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Acoustics of a Stratified Poroelastic Composite

R. P. Gilbert and A. Panchenko

Abstract. In this paper we discuss the acoustic boundary layer problem for a poroelastic seabed abutting onto a liquid half space. The problem is addressed using the method of homogenization where the microscopic equations are modelled after Burridge and Keller [5], Levy [10], and Panasenko [13]. A difference in our approach is that we do not consider the viscosity coefficients to be dependent on the pore size. To achieve continuity of displacement and stress at the interface to an arbitrary asymptotic order, we introduce correctors of two different types on each side. Then correctors of different types are matched across the interface.

Keywords: Homogenization, asymptotic expansions, boundary layers, composite materials

AMS subject classification: 74 Q 10, 35 Q 35

1. Introduction and Remarks

In this paper we discuss the vibrational motion of a porous medium whose pore space is saturated with fluid. The porous medium we propose to study is formed by a periodic arrangement of the pores into cells. The vibrational motion is assumed to be stimulated acoustically by a signal whose wave length is λ. For an averaging procedure to work, we need the wavelength to be large compared to a typical cell size ℓ. Assuming in addition that λ is comparable to the characteristic macroscopic size L of the problem and the fluid phase is incompressible, one can classify different homogenized models, as was done in [2] heuristically, and justified rigorously in [7]. In these works, four different types of possible macroscopic behavior are listed:

- Model I: The acoustics of a fluid in a rigid porous matrix regime. This case was considered previously by Gilbert and Panasenko [6].
- Model II: Diphasic macroscopic behavior of the fluid and solid matrix. This case is considered using the methods of two-scale convergence in [7].
- Model III: Monophasic elastic macroscopic behavior. This case is also discussed in [7].
- Model IV: Monophasic viscoelastic macroscopic behavior.

Model II, the diphasic case corresponds to the Biot model [3, 4].

Both authors: University of Delaware, Dept. Math. Sci., Newark, DE 19716, U.S.A.
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In this paper, we allow the fluid to be compressible and do not assume that λ is comparable to L. Thus the model developed in the paper will also work for λ large compared to ℓ but small compared to L. The homogenized model obtained is close to the "slightly compressible" variant of model III as developed in [7]. The microscopic equations we use are similar to the ones used in papers by Levy [10] and Burridge and Keller [5]. The only, but essential difference is that we do not assume the viscosity coefficients to be comparable to the pore size. We adopt a point of view according to which viscosity characterizes properties of the fluid regardless of the pore geometry. The immediate conclusion then is that the classical Biot model is an approximation valid for pore size lying within a certain range. The latter is determined by the viscosity of the fluid and by the geometric characteristics of the medium. Passing to the limit of "infinitely small" pore size, one should expect to obtain a set of equations different from [3, 5, 10].

Denote the ratio between ℓ and L is given by ε, $\varepsilon L = \ell$. The geometrical structure inside the unit cell $Q = (0,1)^n$ has a solid part Q^s which is a closed subset of $\bar{Q}$, and a fluid part $Q^f = Q \backslash Q^s$. Now we assume Q^s is periodically repeated over $\mathbf{R}^n$ and set $Q^s_k = Q^s + k$ for $k \in \mathbf{Z}^n$. Obviously, the obtained closed set $X^s = \bigcup_{k \in \mathbf{Z}^n} Q^s_k$ is a closed subset of $\mathbf{R}^n$ and $X^f = \mathbf{R}^n \backslash X^s$ in an open set in $\mathbf{R}^n$. We make the following assumptions on Q^f and X^f:

- **(i)** Q^f is an open connected set of strictly positive measure with a smooth boundary, and Q^s has strictly positive measure in $\bar{Q}$ as well.
- **(ii)** X^f and the interior of X^s are open sets with boundary of class C^∞, which are locally situated on one side of their boundary. Moreover, X^f is connected.

Now we see that $X = (0,L)^n$ is covered with a regular mesh of size ε, each cell being a cube Q^ε_i, with $1 \le i \le N(\varepsilon) = |X|(\varepsilon)^{-n}[1 + 0(1)]$. Each cube Q^ε_i is homeomorphic to Q, by a linear homeomorphism Π^ε_i, being composed of translation and an homothety of ratio $\frac{1}{\varepsilon}$. We define

$$Q^\varepsilon_{S_i} = (\Pi^\varepsilon_i)^{-1}(Q^s) \qquad \text{and} \qquad Q^\varepsilon_{f_i} = (\Pi^\varepsilon_i)^{-1}(Q^f).$$

For sufficiently small $\varepsilon > 0$ we consider the sets

$$T_\varepsilon = \{k \in \mathbf{Z}^n |\, Q^\varepsilon_{S_k} \subset X\} \qquad \text{and} \qquad K_\varepsilon = \{k \in \mathbf{Z}^n |\, Q^\varepsilon_{S_k} cap \partial X \neq \emptyset\}$$

and define

$$X^\varepsilon_s = \bigcup_{k \in T_\varepsilon} Q^\varepsilon_{S_k}, \qquad S^\varepsilon = \partial X^\varepsilon_s, \qquad X^\varepsilon_f = X \setminus X^\varepsilon_s.$$

Obviously, $\partial X^\varepsilon_f = \partial X \cup S^\varepsilon$. The domains X^ε_s and X^ε_f represent, respectively, the solid and fluid parts of a porous medium X. For simplicity we suppose $\frac{L}{\varepsilon} \in \mathbf{N}$. Then $K_\varepsilon = \emptyset$.

We construct a full asymptotic expansion for the system of two composites of the above type, separated by a plane interface. The first part of the construction deals with inner expansions. The development here is based on the general method proposed by Panasenko [13]. Then we proceed to take care of the boundary layers at the interface.

The major part of this work is devoted to the investigation of interface effects between two different composites. We consider the simplest model situation of a single plane interface. The goal is again to produce a full asymptotic expansion accurate to an arbitrary power of the parameter, which normally is a difficult task. A simple computation shows that the usual inner expansions can not be matched at the first level, even after the introduction of boundary layer correctors of a type typically employed in the literature. In classical works on homogenization such as [15] this problem is discussed, but the only rigorous work known to the authors is the recent paper by Avellaneda, Berlaynd, and Clouet [1], where the problem of interface matching is solved under the assumption that wave propagation can be described by wave equations at the microscopic level. The authors use the Floquet theory and operator perturbation theory to obtain explicit asymptotics of the Dirichlet-to-Neumann map at the interface.

The main idea that makes matching possible is the introduction of two different correctors on each side. The first corrector is a usual one, so we call it "old". The second, "new" corrector combines the fast variable part of the expansion on this side and the slow variable part of the inner expansion from the opposite side. The matching is done as follows: the old corrector from one side is matched with the new corrector from the opposite side. This is the crucial part of the method. On the output, this "microscopic" matching produces two families of constant matrices used to obtain macroscopic matching conditions. The constants can be determined explicitly from the formula contained in Theorem 4.1, and also from the examination of the proof of Theorem 5.3. As an example, we calculate homogenized transmission conditions for the main term of the expansion. The result shows that the homogenized transmission conditions can be written in terms of homogenized normal stress only. The conditions for subsequent terms contain non-trivial corrections to the homogenized stress, which shows significance of edge effects. The details can be found in Section 4.

In order to control decay of boundary layer correctors, one needs to estimate solutions of the cell problem in unbounded domains. In the present case the cell problem is a system of stationary viscoelasticity. The problem to solve is twofold. First, we need to obtain sufficient conditions for the existence of solutions with derivatives which do not grow too fast. Then, imposing some extra conditions, we show that derivatives will decay exponentially, and the solution itself will stabilize to a constant vector with exponential speed. This is the most difficult and technical part of the paper. It is based on special a priori estimates of Saint-Venant type. Estimates of this kind for the elasticity system were obtained by Oleinik and Yosifian [11]. In the paper, we obtain a generalization of these estimates, since we need to deal with complex-valued coefficients. Moreover, the estimates in [11] are written in a half-space. In order to control the decay of solutions of the transmission problem, we need to estimate the decay of solutions on both sides of the interface, rather than on just one. We point out, however, that most of the technical arguments from [11] still works.

The paper is organized as follows. In Section 2 we discuss the system of equations to be homogenized, and a certain transformation that leads to the elimination of the pressure from the system. In Section 3, the inner expansion is obtained. Section 4 contains the details of the interface matching procedure. Finally, in the technical Section 5 we give proofs of the estimates needed in Section 4.

