Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 4, 827–837

Spatial Riemann Problem for Analytic Functions of Two Complex Variables

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Abstract. The Riemann boundary value problem is discussed for analytic functions in polydiscs. Necessary and sufficient conditions for the existence of a finite number of solutions and a finite number of solvability conditions are derived.

Keywords: *Several complex variables, analytic functions, Riemann problem*

AMS subject classification: 30 E 25, 32 A 07

1. Introduction

Let

$$\begin{aligned}
\mathcal{D}^{++} &= \big\{(z,w) \in \mathbb{C}^2 : |z| < 1 \text{ and } |w| < 1\big\} \\
\mathcal{D}^{-+} &= \big\{(z,w) \in \mathbb{C}^2 : |z| > 1 \text{ and } |w| < 1\big\} \\
\mathcal{D}^{+-} &= \big\{(z,w) \in \mathbb{C}^2 : |z| < 1 \text{ and } |w| > 1\big\} \\
\mathcal{D}^{--} &= \big\{(z,w) \in \mathbb{C}^2 : |z| > 1 \text{ and } |w| > 1\big\}
\end{aligned}$$

and let

$$T^2 = \big\{(t,\omega) \in \mathbb{C}^2 : |t| = |\omega| = 1\big\}$$

be the characteristic boundary. In this paper we shall be concerned with the spatial Riemann problem

Problem $(\mathbf{R}_2)$. Determine four analytic functions $\varphi^{\pm\pm}$ on $\mathcal{D}^{\pm\pm}$ such that

$$\begin{aligned}
\mathcal{R}_2\varphi :&= A(t,\omega)\varphi^{++}(t,\omega) + B(t,\omega)\varphi^{-+}(t,\omega) \\
&\quad + C(t,\omega)\varphi^{+-}(t,\omega) + D(t,\omega)\varphi^{--}(t,\omega) \\
&= F(t,\omega)
\end{aligned} \tag{1}$$

on T^2, where $A, B, C, D \in \mathcal{W}$ and $F \in \mathcal{W}$, $\mathcal{W}$ being the Wiener algebra of functions f defined on T^2 such that

$$f(t,\omega) = \sum_{(p,q)\in\mathbb{Z}^2} f_{n,m} t^p \omega^q$$

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ISSN 0232-2064 / $ 2.50

is continuous on T^2, with norm

$$\|f\|_{T^2} = \sum_{(p,q)\in\mathbb{Z}^2} |f_{p,q}| < +\infty,$$

where $\mathbb{Z}^2 = \{(p,q) : p \text{ and } q \text{ are integers}\}$.

For the Riemann problem for analytic functions of one complex variable, due to a technique of canonical decomposition, complete solutions have been obtained (see, e.g., [9, 13]). This problem is also solved for generalized analytic functions, as well as linear and nonlinear elliptic and other systems of equations in the plane [1, 3, 4, 6, 7, 15, 16]. Spatial Riemann problems were investigated, e.g., in [8, 10 - 12, 14] (for more references one is referred to [2]).

In this paper, we require the unknown functions to satisfy the Cauchy conditions

$$\left.\begin{aligned} \varphi^{\pm -}(z,\infty) &= 0 \qquad \text{for all } |z| \lesseqgtr 1 \\ \varphi^{-\pm}(\infty,w) &= 0 \qquad \text{for all } |w| \lesseqgtr 1 \end{aligned}\right\} \tag{2}$$

so that the homogeneous problem (1) with $F = 0$ and $A = B = C = D = 1$ has only the trivial solution. We restrict ourself to the case that

$$\left.\begin{aligned} A &= 1 \\ B &= t^l \omega^\lambda \\ C &= t^m \omega^\mu \\ D &= t^n \omega^\nu \end{aligned}\right\} \tag{3}$$

where $(l,\lambda), (m,\mu), (n,\nu) \in \mathbb{Z}^2$. In the general case, that is without condition (3), let us first review the main results concerning the one-dimensional Riemann problem

