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Representation Formulas for Non-Symmetric Dirichlet Forms

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Abstract. As well-known, for X a given locally compact separable Hausdorff space, m a positive Radon measure on X with $\operatorname{supp}[m] = X$ and $C_0(X)$ the space of all continuous functions with compact support on X the Beurling and Deny formula states that any regular Dirichlet form $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ on $L^2(X, m)$ can be expressed as

$$\tilde{\mathcal{E}}(u,v) = \tilde{\mathcal{E}}^c(u,v) + \int_X uvk(dx) + \iint_{X\times X-d} (u(x)-u(y))((v(x)-v(y))\tilde{j}(dx,dy)$$

for all $u, v \in D(\tilde{\mathcal{E}}) \cap C_0(X)$ where the symmetric Dirichlet form $\tilde{\mathcal{E}}^c$, the symmetric measure $\tilde{j}(dx,dy)$ and the measure $k(dx)$ are uniquely determined by $\tilde{\mathcal{E}}$. It is our aim to prove this formula in the non-symmetric case. For this we consider certain families of non-symmetric Dirichlet forms of diffusion type and show that these forms admit an integral representation involving a measure that enjoys some important functional properties as well as in the symmetric case.

Keywords: *Non-symmetric Dirichlet forms, Beurling-Deny formula, diffusion forms, energy measures, differentiation formulas, differential operators*

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0. Introduction

It is well known that the Dirichlet forms are suitable tools to describe the variational principles of irregular bodies. In order to formulate such principles, *regular* Dirichlet forms have been studied. For these forms a rich representation theory is available, based on the fundamental formula of Beurling and Deny (see [1, 2]) and extended by Silverstein [9, 10], Fukushima [4] and Le Jean [6].

Let X be a given locally compact separable Hausdorff space and let m be a positive Radon measure on X such that $\operatorname{supp}[m] = X$. Let $C_0(X)$ be the space of all continuous

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functions with compact support on X. The Beurling and Deny formula states that any regular Dirichlet form $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ on $L^2(X,m)$ can be expressed as

$$\tilde{\mathcal{E}}(u,v) = \tilde{\mathcal{E}}^c(u,v) + \int_X uvk(dx) + \iint_{X\times X-d} (u(x)-u(y))((v(x)-v(y))\tilde{j}(dx,dy)$$

for all $u,v \in D(\tilde{\mathcal{E}}) \cap C_0(X)$, the symmetric Dirichlet form $\tilde{\mathcal{E}}^c$, the symmetric measure $\tilde{j}(dx,dy)$ and the measure $k(dx)$ occuring in the formula being uniquely determined by $\tilde{\mathcal{E}}$. These terms are respectively called the *diffusion part*, the *jumping* measure and the *killing* measure of $\tilde{\mathcal{E}}$.

A particular class of regular Dirichlet forms has been recently developed by many authors – it is the family of *diffusion* forms. A regular Dirichlet form is called a *diffusion* if both its killing and jumping measures vanish. In this case the form $\tilde{\mathcal{E}} \equiv \tilde{\mathcal{E}}^c$ admits the integral representation

$$\tilde{\mathcal{E}}^c(u,v) = \int_X \tilde{\mu}(u,v)(dx) \qquad \forall\, u,v \in D(\tilde{\mathcal{E}}^c)$$

where $\tilde{\mu}(u,v)$ is a Radon measure-valued non-negative definite symmetric bilinear form on $D(\tilde{\mathcal{E}})$. Such a measure is called *local energy measure* of $\mathcal{E}$, or briefly *energy measure* of $\mathcal{E}$. The measure μ has indeed a “local character” in X, that is, the restriction of the measure μ to any open subset A of X only depends on the restriction to A of the functions in its argument. Moreover, μ satisfies the following properties:

Leibniz property: For every $u,v,w \in D(\tilde{\mathcal{E}}) \cap C_0(X)$,

$$\tilde{\mu}(uv,w) = v\tilde{\mu}(u,w) + u\tilde{\mu}(v,w)$$

in the sense of measure.

Schwarz inequality: If $\varphi \in L^2(X,\tilde{\mu}(u,u))$ and $\psi \in L^2(X,\tilde{\mu}(v,v))$, then $\varphi\psi$ is integrable with respect to the total variation of $|\tilde{\mu}(u,v)|$ and

$$\int_X |\varphi\psi|\,|\tilde{\mu}(u,v)|\,(dx) \le \left(\int_X \varphi^2\tilde{\mu}(u,u)\,(dx)\right)^{\frac{1}{2}} \left(\int_X \psi^2\tilde{\mu}(v,v)\,(dx)\right)^{\frac{1}{2}}$$

for every $u,v \in D(\tilde{\mathcal{E}}) \cap C_0(X)$.

Chain rule: Let $u \in D(\tilde{\mathcal{E}}) \cap C_0(X)$ and let $f \in C^1(\mathbb{R})$, with bounded derivative and $f(0)=0$. Then $f(u)$ belongs to $D(\tilde{\mathcal{E}}) \cap C_0(X)$ and for every $v \in D(\tilde{\mathcal{E}}) \cap C_0(X)$

$$\tilde{\mu}(f(u),v) = f'(u)\tilde{\mu}(u,v)$$

in the sense of measure.

Truncation rule: For every $u,v \in D(\tilde{\mathcal{E}}) \cap C_0(X)$ we have

$$\tilde{\mu}(u^+,v) = \chi_{\{u>0\}}\tilde{\mu}(u,v)$$

where χ_A denotes the characteristic function of the set A.

The aim of this paper is to show that these results hold without any symmetry assumption on the form $\tilde{\mathcal{E}}$.

Actually, in Section 2 we prove that any regular non-symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(X, m)$ can be expressed for $u, v \in D(\mathcal{E}) \cap C_0(X)$ as

$$\begin{aligned}\mathcal{E}(u,v) = \mathcal{E}^c(u,v) + \int_X uvk(dx) \\ + \iint_{X\times X-d} \Big[u(x)(v(x)-v(y)) + v(y)(u(y)-u(x))\Big] j(dx,dy),\end{aligned}$$

the non-symmetric non-negative definite form $\mathcal{E}^c$, the measures $j(dx, dy)$ and $k(dx)$ occuring in the formula being uniquely determined by $\mathcal{E}$. We point out that the measure k is the Killing measure associated with the symmetric part $\tilde{\mathcal{E}}$ of $\mathcal{E}$. The form $\mathcal{E}^c$ and the measure $j(dx, dy)$ are not symmetric, that is $\mathcal{E}^c(u, v) \neq \mathcal{E}^c(v, u)$ and $j(dx, dy) \neq j(dy, dx)$.

In Section 3 we prove that $\mathcal{E}^c$ admits the integral representation

$$\mathcal{E}^c(u,v) = \int_X \mu(u,v)(dx) \qquad \forall\, u, v \in D(\mathcal{E}^c)$$

where $\mu(u, v)$ is a Radon measures-valued non-negative definite non-symmetric bilinear form on $D(\mathcal{E})$ and we show that such a measure enjoys the same properties as in the symmetric case. This fact allows us to extend many results on diffusion forms to the non-symmetric case, for example in [7] we extend to non-symmetric Dirichlet forms, the results obtained by Dal Maso and A. Garroni in [3] for second order non-symmetric elliptic operators.

We conclude the paper with an example in Section 4.

1. Preliminaries on non-symmetric Dirichlet forms

We shall consider the Hilbert space $H = L^2(X, m)$, where X is a given locally compact separable Hausdorff space and m a positive Radon measure on X such that $\operatorname{supp}[m] = X$, i.e. m is a non-negative Borel measure on X which is finite on compact sets and strictly positive on each non-empty open set. By $(u, v) = \int_X uv\, dm$ we denote the inner product of H, and by $||\cdot||$ the related norm. Let $D(\mathcal{E})$ be a linear subspace of H and let $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ be a bilinear map.

Definition 1.1. A pair $(\mathcal{E}, D(\mathcal{E}))$ is called a *closed form* (on H) if $D(\mathcal{E})$ is a dense linear subspace of H and $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ is a non-negative definite bilinear form which is symmetric and $D(\mathcal{E})$ is complete with respect to the intrinsic norm

$$||u||_{D(\mathcal{E})} = \mathcal{E}_1(u,u)^{\frac{1}{2}} = (\mathcal{E}(u,u) + (u,u))^{\frac{1}{2}}.$$

To give an analogous definition for non symmetric forms, one has to define the

– *symmetric part* of $\mathcal{E}$ as $\tilde{\mathcal{E}}(u,v) = \frac{1}{2}(\mathcal{E}(u,v) + \mathcal{E}(v,u))$

– *antisymmetric part* of $\mathcal{E}$ as $\check{\mathcal{E}}(u,v) = \frac{1}{2}(\mathcal{E}(u,v) - \mathcal{E}(v,u))$.

Clearly, $(\mathcal{E}(u,v), D(\mathcal{E})) = (\tilde{\mathcal{E}}(u,v) + \check{\mathcal{E}}(u,v), D(\mathcal{E}))$.

Then we have the following

Definition 1.2. A pair $(\mathcal{E}, D(\mathcal{E}))$ is called a *coercive closed form* (on H) if $D(\mathcal{E})$ is a dense linear subspace of H and $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ is a bilinear form such that the following conditions hold:

i) Its symmetric part $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ is a closed form on H.

ii) $(\mathcal{E}, D(\mathcal{E}))$ satisfies the following "*weak sector condition*": there exists a constant $K > 0$ (called *continuity constant*) such that

$$|\mathcal{E}_1(u,v)| \leq K\mathcal{E}_1(u,u)^{\frac{1}{2}}\mathcal{E}_1(v,v)^{\frac{1}{2}} \qquad \forall\, u,v \in D(\mathcal{E}),$$

i.e. $(\mathcal{E}_1, D(\mathcal{E}))$ is continuous with respect to the intrinsic norm on $D(\mathcal{E})$.

Let us recall now the main definitions and properties of coercive closed forms which are necessary in the following sections (see [5] for furthers details), while we refer to [4, 8] for properties of symmetric closed form.