2. System of equations for acoustics in a periodic porous medium

Consider an infinite ε-periodic medium composed of an elastic solid and a compressible viscous fluid. Let X^s and X^f denote the domains occupied by the solid and fluid, respectively. Their common boundary S is assumed to be a smooth manifold of codimension one. The displacement vector u satisfies the system of equations (written componentwise):

$$\rho^s \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma^s_{ij}}{\partial x_j} + f_i \qquad \text{in } X^s \tag{2.1}$$

$$\rho^f \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma^f_{ij}}{\partial x_j} + f_i \qquad \text{in } X^f. \tag{2.2}$$

Moreover, on the interface S, the transmission conditions

$$\left.\begin{aligned} [u] &= 0 \\ [\sigma_{ij}\nu_j] &= 0 \end{aligned}\right\} \tag{2.3}$$

hold, where ν_j denote components of unit normal to S pointing inside of X^s. In the solid part, components of the stress tensor σ^s satisfy the Hooke law

$$\sigma^s_{ij} = a^s_{ijkl} e_{kl}(u), \qquad e_{kl}(u) = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big) \tag{2.4}$$

with coefficients $a^s_{ijkl} \in C^\infty(X^s)$ satisfying conditions of symmetry and positivity:

$$\begin{gathered} a^s_{ijkl} = a^s_{jikl} = a_{jilk} = a^s_{klij} \\ k_1 e_{ij}(u) e_{ij}(u) \geq a^s_{ijkl} e_{ij}(u) e_{kl}(u) \geq k_2 e_{ij}(u) e_{ij}(u). \end{gathered} \tag{2.5}$$

In the fluid part X^f, the stress tensor satisfies the Navier-Stokes law

$$\sigma^f_{ij} = -\delta_{ij} P + (\lambda \delta_{ij}\delta_{kl} + 2\mu \delta_{ik}\delta_{jl}) e_{kl}\Big(\frac{\partial u}{\partial t}\Big), \tag{2.6}$$

where the viscosity coefficients μ and λ satisfy

$$\mu > 0, \qquad \frac{\lambda}{\mu} > -\frac{2}{3} k_2, \qquad 0 < k_2 < 1. \tag{2.7}$$

If the displacement is small, we can linearize the equation near the reference state characterized by the known reference densities ρ^s and ρ^f. The linearized equation of state relates pressure P to the perturbation of density ρ:

$$P = c^2 \rho, \tag{2.8}$$

where c is the speed of sound. Moreover, the linearized conservation of mass equation gives a relation between ρ and the velocity $\frac{\partial u}{\partial t}$:

$$\frac{\partial \rho}{\partial t} + \rho^f \operatorname{div} \frac{\partial u}{\partial t} = 0. \tag{2.9}$$

Integrating, we obtain $\rho + \rho^f \operatorname{div} u = 0$. Combining this and (2.8) we have $P = -c^2 \rho^f \operatorname{div} u$, so that the pressure can be eliminated from (2.6):

$$\sigma^f_{ij} = a^f_{ijkl} e_{kl}(u), \tag{2.10}$$

where

$$a^f_{ijkl} = -c^2 \rho^f \delta_{ij}\delta_{kl} + (\lambda \delta_{ij}\delta_{kl} + 2\mu \delta_{ik}\delta_{jl}) \frac{\partial}{\partial t}.$$

Denote by a_{ijkl} the components of thew symmetric fourth-order tensor equal to a^s_{ijkl} in X^s and to a^f_{ijkl} in X^f, and denote by σ the corresponding "unified" symmetric stress tensor. Since $e(u)$ is also symmetric, we can write

$$\sigma_{ij} = a_{ijkl} \frac{\partial u_l}{\partial x_k}.$$

Then

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\Big(M_{kj} \frac{\partial u}{\partial x_k} \Big),$$

where M_{kj} are $n \times n$-matrix operators with components a_{ijkl}. Using matrices M_{kj} we can replace (1.1) and (2.2) by a single system:

$$\rho^o \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x_k}\Big(M_{kj} \frac{\partial u}{\partial x_j} \Big) = f, \tag{2.11}$$

where ρ^0 equals to ρ^s in X^s, and to ρ^f in X^f. The components a_{ijkl} of the matrices M_{kj} satisfy (2.5) and (2.10) in X^s and X^f, respectively. On the interface S, the transmission conditions (2.3) hold.

3. Inner expansion and homogenized system

The purpose of this section is to obtain a complete inner asymptotic expansion for a solution of system (2.11). The word "inner" means that at the moment we prescribe no macroscopic boundary conditions and treat the medium as infinite in all directions. The terms of the expansion will satisfy a chain of problems with coefficients independent of the parameter. The first system in this chain provides the macroscopic homogenized equations.

Consider system (2.11) together with the interface conditions (2.3). Assuming that u is a time-harmonic vector with angular frequency ω, and slightly abusing notation, we replace $u(x,t)$ by $u(x,\omega)e^{i\omega t}$. Then the amplitude $u(x,\omega)$ satisfies

$$-\omega^2 \rho u - \frac{\partial}{\partial x_j}\Big(A_{kj} \frac{\partial u}{\partial x_k} \Big) = f, \tag{3.1}$$

where $u(x)$ is an n-component vector function of x, $A_{kj}(\frac{x}{\varepsilon}, \omega)$ are periodic $n \times n$ matrices with components $A_{kj}^{(il)}$ given by

$$A_{kj}^{(il)} = a_{ijkl}^{s} \tag{3.2}$$

in X^s, and by

$$A_{kj}^{(il)} = (-c^2\rho^f + i\omega\lambda)\delta_{ij}\delta_{kl} + 2i\omega\mu\delta_{ik}\delta_{jl} \tag{3.3}$$

in X^f. By assumption, A_{kj} are smooth everywhere in $\mathbf{R}^n$ except a smooth manifold S of codimension one. On S, the transmission conditions

$$\left.\begin{aligned} [u] &= 0 \\ \Big[A_{kj}^{il}\frac{\partial u_l}{\partial x_k}\nu_j\Big] &= 0 \quad (i = 1, 2, \ldots, n) \end{aligned}\right\}$$

are satisfied, where ν_j are components of the unit normal to S. We look for an asymptotic in ε solution of the form

$$u \sim \sum_{p,l=0}^{\infty} \varepsilon^{p+l} \sum_{|i|=l} N^{p,i}(\xi)D^i v(x). \tag{3.4}$$

By the chain rule, the differential operator

$$L_x = \frac{\partial}{\partial x_j}\Big(A_{kj}\frac{\partial}{\partial x_k}\Big)$$

corresponds to

$$\varepsilon^{-2}L_\xi + \varepsilon^{-1}\Big[A_{kj}\Big(\frac{\partial^2}{\partial x_j \partial\xi_k} + \frac{\partial^2}{\partial x_k \partial \xi_j}\Big) + \frac{\partial A_{kj}}{\partial \xi_j}\frac{\partial}{\partial x_k}\Big] + A_{kj}\frac{\partial^2}{\partial x_j \partial x_k}. \tag{3.5}$$

Substituting (3.4) into system (3.1), and taking (3.5) into account, we obtain

$$\begin{aligned} f \sim &\sum_{l,p=0} \varepsilon^{l+p} \sum_{|i|=l} -\omega^2\rho N^{p,i}D^i v + \sum_{l,p=0} \varepsilon^{l+p-2} \sum_{|i|=l} L_\xi N^{p,i}D^i v \\ &+ \sum_{l,p=0} \varepsilon^{l+p-1} \sum_{|i|=l} A_{kj}\frac{\partial N^{p,i}}{\partial \xi_k}\frac{\partial D^i v}{\partial x_j} + \sum_{l,p=0} \varepsilon^{l+p-1} \sum_{|i|=l} \frac{\partial}{\partial \xi_j}(A_{kj}N^{p,i})\frac{\partial D^i v}{\partial x_k} \\ &+ \sum_{l,p=0} \varepsilon^{l} \sum_{|i|=l} A_{kj}N^{p,i}\frac{\partial^2 D^i v}{\partial x_k \partial x_j}. \end{aligned} \tag{3.6}$$

Next, we shift indices in the sums above to get

$$\sum_{l,p=0} \varepsilon^{l+p} \sum_{|i|=l} -\omega^2\rho N^{p,i}D^i v = \sum_{l=0,p=2} \varepsilon^{l+p-2} \sum_{|i|=l} -\omega^2\rho N^{p-2,i}D^i v$$

$$\sum_{l,p=0} \varepsilon^{l+p-1} \sum_{|i|=l} A_{kj}\frac{\partial N^{p,i}}{\partial \xi_k}\frac{\partial D^i v}{\partial x_j} = \sum_{l=1,p=0} \varepsilon^{l+p-2} \sum_{|i|=l} A_{ki_1}\frac{\partial N^{p,i_2\cdots i_l}}{\partial \xi_k}D^i v$$

$$\sum_{l,p=0} \varepsilon^{l+p-1} \sum_{|i|=l} \frac{\partial}{\partial \xi_j}(A_{kj}N^{p,i})\frac{\partial D^i v}{\partial x_k} = \sum_{l=1,p=0} \varepsilon^{l+p-2} \sum_{|i|=l} \frac{\partial}{\partial \xi_j}(A_{i_1 j}N^{p,i_2\cdots i_l})D^i v$$

$$\sum_{l,p=0} \varepsilon^{l} \sum_{|i|=l} A_{kj}N^{p,i}\frac{\partial^2 D^i v}{\partial x_k \partial x_j} = \sum_{l=2,p=0} \varepsilon^{p+l-2} \sum_{|i|=l} A_{i_1 i_2}N^{p,i_3\cdots i_l}D^i v.$$

Then (3.6) transforms into

$$f \sim \sum_{l,p=0} \varepsilon^{l+p-2} \sum_{|i|=l} H^{p,i}(\xi) D^i v, \tag{3.7}$$

where $H^{p,i}$ depend on $N^{p,i}, A_{kj}, \omega$ and ρ. Since the left-hand side is of order ε^0, we obtain

$$N^{0,0} = N^{1,0} = I,$$

where I denotes the unit matrix. Also, we assume that $N^{p,i} = 0$ if at least one of p and $|i|$ is negative. Then, collecting terms in (3.7), we have for $H^{p,i}$ the expressions

$$\begin{aligned} H^{0,0} &= 0 \\ H^{1,0} &= L_\xi N^{1,0} = 0 \\ H^{p,0} &= -\omega^2 \rho N^{p-2,0} + L_\xi N^{p,0} \quad (p \geq 2) \end{aligned} \tag{3.8}$$

$$H^{0,i_1} = \frac{\partial A_{i_1 j}}{\partial \xi_j} + L_\xi N^{0,i_1} \qquad \text{and} \qquad H^{1,i_1} = \frac{\partial A_{i_1 j}}{\partial \xi_j} + L_\xi N^{1,i_1}$$

and

$$\begin{aligned} H^{p,i} &= -\omega^2 \rho N^{p-2,i} + A_{ki_1} \frac{\partial N^{p,i_2 \cdots i_l}}{\partial \xi_k} \\ &\quad + \frac{\partial}{\partial \xi_j} (A_{i_1 j} N^{p,i_2 \cdots i_l}) + A_{i_1 i_2} N^{p,i_3 \cdots i_l} + L_\xi N^{p,i}. \end{aligned} \qquad (p \geq 0, |i| > 1) \tag{3.9}$$

If we require that $H^{p,i}$ be constant, equations (3.8) and (3.9) can be used to determine $N^{p,i}$ recursively.