$$\mathcal{R}_1\Phi := \Phi^+ + G\Phi^- = g \qquad \text{on } \Gamma, \tag{4}$$

where Γ is a closed smooth curve in $\mathbb{C}$, $G \neq 0$ on Γ and Hölder continuous. The function G has a canonical decomposition (see [9, 13])

$$G(t) = t^\kappa \frac{X^+(t)}{X^-(t)}, \tag{5}$$

where X^+ and X^- are respectively boundary values of functions holomorphic in the interior and exterior domains determined by Γ. Inserting (5) into (4) and setting

$$\Phi_0^{\substack{+\\-}}(z) = \frac{\varphi^{\substack{+\\-}}(z)}{X^{\substack{+\\-}}(z)}$$

problem (4) is reduced to

$$\Phi_0^+(t) + t^\kappa \Phi_0^-(t) = \frac{g(t)}{X^+(t)} \qquad \text{on } \Gamma. \tag{6}$$

Two consequences of (6) are now at hand (see [9, 13]):

(i) Problem $(\mathcal{R}_1, g)$ is solvable in the sense of Noether.

(ii) The number of solutions and of solvability conditions is determined by the index κ, not on other properties of the function G.

Result (i) is, in general, not true for problem (1) - (2). For Result (ii), it was observed in [10] that the solvability of problem (1), i.e. $\mathcal{R}_2\varphi = F$, depends on the coefficients of the operator $\mathcal{R}_2$, which means dependence not only on the partial indices of the coefficients A, B, C, D, but also on other analytical properties, e.g. on being the boundary values of analytic functions. Some necessary and sufficient conditions for the solvability of problem (1) with special coefficients, for example when $B = C = 0$, were obtained in [10: Theorems 3 - 7].

We remark that, for any positive integers a_i, b_i, c_i, d_i $(i = 1, 2)$ and any continuous functions a, b, c, d on T^2, the problem

$$a(t,\omega)f^{++} + b(t,\omega)f^{-+} + c(t,\omega)f^{+-} + d(t,\omega)f^{--} = F_0(t,\omega) \qquad \text{on } \Gamma,$$

where

$$F_0(t,\omega) = a(t,\omega)t^{a_1}\omega^{a_2} + b(t,\omega)t^{-b_1}\omega^{b_2} + c(t,\omega)t^{c_1}\omega^{-c_2} + d(t,\omega)t^{-d_1}\omega^{-d_2},$$

has the solution

$$\begin{aligned} f^{++}(z,w) &= z^{a_1}w^{a_2} && \text{for } (z,w) \in \mathcal{D}^{++} \\ f^{-+}(z,w) &= z^{-b_1}w^{b_2} && \text{for } (z,w) \in \mathcal{D}^{-+} \\ f^{+-}(z,w) &= z^{c_1}w^{-c_2} && \text{for } (z,w) \in \mathcal{D}^{+-} \\ f^{--}(z,w) &= z^{-d_1}w^{-d_2} && \text{for } (z,w) \in \mathcal{D}^{--}. \end{aligned}$$

This example shows that it might be necessary to distinguish statements for the following two subjects:

(a) *The solvability* (*in some sense*) *of the problem* $\mathcal{R}_2\varphi = F$ *for a given function* F

(b) *The solvability* (*in some sense*) *of the problem* $\mathcal{R}_2\varphi = F$ *for any function* F

which are equivalent for the Noether solvability of $\mathcal{R}_1\Phi = g$ and were ignored in [10]. We do not develop any further along this direction in this paper and confine ourselves to case (3).

For problem (1) - (3), in [10: Theorem 2, p. 223] it was shown that its solvability depends on no more than a countable number of necessary and sufficient conditions imposed on the free term. In [11: Theorem 2, p. 36] necessary conditions were found for the homogeneous problem to have a finite set of linearly independent solutions, and for the inhomogeneous problem to have a finite set of solvability conditions. Readers should be aware that the conditions stated in [11: p. 35] in Theorem 2 there are not correct, which might be due to misprints.