Definition 1.3. A coercive closed form $(\mathcal{E}, D(\mathcal{E}))$ (on H) satisfies the *strong sector condition* if there exists a constant $K \in (0, \infty)$ such that

$$|\mathcal{E}(u,v)| \leq K\mathcal{E}(u,u)^{\frac{1}{2}}\mathcal{E}(v,v)^{\frac{1}{2}} \qquad \forall\, u,v \in D(\mathcal{E}). \tag{1.1}$$

Definition 1.4. A family $\{G_\beta\}_{\beta>0}$ of linear operators on H with $D(G_\beta) = H$ for all $\beta \in (0,\infty)$ is called a *strongly continuous contraction resolvent* on H if

i) βG_β is a contraction on H for all $\beta > 0$.

ii) $G_\alpha - G_\beta = (\beta - \alpha)G_\alpha G_\beta$ for all $\alpha, \beta > 0$.

iii) $\lim_{\beta\to\infty} \beta G_\beta u = u$ for all $u \in H$.

For any given coercive closed form $(\mathcal{E}, D(\mathcal{E}))$, always there exists a strongly continuous contraction resolvent $\{G_\beta\}_{\beta>0}$ associated with $(\mathcal{E}, D(\mathcal{E}))$. It is defined as follows: for all $u \in H$, $G_\beta u$ is the unique element in H such that

$$\mathcal{E}_\beta(G_\beta u, v) := \mathcal{E}(G_\beta u, v) + \beta(G_\beta u, v) = (u,v) \qquad \forall\, v \in D(\mathcal{E}).$$

Moreover, the range $R(G_\beta)$ of G_β is contained in $D(\mathcal{E})$. Let us also observe that resolvent is not a self-adjoint operator actually, naming *coresolvent* the adjoint operator of G_β, we have that, for any given coercive closed form $(\mathcal{E}, D(\mathcal{E}))$, always there exists a strongly continuous contraction coresolvent $\{\hat{G}_\beta\}_{\beta>0}$ associated with $(\mathcal{E}, D(\mathcal{E}))$, and it results that, for all $u \in H$, $\hat{G}_\beta u$ is the unique element in H such that

$$\left.\begin{aligned} \mathcal{E}_\beta(v, \hat{G}_\beta u) &:= \mathcal{E}(v, \hat{G}_\beta u) + \beta(v, \hat{G}_\beta u) = (u,v) \\ \mathcal{E}_\beta(v, \hat{G}_\beta u) &= \mathcal{E}_\beta(G_\beta u, v) \end{aligned}\right\} \qquad \forall\, v \in D(\mathcal{E}).$$

Finally, let us recall that, if G_β is the resolvent associated with the coercive closed form $(\mathcal{E}, D(\mathcal{E}))$, then we can define, for $\beta > 0$,

$$\mathcal{E}^{(\beta)}(u,v) := \beta(u - \beta G_\beta u, v) \qquad \forall\, u,v \in H. \tag{1.2}$$

Thus

$$\mathcal{E}^{(\beta)}(u,v) = \mathcal{E}(\beta G_\beta u, v) \qquad \forall\, u \in H, v \in D(\mathcal{E})$$

and

$$\lim_{\beta\to\infty} \mathcal{E}^{(\beta)}(u,v) = \mathcal{E}(u,v) \qquad \forall\, u,v \in D(\mathcal{E}). \tag{1.3}$$

For further details see [5].

We conclude this section giving the definition of *non-symmetric Dirichlet form.* As usually, we put for all $u, v : X \to \mathbb{R}$

$$u \vee v := \sup(u,v), \quad u \wedge v := \inf(u,v), \quad u^+ := u \vee 0, \quad u^- := -(u \wedge 0).$$

Definition 1.5. A coercive closed form $(\mathcal{E}, D(\mathcal{E}))$ is called a *non-symmetric Dirichlet form* (on H) if, for all $u \in D(\mathcal{E})$, it results

$$u^+ \wedge 1 \in D(\mathcal{E}) \qquad \text{and} \qquad \left.\begin{aligned} &\mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0 \\ &\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0 \end{aligned}\right\}. \tag{1.4}$$

Proposition 1.6. *A coercive closed form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(X,m)$ is a non-symmetric Dirichlet form if and only if the Markovian property*

$$\left.\begin{aligned} &\textit{for each } \varepsilon > 0\ \exists \textit{ a function } \varphi_\varepsilon : \mathbb{R} \to [-\varepsilon, 1+\varepsilon] \textit{ such that} \\ &\varphi_\varepsilon(t) = t \quad (t \in [0,1]) \textit{ and } 0 \leq \varphi_\varepsilon(t_2) - \varphi_\varepsilon(t_1) \leq t_2 - t_1 \textit{ if } t_1 \leq t_2 \end{aligned}\right\} \tag{1.5}$$

and, moreover,

$$\left.\begin{aligned} &\varphi_\varepsilon \circ u \in D(\mathcal{E}) \\ &\liminf_{\varepsilon\to 0} \mathcal{E}(\varphi_\varepsilon \circ u, u - \varphi_\varepsilon \circ u) \geq 0 \\ &\liminf_{\varepsilon\to 0} \mathcal{E}(u - \varphi_\varepsilon \circ u, \varphi_\varepsilon \circ u) \geq 0 \end{aligned}\right\} \qquad \forall\, u \in D(\mathcal{E})$$

holds.

Proof. See [5: Proposition I.4.7] ∎

Theorem 1.7. *A coercive closed form $(\mathcal{E}, D(\mathcal{E}))$ is a non-symmetric Dirichlet form (on H) if and only if the resolvent $\{\beta G_\beta\}$ associated with the form is Markovian (i.e. $\{\beta G_\beta\}$ is bounded and $0 \leq \beta G_\beta u \leq 1$ m-a.e., whenever $u \in L^2(X,m), 0 \leq u \leq 1$ m-a.e.).*

Proof. See [5: Theorem 1.4.4] ∎

In the following we study some particular non-symmetric Dirichlet forms, named *regular* non-symmetric Dirichlet forms. From now on we denote by $C_0(X)$ the space of all continuous functions with compact support in X.

Definition 1.8. *A pair $(\mathcal{E}, D(\mathcal{E}))$ is called a regular form if it possesses a core, that is a subset F of $D(\mathcal{E}) \cap C_0(X)$, which is dense in $C_0(X)$ with the uniform norm and in $D(\mathcal{E})$ with the intrinsic norm.*

Proposition 1.9. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a regular non-symmetric Dirichlet form on $H = L^2(X, m)$. Let K be a compact set and let U be an open set of X with $K \subset U$. Then there exists a function Φ_K such that*

$$\left.\begin{aligned} &\Phi_K \in D(\mathcal{E}) \cap C_0(X) \\ &\Phi_K = 1 \ \text{on } K \\ &\operatorname{supp}[\Phi_K] \subset U. \end{aligned}\right\}$$

Proof. Since X is a locally compact Hausdorff space, by the Uryshon lemma, there exists a function $f \in C_0(X)$ such that $f = 1$ on K and $\operatorname{supp}[f] \subset U$. Let us consider $2f \in C_0(X)$. By regularity of $\mathcal{E}$, $D(\mathcal{E}) \cap C_0(X)$ is a dense subspace of $C_0(X)$. Then there exists a sequence $\{\psi_n\} \in D(\mathcal{E}) \cap C_0(X)$ such that

$$|\psi_n(x) - 2f(x)| < ||\psi_n - 2f||_\infty < \frac{1}{n} \qquad \forall\, n \in \mathbb{N},\ x \in X.$$

For any fixed $\bar{n}$, sufficiently large, it results that $\psi_{\bar{n}} - \frac{1}{\bar{n}} \geq 1$ on K and $\psi := \psi_{\bar{n}} - \frac{1}{\bar{n}} \leq 0$ on $X - U$. By (1.4) it follows that $\Phi_K := \psi^+ \wedge 1$ is the required function ∎

2. Beurling-Deny formula in the non-symmetric case

We are concerned now with a general representation theorem on regular non-symmetric Dirichlet forms, firstly presented by Beurling and Deny in the symmetric case in their famous paper [2]. Let us start with two fundamental lemmas.

Lemma 2.1. *The following statements are true:*

(i) *Let S be a positive linear operator on $L^2(X, m)$ (i.e. $Su \geq 0$ m-a.e. whenever $u \in L^2(X, m), u \geq 0$ m-a.e.). Then there exists uniquely a positive Radon measure σ on the product space $X \times X$ such that, for any pair of Borel functions $u, v \in L^2(X, m)$,*

$$(Su, v) = \iint_{X \times X} u(x)v(y)\sigma(dx, dy). \tag{2.1}$$

(ii) *Let $\hat{S}$ the adjoint operator of a positive linear operator S on $L^2(X, m)$ and let $\hat{\sigma}$ be the unique positive Radon measure on the product space $X \times X$ associated with $\hat{S}$ by* (i). *Then for any pair of Borel functions $u, v \in L^2(X, m)$ one has*

$$(Su, v) = \iint_{X \times X} u(y)v(x)\hat{\sigma}(dx, dy). \tag{2.2}$$

(iii) *Let $\mathcal{B}(X)$ be the family of all Borel subset of X. Then*

$$\sigma(E \times F) = \hat{\sigma}(F \times E) \qquad \forall\, E, F \in \mathcal{B}(X). \tag{2.3}$$

(iv) *If, in addition, S is Markovian (i.e. S is bounded and $0 \le Su \le 1$ m-a.e. whenever $u \in L^2(X,m), 0 \le u \le 1$ m-a.e.), one has*

$$\sigma(X \times E) \le m(E) \qquad \forall E \in \mathcal{B}(X) \tag{2.4}$$

and, if $\hat{S}$ is Markovian, one has

$$\sigma(E \times X) \le m(E) \qquad \forall\, E \in \mathcal{B}(X). \tag{2.5}$$

Proof. Firstly we observe that, if S is a symmetric operator, the proof of the lemma is given in [4: Lemma 1.4.1] and it results that σ is a positive symmetric Radon measure.

Let us start to prove that (2.4) follows by (2.1). Let assume, at first, that $m(X) < \infty$ and let be χ_E the characteristic function of the set $E \in \mathcal{B}(X)$. Then

$$\begin{aligned}
\sigma(X \times E) &= \iint_{X \times E} \sigma(dx, dy) \\
&= \iint_{X \times X} \chi_X(x)\chi_E(y)\sigma(dx, dy) \\
&= (S\chi_X, \chi_E) \\
&\le \int_X \chi_E(x)\, dm \\
&= m(E)
\end{aligned}$$

since S is Markovian. Let now be $m(X) = \infty$. By the assumptions on X and m, m is σ-finite, actually let $\{X_n\}$ be a sequence of compact sets such that $X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$ and $\lim_{n\to\infty} X_n = \cup_{n=1}^{\infty} X_n = X$. As before one easily check that

$$\sigma(X_n \times E) \le \int_{X_n} \chi_E(x)\, dm.$$

Letting the limit as $n \to \infty$, we obtain the thesis. Analogously, if $\hat{S}$ is Markovian, (2.2) implies $\hat{\sigma}(X \times E) \le m(E)$ then, by (2.3), (2.5) follows.