All equations above are of the form

$$L_\xi N^{p,i} = -T^{p,i} - H^{p,i},$$

where

$$T^{p,i} = -\omega^2 \rho N^{p-2,i} + A_{ki_1} \frac{\partial N^{p,i_2 \cdots i_l}}{\partial \xi_k} + \frac{\partial}{\partial \xi_j} (A_{i_1 j} N^{p,i_2 \cdots i_l}) + A_{i_1 i_2} N^{p,i_3 \cdots i_l}.$$

Note that $T^{p,i}$ depends on the previously obtained $N^{p',i'}$ with $p' + |i'| < p + |i|$. We specify the constants $H^{p,i}$ to be $\langle T^{p,i} \rangle$, and write

$$L_\xi N^{p,l} = -T^{p,i} + \langle T^{p,i} \rangle. \tag{3.10}$$

This choice of $H^{p,i}$ guarantees that each cell problem is uniquely solvable up to a constant matrix. To show this, consider the variational formulation of cell problems (3.10). Denote by V^1 and V^0 the spaces of 1-periodic vector functions

$$V^1 = \{v \in H^1_{loc} : \langle v \rangle = 0\} \qquad \text{and} \qquad V^0 = \{v \in L^2_{loc} : \langle v \rangle = 0\}.$$

Consider a sesquilinear form b on V^1 given by

$$b(u, v) = \int B(u, v)\, dx, \qquad \text{where} \quad B(u, v)(x) = A^{il}_{kj} \frac{\partial u_i}{\partial x_k} \frac{\partial \bar{v}_l}{\partial x_j}.$$

The variational formulation of a cell problem now reads: find $u \in V^1$ such that

$$b(u, v) = \langle f, v \rangle \tag{3.11}$$

for some $f \in V^0$ and all $v \in V^1$. Of course, the actual cell problem (3.10) is a matrix one, so we have to solve several vector problems and determine columns of an unknown matrix one by one.

Theorem 3.1. *There exists a unique solution of problem* (3.11).

Proof. We note first that the symmetry conditions for A_{kj} remain the same as in the real-valued case. This implies $B(u,u) = e(u)^* Ae(u)$, where A is the fourth order tensor with components $A_{kj}^{(il)}$ and $e(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. This and the ellipticity condition for A implies $|B(u,u)| \geq c_1 \|e(u)\|^2_{L^2(Q)}$. By [13: Theorem 2.8], the second Korn inequality $\|u\|_{H^1(Q)} \leq C \|e(u)\|_{L^2(Q)}$ holds for all $u \in V^1$. Hence, the form B is coercive on V^1, and by the Lax-Milgram lemma, problem (3.11) is uniquely solvable for any $f \in V^0(Q)$ ∎

The asymptotic series for u now takes the form

$$
\begin{aligned}
u &\sim H^{2,0}v(x) + H^{i_1,i_2}\frac{\partial^2 v(x)}{\partial x_{i_1}\partial x_{i_2}} + \sum_{p+l>2} \varepsilon^{l+p-2} \sum_{|i|=l} H^{p,i} D^i v \\
&= -\omega^2 \langle\rho\rangle v(x) + \sum_{i_1,i_2=1}^{n} \Big\langle A_{ki_1}\frac{\partial N^{0,i_2}}{\partial \xi_k} + A_{i_1 i_2}\Big\rangle \frac{\partial^2 v(x)}{\partial x_{i_1}\partial x_{i_2}} \\
&\quad + \sum_{p+l>2} \varepsilon^{l+p-2} \sum_{|i|=l} H^{p,i} D^i v.
\end{aligned}
$$

Representing v as an asymptotic series $v(x) \sim \sum_{q=0} \varepsilon^q v_q(x)$ we obtain a chain of averaged problems for successive determination of v_q:

$$
\hat{L} v_q = f_q, \tag{3.12}
$$

where

$$
\hat{L} v = -\omega^2 \langle\rho\rangle u + \sum_{i_1,i_2=1}^{n} \Big\langle A_{ki_1}\frac{\partial N^{0,i_2}}{\partial \xi_k} + A_{i_1 i_2}\Big\rangle \frac{\partial^2 v(x)}{\partial x_{i_1}\partial x_{i_2}},
$$

and

$$
\begin{aligned}
f_0 &= f \\
f_1 &= -\sum_{l+p=3}\sum_{|i|=l} H^{p,i} D^i v_0 \\
f_2 &= -\sum_{l+p=4}\sum_{|i|=l} H^{p,i} D^i v_0 - \sum_{l+p=3}\sum_{|i|=l} H^{p,i} D^i v_1,
\end{aligned}
$$

and, generally,

$$
f_q = -\sum_{t=0}^{q-1} \sum_{l+p=q-t+2} \sum_{|i|=l} H^{p,i} D^i v_t.
$$

The first equation in chain (3.12) is the homogenized system

$$
-\omega^2 \rho u + \sum_{i_1,i_2=1}^{n} \Big\langle A_{ki_1}\frac{\partial N^{0,i_2}}{\partial \xi_k} + A_{i_1 i_2}\Big\rangle \frac{\partial^2 v(x)}{\partial x_{i_1}\partial x_{i_2}} = f. \tag{3.13}
$$

The matrices N^{0,i_2} above are obtained as solutions of the cell problem

$$L_\xi N^{0,i_2} = -\frac{\partial}{\partial \xi_j} A_{i_2 j} \tag{3.14}$$

satisfying the periodic boundary conditions and the transmission conditions $[N^{0,i_2}] = 0$ and $\left[A_{kj}\frac{\partial N^{p,i}}{\partial \xi_k} n\right] = 0$ on the interface hypersurface S.

4. Interface matching and boundary layers

In the previous section we did not consider boundary conditions, so the construction above applies only locally in $\mathbf{R}^n$. To investigate the nature of the changes needed to incorporate boundary effects, consider the following model problem. Suppose that the plane interface $\{x : x_n = 0\}$ separates two different periodic media. We assume that equations (3.1) together with the constitutive relations (3.2) with possibly different 1-periodic matrices $A^{\pm}_{kj}$ are valid in the halfspaces $K^+ = \{x : x_n > 0\}$ and $K^- = \{x : x_n < 0\}$, respectively. A particular case of this is acoustics in a two-layer media of the type homogeneous fluid above, fluid-saturated sediment below. At this point we prescribe no conditions on u as $|x_n| \to \infty$. Our primary interest is to investigate how the presence of the interface affects homogenization.

For any $z = (z_1, ...z_n) \in \mathbf{R}^n$, let $\hat{z}$ denote the vector $(z_1, ...z_{n-1}, 0)$. In what follows, we use the notations

$$\begin{aligned}
\omega(a,b) &= \{x : a < x_n < b\} \\
\hat{\omega}(a,b) &= \{x :\ 0 < x_j < 1\ (j = 1, \ldots, n-1) \text{ and } a < x_n < b\} \\
\Gamma_t &= \{x : x_n = t\}
\end{aligned}$$

with $\hat{\Gamma}_t$ modified accordingly. We also denote

$$\begin{aligned}
Q &= \{x : x_j \in (0,1)\ (j = 1, \ldots, n)\} \\
\hat{Q} &= \{x : x_j \in (0,1)\ (j = 1, \ldots, n-1) \text{ and } x_n = 0\}.
\end{aligned}$$

Denote by $\hat{H}^1(\omega(a,b))$ the space of locally H^1-functions 1-periodic in $\hat{x}$. We recall that L denotes the differential operator in equations (3.1). A function $u \in \hat{H}^1(\omega(a,b))$ is a weak periodic in $\hat{x}$ solution of the problem:

$$Lu = f$$

in $\omega(a,b)$ if for any $v \in \hat{H}^1(\omega(a,b))$ such that $v = 0$ on $\Gamma_a \cup \Gamma_b$ the relation

$$\int_{\hat{\omega}(a,b)} (A^{hk}\partial_k u, \partial_h v)\, dx = \int_{\hat{\omega}(a,b)} (f, v)\, dx$$

holds.