We shall find necessary and sufficient conditions for the solvability of problem (1) - (3). It should be pointed out that the methods used in this paper could be generalized to higher dimensions. This will be carried out in a forthcoming publication.

2. The homogeneous problem

In this section, we consider the homogeneous problem (R_2), that is, $F = 0$ in (1). We introduce four subsets of $\mathbb{Z}^2$ by

$$\begin{aligned} d_A &= \{(p,q) \in \mathbb{Z}^2 : p \geq 0 \text{ and } q \geq 0\} \\ d_B &= \{(p,q) \in \mathbb{Z}^2 : p < l \text{ and } q \geq \lambda\} \\ d_C &= \{(p,q) \in \mathbb{Z}^2 : p \geq m \text{ and } q < \mu\} \\ d_D &= \{(p,q) \in \mathbb{Z}^2 : p < n \text{ and } q < \nu\}. \end{aligned}$$

Let $f \in \mathcal{W}$ with

$$f(t,\omega) = \sum_{(p,q)\in\mathbb{Z}^2} f_{p,q} t^p \omega^q,$$

and let X be a subset of $\mathbb{Z}^2$. We define an operator $P_X : \mathcal{W} \to \mathcal{W}$ by

$$(P_X f)(t,\omega) = \sum_{(p,q)\in X} f_{p,q} t^p \omega^q. \tag{7}$$

Then we have the following consequences.

Lemma 1. *The operator P_X defined by* (7) *satisfies:*

(i) $P_X^2 = P_X$.

(ii) $P_X P_Y = P_{X\cap Y}$ *for all* $X, Y \in \mathbb{Z}^2$.

Proof. For any $f \in \mathcal{W}$ with $f(t,\omega) = \sum_{(p,q)\in\mathbb{Z}^2} f_{p,q} t^p \omega^q$, we have

$$P_X^2 f = P_X(P_X f) = P_X\left(\sum_{(p,q)\in X} f_{(p,q)} t^p \omega^q\right) = \sum_{(p,q)\in X} f_{p,q} t^p \omega^q = P_X f,$$

hence $P_X^2 = P_X$, which proves (i). Moreover, we have

$$P_X P_Y f = P_X(P_Y f) = P_X\left(\sum_{(p,q)\in Y} f_{p,q} t^p \omega^q\right) = \sum_{(p,q)\in X\cap Y} f_{p,q} t^p \omega^q = P_{X\cup Y} f.$$

Hence $P_X P_X = P_{X\cap Y}$, which proves (ii) ∎

For the homogeneous problem (1) - (3), we are interested in finding non-trivial solutions. We have

Theorem 1. *The homogeneous problem* (1) – (3) *has non-trivial solutions if and only if there exist* $\alpha, \beta \in \{A, B, C, D\}$, $\alpha \neq \beta$, *such that*

$$d_\alpha \cap d_\beta \neq \emptyset. \tag{8}$$

Proof. Assume that (8) holds and let $X = d_\alpha \cap d_\beta$. Without loss of generality, we assume that $\alpha = A$ and $\beta = B$. Then the functions

$$\left.\begin{array}{ll} \varphi^{++}_{p,q}(z,w) = z^p w^q & \text{for } (z,w) \in \mathcal{D}^{++} \\ \varphi^{-+}_{p,q}(z,w) = -z^{p-l} w^{q-\lambda} & \text{for } (z,w) \in \mathcal{D}^{-+} \\ \varphi^{+-}_{p,q}(z,w) = 0 & \text{for } (z,w) \in \mathcal{D}^{+-} \\ \varphi^{--}_{p,q}(z,w) = 0 & \text{for } (z,w) \in \mathcal{D}^{--} \end{array}\right\}$$

for $(p,q) \in X$ are solutions of the homogeneous problem (1) - (3).