To prove statement (i), let us denote by $C_0(X \times X)$ the family of all continuous functions with compact support on $X \times X$. Let us denote by $\tilde{C}_0(X \times X)$ the set of functions $f \in C_0(X \times X)$ such that

$$f(x,y) = \sum_{i=1}^{l} u_i(x)v_i(y) \qquad \forall\, u_i, v_i \in C_0(X), (x,y) \in X \times X$$

and consider the functional

$$I(f) = \sum_{i=1}^{l} (Su_i, v_i) \qquad \forall\, u_i, v_i \in C_0(X).$$

By some analogous arguments to those used in the proof of [4: Lemma 1.4.1], we can show that if $f(x,y) \ge 0$, then $I(f) \ge 0$. Since $\tilde{C}_0(X \times X)$ is dense in $C_0(X \times X)$, we can extend I to a positive linear functional on $C_0(X \times X)$. Then there exists a unique positive Radon measure σ such that

$$I(f) = \iint_{X\times X} f(x,y)\sigma(dx,dy).$$

Choosing $f(x,y) = u(x)\,v(y)$ and $I(f) = (Su, v)$, we obtain

$$(Su,v) = \iint_{X\times X} u(x)v(y)\sigma(dx,dy).$$

Statement (ii) follows by statement (i). Indeed, by (i) there exists uniquely a positive Radon measure $\hat{\sigma}$ on the product space $X \times X$ such that, for any pair of Borel functions $u, v \in L^2(X, m)$, one has

$$(\hat{S}u,v) = \iint_{X\times X} u(x)v(y)\hat{\sigma}(dx,dy) = (v, \hat{S}u).$$

Then, interchanging the role of u and v, we obtain

$$(Su,v) = (u,\hat{S}v) = \iint_{X\times X} u(y)v(x)\hat{\sigma}(dx,dy).$$

It remains to prove statement (iii). Indeed, let $E, F \in \mathcal{B}(X)$. Then one has

$$\begin{aligned}\sigma(E\times F) &= \iint_{X\times X} \chi_E(x)\chi_F(y)\sigma(dx,dy)\\ &= (S\chi_E,\chi_F)\\ &= (\chi_E,\hat{S}\chi_F)\\ &= \iint_{X\times X} \chi_E(y)\chi_F(x)\hat{\sigma}(dx,dy)\\ &= \hat{\sigma}(F\times E)\end{aligned}$$

and the proof is complete ∎

Remark 2.2. Let us remark that, by Lemma 2.1/(iv), the measures $\sigma(\cdot \times X)$ and $\sigma(X\times\cdot)$ are absolutely continuous with respect to m and the Radon-Nikodym derivatives satisfy the relations

$$0 \le \frac{\sigma(dx\times X)}{dm} \le 1 \quad \text{and} \quad 0 \le \frac{\sigma(X\times dx)}{dm} \le 1 \qquad m\text{-a.e.} \tag{2.6}$$

Lemma 2.3. *Let $\{\beta G_\beta\}_{\beta>0}$ or $\{\beta\hat{G}_\beta\}_{\beta>0}$ be a Markovian resolvent or coresolvent in $L^2(X,m)$, respectively (see Theorem 1.7). Then there exist some positive Radon measures σ_β or $\hat{\sigma}_\beta$ for every $\beta > 0$ such that the approximating form $\mathcal{E}^{(\beta)}$ associated with G_β or $\widehat{G}_\beta$ for every $\beta > 0$ (see 1.6) has the representation form*

$$\mathcal{E}^{(\beta)}(u,v) = \beta\int_X uv(1 - s_\beta(x))dm + \beta\iint_{X\times X} u(x)(v(x) - v(y))\sigma_\beta(dx,dy) \tag{2.7}$$

where

$$s_\beta(x) = \frac{\sigma_\beta(dx \times X)}{dm} \tag{2.8}$$

or

$$\mathcal{E}^{(\beta)}(v,u) = \beta\int_X uv(1-\hat{s}_\beta(x))dm + \beta\iint_{X\times X} u(x)(v(x)-v(y))\hat{\sigma}_\beta(dx,dy) \tag{2.9}$$

where

$$\hat{s}_\beta(x) = \frac{\sigma_\beta(X \times dx)}{dm}, \tag{2.10}$$

respectively.

Proof. Firstly we prove that βG_β is a positive linear operator on $L^2(X,m)$. For every $u \in C_0(X)$, $u \geq 0$, let us put $v = \frac{u}{||u||_\infty}$. Since $0 \leq v \leq 1$ and βG_β is Markovian, then $0 \leq \beta G_\beta v \leq 1$. Hence $0 \leq \beta G_\beta u \leq ||u||_\infty$. Let $u \in L^2(X,m)$, $u \geq 0$. Then there exists a sequence $\{u_n\} \in C_0(X)$, $u_n \geq 0$, such that $u_n \to u$ in $L^2(X,m)$. By the continuity of βG_β we have $\beta G_\beta u_n \to \beta G_\beta u$ with $\beta G_\beta u_n \geq 0$, thus $\beta G_\beta u \geq 0$. Now, by Lemma 2.1 one has

$$(\beta G_\beta u, v) = \iint_{X\times X} u(x)v(y)\sigma_\beta(dx,dy).$$

Since, by (1.2), it results that

$$\mathcal{E}^{(\beta)}(u,v) = \beta(u - \beta G_\beta u, v) = \beta(u,v) - \beta(\beta G_\beta u, v),$$

then

$$\begin{aligned}\mathcal{E}^{(\beta)}(u,v) &= \beta\int_X u(x)v(x)\,dm - \beta\iint_{X\times X} u(x)v(y)\sigma_\beta(dx,dy)\\ &= \beta\int_X u(x)v(x)\,dm + \beta\iint_{X\times X} u(x)[v(x)-v(y)]\sigma_\beta(dx,dy)\\ &\quad - \beta\iint_{X\times X} u(x)v(x)\sigma_\beta(dx,dy).\end{aligned}$$

Let us observe now that, by the Fubini Theorem,

$$\iint_{X\times X} u(x)v(x)\sigma_\beta(dx,dy) = \int_X u(x)v(x)\sigma_\beta(dx\times X)$$

hence, by (2.8),

$$\begin{aligned}\mathcal{E}^{(\beta)}(u,v) &= \beta\int_X uv(dm - \sigma_\beta(dx\times X)) + \beta\iint_{X\times X} u(x)[v(x)-v(y)]\sigma_\beta(dx,dy)\\ &= \beta\int_X uv(1-s_\beta(x))dm + \beta\iint_{X\times X} u(x)[v(x)-v(y)]\sigma_\beta(dx,dy)\end{aligned}$$

and (2.7) is proved.

To prove (2.9) it is enough to start with

$$\mathcal{E}^{(\beta)}(v,u) = \beta(u - \beta\hat{G}_\beta u, v)$$

and to make the same arguments used to prove (2.7) ∎

Remark 2.4. Interchanging the role of u and v and with a change of variables x and y in (2.9) and (2.7), we obtain two further representation forms of $\mathcal{E}^{(\beta)}$:

$$\mathcal{E}^{(\beta)}(u,v) = \beta\int_X uv(1-\hat{s}_\beta(x))\,dm + \beta\iint_{X\times X} v(y)(u(y)-u(x))\sigma_\beta(dx,dy) \quad (2.11)$$

$$\mathcal{E}^{(\beta)}(v,u) = \beta\int_X uv(1-s_\beta(x))\,dm + \beta\iint_{X\times X} v(y)(u(y)-u(x))\hat{\sigma}_\beta(dx,dy). \quad (2.12)$$

Remark 2.5. Let $(\mathcal{E}, D(\mathcal{E}))$ be a non-symmetric Dirichlet form. Let $\{\beta G_\beta\}_{\beta>0}$ be the Markovian resolvent and let $\{\beta\hat{G}_\beta\}_{\beta>0}$ be the Markovian coresolvent associated with $\mathcal{E}$. Moreover, let $\{\beta\tilde{G}_\beta\}_{\beta>0}$ be the Markovian symmetric resolvent associated with the symmetric part $\tilde{\mathcal{E}}$ of $\mathcal{E}$. By simple considerations one has that $\tilde{G}_\beta = \frac{\hat{G}_\beta+G_\beta}{2}$, hence denoting by σ_β and $\hat{\sigma}_\beta$ the measures associated with G_β and $\hat{G}_\beta$, respectively, by Lemma 2.3, and by $\tilde{\sigma}_\beta$ the measure associated with $\tilde{G}_\beta$ by [4: Lemma 1.4.1], it results

$$\tilde{\sigma}_\beta(dx,dy) = \frac{\sigma_\beta(dx,dy)+\hat{\sigma}_\beta(dx,dy)}{2}. \quad (2.13)$$

Moreover, putting

$$\tilde{s}_\beta(x) = \frac{\tilde{\sigma}_\beta(dx\times X)}{dm} \quad (2.14)$$

by (2.8), (2.10) and (2.13) easily

$$\tilde{s}_\beta(x) = \frac{s_\beta(x)+\hat{s}_\beta(x)}{2} \quad (2.15)$$

follows. These considerations, (2.7), (2.9), (2.11), (2.12) yield

$$\begin{aligned}\tilde{\mathcal{E}}^{(\beta)}(u,v) &= \frac{1}{2}\big(\mathcal{E}^{(\beta)}(u,v)+\mathcal{E}^{(\beta)}(v,u)\big)\\ &= \beta\int_X uv(1-\tilde{s}_\beta(x))\,dm + \frac{\beta}{2}\iint_{X\times X}(u(x)-u(y))(v(x)-v(y))\tilde{\sigma}_\beta(dx,dy)\end{aligned}$$

according with [4: Formula (1.4.8)] for the symmetric case.

Now we are in a position to prove the Beurling-Deny formula for regular non-symmetric Dirichlet forms.

