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# A Real Inversion Formula for the Laplace Transform in a Sobolev Space

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**Abstract.** For the real-valued Sobolev-Hilbert space on $[0,\infty)$ comprising absolutely continuous functions $F = F(t)$ normalized by $F(0) = 0$ and equipped with the inner product $(F_1, F_2) = \int_0^\infty (F_1(t)F_2(t) + F_1'(t)F_2'(t))\,dt$ we shall establish a real inversion formula for the Laplace transform.

**Keywords:** *Laplace transform, real inversion formula, Sobolev space, reproducing kernel, Mellin transform, Szegö space*

**AMS subject classification:** 44 A 10, 30 C 40

## 1. Introduction and results

The real inversion formulas for the Laplace transform are important in mathematical sciences, but the formulas are, in general, very involved (see, for example, [8, 12, 13]). In [3, 11], new real inversion formulas for some general situations were given by a new method for integral transforms in the framework of Hilbert spaces. Those formulas are translated into computer algorithms in [4, 5]. In some special cases, their error estimates were given in [2]. In the new method, inversion formulas for integral transforms will be, in general, given in terms of strong convergence. For some practical purposes, we wish to obtain inversion formulas in terms of pointwise convergence. For this purpose, we shall establish a real inversion formula for the Laplace transform of a simple Sobolev space, which will be given in terms of pointwise convergence.

Let $S$ be the Sobolev-Hilbert space on $t \geq 0$ comprising absolutely continuous real-valued functions $F$ normalized by $F(0) = 0$ and equipped with the inner product

$$(F_1, F_2)_S = \int_0^\infty \big(F_1(t)F_2(t) + F_1'(t)F_2'(t)\big)dt.$$

We consider the Laplace transform of $F \in S$

$$f(x) = [LF](x) = \int_0^\infty F(t)e^{-xt}dt \qquad (x > 0). \tag{1}$$

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In connection with some general real inversion formulas [3, 11], we would like to consider a more general Sobolev space such that for any positive $q$ the inner product is given by

$$(F_1, F_2)_{S,q} = \int_0^\infty \big(F_1(t)F_2(t) + F_1'(t)F_2'(t)\big)t^{1-2q}dt.$$

However in this general case, its reproducing kernel will be very involved. So, we shall consider the simple Sobolev space $S$. For more general order Sobolev spaces, the circumstances are similar. That is, the Sobolev space $S$ will be a reasonable space for the Laplace transform for our purposes (see Lemmas 1 and 3 for this comment).

Then, we obtain

**Theorem.** *For the Laplace transform* (1) *of the Sobolev-Hilbert space $S$, we have the real inversion formula*

$$F(t) = \lim_{N\to\infty} \int_0^\infty f(x) \int_0^\infty e^{-x\tau} K(\tau,t)\big(P_N(x,\tau) + Q_N(x,\tau)\big)d\tau dx \tag{2}$$

*where*

$$\begin{aligned}
K(\tau,t) &= \frac{1}{2}\big(e^{-|\tau-t|} - e^{-\tau}e^{-t}\big)\\
P_N(x,\tau) &= \sum_{n=0}^{N}\sum_{\nu=n}^{2n}(-1)^{\nu-n+1}\binom{2n}{\nu}\binom{\nu}{n}\frac{1}{(n+1)(\nu+1)!}(\tau x)^{\nu}\\
&\quad\times\Big((2n+1)(\tau x)^2 - (2+5n+\nu+3n\nu)\tau x + n(\nu+1)^2\Big)\\
Q_N(x,\tau) &= \frac{1}{\tau^2}\sum_{n=0}^{N}\sum_{\nu=n}^{2n}(-1)^{\nu-n+1}\binom{2n}{\nu}\binom{\nu}{n}\frac{1}{(n+1)(\nu+2)!}(\tau x)^{\nu+1}\\
&\quad\times\Big((4n^2+6n+2)(\tau x)^3 - (8+3\nu+26n+10n\nu+20n^2+8n^2\nu)(\tau x)^2\\
&\quad+(\nu+2)(2+\nu+8n+4n\nu+9n^2+5n^2\nu)\tau x - n^2(\nu+1)^2(\nu+2)\Big).
\end{aligned}$$

*In the real inversion formula* (2)*, for any $t \geq 0$ the right-hand side converges and its convergence is uniform on* $[0,\infty)$.

We introduce the differential operator

$$D_n = x^n\partial_x^n x\partial_x$$

for any non-negative integer $n$.