Conversely, if (8) does not hold, then $d_\alpha \cap d_\beta = \emptyset$ for all $\alpha, \beta \in \{A, B, C, D\}$, $\alpha \neq \beta$, and, by virtue of Lemma 1, $P_{d_\alpha} P_{d_\beta} = 0$. By applying P_{d_D} to (1) we get $t^n \omega^\nu \varphi^{--} = 0$. It then follows that $\varphi^{--} = 0$. Similarly, we can show that $\varphi^{++} = 0$, $\varphi^{-+} = 0$ and $\varphi^{+-} = 0$. That is the homogeneous problem has only the trivial solution ∎

Let

$$\Sigma = (d_A \cap d_B) \cup (d_A \cap d_C) \cup (d_D \cap d_B) \cup (d_D \cap d_C). \tag{9}$$

From Theorem 1, if, for example, $d_A \cap d_B \neq \emptyset$, since its cardinal number will be infinite, the homogeneous problem (1) - (3) will have an infinite number of linearly independent solutions. We now derive conditions for finiteness of the number of solutions.

Lemma 2. *The number of solutions of the homogeneous problem* $(1) - (3)$ *is finite if and only if* $\Sigma = \emptyset$*, where* Σ *is defined by* (9).

Proof. Let Σ be not empty. Then there exists $(p_0, q_0) \in \Sigma$. Without loss of generality, let $(p_0, q_0) \in d_A \cap d_B$. Then $\{(p,q) \in \mathbb{Z}^2 : p = p_0 \text{ and } q \geq q_0\} \subset d_A \cap d_B$. From the proof of Theorem 1, problem (1) - (3) has an infinite number of linearly independent solutions. Hence $\Sigma = \emptyset$.

Conversely, let $\Sigma = \emptyset$. Since $d_A \cap d_B = \emptyset$ and $d_A \cap d_C = \emptyset$, by applying P_{d_A} to (1) we get $\varphi^{++}(t,\omega) + P_{d_A}(t^n \omega^\nu \varphi^{--}) = 0$ on T^2. From $P_{d_A \setminus d_D}(t^n \omega^\nu \varphi^{--}) = 0$, we get $P_{d_A \setminus d_D} \varphi^{++} = 0$. Noticing that $d_A \cap d_D$ is at most a finite subset of $\mathbb{Z}^2$, the Fourier coefficients of φ^{++} contains only at most finitely many terms. One can argue similarly for φ^{-+}, φ^{+-} and φ^{--}. Hence problem (1) - (3) has at most finitely many solutions ∎

Lemma 3. *Let* Σ *be defined by* (9). *Then* Σ *is empty if and only if*

$$\left.\begin{array}{r} l \leq 0 \\ \mu \leq 0 \\ \lambda \geq \nu \\ m \geq n. \end{array}\right\} \tag{10}$$

Proof. From

$$\begin{aligned} d_A \cap d_B &= \big\{(p,q) \in \mathbb{Z}^2 : 0 \leq p < l \text{ and } q \geq \max(\lambda, 0)\big\} \\ d_A \cap d_C &= \big\{(p,q) \in \mathbb{Z}^2 : p \geq \max(m, 0) \text{ and } 0 \leq q < \mu\big\} \\ d_D \cap d_B &= \big\{(p,q) \in \mathbb{Z}^2 : p < \min(n, l) \text{ and } \lambda \leq q < \nu\big\} \\ d_D \cap d_C &= \big\{(p,q) \in \mathbb{Z}^2 : m \leq p < n \text{ and } q < \min(\mu, \nu)\big\} \end{aligned}$$

condition (10) follows.

Lemma 4. *Let* Σ *be defined by* (9) *and* $\Sigma = \emptyset$. *Then if the homogeneous problem* (1) – (3) *has a non-trivial solution, we have either*

$$d_A \cap d_D \neq \emptyset \tag{11}$$

or

$$d_B \cap d_C \neq \emptyset. \tag{12}$$

Proof. If $d_A \cap d_D = \emptyset$ and $d_B \cap d_C = \emptyset$, by virtue of Theorem 1, the homogeneous problem (1) - (3) would have only the trivial solution ∎

Lemma 5. *Conditions* (11) *and* (12) *are equivalent to*

$$\left.\begin{matrix} n > 0 \\ \nu > 0 \end{matrix}\right\} \tag{13}$$

and

$$\left.\begin{matrix} l > m \\ \mu > \lambda \end{matrix}\right\}, \tag{14}$$

respectively.