Theorem 2.6. *Any regular non-symmetric Dirichlet form on $L^2(X,m)$ can be expressed as*

$$\begin{aligned}\mathcal{E}(u,v) &= \mathcal{E}^c(u,v) + \int_X uvk(dx)\\ &\quad + \iint_{X\times X-d}\Big[u(x)(v(x)-v(y)) + v(y)(u(y)-u(x))\Big]j(dx,dy)\end{aligned} \quad (2.16)$$

for all $u,v\in D(\mathcal{E})\cap C_0(X)$. Here $\mathcal{E}^c$ is a non-symmetric non-negative definite form with domain $D(\mathcal{E}^c) = D(\mathcal{E})\cap C_0(X)$, j is a positive Radon measure on the product space $X\times X$ off the diagonal d, and k is a positive Radon measure on X. Such elements $\mathcal{E}^c$, j and k are uniquely determined by $\mathcal{E}$.

In order to prove this result we give the proof of two further theorems.

Theorem 2.7. *Any regular non-symmetric Dirichlet form on $L^2(X,m)$ can be expressed as*

$$\mathcal{E}(u,v) = \mathcal{G}^c(u,v) + \int_X uvg(dx) + 2\iint_{X\times X-d} u(x)(v(x)-v(y))j(dx,dy) \tag{2.17}$$

for all $u,v \in D(\mathcal{E})\cap C_0(X)$. Here $\mathcal{G}^c$ is a bilinear form with domain $D(\mathcal{G}^c) = D(\mathcal{E})\cap C_0(X)$, such that $\mathcal{G}^c(u,v)=0$ for every $u,v\in D(\mathcal{G}^c)$ with disjoint supports and

$$\mathcal{G}^c(u,v) = 0 \qquad \forall\, u \in D(\mathcal{G}^c), v \in \Theta(u)$$

where $\Theta(u) = \{w \in D(\mathcal{G}^c) : w$ is constant on a neighbourhood of $\operatorname{supp}[u]\}$, j is a positive Radon measure on the product space $X\times X$ off the diagonal d and g is a positive Radon measure on X. Such elements $\mathcal{G}^c, j$ and g are uniquely determined by $\mathcal{E}$.

Proof. Let us suppose that (2.17) holds. The proof of the uniqueness of j,g and $\mathcal{G}^c$ is the same of the symmetric case (see [4: Theorem 2.2.1]). In particular, it results that there exists uniquely j such that

$$\mathcal{E}(u,v) = -2\iint_{X\times X-d} u(x)v(y)j(dx,dy) \tag{2.18}$$

for any $u,v\in D(\mathcal{E})\cap C_0(X)$ with disjoint supports, and there exists uniquely g such that

$$\int_X u(x)g(dx) = \mathcal{E}(u,v) - 2\iint_{X\times X-d} u(x)(v(x)-v(y))j(dx,dy)$$

for any $u,v\in D(\mathcal{E})\cap C_0(X)$, with v identically equal to 1 on a neighbourhood of $\operatorname{supp}[u]$.

Let us prove now the existence of the measure j. By (2.7) and by (1.3), one has

$$\mathcal{E}^{(\beta)}(u,v) = -\beta\iint_{X\times X} u(x)v(y)\sigma_\beta(dx,dy) \to \mathcal{E}(u,v)$$

as $\beta\to\infty$ if $u,v\in D(\mathcal{E})\cap C_0(X)$ and $\operatorname{supp}[u]\cap\operatorname{supp}[v]=\emptyset$. Let K_1,K_2 be arbitrary compacts subset of X such that $K_1\cap K_2=\emptyset$ and let u,v be a pair of functions such that

$$\begin{aligned} 0 \le\, & u\in D(\mathcal{E})\cap C_0(X),\ u=1 \text{ on } K_1 \text{ and } u=0 \text{ on } K_2\\ 0 \le\, & v\in D(\mathcal{E})\cap C_0(X),\ v=0 \text{ on } K_1 \text{ and } v=1 \text{ on } K_2. \end{aligned}$$

It results that

$$\begin{aligned} \beta\sigma_\beta(K_1\times K_2) &= \iint_{K_1\times K_2} \beta\sigma_\beta(dx,dy)\\ &\le \iint_{X\times X} \beta u(x)v(y)\sigma_\beta(dx,dy)\\ &= -\mathcal{E}^{(\beta)}(u,v)\\ &\to -\mathcal{E}(u,v)\\ &< \infty \end{aligned}$$

as $\beta \to \infty$ by (1.3). Then the family of measures $\{\beta\sigma_\beta\}_{\beta>0}$ is uniformly bounded on each compact subset of $X \times X - d$, hence a sequence $\beta_n\sigma_{\beta_n}$ converges as $\beta_n \to \infty$ vaguely on $X \times X - d$ to a positive Radon measure $\overline{j}$. Actually, if we define $j = \frac{\overline{j}}{2}$, we obtain that such a measure j satisfies (2.18), so we get that there exists a unique Radon measure j on $X \times X - d$ such that

$$\beta\sigma_\beta \to 2j \qquad \text{vaguely on } X \times X - d \text{ as } \beta \to \infty. \tag{2.19}$$

Let us prove now the existence of the measure g. Let us fix a metric ρ on X which is compatible with the given topology and choose a sequence of relatively compact open sets $\{G_l\}$ increasing to X and a sequence of numbers $\delta_l \to 0$ such that

$$\Gamma_l = \big\{(x,y) \in G_l \times G_l : \rho(x,y) \geq \delta_l\big\}$$

is a continuous set with respect to the measure j for every l, that is $\lim_{l\to\infty} j(\Gamma_l) = j(\cup_{l=1}^{\infty}\Gamma_l)$. By (2.7), for any $u \in D(\mathcal{E}) \cap C_0(X)$ such that $\operatorname{supp}[u] \subset G_l$, one has

$$\begin{aligned}\mathcal{E}^{(\beta_n)}(u,u) = \beta_n &\int_{G_l} u^2(1 - s^l_{\beta_n}(x))\,dm \\ &+ \beta_n \iint_{G_l\times G_l} u(x)(u(x)-u(y))\sigma_{\beta_n}(dx,dy)\end{aligned} \tag{2.20}$$

where $s^l_{\beta_n}(x) = \frac{\sigma_{\beta_n}(dx\times G_l)}{dm}$ and $\{\beta_n\}$ is such that $\beta_n\sigma_{\beta_n} \to 2j$ vaguely. Moreover, by (2.6) and (2.20)

$$\begin{aligned}0 &\leq \beta_n \int_{G_l} u^2(1 - s^l_{\beta_n}(x))dm \\ &= \mathcal{E}^{(\beta_n)}(u,u) - \beta_n \iint_{G_l\times G_l - d} u(x)^2\sigma_{\beta_n}(dx,dy) \\ &\quad + \beta_n \iint_{G_l\times G_l - d} u(x)u(y)\sigma_{\beta_n}(dx,dy) \\ &\leq \mathcal{E}^{(\beta_n)}(u,u) + \beta_n \iint_{G_l\times G_l - d} u(x)u(y)\sigma_{\beta_n}(dx,dy).\end{aligned}$$

Then, by (2.19),

$$\begin{aligned}0 &\leq \limsup_{n\to\infty} \beta_n \int_{G_l} u^2(1 - s^l_{\beta_n}(x))dm \\ &\leq \mathcal{E}(u,u) + 2\iint_{G_l\times G_l - d} u(x)u(y)j(dx,dy) \\ &< \infty.\end{aligned}$$

This implies, in particular, that the family of measures $\{\beta_n(1 - s^l_{\beta_n}(x))dm\}_{\beta>0}$ is uniformly bounded on each compact subset of G_l. Hence an increasing sequence $\{\beta_n\}$, $\beta_n \to \infty$ and some positive Radon measures g_l on G_l exist in such a way that

$$\beta_n(1 - s^l_{\beta_n}(x))dm \to g_l \qquad \text{vaguely on } G_l \text{ as } \beta_n \to \infty\ \forall\, l. \tag{2.21}$$

In view of (2.19) and (2.21) we have that, for every l such that $\operatorname{supp}[u] \subset G_l$,

$$\begin{aligned}\mathcal{E}(u,v) = \lim_{\beta_n\to\infty} \beta_n \iint_{G_l\times G_l,\rho<\delta_l} u(x)(v(x)-v(y))\sigma_{\beta_n}(dx,dy) \\ + 2\iint_{\Gamma_l} u(x)(v(x)-v(y))j(dx,dy) + \int_{G_l} u(x)v(x)g_l(dx).\end{aligned} \tag{2.22}$$

Let us extend the measures g_l to X, by setting $g_l(E) = g_l(E\cap G_l)$. Then by (2.21), $\{g_l\}$ is non-increasing on each compact set. Let us denote the vague limit by g, i.e.

$$g_l \to g \qquad \text{vaguely on } X \text{ as } l\to\infty.$$

Letting $l\to\infty$ in (2.22), we get the expression (2.17) with

$$\mathcal{G}^c(u,v) = \lim_{l\to\infty}\lim_{\beta_n\to\infty} \beta_n \iint_{G_l\times G_l,\rho<\delta_l} u(x)(v(x)-v(y))\sigma_{\beta_n}(dx,dy). \tag{2.23}$$

By (2.23) it is easy to prove that $\mathcal{G}^c$ has the asked properties ∎

Theorem 2.8. *Any regular non-symmetric Dirichlet form on $L^2(X,m)$ can be expressed as*

$$\mathcal{E}(u,v) = \mathcal{F}^c(u,v) + \int_X uvh(dx) + 2\iint_{X\times X-d} v(y)(u(y)-u(x))j(dx,dy) \tag{2.24}$$

for all $u,v\in D(\mathcal{E})\cap C_0(X)$. Here $\mathcal{F}^c$ is a bilinear form with domain $D(\mathcal{F}^c) = D(\mathcal{E})\cap C_0(X)$, such that $\mathcal{F}^c(u,v)=0$ for every $u,v\in D(\mathcal{F}^c)$ with disjoint supports and

$$\mathcal{F}^c(u,v) = 0 \qquad \forall\, v\in D(\mathcal{F}^c), u\in\Theta(v)$$

where $\Theta(v) = \{w\in D(\mathcal{G}^c) : w$ is constant on a neighbourhood of $\operatorname{supp}[v]\}$, j is a positive Radon measure on the product space $X\times X$ off the diagonal d and h is a positive Radon measure on X. Such elements $\mathcal{F}^c, j$ and h are uniquely determined by $\mathcal{E}$.

