kappa)\in\mathcal{M}$ and $t\in[0,T']$. Thus, by Lemma 5.2, we get for $(\sigma,\kappa)\in\mathcal{M}$ and $t,t'\in[0,T']$ with $t\le t'$:

$$\begin{aligned}\int_{t'}^{t}&\big\|\big(R\circ\mathcal{B}(v,\sigma,\kappa),S\circ\mathcal{B}(v,\sigma,\kappa)\big)(s)\big\|_{\mathcal{Y}_p^{2\times 2}\times\mathcal{Y}_p}\,ds\\&\le C_{13}\big(M+\|\varepsilon_0\|_{\mathcal{Y}_p^{2\times 2}}+\|d_0\|_{\mathcal{Y}_p}+1\big)^{(m\vee n)+1}(t-t')\\&\le\frac{\beta}{2(1+C_1)}\end{aligned}\tag{5.17}$$

where the last inequality follows by the choice of T'. Due to the preceding estimate, the mapping $\mathcal{T}:\mathcal{M}\mapsto C^0([0,T'],\mathcal{Y}_p^{2\times 2}\times\mathcal{Y}_p)$, introduced by

$$\mathcal{T}(\sigma,\kappa)(t)=(\varepsilon_0,d_0)+\int_0^t\big(R\circ\mathcal{B}(v,\sigma,\kappa),S\circ\mathcal{B}(v,\sigma,\kappa)\big)(s)\,ds$$

for $t\in[0,T']$ and $(\sigma,\kappa)\in\mathcal{M}$, is well defined. It further follows from (5.17), for $(\sigma,\kappa)\in\mathcal{M}$ and $t\in[0,T']$,

$$\big\|\mathcal{T}(\sigma,\kappa)(t)-(\varepsilon_0,d_0)\big\|_{\mathcal{Y}_p^{2\times 2}\times\mathcal{Y}_p}\le\frac{\beta}{2(1+C_1)}$$

hence $\mathcal{T}(\mathcal{M})\subset\mathcal{M}$. Moreover, referring to (5.15), (5.16) and Lemma 5.2, we find

$$\begin{aligned}&\big\|\mathcal{T}(\sigma,\kappa)(t)-\mathcal{T}(\widetilde{\sigma},\widetilde{\kappa})(t)\big\|_{\mathcal{Y}_p^{2\times 2}\times\mathcal{Y}_p}\\&\quad\le C_{13}\big(M+2\|\varepsilon_0\|_{\mathcal{Y}_p^{2\times 2}}+2\|d_0\|_{\mathcal{Y}_p}+1\big)^{(m\vee n)+1}\big(\|\sigma-\widetilde{\sigma}\|_{\mathcal{Y}_p^{2\times 2},\infty}+\|\kappa-\widetilde{\kappa}\|_{\mathcal{Y}_p,\infty}\big)T'\\&\quad\le\frac{\beta}{2(1+C_1)}\big(\|\sigma-\widetilde{\sigma}\|_{\mathcal{Y}_p^{2\times 2},\infty}+\|\kappa-\widetilde{\kappa}\|_{\mathcal{Y}_p,\infty}\big)\end{aligned}$$

for $t\in[0,T']$ and $(\sigma,\kappa),(\widetilde{\sigma},\widetilde{\kappa})\in\mathcal{M}$. Since $\frac{\beta}{2+2C_1}\le\frac{1}{4}$, we may conclude the mapping $\mathcal{T}$ is a contraction with respect to the norm of the space $C^0([0,T'],\mathcal{Y}_p^{2\times 2}\times\mathcal{Y}_p)$. Therefore Banach's fixed point theorem yields there is a uniquely determined element $(g,\delta)\in\mathcal{M}$ with $\mathcal{T}(g,\delta)=(g,\delta)$. In other words, there is one and only one pair $(g,\delta)\in C^0([0,T'],\mathcal{Y}_p^{2\times 2}\times\mathcal{Y}_p)$ satisfying (5.8) and (5.9). Note that (5.10) follows from (5.15), and (5.14) from (5.9).