Proof. From

$$\begin{aligned} d_A \cap d_D &= \big\{(p,q) \in \mathbb{Z}^2 : \, 0 \leq p < n \text{ and } 0 \leq q < \nu\big\} \\ d_B \cap d_C &= \big\{(p,q) \in \mathbb{Z}^2 : \, m \leq p < l \text{ and } \lambda \leq q < \mu\big\} \end{aligned}$$

conditions (13) and (14) follow ∎

Lemma 6. *Let* Σ *be defined by* (9) *and* $\Sigma = \emptyset$. *Then if* (13) *or* (14) *hold, the homogeneous problem* (1)–(3) *has respectively* $n\nu$ *and* $(l-m)(\mu-\lambda)$ *linearly independent solutions, which are given by*

$$\left.\begin{array}{ll} \varphi^{++}_{p,q}(z,w) = z^p w^q & \textit{for } (z,w) \in \mathcal{D}^{++} \\ \varphi^{-+}_{p,q}(z,w) = 0 & \textit{for } (z,w) \in \mathcal{D}^{-+} \\ \varphi^{+-}_{p,q}(z,w) = 0 & \textit{for } (z,w) \in \mathcal{D}^{+-} \\ \varphi^{--}_{p,q}(z,w) = -z^{p-n} w^{q-\nu} & \textit{for } (z,w) \in \mathcal{D}^{--} \end{array}\right\} \tag{15}$$

for all $(p,q) \in d_A \cap d_D$, *or by*

$$\left.\begin{array}{ll} \varphi^{++}_{p,q}(z,w) = 0 & \textit{for } (z,w) \in \mathcal{D}^{++} \\ \varphi^{-+}_{p,q}(z,w) = z^{p-l} w^{q-\lambda} & \textit{for } (z,w) \in \mathcal{D}^{-+} \\ \varphi^{+-}_{p,q}(z,w) = -z^{p-m} w^{q-\mu} & \textit{for } (z,w) \in \mathcal{D}^{+-} \\ \varphi^{--}_{p,q}(z,w) = 0 & \textit{for } (z,w) \in \mathcal{D}^{--} \end{array}\right\} \tag{16}$$

for all $(p,q) \in d_B \cap d_C$.

Proof. We prove only (15). When (13) holds, we have for $(p,q) \in d_A \cap d_D$

$$p \geq 0, q \geq 0 \qquad \text{and} \qquad p-n<0, q-\nu<0.$$

Hence the functions $z^p w^q$ and $z^{p-n} w^{q-\nu}$ are holomorphic in $\mathcal{D}^{++}$ and $\mathcal{D}^{--}$, respectively. Moreover, we have for $(t,\omega) \in T^2$

$$\begin{aligned} 1 \cdot \varphi^{++}_{p,q}(t,\omega) &+ t^l \omega^\lambda \varphi^{-+}_{p,q}(t,\omega) + t^m \omega^\mu \varphi^{+-}_{p,q}(t,\omega) + t^n \omega^\nu \varphi^{--}_{p,q}(t,\omega) \\ &= 1 \cdot t^p \omega^q + t^l \omega^\lambda \cdot 0 + t^m \omega^\mu \cdot 0 + t^n \omega^\nu (-t^{p-n} \omega^{q-\nu}) \\ &= 0. \end{aligned}$$

Hence (15) is a solution of the homogeneous problem (1) - (3).