## 2. Preliminaries for Theorem

At first we note

**Lemma 1.** *The reproducing kernel $K$ for the Sobolev-Hilbert space $S$ is given by*

$$K(t,\hat{t}\,) = \frac{2}{\pi}\int_0^\infty \frac{\sin(t\xi)\sin(\hat{t}\xi)}{\xi^2+1}\,d\xi = \frac{1}{2}\big(e^{-|t-\hat{t}|} - e^{-t}e^{-\hat{t}}\big). \tag{3}$$

**Proof.** For the positive matrix $K(t,\hat{t})$ defined by (3) we shall show that the reproducing kernel Hilbert space $H_K$ admitting the reproducing kernel $K$ coincides with $S$. From (3), we see that any member $F$ of $H_K$ is expressible in the form

$$F(t) = \frac{2}{\pi}\int_0^\infty \frac{H(\xi)\sin(t\xi)}{\xi^2+1}\,d\xi \tag{4}$$

for a (of course, uniquely determined) function $H$ satisfying $\frac{2}{\pi}\int_0^\infty \frac{H(\xi)^2}{\xi^2+1}\,d\xi < \infty$ and we have the isometrical identity $\|F\|_{H_K}^2 = \frac{2}{\pi}\int_0^\infty \frac{H(\xi)^2}{\xi^2+1}\,d\xi$ (for this argument see [9, 10] or [11]). From (4)

$$H(\xi) = (\xi^2+1)\int_0^\infty F(t)\sin(t\xi)\,dt \tag{5}$$

in the $L_2$ space and so, from (5) we obtain $\|F\|_{H_K}^2 = \int_0^\infty \big(F(t)^2 + F'(t)^2\big)dt$. From the uniqueness of reproducing kernels, we have the desired result ∎

**Lemma 2.** *In the Laplace transform* (1) *of $S$, we have the isometrical identity*

$$\|F\|_S^2 = \sum_{n=0}^\infty \frac{1}{n!(n+1)!}\int_0^\infty \big\{(D_n f(x))^2 + (D_n(xf(x)))^2\big\}dx. \tag{6}$$

**Proof.** In general, for $F \in L_2(0,\infty)$ we have the isometrical identity

$$\int_0^\infty F(t)^2 dt = \sum_{n=0}^\infty \frac{1}{n!(n+1)!}\int_0^\infty (D_n f(x))^2 dx \tag{7}$$

[11: Chapter 5]. Since $F(0)=0$ and by integration by parts, $\int_0^\infty F'(t)e^{-xt}dt = xf(x)$. Hence, from (7) we have the desired isometrical identity (6) ∎

**Lemma 3.** *In the Laplace transform* (1) *of* $S$*, we have the real inversion formula*

$$\begin{aligned} F(t) &= \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \int_0^\infty \Bigg[ D_n f(x) \cdot D_n \int_0^\infty e^{-\tau x} K(\tau, t)\, d\tau \\ &\quad + D_n(xf(x)) \cdot D_n \left( x \int_0^\infty e^{-\tau x} K(\tau, t)\, d\tau \right) \Bigg] dx \\ &= \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \int_0^\infty \Bigg[ D_n f(x) \cdot D_n \Big( \frac{e^{-t} - e^{-xt}}{x^2 - 1} \Big) \\ &\quad + D_n(xf(x)) \cdot D_n \Big( x \Big( \frac{e^{-t} - e^{-xt}}{x^2 - 1} \Big) \Big) \Bigg] dx. \end{aligned} \tag{8}$$

*The convergence of this series is uniform on* $[0, \infty)$*.*

**Proof.** First we have

$$\begin{aligned} (LK(\cdot, t))(x) &= \int_0^\infty e^{-x\tau} K(\tau, t)\, d\tau \\ &= \int_0^\infty e^{-x\tau} \left( \frac{2}{\pi} \int_0^\infty \frac{\sin(\tau\xi)\sin(t\xi)}{\xi^2 + 1}\, d\xi \right) d\tau \\ &= \frac{2}{\pi} \int_0^\infty \frac{\xi \sin(t\xi)}{(\xi^2 + 1)(\xi^2 + x^2)}\, d\xi \\ &= \frac{e^{-t} - e^{-xt}}{x^2 - 1} \end{aligned}$$

[1: page 410]. Hence, by using the reproducing property $F(t) = (F(\cdot), K(\cdot, t))_S$ of $K(\cdot, t)$ in $S$ and the isometrical identity (6) we have the desired result (8). The uniform convergence of (8) on $[0, \infty)$ follows from the general property of reproducing kernel Hilbert spaces (see [11: page 35/Theorem 1]) and the boundedness of the reproducing kernel (3) for $S$ on $[0, \infty)$ ∎

For the property of $f$ satisfying (7) we note

**Proposition 1** [11: Chapter 5]. *For a function* $f$ *satisfying* (7) *we have the isometrical identity*

$$\sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \int_0^\infty (D_n f(x))^2 dx = \lim_{x \to 0+} \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x + iy)|^2 dy \tag{9}$$