Below quantities with sub- or superscript $+$ are defined in K^+, and similarly $-$ refers to a quantity defined in K^-. In K^+, we look for asymptotic expansions of the form

$$u_+ \sim \sum_{p,l=0}^{\infty} \varepsilon^{p+l} \sum_{|i|=l} (N_+^{p,i} + M_+^{p,i})(\xi) D^i v_+(x) + S_+^{p,i} D^i v_-(x), \tag{4.1}$$

where $v_\pm$ are asymptotic series formed by (so far) arbitrary solutions of the chain of homogenized problems (3.12) in $K^\pm$. In K^- we look for a similar expansion with all pluses replaced by minuses and vice versa. The matrices $N^{p,i}$ are as above, and $M_+^{p,i}$ and $S_+^{p,i}$ are matrices 1-periodic in $\hat{\xi}$. Substituting (4.1) into the original equations (3.1) and repeating the calculations of Section 3, we obtain the identical equations for $M_+^{p,i}$ and $S_+^{p,i}$, written explicitly only for $M_+^{p,i}$:

$$L_\xi^+ M_+^{p,i} = MM_+^{p,i} \tag{4.2}$$

in K^+, where $MM_+^{p,i}$ are of the form

$$MM_+^{p,i} = MM_{+,0}^{p,i} + \frac{\partial MM_{+,j}^{p,i}}{\partial \xi_j},$$

where

$$\begin{aligned} MM_{+,0}^{p,i} &= -\omega^2 \rho M_+^{p-2,i} + A_{i_1 i_2}^+ M_+^{p,i_3\cdots i_l} \\ MM_{+,j}^{p,i} &= A_{j i_1}^+ M_+^{p,i_1..1_l} + A_{i_1 j}^+ M_+^{p,i_2\cdots i_l}. \end{aligned}$$

To start the chain, we set $M_+^{0,0} = M_+^{1,0} = S_+^{0,0} = S_+^{1,0} = 0$. On the interface $x_n = 0$ we impose the transmission conditions

$$\left.\begin{aligned} u^+ &= u^- \\ \sigma^+(u^+)_n &= \sigma^-(u^-)_n \end{aligned}\right\} \tag{4.3}$$

where

$$\sigma^\pm(u^\pm)_n = A_\pm^{nj} \frac{\partial}{\partial x_j} u^\pm.$$

These conditions arise due to requirements of continuity of displacements and stresses. Differentiating $u^\pm$ and shifting indices in the sums in the same fashion as in Section 3, we obtain

$$\begin{aligned} \sigma^+(u^+)_N = {} & \sum_{p,l=0}^{\infty} \varepsilon^{p+l-1} \\ & \times \sum_{|i|=l} \left[A_{nj}^+ \frac{\partial}{\partial \xi_j} (N_+^{p,i} + M_+^{p,i}) + A_{ni_1}^+ (N_+^{p,i_2\cdots i_l} + M_+^{p,i_2\cdots i_l}) \right] D^i v^+ \\ & + \sum_{p,l=0}^{\infty} \varepsilon^{p+l-1} \sum_{|i|=l} \left[A_{nj}^+ \frac{\partial S_+^{p,i}}{\partial \xi_j} + A_{ni_1}^+ S_+^{p,i_2\cdots i_l} \right] D^i v^- \end{aligned}$$

and a similar expression for $\sigma^-(u^-)_n$. Substituting into the second equation in (4.3), collecting terms and combining with equations (4.2) we obtain the following transmission problems for determination of the pair of matrices $M_+^{p,i}$ and $S_-^{p,i}$:

$$\left.\begin{aligned} L_\xi^+ M_+^{p,i} &= M M_+^{p,i} \quad \text{in } K^+ \\ L_\xi^- S_-^{p,i} &= S S_-^{p,i} \quad \text{in } K^- \end{aligned}\right\} \tag{4.4}$$

with the interface conditions

$$N_+^{p,i} + M_+^{p,i} = S_-^{p,i} + k_+^{p,i} \tag{4.5}$$

and

$$A_{nj}^+ \frac{\partial}{\partial \xi_j}(N_+^{p,i} + M_+^{p,i}) + A_{ni_1}^+ (N_+^{p,i_2\cdots i_l} + M_+^{p,i_2\cdots i_l}) = A_{nj}^- \frac{\partial S_-^{p,i}}{\partial \xi_j} + A_{ni_1}^- S_-^{p,i_2\cdots i_l} + t_+^{p,i},$$

where $k_+^{p,i}$ and $t_+^{p,i}$ are constant matrices. We look for solution of this problem in the class of 1-periodic in $\hat{\xi}$ matrices which decay exponentially as $|\xi_n| \to \infty$. Similarly, the pair $M_-^{p,i}, S_+^{p,i}$ should be a solution to the problem

$$\left.\begin{aligned} L_\xi^- M_-^{p,i} &= M M_-^{p,i} \quad \text{in } K^- \\ L_\xi^+ S_+^{p,i} &= S S_+^{p,i} \quad \text{in } K^+ \end{aligned}\right\} \tag{4.6}$$

with the interface conditions

$$N_-^{p,i} + M_-^{p,i} = S_+^{p,i} + k_-^{p,i} \tag{4.7}$$

and

$$A_{nj}^- \frac{\partial}{\partial \xi_j}(N_-^{p,i} + M_-^{p,i}) + A_{ni_1}^- (N_-^{p,i_2\cdots i_l} + M_-^{p,i_2\cdots i_l}) = A_{nj}^+ \frac{\partial S_+^{p,i}}{\partial \xi_j} + A_{ni_1}^+ S_+^{p,i_2..i_l} + t_-^{p,i}$$

at the interface. Let us define the operator L to be L^+ in K^+ and L^- in K^-. Then the problems above can be written in the common form

$$\begin{aligned} Lu &= f && \text{in } K^+ \cup K^- \\ [u] &= \Phi(\hat{x}) && \text{at } x_n = 0 \\ [\sigma(u)_n] &= \Psi(\hat{x}) + t && \text{at } x_n = 0. \end{aligned} \tag{4.8}$$

In order to formulate the solvability theorem, we first introduce some definitions.

Definition 4.1. Let $u(\hat{x}, x_n) \in L^2_{loc}$ be a vector function 1-periodic in $\hat{x}$. We say that u has *one-sided exponential decay*, if the estimate

$$\|u\|_{L^2(Q_s)} \le C e^{-a|s|} \tag{4.9}$$

holds either for $s \in \mathbf{Z}^+$ or $s \in \mathbf{Z}^-$, with constants $C > 0$ and $a > 0$ independent of s. If (4.9) holds for all $s \in \mathbf{Z}$, we will say that u has *two-sided* exponential decay.

To describe the behavior at infinity we will use

Definition 4.2. A vector function u will be called *one-sided exponentially stabilizing* if there exist a constant vector w such that the function $u - w$ satisfies estimate (4.9) either for $s \in \mathbf{Z}^+$ or $s \in \mathbf{Z}^-$. If there is a pair of constant vectors w^+ and w^- such that one-sided estimates (4.9) hold for both respective differences, we will call u *two-sided* exponentially stabilizing to w^+ and w^-.

Theorem 4.1. *Suppose that f in (4.8) has two-sided exponential decay. Then there exist constant vectors t, w^+, w^- such that problem (4.8) has a 1-periodic in ξ solution u such that $e(u)$ has two-sided exponential decay, and u is two-sided exponentially stabilizing to w^+ and w^-. Moreover,*

$$t = \int_{K^-} f(x)\,dx - \int_{K^+} f(x)\,dx - \int_{\hat{Q}} \Psi(\hat{x})\,d\hat{x}. \tag{4.10}$$

Proof. First, we prove the theorem under the assumption that A^{jk}, f, Φ, Ψ are smooth functions of their arguments. Consider a vector function U defined in K^+ and satisfying the following conditions:

i) $U(\hat{x}, 0) = -\Phi(\hat{x})$.

ii) $\sigma(U)_n(\hat{x}, 0) = -\Psi(\hat{x}) - t$.

iii) U and $e(U)$ have one-sided exponential decay.

The existence of U follows from Borel's theorem (see, for instance, [8: Theorem 1.2.6]). Next, consider the function $v = u - U$, and let $\tilde{F}$ denote the function equal to $f - F$ in K^+, and equal to f in K^-. The function v is a solution to the problem

$$\left.\begin{aligned} Lv &= \tilde{F} && \text{in } K^+ \cup K^- \\ [v] &= 0 && \text{when } x_n = 0 \\ [\sigma(v)_n] &= 0 && \text{when } x_n = 0. \end{aligned}\right\} \tag{4.11}$$

In other words, v must be a global solution of the system of equations in (4.11). By Theorem 5.4 from Section 5, for any constant vector q this system has a solution v such that $P(v, 0) = q$, where $P(v, 0)$ denotes a generalized moment of v at $x_n = 0$. For the definition and properties of momenta, we refer to Section 5 below, and to [12]. Theorem 5.4 also implies that v satisfies the a priori estimate

$$\|e(v)\|^2_{L^2(\hat{\omega}(-k,k))} \le C\left(M e^{(A-\delta_1)|k|} + |k| \left\| q - \int_{\hat{\Gamma}_0} f_n d\hat{x} \right\|^2 \right).$$

Now apply Theorem 5.3 from Section 5. We see that v is exponentially stabilizing in K^+ if $P(v, 0) = -\int_{K^-} f(x)\,dx$. Also, v will be exponentially stabilizing in K^- provided $P(v, 0) = -\int_{K^-} f(x)\,dx$. Combined, these equalities imply that v will be two-sided exponentially stabilizing if we set

$$P(v, 0) = -\int_{K^-} f(x)\,dx = -\int_{K^-} f(x)\,dx.$$

Hence we obtain that

$$\int_{K^+} F(x)\,dx = \int_{K^+} f(x)\,dx - \int_{K^-} f(x)\,dx. \tag{4.12}$$

Consider the left-hand side separately. Since $F = \operatorname{div}(\sigma(U))$, we can integrate by parts. Due to periodicity, contributions of the derivatives with respect to $\hat{x}_j$ vanish. Hence,

$$\int_{K^+} F(x)\,dx = \int_0^\infty \partial_x \int_{\hat{Q}} \sigma(U)_n d\hat{x} dx_n.$$

The choice of U makes the contribution of the upper limit in the x_n-integral zero, and we obtain

$$\int_{K^+} F(x)\,dx = -\int_{\hat{Q}} \sigma(U)_n(\hat{x},0) = -\int_{\hat{Q}} \Psi(\hat{x})\,d\hat{x}.$$

Combining this with (4.12) we obtain (4.10). The existence of the constant vectors w^+ and w^- as well as two-sided exponential decay of $e(v)$ follow from Theorem 5.3. Now, using condition iii) for U we find that u is two-sided exponentially stabilizing to w^+ and w^-, and that $e(u)$ has two-sided exponential decay.