In order to obtain (5.11) and (5.12), we have to check whether the mapping $\mathcal{S}:[0,T']\mapsto\mathcal{Y}_p^{2\times 2}\times\mathcal{Y}_p$, defined by

$$\mathcal{S}(s)=\big(R\circ\mathcal{B}(v,g,\delta),S\circ\mathcal{B}(v,g,\delta)\big)(s)\qquad(s\in[0,T'])$$

is continuous. To this end, we note that by (5.8)

$$\|g(s)-g(t)\|_{\mathcal{Y}_p^{2\times 2}}+\|\delta(s)-\delta(t)\|_{\mathcal{Y}_p}\le\frac{2\beta}{2(1+C_1)}\le 1\qquad(s,t\in[0,T']).$$

So we may use Lemma 5.2 in order to estimate differences of the form $\mathcal{S}(t)-\mathcal{S}(s)$ in the norm of the space $\mathcal{Y}_p^{2\times 2}\times\mathcal{Y}_p$. The continuity of S then follows by an easy computation, which is omitted here. Now the relations in (5.11) and (5.12) readily follow from (5.9). We finally remark that (5.13) may easily be reduced to (5.12) by referring to Theorem 3.1 ∎

Corollary 5.1. *Let $M \in (0,\infty), K \in [1,\infty), \varepsilon_0 \in \mathcal{Y}_p^{2\times 2}$ and $d_0 \in \mathcal{Y}_p$ with $d_0(x) \le 1-\beta$ for $x \in V$. Put*

$$T_1 = \Big[C_{13}\big(M + 2\|\varepsilon_0\|_{\mathcal{Y}_p^{2\times 2}} + 2\|d_0\|_{\mathcal{Y}_p} + 1\big)^{(m\vee n)+1}\Big]^{-1} \min\Big\{\frac{1}{2K}, \frac{\beta}{2(1+C_1)}\Big\}.$$

Take $T' \in (0,T_1]$ and $v^{(1)}, v^{(2)} \in C^0([0,T'],\mathcal{V}_p)$ with $\|v^{(\tau)}\|_{\mathcal{V}_p,\infty} \le M \quad (\tau \in \{1,2\})$. Assume that for $\tau \in \{1,2\}$, the mapping $(g^{(\tau)},\delta^{(\tau)}) \in C^0([0,T'],\mathcal{Y}_p^{2\times 2}\times\mathcal{Y}_p)$ satisfies (5.8) and (5.9) (and hence (5.10)) with v,g,δ replaced by $v^{(\tau)}, g^{(\tau)}, \delta^{(\tau)}$, respectively. Then

$$\|g^{(1)}-g^{(2)}\|_{\mathcal{Y}_p^{2\times 2},\infty} + \|\delta^{(1)}-\delta^{(2)}\|_{\mathcal{Y}_p,\infty} \le \frac{1}{K}\|v^{(1)}-v^{(2)}\|_{\mathcal{V}_p,\infty}. \tag{5.18}$$

Proof. Abbreviate

$$\mathcal{C} = C_{13}\big(M + 2\|\varepsilon_0\|_{\mathcal{Y}_p^{2\times 2}} + 2\|d_0\|_{\mathcal{Y}_p} + 1\big)^{(m\vee n)+1}.$$

Then we get by (5.8), (5.10) and Lemma 5.2, for $s \in [0,T']$:

$$\begin{aligned}
&\Big\|R\circ\mathcal{B}(v^{(1)},g^{(1)},\delta^{(1)})(s) - R\circ\mathcal{B}(v^{(2)},g^{(2)},\delta^{(2)})(s)\Big\|_{\mathcal{Y}_p^{2\times 2}} \\
&\qquad + \Big\|S\circ\mathcal{B}(v^{(1)},g^{(1)},\delta^{(1)})(s) - S\circ\mathcal{B}(v^{(2)},g^{(2)},\delta^{(2)})(s)\Big\|_{\mathcal{Y}_p} \\
&\qquad \le \mathcal{C}\Big(\|v^{(1)}-v^{(2)}\|_{\mathcal{V}_p,\infty} + \|g^{(1)}-g^{(2)}\|_{\mathcal{Y}_p^{2\times 2},\infty} + \|\delta^{(1)}-\delta^{(2)}\|_{\mathcal{Y}_p,\infty}\Big)
\end{aligned}$$