For different $(p_1,q_1),(p_2,q_2) \in d_A \cap d_D$ it is clear that $\varphi_{p_1,q_1}, \varphi_{p_2,q_2}$ are linearly independent. By calculating the cardinality of $d_A \cap d_D$, the homogeneous problem $(1)-(3)$ has therefore $n\nu$ linearly independent solutions ∎

Theorem 2. *The homogeneous problem* $(1)-(3)$ *has a finite number of non-trivial solutions if and only if*

$$\left.\begin{aligned} l &\leq 0 \\ \mu &\leq 0 \\ \lambda \geq \nu &> 0 \\ m \geq n &> 0 \end{aligned}\right\} \tag{17}$$

or

$$\left.\begin{aligned} 0 \geq l > m \geq n \\ 0 \geq \mu > \lambda \geq \nu. \end{aligned}\right\} \tag{18}$$

Moreover, if $(17)-(18)$ *are satisfied, the linearly independent solutions are given by* (15) *and* (16), *respectively.*

Proof. Combining Lemmas 2 - 5, we get (17) and (18). The rest follows from Lemma 6 ∎

Remarks. It seems worth of mentioning the symmetry of the roles of t and ω in problem (1) - (3). We denote the indicies in (3) by $(0,0),(l,\lambda),(m,\mu)$ and (n,ν). After the transform $t \to \omega, \omega \to t$, the indicies become correspondingly $(0,0),(\mu,m),(\lambda,l)$ and (ν,n) since B and C are interchanged with one another. In conditions (17) and (18) this symmetry is well-preserved. This remark applies also to (26) and (27) below.

3. The inhomogeneous problem

For the solvability of problem (R_2), we have the following necessary condition.

Lemma 7. *If problem* (R_2) *is solvable for any* $F \in \mathcal{W}$, *then*

$$d_A \cup d_B \cup d_C \cup d_D = \mathbb{Z}^2. \tag{19}$$

Proof. Let

$$X = \mathbb{Z}^2 \backslash (d_A \cup d_B \cup d_C \cup d_D), \tag{20}$$

suppose that $X \neq \emptyset$ and consider the operator P_X. From

$$X = (\mathbb{Z}^2 \backslash d_A) \cap (\mathbb{Z}^2 \backslash d_B) \cap (\mathbb{Z}^2 \backslash d_C) \cap (\mathbb{Z}^2 \backslash d_D),$$

by virtue of Lemma 1, we have

$$P_X = P_{\mathbb{Z}^2 \backslash d_A} P_{\mathbb{Z}^2 \backslash d_B} P_{\mathbb{Z}^2 \backslash d_C} P_{\mathbb{Z}^2 \backslash d_D}$$

and

$$P_{\mathbb{Z}^2 \backslash d_A} \varphi^{++} = P_{\mathbb{Z}^2 \backslash d_B} (t^l \omega^\lambda \varphi^{-+}) = P_{\mathbb{Z}^2 \backslash d_C} (t^m \omega^\mu \varphi^{+-}) = P_{\mathbb{Z}^2 \backslash d_D} (t^n \omega^\nu \varphi^{--}) = 0.$$

Applying P_X to both sides of (1), we get $0 = P_X F$. Thus problem (R_2) is not solvable for F satisfying $P_X F \neq 0$.

Supposing now that (19) holds we seek a particular solution to problem (R_2). We shrink the sets d_A, d_B, d_C, d_D to d'_A, d'_B, d'_C, d'_D so that $d'_A \cup d'_B \cup d'_C \cup d'_D = \mathbb{Z}^2$ and $d'_\alpha \cap d'_\beta = \emptyset$ for all $\alpha, \beta \in \{A, B, C, D\}, \alpha \neq \beta$. Let the free term F have the Fourier series representation