Proof. The proof is the same of the previous one using (2.11) in place of (2.7). It is clear that

$$\lim_{\beta_n\to\infty}\lim_{l\to\infty} \beta_n(1-\hat{s}^l_{\beta_n}(x))dm = h \qquad \text{vaguely on } X \tag{2.25}$$

$$\mathcal{F}^c(u,v) = \lim_{l\to\infty}\lim_{\beta_n\to\infty} \beta_n \iint_{G_l\times G_l,\rho<\delta_l} v(y)(u(y)-u(x))\sigma_{\beta_n}(dx,dy) \tag{2.26}$$

and j is the same of Theorem 2.7 since is obtained as in (2.19) ∎

We are now in a position to prove Theorem 2.6.

Proof of Theorem 2.6. Expression (2.16) easily follows by Theorems 2.7 and 2.8 putting

$$\begin{aligned}\mathcal{E}^c(u,v) &= \frac{\mathcal{F}^c(u,v)+\mathcal{G}^c(u,v)}{2}\\ &= \lim_{l\to\infty}\lim_{\beta_n\to\infty}\frac{\beta_n}{2}\iint_{G_l\times G_l,\rho<\delta_l}\\ &\quad\times\Big[u(x)(v(x)-v(y))+v(y)(u(y)-u(x))\Big]\sigma_{\beta_n}(dx,dy)\end{aligned}$$

and

$$k(dx)=\frac{h(dx)+g(dx)}{2}.$$

$\mathcal{E}^c$ is a non-symmetric Dirichlet form. Indeed,

$$\mathcal{E}^c(u,u)=\lim_{l\to\infty}\lim_{\beta_n\to\infty}\frac{\beta_n}{2}\iint_{G_l\times G_l,\rho<\delta_l}(u(x)-u(y))^2\sigma_{\beta_n}(dx,dy)\geq 0$$

and the statement is proved ∎

Remark 2.9. Let us denote by $\hat{j}$ the positive Radon measure on the product space $X\times X-d$ such that $j(dx,dy)=\hat{j}(dy,dx)$. We can also obtain $\hat{j}$ as the vague limit of the sequence $\{\beta\hat{\sigma}_\beta\}_{\beta>0}$ that appear in (2.9). Moreover, using representation (2.9) in the proof of Theorem 2.6, we obtain

$$\begin{aligned}\mathcal{E}(v,u) &= \mathcal{E}^c(v,u)+\int_X uvk(dx)\\ &\quad+\iint_{X\times X-d}\Big[u(x)(v(x)-v(y))+v(y)(u(y)-u(x))\Big]\hat{j}(dx,dy)\end{aligned}$$

with

$$\begin{aligned}\mathcal{E}^c(v,u) &= \lim_{l\to\infty}\lim_{\beta_n\to\infty}\frac{\beta_n}{2}\iint_{G_l\times G_l,\rho<\delta_l}\\ &\quad\times\Big[u(x)(v(x)-v(y))+v(y)(u(y)-u(x))\Big]\hat{\sigma}_{\beta_n}(dx,dy).\end{aligned}$$

Moreover, if $(\mathcal{E},D(\mathcal{E}))$ is a symmetric Dirichlet form, (2.16) becames the well known Beurling-Deny formula.

3. Non-symmetric diffusion forms

By using the terminology of the symmetric case, we give the following

Definition 3.1. Let us call *non-symmetric diffusion form* a regular non-symmetric Dirichlet form when the positive Radon measures j and k in (2.16) vanish.

By Definition 3.1 and Theorem 2.6 it results that $\mathcal{E}\equiv\mathcal{E}^c$. Moreover, we observe that the positive Radon measure k vanishes if and only if the positive Radon measures g and h of (2.17) and (2.24) vanish; then $\mathcal{E}\equiv\mathcal{E}^c\equiv\mathcal{G}^c\equiv\mathcal{F}^c$.

Proposition 3.2. *A non-symmetric diffusion form $(\mathcal{E}, D(\mathcal{E}))$ is strongly local, i.e.*

$$\begin{aligned} \mathcal{E}(u,v) = 0 \qquad & \forall\, u \in D(\mathcal{E}), v \in \Theta(u) \\ \mathcal{E}(u,v) = 0 \qquad & \forall\, v \in D(\mathcal{E}), u \in \Theta(v) \end{aligned}$$

where $\Theta(w) = \{f \in D(\mathcal{E}) : f$ is constant on a neighbourhood of $supp[w]\}$.

Proof. The thesis easily follows by the identities $\mathcal{E} \equiv \mathcal{E}^c \equiv \mathcal{G}^c \equiv \mathcal{F}^c$ and by their integral representation forms (2.23) and (2.26) ∎

In this section we are interested to consider non-symmetric diffusion forms. Our aim is to prove that, as well as in the symmetric case, these forms admit an integral representation.

From now on, let us denote

$$A_l := \big\{(x,y) \in G_l \times G_l : \rho(x,y) < \delta_l\big\}.$$

Let $(\mathcal{E}, D(\mathcal{E}))$ be a non-symmetric diffusion form. Then $\mathcal{E} \equiv \mathcal{E}^c \equiv \mathcal{G}^c \equiv \mathcal{F}^c$. By (2.23) and (2.26),

$$\begin{aligned} \mathcal{E}^c(u,v) &= \lim_{l\to\infty} \lim_{\beta\to\infty} \Big[\tfrac{\beta}{2} \iint_{A_l} u(x)(v(x)-v(y))\sigma_\beta(dx,dy) \\ &\quad + \tfrac{\beta}{2} \iint_{A_l} u(y)(v(y)-v(x))\hat{\sigma}_\beta(dx,dy)\Big] \\ &= \lim_{l\to\infty} \lim_{\beta\to\infty} \Big[\tfrac{\beta}{2} \iint_{A_l} v(y)(u(y)-u(x))\sigma_\beta(dx,dy) \\ &\quad + \tfrac{\beta}{2} \iint_{A_l} v(x)(u(x)-u(y))\hat{\sigma}_\beta(dx,dy)\Big]. \end{aligned}$$

Then using (2.13), the symmetric part $\tilde{\mathcal{E}}^c$ of $\mathcal{E}^c$ has the expression

$$\tilde{\mathcal{E}}^c(u,v) = \lim_{l\to\infty} \lim_{\beta\to\infty} \tfrac{\beta}{2} \iint_{A_l} (u(x)-u(y))(v(x)-v(y))\tilde{\sigma}_\beta(dx,dy).$$

Moreover, the antisymmetric part $\breve{\mathcal{E}}^c$ of $\mathcal{E}^c$ has the expression

$$\begin{aligned} \breve{\mathcal{E}}^c(u,v) &= \mathcal{E}^c(u,v) - \tilde{\mathcal{E}}^c(u,v) \\ &= \lim_{l\to\infty} \lim_{\beta\to\infty} \tfrac{\beta}{4} \Big[\iint_{A_l} u(x)(v(x)-v(y))\sigma_\beta(dx,dy) \\ &\quad + \iint_{A_l} u(y)(v(y)-v(x))\hat{\sigma}_\beta(dx,dy) \end{aligned}$$

$$
\begin{aligned}
&- \iint_{A_l} u(x)(v(x)-v(y))\hat{\sigma}_\beta(dx,dy) \\
&- \iint_{A_l} u(y)(v(y)-v(x))\sigma_\beta(dx,dy)\Big] \\
&= \lim_{l\to\infty}\lim_{\beta\to\infty} \tfrac{\beta}{4}\Big[\iint_{A_l}(u(x)+u(y))(v(x)-v(y))\sigma_\beta(dx,dy) \\
&- \iint_{A_l}(u(x)+u(y))(v(x)-v(y))\hat{\sigma}_\beta(dx,dy)\Big] \\
&= \lim_{l\to\infty}\lim_{\beta\to\infty} \tfrac{\beta}{2}\iint_{A_l}(u(x)+u(y))(v(x)-v(y))\Big(\frac{\sigma_\beta-\hat{\sigma}_\beta}{2}\Big)(dx,dy).
\end{aligned}
$$

Let us define now

$$\breve{\sigma}_\beta(dx,dy) = \frac{\sigma_\beta-\hat{\sigma}_\beta}{2}(dx,dy).$$

Then it results that

$$\breve{\mathcal{E}}^c(u,v) = \lim_{l\to\infty}\lim_{\beta\to\infty} \tfrac{\beta}{2}\iint_{A_l}(u(x)+u(y))(v(x)-v(y))\breve{\sigma}_\beta(dx,dy). \tag{3.1}$$

Remark 3.3. Let $\mathcal{E}(u,v)$ be a non-symmetric diffusion form. Then by the identity $\mathcal{F}^c \equiv \mathcal{G}^c$ and by their integral representation forms, it follows that

$$\lim_{l\to\infty}\lim_{\beta\to\infty}\beta\iint_{A_l} u(x)v(x)\sigma_\beta(dx,dy) = \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\iint_{A_l} u(y)v(y)\sigma_\beta(dx,dy).$$

Then, obviously, since $\tilde{\sigma}_\beta(dx,dy)=\tilde{\sigma}_\beta(dy,dx)$ one has

$$\lim_{l\to\infty}\lim_{\beta\to\infty}\beta\iint_{A_l} u(x)v(x)\tilde{\sigma}_\beta(dx,dy) = \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\iint_{A_l} u(y)v(y)\tilde{\sigma}_\beta(dx,dy)$$

and, since

$$\left.\begin{aligned}
\sigma_\beta(dx,dy) &= \tilde{\sigma}_\beta(dx,dy)+\breve{\sigma}_\beta(dx,dy) \\
\breve{\sigma}_\beta(dx,dy) &= -\breve{\sigma}_\beta(dy,dx)
\end{aligned}\right\}$$

it results that

$$
\begin{aligned}
\lim_{l\to\infty}\lim_{\beta\to\infty}\beta\iint_{A_l} u(x)v(x)\breve{\sigma}_\beta(dx,dy) &= \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\iint_{A_l} u(y)v(y)\breve{\sigma}_\beta(dx,dy) \\
&= -\lim_{l\to\infty}\lim_{\beta\to\infty}\beta\iint_{A_l} u(x)v(x)\breve{\sigma}_\beta(dx,dy).
\end{aligned}
$$

Then

$$
\begin{aligned}
\lim_{l\to\infty}\lim_{\beta\to\infty}\beta\iint_{A_l} u(x)v(x)\breve{\sigma}_\beta(dx,dy) &= \lim_{l\to\infty}\lim_{\beta\to\infty}\beta\iint_{A_l} u(y)v(y)\breve{\sigma}_\beta(dx,dy) \\
&= 0.
\end{aligned}
$$

With these considerations there follows easily that the symmetric and antisymmetric part of a non-symmetric diffusion form are strongly local forms. Moreover, it is easy to check that $\breve{\mathcal{E}}^c(u,v) = -\breve{\mathcal{E}}^c(v,u)$ hence (3.1) is a "good" representation.