hence by (5.9)

$$\begin{aligned}
&\|g^{(1)}-g^{(2)}\|_{\mathcal{Y}_p^{2\times 2},\infty} + \|\delta^{(1)}-\delta^{(2)}\|_{\mathcal{Y}_p,\infty} \\
&\quad \le \mathcal{C}T'\Big(\|v^{(1)}-v^{(2)}\|_{\mathcal{V}_p,\infty} + \|g^{(1)}-g^{(2)}\|_{\mathcal{Y}_p^{2\times 2},\infty} + \|\delta^{(1)}-\delta^{(2)}\|_{\mathcal{Y}_p,\infty}\Big) \qquad (5.19)\\
&\quad \le \frac{1}{2K}\|v^{(1)}-v^{(2)}\|_{\mathcal{V}_p,\infty} + \frac{1}{2}\Big(\|g^{(1)}-g^{(2)}\|_{\mathcal{Y}_p^{2\times 2},\infty} + \|\delta^{(1)}-\delta^{(2)}\|_{\mathcal{Y}_p,\infty}\Big)
\end{aligned}$$

with the last inequality being a consequence of the choice of T'. Now inequality (5.18) follows from (5.19) ∎

6. A fixed point argument

In the following, we shall exploit the results of the preceding sections in order to solve problem (1.4) - (1.9). Our main result is

Theorem 6.1. *Let $u_0 \in \mathcal{V}_p, T \in (0,\infty), q \in C^0([0,T],\mathcal{X}_p), \varepsilon_0 \in \mathcal{Y}_p^{2\times 2}$, and $d_0 \in \mathcal{Y}_p$ with $d_0(x) \le 1 - \frac{3}{2}\beta$ for $x \in V$, where β was fixed at the beginning of Section 5. Put*

$$\left.\begin{aligned}
K &= \frac{2}{2C_{11}} \\
M &= C_{11}\left(\|u_0\|_{\mathcal{V}_p} + \|\varepsilon_0\|_{\mathcal{Y}_p^{2\times 2}} + \|q\|_{\mathcal{X}_p,\infty} + 1\right) + \|u_0\|_{\mathcal{V}_p}
\end{aligned}\right\}$$

with C_{11} from Corollary 4.2. Choose T_1 as in Corollary 5.1, and let $T' \in (0, T_1]$. Then there is a uniquely determined mapping

$$(v, \varepsilon, d) \in C^0\big([0, T'], \mathcal{V}_p^0 \times \mathcal{Y}_p^{2\times 2} \times \mathcal{Y}_p\big)$$

such that

$$d(x,t) < 1 \quad (x \in V) \tag{6.1}$$

$$\mathcal{L}(v(t)) = -\mathcal{L}(u_0) + \mathcal{A}_p(\varepsilon(t)) + q(t) \tag{6.2}$$

$$(\varepsilon, d)(t) = (\varepsilon_0, d_0) + \int_0^t \big(R \circ \mathcal{B}(v + u_0, \varepsilon, d), S \circ \mathcal{B}(v + u_0, \varepsilon, d)\big)(s)\, ds \tag{6.3}$$

for $t \in [0, T']$, where the operators $\mathcal{L}$ and $\mathcal{A}_p$ were introduced in Definitions 4.4 and 4.3, respectively. For the definition of $R, S, \mathcal{B}$ see Definition 5.1 and (5.2).