$$F(t, \omega) = \sum_{(p,q) \in \mathbb{Z}^2} F_{p,q} t^p \omega^q.$$

We define

$$\left.\begin{aligned}
\varphi_0^{++}(z, w) &= \textstyle\sum_{(p,q) \in d'_A} F_{p,q} z^p w^q & &\text{for } (z, w) \in \mathcal{D}^{++} \\
\varphi_0^{-+}(z, w) &= \textstyle\sum_{(p,q) \in d'_B} F_{p,q} z^{p-l} w^{q-\lambda} & &\text{for } (z, w) \in \mathcal{D}^{-+} \\
\varphi_0^{+-}(z, w) &= \textstyle\sum_{(p,q) \in d'_C} F_{p,q} z^{p-m} w^{q-\mu} & &\text{for } (z, w) \in \mathcal{D}^{+-} \\
\varphi_0^{--}(z, w) &= \textstyle\sum_{(p,q) \in d'_D} F_{p,q} z^{p-n} w^{q-\nu} & &\text{for } (z, w) \in \mathcal{D}^{--}.
\end{aligned}\right\}$$

From $d'_B \subset d_B$, it follows that, for all $(p, q) \in d'_B$, we have $p < l$ and $q \geq \lambda$. Hence φ_0^{-+} satisfies (2). Similarly, we can show that the rest meet with (2), too. Thus $\{\varphi_0^{\pm\pm}\}$ is a solution of problem (R_2) ∎

From Lemma 7, we therefore have

Theorem 3. *Problem* (R_2) *is solvable for any* $F \in \mathcal{W}$ *if and only if* (19) *holds.*

If the set X defined by (20) is not empty, then by virtue of Lemma 7 problem (R_2) is not solvable. Moreover, when X is an infinite set, the free term F has to satisfy an infinite number of solvability conditions. We now seek conditions so that only a finite number of solvability conditions is needed.

Lemma 8. *If problem* (R_2) *has only a finite number of solvability conditions, then*

$$\left.\begin{aligned} l &\geq 0 \\ \nu &\geq \lambda \\ n &\geq m \\ \mu &\geq 0. \end{aligned}\right\} \tag{21}$$

Proof. From the proof of Lemma 7, the set

$$X = \mathbb{Z}^2 \backslash (d_A \cup d_B \cup d_C \cup d_D)$$

must be finite. If $l < 0$, there would be an infinite gap

$$\{(p, q) \in \mathbb{Z}^2 : l \leq p < 0 \text{ and } q > \max(\lambda, 0)\}$$

between d_A and d_B for whatever λ is. This gap can not be filled up by d_C and d_D since for $(p, q) \in d_C$ or $(p, q) \in d_D$, we have $q < \mu$ or $q < \nu$. Hence we must have $l \geq 0$. By considering $d_B \cap d_D$, $d_C \cap d_D$ and $d_C \cap d_A$, respectively, we get the rest of (21) ∎

Lemma 9. *If problem* (R_2) *has a non-empty finite set of solvability conditions, then*

$$\left.\begin{aligned} \lambda &\neq 0 \\ m &\neq 0 \\ n &\neq 0. \end{aligned}\right\} \tag{22}$$

Proof. If $\lambda = 0$, by virtue of Lemma 8, we have

$$d_A \cup d_B = \{(p, q) \in \mathbb{Z}^2 : -\infty < p < +\infty \text{ and } q \geq 0\} = \mathbb{Z}^2_+. \tag{23}$$

Let $\mathbb{Z}^2_- = \mathbb{Z} \backslash \mathbb{Z}^2_+$. Then from

$$\begin{aligned} d''_C = d_C \cap \mathbb{Z}^2_- &= \{(p, q) \in \mathbb{Z}^2 : p \geq m \text{ and } q < \min(\mu, 0)\} \\ &= \{(p, q) \in \mathbb{Z}^2 : p \geq m \text{ and } q < 0\} \end{aligned}$$

and

$$\begin{aligned} d''_D = d_D \cap \mathbb{Z}^2_- &= \{(p, q) \in \mathbb{Z}^2 : p < n \text{ and } q < \min(\nu, 0)\} \\ &= \{(p, q) \in \mathbb{Z}^2 : p < n \text{ and } q < 0\} \end{aligned}$$

we have, in view of (21),

$$d''_C \cup d''_D = \mathbb{Z}^2_-. \tag{24}$$

From (23) - (24) we get

$$d_A \cup d_B \cup d_C \cup d_D = \mathbb{Z}^2. \tag{25}$$

By Lemma 7, (25) implies that the set of solvability conditions is empty. Hence $\lambda \neq 0$. Similarly, we have $m \neq 0$ and $n \neq 0$ ∎

Lemma 10. *Under the assumption of Lemma* 9, *we have:*

(i) *If* $\lambda > 0$, *then* $n < 0$ *and* $\lambda > \mu$ (*and* $m \le n < 0$).