Theorem 3.4. *Let $\mathcal{E}^c(u,v)$ be a non-symmetric diffusion form and let $\check{\mathcal{E}}^c(u,v)$ be its antisymmetric part. There exists a signed Radon measure $\check{\mu}(u,v)$ such that*

$$\check{\mathcal{E}}^c(u,v) = \int_X \check{\mu}(u,v)(dx) \qquad \forall\, u,v \in D(\mathcal{E}^c). \tag{3.2}$$

Proof. Let us consider, for any fixed pair of functions $u,v \in D(\mathcal{E}^c) \cap C_0(X)$, the linear functional

$$\langle \Psi(u,v), \varphi \rangle := \lim_{l\to\infty} \lim_{\beta\to\infty} \frac{\beta}{2} \iint_{A_l} \varphi(x)(u(x)+u(y))(v(x)-v(y))\check{\sigma}_\beta(dx,dy) \tag{3.3}$$

on $C_0(X)$. If $\varphi \equiv 1$ on $\mathrm{supp}\,[u] \cap \mathrm{supp}\,[v]$, the functional $\Psi(u,v)$ is well defined by (3.1). Moreover, by simple calculation using Cauchy criterion, the functional $\Psi(u,v)$ is well defined for all $\varphi \in C_0(X)$ and is bounded in $C_0(X)^*$ (the dual space of $C_0(X)$), hence by the Riesz representation theorem, for any fixed $u,v \in D(\mathcal{E}^c) \cap C_0(X)$ there exists a unique signed bounded Radon measure $\check{\mu}(u,v)$ such that

$$\langle \Psi(u,v), \varphi \rangle = \int_X \varphi(x)\check{\mu}(u,v)(dx) \qquad \forall\, \varphi \in C_0(X). \tag{3.4}$$

Choosing now $\varphi \equiv 1$ on $\mathrm{supp}\,[u] \cap \mathrm{supp}\,[v]$, by (3.1) $\check{\mathcal{E}}^c(u,v) = \int_X \check{\mu}(u,v)(dx)$ ∎

At this point let us define $\mu(u,v) := \tilde{\mu}(u,v) + \check{\mu}(u,v)$, where $\tilde{\mu}(u,v)$ is the *energy* measure associated with the symmetric form $\tilde{\mathcal{E}}^c$ (see [6: Proposition 1.4.1]). Hence we have proved the following

Theorem 3.5. *Let $\mathcal{E}^c(u,v)$ be a non-symmetric diffusion form. Then there exists a signed bounded Radon measure $\mu(u,v)$ such that*

$$\mathcal{E}^c(u,v) = \int_X \mu(u,v)(dx). \tag{3.5}$$

Such a Radon measure $\mu(u,v)$ enjoys some important functional properties that will be describe below. The following proposition shows one of the most important properties of μ: its local character.

Proposition 3.6. *Let $\mathcal{E}(u,v)$ be a non-symmetric diffusion form and $\mu(u,v)$ the measure associated with $\mathcal{E}$ by* (3.5)*. Let $u_1,u_2,v_1,v_2 \in D(\mathcal{E}) \cap C_0(X)$ such that $u_1 = u_2$ and $v_1 = v_2$ m-a.e. on A open subset of X. Then it results*

$$\chi_A(x)\mu(u_1,v_1) = \chi_A(x)\mu(u_2,v_2) \qquad \text{on } X.$$

Proof. The proof in the symmetric case is given in [6: Proposition 1.5.2] and in [4: Lemma 5.4.6]. Firstly let us observe that, by (3.3) and by the antisymmetry of the measure $\check{\sigma}_\beta(u,v)$, if $v = 0$ on A or $u = 0$ on A, then

$$\int_X \chi_A(x)\check{\mu}(u,v) = 0.$$

Then, since

$$\begin{aligned} &\left| \int_X \chi_A(x)\check{\mu}(u_1,v_1) - \chi_A(x)\check{\mu}(u_2,v_2) \right| \\ &\qquad \le \left| \int_X \chi_A(x)\check{\mu}(u_1-u_2,v_1) \right| + \left| \int_X \chi_A(x)\check{\mu}(u_2,v_1-v_2) \right| \end{aligned}$$

the thesis easily follows ∎

The previous proposition states that for any open subset A of X, the restriction $\mu(u,v)$ to A depends only on the restriction of u and v to A. This allows us to define the domain of the form restricted to A, denoted by $D_0(\mathcal{E},A)$ as the closure of $D(\mathcal{E})\cap C_0(A)$ in $D(\mathcal{E})$.

Lemma 3.7. *Let Ω be a relatively compact subset of X and let A be an open subset of Ω. Let $(\mathcal{E},D_0(\mathcal{E},\Omega))$ be a regular non-symmetric Dirichlet form on $L^2(X,m)$. Then there exists a sequence $\{\varphi_n\}\in D(\mathcal{E})\cap C_0(\Omega)$ such that $\varphi_n\geq 0$ for any $n\in\mathbb{N}$ and $\varphi_n\nearrow\chi_A$ in Ω.*

Proof. Let us fix a metric ρ on X which is compatible with the given topology and let us consider

$$A_n := \left\{y\in A:\ \rho(y,\partial\Omega)\geq\frac{1}{n}\right\}$$

where $\partial\Omega$ denotes the boundary of Ω. For any $n\in\mathbb{N}$, A_n is closed and $A_n\subset\Omega$, so A_n is compact. By Proposition 1.9 there exists a sequence $\Phi_n\in D(\mathcal{E})\cap C_0(\Omega)$ such that, for any $n\in\mathbb{N}$, $\Phi_n=1$ on A_n and $\operatorname{supp}[\Phi_n]\subset A$. Let us define $\varphi_n:=\varphi_{n-1}\vee\Phi_n$ and $\varphi_0:=\Phi_0$. Then $\varphi_n\in D(\mathcal{E})\cap C_0(\Omega)$, $\varphi_n=1$ on A_n, $\varphi_n\geq\varphi_{n-1}$ and $\varphi_n\nearrow\chi_A$. Indeed, for all $x\in A$, $\rho(x,\partial\Omega)\geq\rho(x,\partial A)=\delta>0$, hence $x\in A_n$ for each $n>\delta^{-1}$ and $\varphi_n(x)=1\to 1$. Moreover, for all $x\notin A$, $\varphi_n(x)=0\to 0$, then $\varphi_n(x)\to\chi_A(x)$ ∎

Proposition 3.8. *Let $\mathcal{E}(u,v)$ be a non-symmetric diffusion form and $\mu(u,v)$ the measure associated with $\mathcal{E}$ by (3.5). For every $u,v\in D(\mathcal{E})\cap C_0(X)$, it results that*

$$\breve{\mu}(u,v)=-\breve{\mu}(v,u)$$

in the sense of measure, that is

$$\int_E\breve{\mu}(u,v)(dx)=-\int_E\breve{\mu}(v,u)(dx)$$

for every Borel set E of X.

Proof. *Step I.* Let Ω be an open set such that $\operatorname{supp}[u]\cap\operatorname{supp}[v]\subset\overline{\Omega}$. By Lemma 3.7, for any open subset A of Ω there exists a sequence $\{\varphi_n\}\in D(\mathcal{E})\cap C_0(\Omega)$ such that $\varphi_n\to\chi_A$. By (3.2) and by the Lebesgue convergence theorem, we have

$$\begin{aligned}0&=\lim_{n\to\infty}\lim_{l\to\infty}\lim_{\beta\to\infty}\iint_{A_l}\varphi_n(x)u(x)v(x)\breve{\sigma}_\beta(dx,dy)\\&=\lim_{l\to\infty}\lim_{\beta\to\infty}\iint_{A_l}\chi_A(x)u(x)v(x)\breve{\sigma}_\beta(dx,dy)\\&=\lim_{l\to\infty}\lim_{\beta\to\infty}\iint_{\{A\times G_l,\rho(x,y)<\delta_l\}}u(x)v(x)\breve{\sigma}_\beta(dx,dy)\\&=\lim_{l\to\infty}\lim_{\beta\to\infty}\iint_{\{A\times G_l,\rho(x,y)<\delta_l\}}u(y)v(y)\breve{\sigma}_\beta(dy,dx).\end{aligned}$$

Step II. We show now that for the sequence $\{\varphi_n\}$ as before, we have

$$\lim_{n\to\infty}\int_X\varphi_n(x)\breve{\mu}(u\varphi_n,v\varphi_n)-\varphi_n(x)\breve{\mu}(u,v)=0.$$

Indeed, for any $n \in \mathbb{N}$

$$\left| \int_X \varphi_n(x)[\breve{\mu}(u\varphi_n, v\varphi_n) - \breve{\mu}(u,v)] \right| \\ \leq \left| \int_X \varphi_n(x)\breve{\mu}(u\varphi_n - u, v\varphi_n) \right| + \left| \int_X \varphi_n(x)\breve{\mu}(u, v\varphi_n - v) \right|$$

and, by the Lebesgue convergence theorem,

$$\lim_{n\to\infty} \int_X \varphi_n(x)\breve{\mu}(u\varphi_n - u, v\varphi_n) \\ = \lim_{l\to\infty} \lim_{\beta\to\infty} \tfrac{\beta}{2} \iint_{A_l} \chi_A(x)\big(u\chi_A(x) - u(x) + u\chi_A(y) - u(y)\big) \\ \times \big(v\chi_A(x) - v\chi_A(y)\big)\breve{\sigma}_\beta(dx, dy) \\ = 0$$

since for l sufficiently large and for all $\beta > 0$

$$\tfrac{\beta}{2} \iint_{\{A\times G_l, \rho(x,y)<\delta_l\}} (u\chi_A(y) - u(y))(v(x) - v\chi_A(y))\breve{\sigma}_\beta(dx, dy) \\ = \tfrac{\beta}{2} \iint_{\{A\times G_l, \rho(x,y)<\delta_l\}} (u\chi_A(y) - u(y))(v\chi_A(y) - v(x))\breve{\sigma}_\beta(dy, dx) \\ = 0.$$

With the same arguments we have

$$\lim_{n\to\infty} \int_X \varphi_n(x)\breve{\mu}(u, v\varphi_n - v) = 0$$

and the thesis follows.