In addition, for $t \in [0, T']$,

$$(\varepsilon, d) \in C^1\big([0, T'], \mathcal{Y}_p^{2\times 2} \times \mathcal{Y}_p\big) \tag{6.4}$$

$$\left.\begin{aligned} \varepsilon'(t) &= R \circ \mathcal{B}(v + u_0, \varepsilon, d)(t) \\ d'(t) &= S \circ \mathcal{B}(v + u_0, \varepsilon, d)(t) \end{aligned}\right\} \tag{6.5}$$

$$\left.\begin{aligned} \varepsilon(0) &= \varepsilon_0 \\ d(0) &= d_0 \end{aligned}\right\} \tag{6.6}$$

$$\|\varepsilon(t) - \varepsilon_0\|_{\mathcal{Y}_p^{2\times 2}} + \|d(t) - d_0\|_{\mathcal{Y}_p} \le \frac{\beta}{2(1 + C_1)}, \tag{6.7}$$

the functions $\varepsilon : V \times [0, T'] \mapsto \mathbb{R}^{2\times 2}$ and $d : V \times [0, T'] \mapsto \mathbb{R}$ are differentiable with respect to the variable $t \in [0, T']$, and

$$\left.\begin{aligned} \frac{\partial}{\partial t}\varepsilon(x,t) &= R\big(\mathcal{B}(v + u_0, \varepsilon, d)(x,t)\big) \\ \frac{\partial}{\partial t} d(x,t) &= S\big(\mathcal{B}(v + u_0, \varepsilon, d)(x,t)\big) \end{aligned}\right\} \tag{6.8}$$

for $x \in V$ and $t \in [0, T']$.

Proof. Put

$$\mathcal{M} = \Big\{ w \in C^0([0, T'], \mathcal{V}_p^0) : \|w + u_0\|_{\mathcal{V}_p, \infty} \le M \Big\}.$$

Define $\mathcal{T} : \mathcal{M} \mapsto C^0([0, T'], \mathcal{V}_p^0)$ by

$$\mathcal{T}(w) = U\big(u_0, g(w + u_0, \varepsilon_0, d_0), q\big) \qquad (w \in \mathcal{M})$$

with $g(w + u_0, \varepsilon_0, d_0)$ introduced in Theorem 5.1, and $U\big(u_0, g(w + u_0, \varepsilon_0, d_0), q\big)$ in Corollary 4.2. According to Theorem 5.1, we have

$$g(w + u_0, \varepsilon_0, d_0) \in C^0([0, T'], \mathcal{Y}_p^{2\times 2}) \qquad (w \in \mathcal{M})$$

hence $\mathcal{T}(w) \in C^0([0,T'], \mathcal{V}_p^0)$ by Corollary 4.2. Therefore the mapping $\mathcal{T}$ is well defined. By first using Corollary 4.2 and then (5.8), we get for $w \in \mathcal{M}$

$$\begin{aligned}\|\mathcal{T}(w) + u_0\|_{\mathcal{V}_p,\infty} &\le C_{11}\Big(\|u_0\|_{\mathcal{V}_p} + \|g(w+u_0,\varepsilon_0,d_0)\|_{\mathcal{Y}_p^{2\times 2},\infty} + \|q\|_{\mathcal{X}_p,\infty}\Big) + \|u_0\|_{\mathcal{V}_p}\\ &\le M\end{aligned}$$

hence $\mathcal{T}(\mathcal{M}) \subset \mathcal{M}$. We further deduce from Corollary 4.2 and 5.1 and the choice of T'

$$\begin{aligned}\|\mathcal{T}(w) - \mathcal{T}(\widetilde{w})\|_{\mathcal{V}_p,\infty} &\le C_{11}\big\|g(w+u_0,\varepsilon_0,d_0) - g(\widetilde{w}+u_0,\varepsilon_0,d_0)\big\|_{\mathcal{Y}_p^{2\times 2},\infty}\\ &\le \frac{C_{11}}{K}\|w-\widetilde{w}\|_{\mathcal{V}_p,\infty}\end{aligned}$$

for $w, \widetilde{w} \in \mathcal{M}$. Since $\frac{C_{11}}{K} \le \frac{1}{2}$ by the choice of K, we see the mapping $\mathcal{T}$ is a contraction. Now we are in a position to apply Banach's fixed point theorem, which implies there is a uniquely determined mapping $v \in \mathcal{M}$ with $\mathcal{T}(v) = v$. Putting

$$\left.\begin{aligned}\varepsilon &= g(v + u_0(\varepsilon_0, d_0)\\ d &= \delta(v+u_0,\varepsilon_0,d_0)\end{aligned}\right\}$$

(see Theorem 5.1), we obtain a mapping (v,ε,d) which satisfies (6.1) - (6.8).