*where* $f$ *is analytic on the right half complex plane* $R^+ = \{\operatorname{Re} z > 0\}$ *and belongs to the Szegö space on* $R^+$ *with a finite norm* (9)*. Furthermore, then we have, for* $n \geq 1$ *and* $0 \leq m \leq n - 1$*,*

$$\left. \begin{aligned} \partial_x^m [xf'(x)] x^{n+m+1} &= o(1) \\ f(x) x^{\frac{1}{2}} &= O(1) \end{aligned} \right\} \qquad (x \to 0+)$$

*and for* $n \geq 0$,

$$\partial_x^n f(x) \to 0 \qquad (x \to \infty).$$

## 3. Proof of Theorem

For $n \geq 1$, by integration by parts and by using Proposition 1, we have

$$\begin{aligned}
&\int_0^\infty D_n f(x) \cdot D_n e^{-x\tau}\, dx \\
&= \tau^n \int_0^\infty (xf'(x))\partial_x^n \big((nx^{2n} - \tau x^{2n+1})e^{-x\tau}\big) dx \\
&= -\tau^n \int_0^\infty f(x)\partial_x x\partial_x^n \big((nx^{2n} - \tau x^{2n+1})e^{-x\tau}\big) dx \\
&= -\tau^n \int_0^\infty f(x)\Big(\partial_x^n \big((nx^{2n} - \tau x^{2n+1})e^{-x\tau}\big) + x\partial_x^{n+1}\big((nx^{2n} - \tau x^{2n+1})e^{-x\tau}\big)\Big) dx \\
&= -\tau^n \int_0^\infty f(x)\Big(\partial_x^n \big((nx^{2n} - \tau x^{2n+1})e^{-x\tau}\big) \\
&\quad - x\tau\partial_x^n \big((nx^{2n} - \tau x^{2n+1})e^{-x\tau}\big) + x\partial_x^n \big(2n^2x^{2n-1} - (2n+1)\tau x^{2n}\big)e^{-x\tau}\Big) dx \\
&= -\tau^n \int_0^\infty f(x)e^{-\tau x} \sum_{\nu=0}^{n} \binom{n}{\nu}(-\tau)^\nu \Big(\partial_x^{n-\nu}(nx^{2n} - \tau x^{2n+1}) \\
&\quad - x\tau\partial_x^{n-\nu}(nx^{2n} - \tau x^{2n+1}) + x\partial_x^{n-\nu}\big(2n^2x^{2n-1} - (2n+1)\tau x^{2n}\big)\Big) dx \\
&= \int_0^\infty f(x)e^{-x\tau} \sum_{\nu=0}^{n} (-1)^{\nu+1}\binom{n}{\nu}(\tau x)^{n+\nu} \frac{\Gamma(2n+1)}{\Gamma(n+\nu+1)} \\
&\quad \times \Big(\frac{2n+1}{n+\nu+1}(\tau x)^2 - \Big(\frac{2n+1}{n+\nu+1} + 3n + 1\Big)\tau x + n(n+\nu+1)\Big) dx.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^\infty D_n(xf(x)) \cdot D_n(xe^{-x\tau})\, dx \\
&= (-\tau)^{(n-1)} \int_0^\infty \partial_x^n \big(x(xf(x))'\big)\big(x^2\tau^2 - (2n+1)x\tau + n^2\big)e^{-x\tau}x^{2n} dx \\
&= -\tau^{(n-1)} \int_0^\infty \big(xf(x) + x^2 f'(x)\big)\partial_x^n \big(x^{2n+2}\tau^2 - (2n+1)x^{2n+1}\tau + n^2x^{2n}\big)e^{-x\tau} dx
\end{aligned}$$