Step III. We show now that for the sequence $\{\varphi_n\}$ as before, we have

$$\lim_{n\to\infty} \int_X \varphi_n(x)\breve{\mu}(u\varphi_n, v\varphi_n) + \varphi_n(x)\breve{\mu}(v\varphi_n, u\varphi_n) = 0.$$

By the Lebesgue convergence theorem we have

$$\lim_{n\to\infty} \int_X \varphi_n(x)\breve{\mu}(u\varphi_n, v\varphi_n) + \varphi_n(x)\breve{\mu}(v\varphi_n, u\varphi_n) \\ = \lim_{l\to\infty} \lim_{\beta\to\infty} \beta \iint_{A_l} \chi_A(x)\Big(u(x)v(x)\chi_A(x) - u(y)v(y)\chi_A(y)\Big)\breve{\sigma}_\beta(dx, dy).$$

For l, β sufficiently large, we have

$$\beta \iint_{\{A\times G_l, \rho(x,y)<\delta_l\}} \big(u(x)v(x) - u(y)v(y)\chi_A(y)\big)\breve{\sigma}_\beta(dx, dy) \\ = \beta \iint_{\{A\times G_l, \rho(x,y)<\delta_l\}} \big(-u(x)v(x) + u(y)v(y)\big)\breve{\sigma}_\beta(dy, dx) \\ \to 0$$

by Step I.

Step IV. By Steps II and III and the Lebesgue convergence theorem, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty}\int_X \varphi_n(x)\big[\breve\mu(u\varphi_n, v\varphi_n)+\breve\mu(v\varphi_n, u\varphi_n)\big] \\ &= \lim_{n\to\infty}\int_X \varphi_n(x)\big[\breve\mu(u,v)+\breve\mu(v,u)\big] \\ &= \int_A \breve\mu(u,v)+\breve\mu(v,u). \end{aligned}$$

We have so proved the thesis for all open subset A of Ω subset of X. By the regularity of measures $\breve\mu$ the thesis holds for every Borel subset E of $\overline{\Omega}$. To conclude, we just note that since $\operatorname{supp}[u]\cap\operatorname{supp}[v]\subset\overline{\Omega}$ we have for every Borel subset E of X

$$\int_E \breve\mu(u,v)=\int_{E\cap\overline{\Omega}}\breve\mu(u,v)=-\int_{E\cap\overline{\Omega}}\breve\mu(v,u)=-\int_E\breve\mu(v,u)$$

and the proposition is proved ∎

Remark 3.9. We point out that it is just the locality property of $\breve\mu$ which enables to state some general relations holding "in the sense of measure" as consequence of the suitable analogous relations involving the measure of the all space.

Proposition 3.10 (Leibniz property). *Let $\mathcal{E}$ be a non-symmetric diffusion form and $\mu(u,v)$ the measure associated with $\mathcal{E}$ by* (3.5). *Then for any $u,v,w\in D(\mathcal{E})\cap C_0(X)$ the relations*

$$\left.\begin{aligned} \mu(uv,w) &= v\mu(u,w)+u\mu(v,w) \\ \mu(w,uv) &= v\mu(w,u)+u\mu(w,v) \end{aligned}\right\} \tag{3.6}$$

hold in the sense of measure.

Proof. Since the Leibniz property holds for the symmetric part $\tilde\mu(u,v)$ (see [4: Lemma 5.4.2]), it is enough to prove (3.6) for the measure $\breve\mu(u,v)$. Moreover, by Proposition 3.8 and Remark 3.9, we have just to prove that

$$\int_X\breve\mu(w,uv)(dx)=\int_X u(x)\breve\mu(w,v)(dx)+\int_X v(x)\breve\mu(w,u)(dx).$$

Actually, by (3.1) and (3.2) one has

$$\begin{aligned} &\int_X\breve\mu(w,uv)(dx) \\ &\quad= \lim_{l\to\infty}\lim_{\beta\to\infty}\tfrac{\beta}{2}\iint_{A_l}\big(w(x)+w(y)\big)\big(u(x)v(x)-u(y)v(y)\big)\breve\sigma_\beta(dx,dy) \\ &\quad= \lim_{l\to\infty}\lim_{\beta\to\infty}\tfrac{\beta}{2}\iint_{A_l}u(x)(w(x)+w(y))(v(x)-v(y))\breve\sigma_\beta(dx,dy) \\ &\qquad+\tfrac{\beta}{2}\iint_{A_l}v(y)(w(x)+w(y))(u(x)-u(y))\breve\sigma_\beta(dx,dy) \\ &\quad= \int_X u(x)\breve\mu(w,v)+\int_X v(x)\breve\mu(w,u) \end{aligned}$$

where the last equality holds by (3.4) and (3.3) ∎

Proposition 3.11 (Schwarz inequality). *Let $\mathcal{E}$ be a non-symmetric diffusion form such that* (1.1) *holds and let $\mu(u,v)$ be the measure associated with $\mathcal{E}$ by* (3.5). *We suppose that there exist the two Radon-Nikodym derivatives*

$$\tilde{\mu}(u,v)(\cdot) = \frac{d\tilde{\mu}(u,v)}{dm} \in L^1(X,m) \tag{3.7}$$

$$\breve{\mu}(u,v)(\cdot) = \frac{d\breve{\mu}(u,v)}{dm} \in L^1(X,m) \tag{3.8}$$

for every $u,v \in D(\mathcal{E}) \cap C_0(X)$. If $\varphi \in L^2(X, \tilde{\mu}(u,u))$ and $\psi \in L^2(X, \tilde{\mu}(v,v))$, then $\varphi\psi$ is integrable with respect to the total variation of $|\mu(u,v)|$ and one has for every $u,v \in D(\mathcal{E}) \cap C_0(X)$

$$\int_X |\varphi\psi|\, |\mu(u,v)|\,(dx) \le K \left(\int_X |\varphi|^2 \tilde{\mu}(u,u)\,(dx)\right)^{\frac12} \left(\int_X |\psi|^2 \tilde{\mu}(v,v)\,(dx)\right)^{\frac12}$$

where K is the constant that appears in (1.1).

Proof. The proof of the proposition in the symmetric case is proved in [4: Lemma 5.4.3]. By the strong sector condition (1.3) it results that, for every $u,v \in D(\mathcal{E}) \cap C_0(X)$,

$$|\mathcal{E}(u,v)| \le \tfrac{K}{2}(\mathcal{E}(u,u) + \mathcal{E}(v,v)),$$

that is, by (3.2), (3.7) and (3.8),

$$\left|\int_X \mu(u,v)(x)dm\right| \le \tfrac{K}{2} \int_X \big[\tilde{\mu}(u,u)(x) + \tilde{\mu}(v,v)(x)\big]dm.$$

By the local character of the measures $\mu(u,v)$ and $\tilde{\mu}(u,v)$ it results that

$$\left|\int_E \mu(u,v)(x)\,dm\right| \le \tfrac{K}{2} \int_E \left[\tilde{\mu}(u,u)(x) + \tilde{\mu}(v,v)(x)\right]dm$$

for every Borel subset E of X. Choosing $v = \frac{\mu^{\frac14}(u,u)}{\mu^{\frac14}(v,v)}(x)v$ and $u = \frac{\mu^{\frac14}(v,v)}{\mu^{\frac14}(u,u)}(x)u$ we obtain

$$\left|\int_E \mu(u,v)(x)\,dm\right| \le K \int_E \left[\tilde{\mu}^{\frac12}(u,u)(x)\tilde{\mu}^{\frac12}(v,v)(x)\right]dm$$

for every Borel subset E of X. Thus, we have

$$|\mu(u,v)(x)| \le K\big(\tilde{\mu}^{\frac12}(u,u)(x)\,\tilde{\mu}^{\frac12}(v,v)(x)\big).$$

Then, as $\varphi\tilde{\mu}^{\frac12}(u,u)$ and $\psi\tilde{\mu}^{\frac12}(v,v)$ belong to $L^2(X,m)$, by Hölder we obtain

$$\int_X |\varphi\psi|\,|\mu(u,v)(dx)| \le K\left(\int_X |\varphi|^2\tilde{\mu}(u,u)(x)\,dm\right)^{\frac12} \left(\int_X |\psi|^2\tilde{\mu}(v,v)(x)\,dm\right)^{\frac12}$$

and the proposition is proved ∎

Proposition 3.12 (Chain rule). *Let $\mathcal{E}$ be a non-symmetric diffusion form and let $\mu(u,v)$ be the measure associated with $\mathcal{E}$ by* (3.5). *Let $u \in D(\mathcal{E}) \cap C_0(X)$ and $f \in C^1(\mathbb{R})$, with bounded derivative and $f(0) = 0$. Then $f(u)$ belongs to $D(\mathcal{E}) \cap C_0(X)$ and the chain rule holds for every $v \in D(\mathcal{E}) \cap C_0(X)$, i.e.*

$$\left.\begin{aligned} \mu(f(u), v) &= f'(u)\mu(u,v) \\ \mu(v, f(u)) &= f'(u)\mu(v,u) \end{aligned}\right\} \tag{3.9}$$

in the sense of measure.