Although the triple (v,ε,d) chosen in this way is unique in the sense that there is only one suitable element v in $\mathcal{M}$, we still have to show uniqueness of (v,ε,d) in the wider class of mappings verifying (6.1) - (6.3). Therefore let us take

$$(\widetilde{v},\widetilde{\varepsilon},\widetilde{d}) \in C^0\big([0,T'], \mathcal{V}_p^0 \times \mathcal{Y}_p^{2\times 2} \times \mathcal{Y}_p\big)$$

with the property that the relations in (6.1) - (6.3) are valid with v, ε and d replaced by $\widetilde{v}, \widetilde{\varepsilon}$ and $\widetilde{d}$, respectively. Assume for a contradiction that $(v,\varepsilon,d) \ne (\widetilde{v},\widetilde{\varepsilon},\widetilde{d})$ and put

$$t_0 = \max\Big\{r \in [0,T'] : (\widetilde{v},\widetilde{\varepsilon},\widetilde{d})(t) = (v,\varepsilon,d)(t) \quad \text{for } t \in [0,r]\Big\}.$$

Obviously,

$$(\widetilde{v},\widetilde{\varepsilon},\widetilde{d})(t) = (v,\varepsilon,d)(t) \qquad \text{for } t \in [0,t_0]. \tag{6.9}$$

Recalling our assumption, we conclude $t_0 < T'$. Relation (6.9) further implies we may choose $\widetilde{T} \in (0,T']$ so close to t_0 that

$$\|g(t) - \varepsilon(t_0)\|_{\mathcal{Y}_p^{2\times 2}} + \|\delta(t) - d(t_0)\|_{\mathcal{Y}_p} \le \frac{\beta}{2(1+C_1)} \tag{6.10}$$

for $t \in [t_0, \widetilde{T}]$ and $(g,\delta) \in \{(\varepsilon,d),(\widetilde{\varepsilon},\widetilde{d})\}$. Due to (6.7), (6.9) and Theorem 3.1, we get for $\delta \in \{d,\widetilde{d}\}$ and $x \in V$

$$\begin{aligned}\delta(x,t_0) &= d(x,t_0)\\ &\le |d(x,t_0) - d_0(x)| + d_0(x)\\ &\le C_1\|d(\cdot,x_3,t_0) - d_0(\cdot,x_3)\|_{1,p} + 1 - \tfrac{3}{2}\beta\\ &\le C_1\|d(t_0) - d_0\|_{\mathcal{Y}_p} + 1 - \tfrac{3}{2}\beta\\ &\le 1-\beta.\end{aligned} \tag{6.11}$$

Put

$$\widetilde{M} = \max\{\|v+u_0\|_{\mathcal{V}_p,\infty}, \|\widetilde{v}+u_0\|_{\mathcal{V}_p,\infty}\}$$

$$\gamma = \Big[C_{13}\big(\widetilde{M}+2\|\varepsilon(t_0)\|_{\mathcal{Y}_p^{2\times 2}}+2\|d(t_0)\|_{\mathcal{Y}_p}+1\big)^{(m\vee n)+1}\Big]^{-1} \times \min\Big\{\frac{4}{1+C_{11}}, \frac{\beta}{2(1+C_1)}\Big\}$$

$$T'' = \min\{T', \widetilde{T}, t_0+\gamma\}$$

with C_{13} and C_{11} introduced in Lemma 5.2 and Corollary 4.2, respectively. Then, combining (6.9) - (6.11), (6.3) and Corollary 5.1, we obtain for $t \in [t_0, T'']$

$$\|\varepsilon(t)-\widetilde{\varepsilon}(t)\|_{\mathcal{Y}_p^{2\times 2}} + \|d(t)-\widetilde{d}(t)\|_{\mathcal{Y}_p} \le \frac{2}{1+C_{11}}\|(v-\widetilde{v})\,|\,[t_0,T'']\|_{\mathcal{V}_p,\infty}. \tag{6.12}$$