Finally, since constants in the estimates depend only on dimension and bounds for coefficient matrices, we can approximate the actual A^{jk}, f, Φ and Ψ by sequences of smooth functions and pass to the limit in the estimates ∎

Once $M^{p,i}$ and $S^{p,i}$ are found, we can obtain a sequence of macroscopic transmission problems. We use the notation $v_q(x)^\pm, L^\pm$ and so on to refer to vectors and operators defined in K^+ and K^-, respectively. Using these notations and the above transmission conditions for $M^{p,i}$ and $S^{p,i}$, we can write

$$[u] \sim \sum_{p,l=0}^{\infty} \varepsilon^{p+l} \sum_{|i|=l} k_+^{p,i} D^i v_+(x) - k_-^{p,i} D^i v_-(x). \tag{4.13}$$

Similarly, the jump of the normal stress at the interface can be written as

$$[\sigma(u)_n] \sim \sum_{p,l=0}^{\infty} \varepsilon^{p+l-1} \sum_{|i|=l} t_+^{p,i} D^i v_+(x) - t_-^{p,i} D^i v_-(x). \tag{4.14}$$

In order to satisfy the original continuity requirements, we need expressions on the right of these equations to be zero. Now, representing $v(x)$ as an asymptotic series $v = \sum \varepsilon^q v_q$ and collecting terms in (4.13) and (4.14) we obtain the chain of transmission problems

$$\hat{L}^\pm v_q^\pm = f_q^\pm \tag{4.15}$$

in $K^\pm$, where the homogenized operators $\hat{L}^\pm$ and the right-hand sides f_q are defined as in Section 3. On the interface $x_n = 0$ the transmission conditions

$$\begin{aligned} [k^{0,0} v_q] &= \sum_{l+p+k=q,k<q} \sum_{|i|=l} k_+^{p,i} D^i v_+(x) - k_-^{p,i} D^i v_-(x) \\ [t^{0,i_1} D^{i_1} v_q] &= \sum_{l+p+k-1=q,k<q} \sum_{|i|=l} t_+^{p,i} D^i v_+(x) - t_-^{p,i} D^i v_-(x) \end{aligned} \tag{4.16}$$

are fulfilled.

Let us now compute explicitly transmission conditions for the first two problems in this chain. Recall that $N^{0,0} = I$ and $M^{0,0} = S^{0,0} = 0$. This, together with (4.5) and (4.7) immediately gives

$$k_+^{0,0} = k_-^{0,0} = I \qquad \text{and} \qquad t_+^{0,0} = t_-^{0,0} = t_+^{1,0} = t_-^{1,0} = 0.$$

Next, consider equations (4.15), (4.16) for $p = 0$ and some fixed $l = i_1$. The first observation is that $MM^{0,i_1} = SS^{0,i_1} = 0$ which yields $k_\pm^{0,i_1} = 0$ after applications of Theorem 5.3. Next, we apply the formula for t from Theorem 4.1 to get

$$t_+^{0,i_1} = \int_{\hat{Q}} \left(A_{nj}^+ \frac{\partial N_+^{p,i_1}}{\partial \xi_j} + A_{ni_1}^+ \right) d\hat{x}. \tag{4.17}$$

Let $\hat{A}_{nj}$ denote the homogenized matrices

$$\hat{A}_{ni_1} = \int_Q \left(A_{nj} \frac{\partial N^{0,i_1}}{\partial \xi_j} + A_{ni_1} \right) dx.$$

Moreover, denote

$$\tilde{A}_{ni_1} = A_{nj} \frac{\partial N^{0,i_1}}{\partial \xi_j} + A_{ni_1} - \hat{A}_{ni_1}.$$

Now we can write (4.17) as

$$t_+^{0,i_1} = \hat{A}_{ni_1}^+ + \int_{\hat{Q}} \tilde{A}_{ni_1}^+ \, d\hat{x}. \tag{4.18}$$

We want to show that the integral is zero. Note that the cell system for N_+^{0,i_1} can be written as $\frac{\partial}{\partial \xi_k} \tilde{A}_{ki_1}^+ = 0$. If $\tilde{A}_{kj}$ are smooth, we can integrate this equation from zero to $t < 1$ with respect to x_n, and then integrate over $\hat{Q}$. The integrals containing $A_{ki_1}^+$ with $k \neq n$ vanish because of periodicity, and we get

$$\int_{\hat{Q}} \tilde{A}_{ni_1}^+(t, \hat{x}) \, d\hat{x} = \int_{\hat{Q}} \tilde{A}_{ni_1}^+(0, \hat{x}) \, d\hat{x}$$

for all $t \in [0, 1]$. In other words, the integral on the right of (4.18) is a constant. Integrating this constant on the interval $(0, 1)$, we will obtain the average of $\tilde{A}_{ni_1}^+$ which is zero by the definition of $\tilde{A}_{ni_1}^+$. Hence the original constant must be zero. In the case of a general $\tilde{A}_{ni_1}^+$ the same conclusion is obtained by approximating $\tilde{A}_{ni_1}^+$ by a sequence of smooth matrices and passing to the limit. Putting everything together we obtain

Proposition 4.1. *The first non-trivial homogenized problem in the chain has the form $\hat{L}v_0 = f$ in $K^+ \cup K^-$, with the transmission conditions $[v_0] = 0$ and $[\hat{A}_{ni_1} D^{i_1} v_0] = 0$ at the interface $x_n = 0$.*

This shows that microstructure does not affect homogenized transmission conditions at this level.

5. A priori estimates of Saint-Venant type

In this section the technical results needed to construct boundary layers are collected. The most important part here is an a priori estimate of Saint-Venant type obtained in Theorem 5.2. The importance of estimates of this type for investigation of behavior of solutions at infinity and proving existence theorems in unbounded domains has been recognized by Oleinik and Yosifian [11] in the context of linear elasticity. The development here follows closely that in the book [12: Chapter II/Sections 7 and 8]. In the present case, we need estimates for the system of viscoelasticity (complex-valued coefficients are allowed). Moreover, we need to treat the transmission problem rather than Dirichlet or Neumann ones. This makes it necessary to derive estimates valid on both sides of the interface.

Recall that a function $u \in \hat{H}^1(\omega(a,b))$ is a weak periodic in $\hat{x}$ solution of the problem

$$Lu = f \qquad \text{in } \ \omega(a,b)$$

if for any $v \in \hat{H}^1(\omega(a,b))$ such that $v = 0$ on $\Gamma_a \cup \Gamma_b$ the relation

$$\int_{\hat{\omega}(a,b)} (A^{hk}\partial_k u, \partial_h v)\, dx = \int_{\hat{\omega}(a,b)} (f, v)\, dx$$

holds. Next, we introduce generalized momenta $P(t,u)$ defined by

$$P(t,u) = \lim_{s \to 0^+} \int_{\hat{\omega}(t,t+s)} A^{nk}\partial_k u\, dx.$$

The existence of momenta is proved in [12]. Moreover, we have

$$P(t_1,u) - P(t_2,u) = \int_{\hat{\omega}(t_2,t_1)} f\, dx$$

for $b > t_1 > t_2 > b$, and if A^{hk} and f are sufficiently smooth, then

$$P(t,u) = \int_{\hat{\Gamma}_t} A^{nk}\partial_k u\, d\hat{x}.$$

The following theorem is a basic version of the Saint-Venant principle slightly modified from [12].

Theorem 5.1. *Let $s > h > 0$ be integers, and let u be a periodic in $\hat{x}$ solution of $Lu = 0$ in $\omega(s-h, s+h+1)$. Suppose that $P(s-1,u) = 0$. Then*

$$\int_{\hat{\omega}(s,s+1)} |B(u,u)|\, dx \le e^{-Ah} \int_{\hat{\omega}(s-h,s+h+1)} |B(u,u)|\, dx, \tag{5.1}$$

where A is a positive constant independent of s and h.