Proof. The first part of the proposition is well known (see [8]), since it only concerns the symmetric part of $\mathcal{E}$. Let us prove equalities (3.9), that is, by Proposition 3.8 and Remark 3.9,

$$\int_X \mu(f(u), v)\,(dx) = \int_X f'(u)\mu(u,v)\,(dx).$$

Let $v \in D(\mathcal{E}) \cap C_0(X)$. Firstly, we can observe that if $f(v) = v^2$, by the Leibniz property, (3.9) holds. Now if $f(v) = v^3$, we can apply the Leibniz rule on $v^2 \cdot v$ to have (3.9) and in general, by the linearity of the form, we obtain that (3.9) holds when $f(v)$ is a polynomial such that $f(0) = 0$. Since $v \in D(\mathcal{E}) \cap C_0(X)$, hence $-M \le v \le M$, then by the Weierstrass theorem, we know that for every $f \in C^1([-M, M])$ with $f(0) = 0$ there exists a sequence f_n of polynomials such that $f_n(0) = 0$ and $f_n \to f$ in $C^1([-M, M])$. Then we have

$$\int_X \mu(f_n(u), v)\,(dx) = \int_X f_n'(u)\mu(u,v)\,(dx).$$

By the Lebesgue convergence theorem, taking into account that the integrals are taken on the compact set supp $[u]$ where $\{f_n'(u)\}$ is uniformly bounded by a constant, it results that

$$\int_X f_n'(u)\mu(u,v)\,(dx) \to \int_X f'(u)\mu(u,v)\,(dx) \qquad \forall\, u, v \in D(\mathcal{E}) \cap C_0(X).$$

We shall prove that

$$\int_X \mu(f_n(u), v)\,(dx) \to \int_X \mu(f(u), v)\,(dx)$$

hence, by the uniqueness of the limit, the thesis shall follows. Actually, one has

$$\begin{aligned} &\left| \int_X \left[\mu(f_n(u), v) - \mu(f(u), v)\right](dx) \right| \\ &\quad = \left| \int_X \mu(f_n(u) - f(u), v)\,(dx) \right| \\ &\quad \le K \mathcal{E}_1\big(f_n(u) - f(u), f_n(u) - f(u)\big)^{\frac{1}{2}} \mathcal{E}_1(v,v)^{\frac{1}{2}}. \end{aligned}$$

Since

$$\int_X |f_n(u) - f(u)|^2 dx \to 0 \qquad \text{as } n \to \infty$$

it is enough to prove that

$$\mathcal{E}(f_n(u) - f(u), f_n(u) - f(u)) = \int_X \mu\big(f_n(u) - f(u), f_n(u) - f(u)\big)(dx) \\ \to 0$$

as $n \to \infty$. Applying the chain rule on functions f_n, we have

$$\begin{aligned}
&\int_X \mu\big(f_n(u) - f(u), f_n(u) - f(u)\big)(dx) \\
&\quad = \int_X \mu(f_n(u), f_n(u))(dx) - 2\int_X \tilde{\mu}(f_n(u), f(u))(dx) + \int_X \mu(f(u), f(u))(dx) \\
&\quad = \int_X (f_n')^2(u)\mu(u,u)(dx) - 2\int_X f_n'(u)\tilde{\mu}(u, f(u))(dx) + \int_X \mu(f(u), f(u))(dx).
\end{aligned}$$

Taking the limit as $n \to \infty$, by the Lebesgue convergence theorem it follows that

$$\begin{aligned}
&\mathcal{E}\big(f_n(u) - f(u), f_n(u) - f(u)\big) \\
&\quad \to \int_X (f')^2(u)\mu(u,u)(dx) - 2\int_X f'(u)\tilde{\mu}(u, f(u))(dx) + \int_X \mu(f(u), f(u))(dx) \\
&\quad = \int_X \mu(f(u), f(u))(dx) - 2\int_X \mu(f(u), f(u))(dx) + \int_X \mu(f(u), f(u))dx \\
&\quad = 0
\end{aligned}$$

where we have applied the chain rule in the symmetric case. Thus (3.9) is proved ∎

The above Proposition can be also proved for a Lipschitz continuous function f, with $f(0) = 0$. Then it is simple to prove the *truncation rule*, that is

$$\tilde{\mu}(u^+, v) = \chi_{\{u>0\}}\tilde{\mu}(u, v)$$

for every $u, v \in D(\mathcal{E}) \cap C_0(X)$.

4. Example

Let $X := \Omega$ be an open subset of $\mathbb{R}^n$ $(n \geq 3)$ and let $m = dx$ be the Lebesgue measure on Ω. Letting $a_{ij} \in L^1_{\mathrm{loc}}(\Omega, dx)$, we define

$$\tilde{a}_{ij} = \tfrac{1}{2}(a_{ij} + a_{ji}) \quad \text{and} \quad \breve{a}_{ij} = \tfrac{1}{2}(a_{ij} - a_{ji}) \qquad (1 \leq i, j \leq n).$$

Let us suppose that there exist two constant $\lambda, M > 0$ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} \tilde{a}_{ij}(x)\xi_i\xi_j \qquad \forall\, \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^n$$

and

$$|\breve{a}_{ij}| \leq M \qquad \forall\, 1 \leq i, j \leq n.$$

Let $b_i, d_i \in L^n_{\mathrm{loc}}(\Omega, dx)$ and $c \in L^{\frac{n}{2}}_{\mathrm{loc}}$ such that $b_i - d_i \in L^n(\Omega, dx) \cup L^\infty(\Omega, dx)$ and

$$\left.\begin{array}{l} \displaystyle\int \left(cu + \sum_{i=1}^n d_i \frac{\partial u}{\partial x_i} \right) dx \\ \displaystyle\int \left(cu + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} \right) dx \end{array}\right\} \geq 0 \qquad \forall\, u \in C_0^\infty(\Omega), u \geq 0.$$

Let us define for $u, v \in C_0^\infty(\Omega)$

$$\mathcal{E}(u,v) = \sum_{i,j=1}^n \int a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \sum_{i=1}^n \int d_i u \frac{\partial v}{\partial x_i}\, dx + \sum_{i=1}^n \int b_i v \frac{\partial u}{\partial x_i}\, dx + \int cuv\, dx.$$

The *closure* of $(\mathcal{E}, C_0^\infty(\Omega))$ is a regular non-symmetric Dirichlet form on $H = L^2(X, m)$ (see [5: Proposition 2.11]).

Let us observe that

$$\sum_{i,j=1}^n \int a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \sum_{i=1}^n \int d_i u \frac{\partial v}{\partial x_i}\, dx + \sum_{i=1}^n \int b_i v \frac{\partial u}{\partial x_i}\, dx$$

is not a non-negative definite form. Actually, it results that

$$\begin{aligned} \mathcal{E}(u,v) = &\sum_{i,j=1}^n \int a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \sum_{i=1}^n \int \frac{d_i - b_i}{2} u \frac{\partial v}{\partial x_i}\, dx \\ &+ \sum_{i=1}^n \int \frac{b_i - d_i}{2} v \frac{\partial u}{\partial x_i}\, dx + \sum_{i=1}^n \int \left(c - \frac{1}{2} \frac{\partial (b_i + d_i)}{\partial x_i} \right) uv\, dx, \end{aligned}$$

that is (2.16) holds with $j = 0$, $k = c - \frac{1}{2} \sum_{i=1}^n \frac{\partial (b_i + d_i)}{\partial x_i}$ and

$$\mathcal{E}^c(u,v) = \sum_{i,j=1}^n \int a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \sum_{i=1}^n \int \frac{d_i - b_i}{2} u \frac{\partial v}{\partial x_i}\, dx + \sum_{i=1}^n \int \frac{b_i - d_i}{2} v \frac{\partial u}{\partial x_i}\, dx.$$

Moreover, let us observe that Theorems 2.7 and 2.8 hold with $j = 0$ and

$$\begin{aligned} \mathcal{G}^c(u,v) &= \sum_{i,j=1}^n \int a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \sum_{i=1}^n \int (d_i - b_i) u \frac{\partial v}{\partial x_i}\, dx \\ \mathcal{F}^c(u,v) &= \sum_{i,j=1}^n \int a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \sum_{i=1}^n \int (b_i - d_i) v \frac{\partial u}{\partial x_i}\, dx \\ g &= c - \sum_{i=1}^n \frac{\partial b_i}{\partial x_i} \\ h &= c - \sum_{i=1}^n \frac{\partial d_i}{\partial x_i}. \end{aligned}$$

In the case that $d_i = b_i$ and $c = \frac{\partial d_i}{\partial x_i} = \frac{\partial b_i}{\partial x_i}$, it results that $k = 0$. Then

$$\mathcal{E}^c(u,v) = \sum_{i,j=1}^{n} \int a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx$$

is a non-symmetric diffusion form. This form satisfyes the strong sector condition (1.1) with $K = \frac{M}{\lambda} + 1$:

$$|\mathcal{E}(u,v)| \leq \big(\tfrac{M}{\lambda} + 1\big)\mathcal{E}^{\frac{1}{2}}(u,u)\mathcal{E}^{\frac{1}{2}}(v,v). \tag{4.1}$$

Let us observe that Theorem 3.4 holds with

$$\mu(u,v)dx = a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx.$$

For such a measure the Leibniz property and the Chain rule can be easily proved. Moreover, the Schwarz inequality holds:

$$\begin{aligned}
&\int_\Omega |\varphi\psi| \left| \sum_{i,j=1}^{n} \breve{a}_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\right|(dx) \\
&\quad\leq \int_\Omega \sum_{i,j=1}^{n} |\varphi\psi|\, |\breve{a}_{ij}| \left|\frac{\partial u}{\partial x_i}\right| \left|\frac{\partial v}{\partial x_j}\right|(dx) \\
&\quad\leq M \int_\Omega \left(\sum_{i=1}^{n} |\varphi|^2 \left|\frac{\partial u}{\partial x_i}\right|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |\psi|^2 \left|\frac{\partial v}{\partial x_i}\right|^2 \right)^{\frac{1}{2}} \\
&\quad\leq M \left(\sum_{i=1}^{n} \int_\Omega |\varphi|^2 \left|\frac{\partial u}{\partial x_i}\right|^2 dx \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \int_\Omega |\psi|^2 \left|\frac{\partial v}{\partial x_i}\right|^2 dx \right)^{\frac{1}{2}} \\
&\quad\leq \frac{M}{\lambda} \left(\sum_{i,j=1}^{n} \int_\Omega |\varphi|^2 a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, dx \right)^{\frac{1}{2}} \left(\sum_{i,j=1}^{n} \int_\Omega |\psi|^2 a_{ij} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Moreover, since the Schwarz inequality holds for the symmetric part, it results that

$$\begin{aligned}
&\int_\Omega |\varphi\psi| \left| \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\right|(dx) \\
&\leq \big(\tfrac{M}{\lambda} + 1\big) \left(\sum_{i,j=1}^{n} \int_\Omega |\varphi|^2 a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, dx \right)^{\frac{1}{2}} \left(\sum_{i,j=1}^{n} \int_\Omega |\psi|^2 a_{ij} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx \right)^{\frac{1}{2}}
\end{aligned}$$

and $\frac{M}{\lambda} + 1$ is the constant that appears in the strong sector condition (4.1).

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