Now we may conclude from (6.2), (6.12) and Corollary 4.2

$$\begin{aligned}\|(v-\widetilde{v})\,|\,[t_0,T'']\|_{\mathcal{V}_p,\infty} &\le C_{11}\|(\varepsilon-\widetilde{\varepsilon})\,|\,[t_0,T'']\|_{\mathcal{Y}_p^{2\times 2},\infty}\\ &\le \tfrac{1}{2}\|(v-\widetilde{v})\,|\,[t_0,T'']\|_{\mathcal{V}_p,\infty}\end{aligned}$$

hence $v(t)=\widetilde{v}(t)$ for $t \in [t_0, T'']$. It follows with (6.12)

$$\left.\begin{aligned} v(t) &= \widetilde{v}(t)\\ \varepsilon(t) &= \widetilde{\varepsilon}(t)\\ d(t) &= \widetilde{d}(t)\end{aligned}\right\} \qquad (t \in [t_0, T'']). \tag{6.13}$$

But these equations imply a contradiction to the choice of t_0, so $(v,\varepsilon,d) = (\widetilde{v},\widetilde{\varepsilon},\widetilde{d})$ must hold ∎

Due to the previous theorem, we may solve problem (1.4) - (1.9) under appropriate conditions on the data:

Corollary 6.1. *Let $v_0 \in W^{2-\frac{1}{p},p}(\partial A)^3$ with $v_{0,3} \in W^{3-\frac{1}{p},p}(\partial A)$. Moreover, let*

$$w_0 \in W^{2-\frac{1}{p},p}(\partial A), \quad T \in (0,\infty), \quad q \in C^0([0,T],\mathcal{X}_p), \quad \varepsilon_0 \in \mathcal{Y}_p^{2\times 2}, \quad d_0 \in \mathcal{Y}_p$$

with $d_0(x) \le 1-\frac{3}{2}\beta$ for $x \in V$. Then there are $T' \in (0,T]$ and uniquely determined mappings

$$u \in C^0([0,T'],\mathcal{V}_p), \qquad \varepsilon \in C^1([0,T'],\mathcal{Y}_p^{2\times 2}), \qquad d \in C^1([0,T'],\mathcal{Y}_p)$$

such that, for $t \in [0,T']$,

$$\mathcal{L}(u(t)) = \mathcal{A}_p(\varepsilon(t)) + q(t) \tag{6.14}$$

$$d(x,t) < 1 \quad (x \in V) \tag{6.15}$$

$$\varepsilon'(t) = R\circ\mathcal{B}(u,\varepsilon,d)(t), \;\; d'(t) = S\circ\mathcal{B}(u,\varepsilon,d)(t) \tag{6.16}$$

$$u|\partial A = v_0, \quad \sum_{i=1}^{2} D_i u_3|\partial A\, n_i^{(A)} = w_0 \tag{6.17}$$

$$\varepsilon(0) = \varepsilon_0, \quad d(0) = d_0. \tag{6.18}$$

In particular, the pair of functions (u, ε) *solves the boundary value problem* (1.4), (1.5), (1.8) *with* $\varepsilon^{cr} = \varepsilon$.

The functions $\varepsilon : V \times [0, T'] \mapsto \mathbb{R}^{2\times 2}$ *and* $d : V \times [0, T'] \mapsto \mathbb{R}$ *are differentiable with respect to* $t \in [0, T']$, *and*

$$\left.\begin{aligned} \frac{\partial}{\partial t}\varepsilon(x,t) &= R(\mathcal{B}(u,\varepsilon,d)(x,t)) \\ \frac{\partial}{\partial t}d(x,t) &= S(\mathcal{B}(u,\varepsilon,d)(x,t)) \end{aligned}\right\} \tag{6.19}$$

for $x \in V$ *and* $t \in [0, T']$. *Thus the triple* (u, ε, d) *solves the initial value problem* (1.6), (1.7), (1.9) *with* $\varepsilon^{cr} = \varepsilon$.