Proof. Denote

$$g = \hat{\omega}(s-h, s+1+h), \qquad g^1 = \hat{\omega}(s-h, s), \qquad g^2 = \hat{\omega}(s+1, s+1+h).$$

Pick a sequence u^m of smooth functions converging to u. Then define a function $\Phi(x_n)$ by

$$\Phi(x_n) = \begin{cases} \exp(A[x_n - (s-h)] & \text{for } x \in g^1 \\ \exp(A[s+1+h-x_n]) & \text{for } x \in g^2 \\ \exp(Ah) & \text{for all other } x \in g. \end{cases}$$

Here A is a positive constant to be specified. Then choose a test function $v = (\Phi - 1)u^m$ and plug into the integral identity. We obtain

$$\begin{aligned} \int_g \big(A^{hk}\partial_k u, (\Phi-1)\partial_h u^m\big)dx &= -\int_g \big(A^{hk}\partial_k u, \partial_h \Phi u^m\big)dx \\ &= -\int_{g^1\cup g^2} \big(A^{nk}\partial_k u, \partial_n \Phi u^m\big)dx. \end{aligned}$$

Next, we write

$$g^1 = \bigcup_{d=0}^{h-1} w_d \quad \text{where} \quad w_d = \big\{x : s-h+d < x_n < s-h+d+1\big\}.$$

Fix d temporarily and choose a constant vector C such that $\int_{w_d}(u^m + C)\,dx = 0$. Then, using the second Korn inequality and the estimate

$$\|e(u)\|_{L^2(w_k)} \le c\|B(u,u)\|_{L^2(w_k)},$$

we obtain

$$\|(u^m + C)\|_{L^2(w_k)} \le M\|B(u^m, u^m)\|_{L^2(w_k)},$$

where M is independent of m and d. Next we observe that

$$\int_{w_k} (A^{nk}\partial_k u, \partial_n \Phi C)\,dx = \int \partial_n \Phi \int (A^{nk}\partial_k u, C)\,d\hat{x}dx_n = 0,$$

since $P(s+1,u) = 0$ implies $P(t,u) = 0$ for all $s-h < t < s+1+h$. Hence,

$$\begin{aligned} &\left|\int_{w_k} \big(A^{nk}\partial_k u, \partial_n \Phi u^m\big)\,dx\right| \\ &\quad = \left|\int_{w_k} \big(A^{nk}\partial_k u, \partial_n \Phi(u^m + C)\big)\,dx\right| \\ &\quad = \left|\int_{w_k} \big(A^{nk}\partial_k u, A\Phi(u^m + C)\big)dx\right| \\ &\quad \le C_1 M A e^{A(s-h+d+1)} \left(\int_{w_k} |B(u,u)|dx\right)^{\frac{1}{2}} \left(\int_{w_k} |B(u^m,u^m)|dx\right)^{\frac{1}{2}} \\ &\quad \le C_2 M A e^{A} \int_{w_k} |B(u,u)|\Phi\,dx + R_m, \end{aligned}$$

where $R_m \to 0$ as $m \to \infty$. Summing over all w_k we obtain

$$\left| \int_{g^1} \left(A^{nk}\partial_k u, \partial_n \Phi u^m\right) dx \right| \leq C_3 M A e^A \int_{g^1} |B(u,u)| \Phi \, dx + hR_m.$$

Repeating the argument we obtain a similar estimate for g^2. Finally, let $m \to \infty$. Then

$$\int_g |B(u,u)|(\Phi - 1)\, dx \leq C_4 M A e^A \int_{g^1 \cup g^2} |B(u,u)| \Phi \, dx.$$

Choose A so that the constant on the right is equal to one. Then

$$\int_{g \setminus (g^1 \cup g^2)} |B(u,u)| \Phi \, dx \leq \int_g |B(u,u)|\, dx.$$

Multiplying by e^{-Ah} we obtain the estimate desired ∎

Next, we need to generalize this to the case when f and $P(s-h,u)$ are non-zero. The prototype of the main estimate is given by the

Lemma 5.1. *Let N be a positive integer. The system*

$$LU = f_0 + \partial_i f_i \qquad in \ \ \omega(-N,N) \tag{5.2}$$

with boundary conditions

$$\sigma(U)_n = \begin{cases} \phi + f_i & on \ \Gamma_{-N} \\ \psi - f_i & on \ \Gamma_N \end{cases} \tag{5.3}$$

satisfying the compatibility condition

$$\int_{\hat\Gamma_{-N}} \phi(\hat x)\, d\hat x + \int_{\hat\Gamma_N} \psi(\hat x)\, d\hat x = \int_{\hat\omega(-N,N)} f(x)\, dx \tag{5.4}$$

has a unique solution U satisfying the estimate

$$\|e(U)\|^2_{L^2(\hat\omega(-N,N))} \leq C\left[\sum_{i=0}^{n} \|f\|^2_{L^2(\hat\omega(-N,N))} + \sum_{m=-N}^{N} \|\psi_m\|^2_{L^2(\Gamma_m)} \right] \tag{5.5}$$

where

$$\left.\begin{aligned} &\psi_{-N} = \phi \\ &\quad \psi_m = \int_{\hat\omega(m,N)} f(x)\, dx - \int_{\Gamma_N} \psi(\hat x)\, d\hat x \qquad (m = -N+1, ..., N-1) \\ &\quad \psi_N = -\psi. \end{aligned}\right\} \tag{5.6}$$

Proof. Consider the problem

$$\left.\begin{aligned} LV^m &= f && \text{in} \ \ \omega(m-1,m) \\ \sigma(V^m) &= -\psi_m && \text{on} \ \ \Gamma_m \\ \sigma(V^m) &= \psi_{m-1} && \text{on} \ \ \Gamma_{m-1}. \end{aligned}\right\} \tag{5.7}$$

By [12: Theorem 6.3] modified slightly for the case of complex-valued coefficients, problem (5.7) is uniquely solvable if and only if the compatibility condition

$$\int_{\hat{\Gamma}_{m-1}} \psi_{m-1}(\hat{x})\,d\hat{x} - \int_{\Gamma_m} \psi_m(\hat{x})\,d\hat{x} = \int_{\hat{\omega}(m-1,m)} f(x)\,dx \tag{5.8}$$

is satisfied. Let us check these conditions for problem (5.7). Using (5.6) we have

$$\begin{aligned}
&\int_{\hat{\Gamma}_{m-1}} \psi_{m-1}(\hat{x})\,d\hat{x} - \int_{\Gamma_m} \psi_m(\hat{x})d\,\hat{x} \\
&\quad = -\int_{\hat{\Gamma}_{-N}} \psi_{m-1}(\hat{x})\,d\hat{x} + \int_{\Gamma_N} \psi(\hat{x})\,d\hat{x} + \int_{\hat{\omega}(m-1,N)} f(x)\,dx - \int_{\Gamma_N} \psi(\hat{x})\,d\hat{x} \\
&\quad = \int_{\hat{\omega}(m-1,m)} f(x)\,dx.
\end{aligned}$$

Also, for $m = -N+1$ we get

$$\begin{aligned}
&\int_{\hat{\Gamma}_{-N}} \phi(\hat{x})\,d\hat{x} - \int_{\Gamma_{-N+1}} \psi_{-N+1}(\hat{x})\,d\hat{x} \\
&\quad = \int_{\hat{\Gamma}_{-N}} \phi(\hat{x})\,d\hat{x} - \int_{\hat{\omega}(-N+1,N)} f(x)\,dx + \int_{\hat{\Gamma}_N} \psi(\hat{x})\,d\hat{x} \\
&\quad = \int_{\hat{\Gamma}_{-N}} \phi(\hat{x})\,d\hat{x} + \int_{\hat{\Gamma}_N} \psi(\hat{x})\,d\hat{x} - \int_{\hat{\omega}(-N,N)} f(x)\,dx + \int_{\hat{\omega}(-N,-N+1)} f(x)\,dx \\
&\quad = \int_{\hat{\omega}(-N,-N+1)} f(x)\,dx,
\end{aligned}$$

and similarly, the condition also checks for $m = N$. By [12: Theorem 6.3] cited problems (5.7) are uniquely solvable. Moreover, the estimate

$$\begin{aligned}
&\|e(V^m)\|^2_{L^2(\hat{\omega}(m-1,m))} \\
&\qquad \le C\Big[\sum_{i=0}^{n} \|f_i\|^2_{L^2(\hat{\omega}(m-1,m))} + \|\psi_m\|^2_{L^2(\Gamma_m)} + \|\psi_{m-1}\|^2_{L^2(\Gamma_{m-1})}\Big]
\end{aligned} \tag{5.9}$$

holds. Next, consider the sesquilinear forms

$$\begin{aligned}
I_m &= -\int_{\hat{\omega}(m-1,m)} B(V^m, U)\,dx \\
&= \int_{\hat{\omega}(m-1,m)} (f_0, U)\,dx - \int_{\hat{\omega}(m-1,m)} (f_i, \partial_i U)\,dx \\
&\qquad + \int_{\Gamma_m} (\psi_m, U)\,d\hat{x} - \int_{\Gamma_{m-1}} (\psi_{m-1}, U)\,d\hat{x}.
\end{aligned} \tag{5.10}$$