(ii) *If* $\lambda < 0$ *then* $m > l (\ge 0)$ *and* $\nu < 0$ (*and* $n \ge m > 0$).

Proof. We prove only the first statement. Suppose that $n \ge 0$. From $\nu \ge \lambda$, we have

$$\begin{aligned} d_B \cup d_D \supset & \big\{(p,q) \in \mathbb{Z}^2 : p < \min(l,n) \text{ and } q \ge \lambda\big\} \\ & \cup \big\{(p,q) \in \mathbb{Z}^2 : p < \min(l,n) \text{ and } q < \nu\big\} \\ & = \big\{(p,q) \in \mathbb{Z}^2 : p < \min(l,n) \text{ and } -\infty < q < +\infty\big\} \\ & \supset \big\{(p,q) \in \mathbb{Z}^2 : p < 0 \text{ and } -\infty < q < +\infty\big\}. \end{aligned}$$

Since $\mu \ge 0$ and $\nu \ge \lambda > 0$, we have

$$\begin{aligned} d_D \cup d_C \supset & \big\{(p,q) \in \mathbb{Z}^2 : p < n \text{ and } q < 0\big\} \cup \big\{(p,q) \in \mathbb{Z}^2 : p \ge m \text{ and } q < 0\big\} \\ & \supset \big\{(p,q) \in \mathbb{Z}^2 : -\infty < p < +\infty \text{ and } q < 0\big\}. \end{aligned}$$

Hence $d_A \cup d_B \cup d_C \cup d_D = \mathbb{Z}^2$, which shows that $n < 0$. Similarly, we can show that $\lambda > \mu$ ∎

From Lemmas 8 - 11, we get

Lemma 11. *Problem* (R_2) *has a finite number of solvablity conditions only if*

$$\left.\begin{array}{c} l \ge 0 \\ \nu \ge \lambda > \mu \ge 0 > n \ge m \end{array}\right\} \tag{26}$$

or

$$\left.\begin{array}{c} n \ge m > l \ge 0 > \nu \ge \lambda \\ \mu \ge 0. \end{array}\right\} \tag{27}$$

Let $X_1, X_2 \subset \mathbb{Z}^2$ be defined by

$$X_1 = \big\{(p,q) \in \mathbb{Z}^2 : n \le p < 0 \text{ and } \mu \le q < \lambda\big\} \tag{28}$$

$$X_2 = \big\{(p,q) \in \mathbb{Z}^2 : l \le p < m \text{ and } \nu \le q < 0\big\}. \tag{29}$$

The cardinalities of these sets are respectively $(\mu - \lambda)n$ and $(l - m)\nu$. Moreover, we have

Lemma 12. *Let conditions* (26) *or* (27) *be satisfied. Then*

$$\mathbb{Z}^2 \backslash (d_A \cup d_B \cup d_C \cup d_D) = \begin{cases} X_1 & \textit{if } (26) \textit{ is satisfied} \\ X_2 & \textit{if } (27) \textit{ is satisfied} \end{cases}$$

where X_1 *and* X_2 *are defined by* (28) *and* (29).

From Lemmas 11 and 12, we get

Theorem 4. *Problem* (R_2) *has a finite number of solvability conditions if and only if* (26) *or* (27) *hold, i.e.* $P_{X_1}F = 0$ *or* $P_{X_2}F = 0$.

Acknowledgments. The second-named author is partly supported by the Alexander von Humboldt Stiftung and the National Science Foundation of China.

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Received 04.02.1999; in revised form 24.06.1999