Proof. Choose u_0 in such a way that

$$u_0 \in \mathcal{V}_p, \qquad u_0|\partial A = v_0, \qquad \sum_{i=1}^{2} D_i u_3 \,|\, \partial A\; n_i^{(A)} = w_0. \tag{6.20}$$

According to Theorem 3.6, such a choice is possible due to our assumptions on v_0 and w_0. Note that in order to satisfy the relation $u_0 \in \mathcal{V}_p$, the assumptions

$$v_0 \in W^{1-\frac{1}{p},p}(\partial A)^3, \qquad v_{0,3} \in W^{2-\frac{1}{p},p}(\partial A), \qquad w_0 \in W^{1-\frac{1}{p},p}(\partial A)$$

would not be sufficient. For u_0 as in (6.20) and for q, ε_0, d_0 as in the corollary, Theorem 6.1 yields some $T' \in (0, T]$ and a triplet $(v, \varepsilon, d) \in C^0\big([0, T'], \mathcal{V}_p^0 \times \mathcal{Y}_p^{2\times 2} \times \mathcal{Y}_p\big)$ satisfying (6.1) - (6.8). Therefore the mapping (u, ε, d), with $u = v + u_0$, verifies (6.14) - (6.19). In particular, the equations in (6.17) are valid due to Theorem 3.7, the choice of u_0 and because $v(t) \in \mathcal{V}_p^0$ for $t \in [0, T']$ ∎

Note that system (6.14) of partial differential equations, which corresponds to (1.4), (1.5) consists of three equations; two of them are solved in the strong sense, the third one in a weak sense.

The variational form of (1.4), (1.5), (1.8) coupled with (1.6), (1.7), (1.9) – this problem is considered in [8] – may now be solved as well. We state this conclusion in

Corollary 6.2. *Let* $v_0, w_0, T, q, \varepsilon_0, d_0$ *be given as in Corollary* 6.1. *Define the form* $a(v, w)$ *in the same way as* $a_1(v, w)$ *in Definition* 4.1, *but with the domain of the former form enlarged to*

$$\Big\{(v, w) \in W^{1,2}(A)^3 \times W^{1,2}(A)^3 : \, v_3, w_3 \in W^{2,2}(A)\Big\}.$$

Then there are $T' \in (0, T]$ *and uniquely determined mappings*

$$u \in C^0([0, T'], \mathcal{V}_p), \qquad \varepsilon \in C^1([0, T'], \mathcal{Y}_p^{2\times 2}), \qquad d \in C^1([0, T'], \mathcal{Y}_p)$$

such that

$$\begin{aligned} a(u(t), w) = \int_A q(t)\, w\, d(x_1, x_2) + \int_V \sum_{i,j,k,l=1}^{2} & C_{ijkl}\, \varepsilon_{kl}(x, t) \\ \times \Big(\tfrac{1}{2} D_i w_j(x_1, x_2, t) &+ \tfrac{1}{2} D_j w_i(x_1, x_2, t) \\ + \kappa_{ij}(x_1, x_2)\, w_3(x_1, x_2) &- x_3\, D_i D_j w_3(x_1, x_2)\Big) dx \end{aligned} \tag{6.21}$$

for $w \in \mathcal{W}$*, and such that the relations in* (6.15) – (6.19) *are valid.*

Proof. Combine Corollary 6.1 with Lemma 4.4 ∎

In Corollary 6.2, the variational problem (6.21) coupled with system (6.16), under suitable side conditions, is solved in the space $C^0\big([0,T'],\mathcal{V}_p \times \mathcal{Y}_p^{2\times 2} \times \mathcal{Y}_p\big)$. It would be more natural to look for a solution (u,ε,d) with $u(t) \in \mathcal{W}$. However, if such a mapping u were inserted into system (6.16), the solution (ε,d) of this system would, in general, exhibit such a low regularity that ε would not yield a right-hand side in (6.21) which belonged to the appropriate space $W^{-1,2}(A)^2 \times W^{-2,2}(A)$. This is the reason why we replaced (6.21) by (1.4), (1.5) (or, equivalently, (6.14)), and chose a L^p-framework with $p > 2$ for our theory.

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