Summing up we obtain

$$\begin{aligned}
-\sum_{m=-N}^{N} I_m = \int_{\hat{\omega}(-N,N)} (f, U)\,dx &- \int_{\hat{\omega}(m-1,m)} (f_i, \partial_i U)\,dx \\
&+ \int_{\Gamma_{-N}} (\phi, U)\,d\hat{x} - \int_{\Gamma_N} (\psi, U)\,d\hat{x},
\end{aligned}$$

which shows that

$$-\sum_{m=-N}^{N} I_m = \int_{\hat\omega(-N,N)} B(U,U)\,dx.$$

Estimating the left-hand side in the last equation we get

$$\begin{aligned}
&\int_{\hat\omega(-N,N)} B(U,U)\,dx \\
&\quad\le \sum_{m=-N}^{N} \int_{\hat\omega(m-1,m)} |B(V^m,U)|\,dx \\
&\quad\le \sum_{m=-N}^{N} \Big(\int_{\hat\omega(m-1,m)} B(V^m,V^m)dx\Big)^{\frac12}\Big(\int_{\hat\omega(m-1,m)} B(U,U)\,dx\Big)^{\frac12} \\
&\quad\le \Big(\int_{\hat\omega(-N,N)} B(U,U)\,dx\Big)^{\frac12}\Big(\sum_{m=-N}^{N}\int_{\hat\omega(m-1,m)} B(V^m,V^m)\,dx\Big)^{\frac12}.
\end{aligned}$$

This implies

$$\begin{aligned}
\Big(\int_{\hat\omega(-N,N)} B(U,U)\,dx\Big)^{\frac12} &\le \Big(\sum_{m=-N}^{N}\int_{\hat\omega(m-1,m)} B(V^m,V^m)\,dx\Big)^{\frac12} \\
&\le C_1\Big(\sum_{m=-N}^{N} \|e(V^m)\|^2_{L^2(\hat\omega(m-1,m))}\Big)^{\frac12}.
\end{aligned}$$

Applying estimates (5.9) we see that the sum above is bounded by

$$C_2\Big(\|f\|^2_{L^2(\hat\omega(m-1,m))} + \sum_{m=-N}^{N} \|\psi_m\|^2_{L^2(\hat\Gamma_m)}\Big)^{\frac12}.$$

Estimating the left-hand side from below by the norm of $e(U)$ we finish the proof ∎

Using the lemma above we prove the following

Theorem 5.2. *Let u be a periodic in $\hat x$ solution of*

$$Lu = f_0 + \partial_i f_i$$

in $\omega(t_1,t_2)$, where $t_2 > t_1 + 2$ for integers $t_1 < 0$ and $t_2 > 0$. Then for any integer $s, h > 0$ such that $s - h > t_1$ and $s + 1 + h < t_1$ the estimate

$$\begin{aligned}
&\int_{\hat\omega(s,s+1)} |e(u)|^2 dx \\
&\quad\le C\Big[e^{-Ah}\int_{\hat\omega(s-h,s+1+h)} |e(u)|^2dx + \sum_{m=0}^{2h+1} + \int_{\hat\omega(s-h,s-h+m)} (|f_0|^2 + (f_i,f_i))\,dx \\
&\qquad + \Big|P(s-h,u) + \int_{\hat\omega(s-h,s-h+m)} f_0 dx - \int_{\hat\Gamma_{s-h}} f_i d\hat x s\Big|^2\Big]
\end{aligned}$$

holds with C independent of s and h. Here A is the constant from Theorem 1.

Proof . Let U be a periodic in $\hat{x}$ solution to the problem

$$\left.\begin{array}{ll} LU = f_0 + \partial_i f_i & \text{in } \ \omega(s-h, s+h+1) \\ \sigma(U) = \phi + f_n & \text{on } \ \Gamma_{s-h} \\ \sigma(U) = \psi - f_n & \text{on } \ \Gamma_{s+h+1} \end{array}\right\}$$

where ψ and ϕ are constant vectors chosen as

$$\int_{\hat{\Gamma}_{s+h+1}} \psi + \int_{\hat{\Gamma}_{s+h+1}} f_n d\hat{x} = P(s+h+1, u)$$

$$-\int_{\hat{\Gamma}_{s-h}} \phi + \int_{\hat{\Gamma}_{s-h}} f_n d\hat{x} = P(s-h, u).$$

These equations yield

$$\left.\begin{array}{l} \psi = P(s+h+1, u) - \displaystyle\int_{\hat{\Gamma}_{s+h+1}} f_n d\hat{x} \\ \phi = -P(s-h, u) + \displaystyle\int_{\hat{\Gamma}_{s-h}} f_n d\hat{x}. \end{array}\right\} \tag{5.11}$$

With this choice of ϕ and ψ the solvability condition from Lemma 5.1 is satisfied, so there exists a unique U satisfying the a priori estimate

$$\|e(U)\|^2_{L^2(\hat{\omega}(s-h,s+h+1))}$$
$$\leq C\left[\int_{\hat{\omega}(s-h,s+h+1)} \left(|f_0|^2 + (f_i, f_i)\right) dx + \sum_{m=0}^{2h+1} \int_{\hat{\Gamma}_{s-h+m}} |\psi_m|^2 d\hat{x}\right]$$

with C independent of s and h. The functions ψ_m are defined by

$$\psi_m = \int_{\hat{\omega}(s-h,s+h+1)} f_0 dx - \int_{\hat{\Gamma}_{s-h+m}} \psi \, d\hat{x},$$

where $m = 0, 1, ..., 2h+1$. Using (5.11) together with the formula

$$P(s+h+1, u) = P(s-h, u) + \int_{\hat{\omega}(s-h,s+h+1)} f_0 dx + \int_{\hat{\Gamma}_{s+h+1}} f_n d\hat{x} - \int_{\hat{\Gamma}_{s-h}} f_n d\hat{x}$$

we get

$$\psi_m = -\int_{\hat{\omega}(s-h,s+h+1)} f_0 dx - P(s-h, u) - \int_{\hat{\Gamma}_{s-h}} f_n d\hat{x}.$$

Moreover, $u - U$ is a solution of the viscoelasticity system satisfying conditions of Theorem 5.1. Hence,

$$\int_{\hat{\omega}(s,s+1)} |B(u-U, u-U)| \, dx \leq e^{-Ah} \int_{\hat{\omega}(s-h,s+h+1)} |B(u-U, u-U)| \, dx.$$

Estimating $B(u-U, u-U)$ from below by $c\,|e(u-U)|^2$ and then using the triangle inequality we get

$$\int_{\hat{\omega}(s,s+1)} |e(u)|^2 dx \leq C\left[e^{-Ah} \int_{\hat{\omega}(s-h,s+h+1)} |e(u)|^2 dx + \int_{\hat{\omega}(s-h,s+h+1)} |e(U)|^2 dx\right].$$

Application of the apriori estimate for $e(U)$ from Lemma 5.1 completes the proof ∎

As a consequence, we derive

Theorem 5.3. *Let f_i be vector function satisfying the inequality*

$$\sum_{i=0}^{n} \|f_i\|_{L^2(\hat\omega(s,s+1))} + \|f_n\|_{L^2(\hat\Gamma_s)} \le c\, e^{-as} \qquad (s \in \mathbf{N}) \tag{5.12}$$

where c and a are positive constants independent of s. Let u be a periodic solution of the system $Lu = f_0 + \partial_i f_i$ in $\omega(0,\infty)$ such that $P(0,u) = -\int_{\hat\omega(0,\infty)} f_0 dx + \int_{\hat\Gamma_0} f_n$,

$$\|e(u)\|_{L^2(\hat\omega(0,\frac{3s+1}{2}))} \le c\, e^{\frac{(A-\delta)s}{2}} \qquad (s \in \mathbf{N}) \tag{5.13}$$

where c is a constant independent of s, A is the constant from Theorem 5.1, and δ is a constant such that $0 < \delta \le A$. Then there exist constants C_1, C_2 and a_1, a_2 independent of s and a constant vector w such that

$$\|e(u)\|_{L^2(\hat\omega(s,s+1))} \le C_1 e^{-a_1 s} \tag{5.14}$$

$$\|u - w\|_{L^2(\hat\omega(s,s+1))} \le C_2 e^{-a_2 s}. \tag{5.15}$$

Proof. Integrating by parts, we have

$$\begin{aligned} P(s,u) &= P(0,u) + \int_{\hat\omega(0,s)} f_0 dx - \int_{\hat\Gamma_0} f_n d\hat x + \int_{\hat\Gamma_s} f_n d\hat x \\ &= -\int_{\hat\omega(s,\infty)} f dx + \int_{\hat\Gamma_s} f_n d\hat x. \end{aligned}$$

Hence, by virtue of (5.12), $|P(s,u)| \le c\, e^{-bs}$. Using the apriori estimate from Theorem 5.2, (5.12) and (5.13), we obtain (5.14).