$$
\begin{aligned}
&= -\tau^{(n-1)} \int_0^\infty f(x)\Big(x\partial_x^n\big(x^{2n+2}\tau^2 - (2n+1)x^{2n+1}\tau + n^2x^{2n}\big)e^{-x\tau} \\
&\quad - \partial_x\big(x^2\partial_x^n\big(x^{2n+2}\tau^2 - (2n+1)x^{2n+1}\tau + n^2x^{2n}\big)e^{-x\tau}\big)\Big)dx \\
&= \tau^{(n-1)} \int_0^\infty f(x)\Big(x\partial_x^n\big(x^{2n+2}\tau^2 - (2n+1)x^{2n+1}\tau + n^2x^{2n}\big)e^{-x\tau} \\
&\quad + x^2\partial_x^n\partial_x\big(x^{2n+2}\tau^2 - (2n+1)x^{2n+1}\tau + n^2x^{2n}\big)e^{-x\tau}\Big)dx \\
&= \tau^{(n-1)} \int_0^\infty f(x)\Big(x\partial_x^n\big(x^{2n+2}\tau^2 - (2n+1)x^{2n+1}\tau + n^2x^{2n}\big)e^{-x\tau} \\
&\quad - x^2\partial_x^n\big(\tau^3x^{2n+2} - (4n+3)\tau^2x^{2n+1} + (5n^2+4n+1)\tau x^{2n} - 2n^3x^{2n-1}\big)e^{-x\tau}\Big)dx \\
&= \tau^{(n-1)} \int_0^\infty f(x)e^{-x\tau}\sum_{\nu=0}^{n}\binom{n}{v}(-\tau)^\nu \\
&\quad \times \Big(x\partial_x^{n-\nu}\big(x^{2n+2}\tau^2 - (2n+1)x^{2n+1}\tau + n^2x^{2n}\big) \\
&\quad - x^2\partial_x^{n-\nu}\big(\tau^3x^{2n+2} - (4n+3)\tau^2x^{2n+1} + (5n^2+4n+1)\tau x^{2n} - 2n^3x^{2n-1}\big)\Big)dx \\
&= \int_0^\infty f(x)e^{-x\tau}\sum_{\nu=0}^{n}(-1)^{\nu+1}\binom{n}{v}\frac{\Gamma(2n+1)}{\Gamma(n+v+1)}x^2(x\tau)^{n+\nu-1} \\
&\quad \times \Big(\frac{4n^2+6n+2}{(n+\nu+1)(n+\nu+2)}(\tau x)^3 - \frac{8+3\nu+29n+10n\nu+30n^2+8n^2\nu+8n^3}{(n+\nu+1)(n+\nu+2)}(\tau x)^2 \\
&\quad + \frac{2+\nu+9n+4n\nu+13n^2+5n^2\nu+5n^3}{n+\nu+1}\tau x - n^2(n+\nu+1)\Big)dx.
\end{aligned}
$$

Therefore, from Lemma 3 we have the desired real inversion formula (2) ∎

## 4. Concluding remark

The integrals (6) are effectively computable by using the Mellin transform

$$(Mf)(q-it) = \int_0^\infty f(x)x^{q-it-1}dx.$$

Indeed, note the identity

$$2\pi\int_0^\infty |D_nf(x)|^2x^{2q-1}dx$$

$$= \int_{-\infty}^{\infty} |(Mf)(q-it)|^2(q^2+t^2)^2\{(q+1)^2+t^2\}\cdots\{(q+n-1)^2+t^2\}dt \quad (q>0)$$

[11: Page 207/Formula (28)]. Hence,

$$2\pi \int_0^{\infty} |D_n f(x)|^2 dx =$$

$$\int_{-\infty}^{\infty} \left|(Mf)\left(\frac{1}{2}-it\right)\right|^2 \left\{\left(\frac{1}{2}\right)^2+t^2\right\}^2 \left\{\left(\frac{1}{2}+1\right)^2+t^2\right\}\cdots\left\{\left(\frac{1}{2}+n-1\right)^2+t^2\right\}dt,$$

and so the first part of (6) is

$$I = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \int_0^{\infty} |D_n f(x)|^2 dx$$

$$= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \int_{-\infty}^{\infty} \left|(Mf)\left(\frac{1}{2}-it\right)\right|^2 \left\{\left(\frac{1}{2}\right)^2+t^2\right\} \frac{|\Gamma(\frac{1}{2}+n+it)|^2}{|\Gamma(\frac{1}{2}+it)|^2} dt.$$

However, by using the famous Gauss formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \big(c \notin -\mathbb{N}_0, \mathrm{Re}(c-a-b)>0\big)$$

with

$$F(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{n!\,\Gamma(c+n)}$$

[1: p. 556/15.1.10 and 15.1.1], we have

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|(Mf)\left(\frac{1}{2}-it\right)\right|^2 \left\{\left(\frac{1}{2}\right)^2+t^2\right\}^2 \frac{dt}{\left|\Gamma\left(\frac{1}{2}+it\right)\right|^2} \tag{10}$$

[11: p. 205/Formula (22)]. The second part of (6) can be handled similarly by using the transformation rule in the Mellin transform $M(xf(x))(q-it) = (Mf)(q+1-it)$. The series in (10) is estimated by the behavior of the Mellin transform $(Mf)(\frac{1}{2}-it)$ at infinity, in some cases. For example, if $(Mf)(\frac{1}{2}-it)$ is a continuous function in $t \in \mathbb{R}$ and $\left|(Mf)\left(\frac{1}{2}-it\right)\right| = O\left(|t|^b e^{-\frac{\pi}{2}|t|}\right)$ $(|t| \to \infty)$ for any fixed $b < -\frac{1}{2}$, then (10) converges, from $|\Gamma(x+iy)| \sim \sqrt{2\pi}|y|^{x-\frac{1}{2}} e^{-\frac{\pi}{2}|y|} e^{-x}$ $(|y| \to \infty)$ [6].

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