Let χ_S denote a characteristic function fo the set S. Consider a periodic in $\hat x$ solution of the system

$$\left.\begin{aligned} Lv &= \chi_{\hat\omega(s,s+1)} - \chi_{\hat\omega(s+1,s+2)} && \text{in } \ \omega(s,s+2) \\ \sigma(v) &= 0 && \text{on } \ \partial\omega(s,s+2). \end{aligned}\right\} \tag{5.16}$$

Also, set

$$w_s = \int_{\hat\omega(s,s+1)} u\, dx.$$

Choosing v as test function in the basic integral identity and using the Schwarz inequality, we have

$$\begin{aligned} |w_s - w_{s+1}| &= \left| \int_{\hat\omega(s,s+2)} B(u,v)\, dx \right| \\ &\le \left(\int_{\hat\omega(s,s+2)} |B(v,v)|\, dx \right)^{\frac12} \left(\int_{\hat\omega(s,s+2)} |B(u,u)|\, dx \right)^{\frac12}. \end{aligned}$$

The application of the second Korn inequality yields $\|B(v,v)\|_{L^2(\hat\omega(s,s+2))} \le C$ where C is independent of s. Then

$$|w_s - w_{s+1}| \le c\, e^{-a_0 s}$$

where a_0 is positive and independent of s. This implies the existence of $w_\infty = \lim_{s\to\infty}$. Also, we have

$$|w_s - w_{s+t}| \le k\, e^{-a_0 s}$$

where k, a_0 are independent of s, t. Therefore, we can let $t \to \infty$ and obtain

$$|w_s - w_\infty| \le k\, e^{-a_0 s}.$$

Now

$$\begin{aligned}\|u - w_\infty\|_{L^2(\hat\omega(s,s+1))} &\le \|u - w_s\|_{L^2(\hat\omega(s,s+1))} + \|w_s - w_\infty\|_{L^2(\hat\omega(s,s+1))}\\ &\le k_1\left[\|e(u)\|_{L^2(\hat\omega(s,s+1))} + e^{-a_0 s}\right]\end{aligned}$$

where k_1 is independent of s. Now application of (5.14) yields (5.15) ∎

The next result is an existence theorem of the type needed for the construction of the boundary layer. Consider the problem

$$Lu = f_0 + \partial_i f_i \tag{5.17}$$

in $\omega(-\infty,\infty)$. We assume that $f_i \in L^2(\hat\omega(t_2,t_1)$ for all $t_1 < 0$ and $t_2 > 0$, and periodic in $\hat x$.

Theorem 5.4. *Suppose*

$$\sum_{i=0}^{n} \|f_i\|^2_{L^2(\hat\omega(-s,s))} + \|f_n\|^2_{L^2(\hat\Gamma_s)} + \|f_n\|^2_{L^2(\hat\Gamma_s)} \le M e^{(A-\delta)|s|} \qquad (s \in \mathbf{N}) \tag{5.18}$$

where M and δ are constants from Theorem, $0 < \delta \le A$. Then for any constant vector q there exists a solution of problem (5.17) such that $P(0,u) = q$ and the estimate

$$\|e(u)\|^2_{L^2(\hat\omega(-k,k))} \le C\Big(M e^{(A-\delta_1)|k|} + |k|\big|q - \int_{\hat\Gamma_0} f_n d\hat x\big|\Big)^2 \qquad (k \in \mathbf{N}) \tag{5.19}$$

holds where C is independent of k and $\delta_1 \in (0,\delta)$.

Proof. Let v^N be a solution to the Neumann problem (5.3) with

$$\begin{aligned}\psi &= q + \int_{\hat\omega(0,N)} f_0 dx - \int_{\hat\Gamma_0} f_n d\hat x\\ \phi &= -q + \int_{\hat\omega(0,-N)} f_0 dx + \int_{\hat\Gamma_0} f_n d\hat x.\end{aligned}$$

The vectors ψ_m from Lemma 5.1 are determined by

$$\begin{aligned}
\psi_{-N} &= \phi \\
\psi_m &= \int_{\hat\omega(m,N)} f_0 dx - \int_{\hat\Gamma_N} \psi d\hat x = -q + \int_{\hat\omega(m,0)} f_0 dx + \int_{\hat\Gamma_0} f_n d\hat x \\
\psi_N &= -\psi
\end{aligned}$$

where $m = -N+1, \ldots, n-1$. By Lemma 5.1, there exists a unique v^N satisfying the estimate

$$\|e(u)\|^2_{L^2(\hat\omega(-N,N))} \le c\Big[\sum_{i=0}^{n} \|f_i\|_{L^2_{\hat\omega(-N,N)}} + \sum_{m=-N}^{N} \Big|q - \int_{\hat\Gamma_0} f_n d\hat x - \int_{\hat\omega(m,0)} f_0 dx\Big|^2\Big].$$

The function $v^{N+k+1} - v^{N+k}$ satisfies the conditions of Theorem 5.1. Hence, we have an estimate

$$\begin{aligned}
&\big\|e(v^{N+k+1} - v^{N+k})\big\|^2_{L^2(\hat\omega(-k,k))} \\
&\quad \le C\, e^{-AN}\Big[\sum_{i=0}^{n} \|f_i\|_{L^2_{\hat\omega(-N-k-1,N+k+1)}} \\
&\qquad + \sum_{m=-N-k-1}^{N+k+1} \Big|q - \int_{\hat\Gamma_0} f_n d\hat x - \int_{\hat\omega(m,0)} f_0 dx\Big|^2\Big].
\end{aligned} \tag{5.20}$$

Let us estimate the last sum separately:

$$\begin{aligned}
&\sum_{m=-N-k-1}^{N+k+1} \Big|q - \int_{\hat\Gamma_0} f_n d\hat x - \int_{\hat\omega(m,0)} f_0 dx\Big|^2 \\
&\quad \le 2(N+k+1)\Big|q - \int_{\hat\Gamma_0} f_n d\hat x\Big|^2 + 2\sum_{m=-N-k-1}^{N+k+1} |\hat\omega(0,m)| \int_{\hat\omega(0,m)} |f_0|^2 dx \\
&\quad \le 2(N+k+1)\Big|q - \int_{\hat\Gamma_0} f_n d\hat x\Big|^2 + 2\sum_{m=-N-k-1}^{N+k+1} cmM\, e^{(A-\delta)|m|} \\
&\quad \le 2(N+k+1)\Big|q - \int_{\hat\Gamma_0} f_n d\hat x\Big|^2 + 2CMe^{(A-\delta_1)(N+k+1)}.
\end{aligned}$$

The constant δ_1 above is any number from the interval $(0, \delta)$. To obtain the last inequality, we used the estimates for f_i from the statement of the theorem, and the fact that $|\hat\omega(0,m)| \le cm$ where c depends only on the dimension of the space. Then substitution into (5.20) yields

$$\begin{aligned}
&\|e(v^{N+k+1} - v^{N+k})\|^2_{L^2(\hat\omega(-k,k))} \\
&\quad \le M_2\Big[M\, e^{Ak} e^{-\delta_1(N+k)} + e^{-AN}(k+N+1)\Big|q - \int_{\hat\Gamma_0} f^n d\hat x\Big|^2\Big].
\end{aligned} \tag{5.21}$$

The constant M_2 is independent of N and k.

Next, we use this inequality to show that $e(v^{k+s})$ for any fixed k is a Cauchy sequence in $L^2(\hat{\omega}(-k,k))$. For any $t \geq 0$ and any $s \geq 0$ consider

$$\begin{aligned}
&\left\|e(v^{k+s} - v^{k+s+t})\right\|^2_{L^2(\hat{\omega}(-k,k))} \\
&\quad \leq \sum_{l=0}^{t-1} \left\|e(v^{k+s+l} - v^{k+s+l+1})\right\|^2_{L^2(\hat{\omega}(-k,k))} \\
&\quad \leq \sum_{l=0}^{t-1} \left[M_2 M e^{Ak} e^{-\delta_1(s+l+k)} + e^{-A(s+l)}(k+s+l+1) \left| q - \int_{\hat{\Gamma}_0} f^n \, d\hat{x} \right|^2 \right]^{\frac{1}{2}} \qquad (5.22) \\
&\quad \leq (M_2 M)^{\frac{1}{2}} e^{(A-\delta_1)\frac{k}{2}} e^{-\delta_1 \frac{s}{2}} \sum_{l=0}^{\infty} e^{-\delta_1 \frac{l}{2}} + M_3 k^{\frac{1}{2}} \left| q - \int_{\hat{\Gamma}_0} f^n \, d\hat{x} \right| \sum_{l=0}^{\infty} e^{-\frac{A-\delta_1}{2(s+l)}}.
\end{aligned}$$

This shows that the sequence in question is of Cauchy type.

Next we note that all v^{k+s} for k fixed are solutions to the Neumann problems. Hence, they are orthogonal to all rigid displacements. For such functions, the second Korn inequality can be written in the form (see, for instance, [12: Theorem 2.9])

$$\|v^{k+s}\|_{H^1(\hat{\omega}(-k,k))} \leq C_k \|e(v^{k+s})\|_{L^2(\hat{\omega}(-k,k))}.$$

Together, the last two inequalities imply the existence of a function u such that $v^s \to u$ in $H^1(\hat{\omega}(-k,k))$ for any fixed k as $s \to \infty$. Substituting v^s into the integral identity and taking the limit as $s \to \infty$ we obtain that u is a solution to the original problem. Setting $s = 0$ in (5.22) and taking the limit as $t \to \infty$ we obtain estimate (5.19). Finally, integrate $\sigma(u - v^s)$ over a thin slab $\omega(-\frac{1}{t}, \frac{1}{t})$ including the plane $x_n = 0$. Then we use the second Korn inequality to estimate this integral in terms of $e(u - v^s)$. Next, divide both sides by the measure of the slab wish equals to $c\frac{1}{t}$ with c depending only on the dimension of the space. Finally, applying diagonal argument and the Lebesgue differentiation theorem, we select a subsequence v^{st} such that

$$\frac{1}{ct} \int_{\hat{\omega}(-1/t,1/t)} \sigma(u - v^{st}) \, dx \ \to 0 \qquad \text{as} \ \ s, t \to \infty.$$

By definition of the momenta, this yields $P(0,u) = q$ ∎

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