Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 4, 953–975

Homogenization of the Poisson Equation in a Thick Periodic Junction

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Abstract. A convergence theorem and asymptotic estimates as $\varepsilon \to 0$ are proved for a solution to a mixed boundary-value problem for the Poisson equation in a junction Ω_ε of a domain Ω_0 and a large number N^2 of ε-periodically situated thin cylinders with thickness of order $\varepsilon = O(\frac{1}{N})$. For this junction, we construct an extension operator and study its properties.

Keywords: *Homogenization, asymptotic estimates, extension operators*

AMS subject classification: Primary 35 B 27, secondary 35 B 40, 35 B 25

0. Introduction

Let D_ε be a domain in $\mathbb{R}^n$ which depends on a small parameter $\varepsilon > 0$ and, by the limit as $\varepsilon \to 0$, is transformed to a submanifold S of dimension m. The number m is called the limit dimension of the domain D_ε. If $m < n$, then the domain D_ε is called *thin*. Asymptotic methods for thin domains are well-known.

Some years ago several papers appeared that deal with the asymptotic investigation of boundary-value problems in junctions consisting of a finite number of domains with different limit dimensions [1, 5, 11, 12, 22, 23]. Boundary-value problems in thick periodic junctions whose number of components increases as $\varepsilon \to 0$ have own specific difficulties (see below), and until recently, there were no full asymptotic investigations of these problems. For these junctions we give the following classification:

A *thick periodic junction* Ω_ε of type $m : k : d$ is a domain in $\mathbb{R}^n$ that is obtained by joining a large number of ε-periodically situated thin domains with limit dimensions d to an external part of the boundary (which is the contact zone with limit dimension $k \le m$) of a domain Ω_0 (which is the junction's "body" with limit dimension $m \le n$). Here ε is a small parameter which depends on the number of the joined thin domains. The junction can have two or more "bodies".

These junctions are prototypes of widely adapted engineering constructions such as long bridges on supports, frameworks of houses, industrial installations, spaceship grids as well as other physical systems with very distinct characteristic scales. The objective of studying boundary-value problems in thick periodic junctions is to describe

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ISSN 0232-2064 / \$ 2.50

the asymptotic behaviour of solutions as $\varepsilon \to 0$, i.e. when the number of joined thin domains increases and their thickness decreases.

In the papers [10,29] the asymptotic behaviour of Green's function of the Neumann problem for the Helmholtz equation in the unbounded junction of type $3 : 2 : 1$ was studied. The limit equations describing acoustic vibrations in a porous medium, made by narrow parallel channels or thin parallel sheets in a solid body (the junctions of the type 3:2:1 or 3:2:2, respectively) were obtained in the papers [2, 7, 27]. In these articles the authors established some new qualitative properties of the homogenized equations: The corresponding Helmholtz equation is no longer elliptic and the operator which corresponds to the spectral limit problem is non-compact. These properties were the main difficulties in the asymptotic investigation of boundary-value problems in thick periodic junctions. As a result of these difficulties convergence theorems and asymptotic estimates were not obtained.

In the articles [15 - 20], using some results on the spectrum of discontinuous operator functions (see [8, 20]) and constructing special extension operators, the asymptotic behaviour of eigenvalues and eigenfunctions of the Neumann problems for the Laplace operator in junctions of different types $2 : 1 : 1$, $3 : 2 : 2$, $3 : 2 : 1$, $3 : 1 : 1$ was studied. The problem in the junction $3 : 1 : 1$ was examined in the case when the limiting process as $\varepsilon \to 0$ is accompanied by a concentration of masses on the joined thin domains [17]. These papers show that the type of a thick periodic junction defines a resulting boundary-value problem and junction-layer problems in the contact zone of this junction. However, the junction layers behave as powers at infinity and do not decrease exponentially. Therefore, they influence directly the principal terms of the asymptotics of the solution to the initial problem. The cause of this effect is in the modification of the geometrical structure of the domain, where junction-layer problems are considered. The type of a thick periodic junction defines also the construction scheme of an extension operator $\mathbf{P}_\varepsilon : H^1(\Omega_\varepsilon) \to H^1(\Omega_0 \cup D_0)$, where D_0 is a domain that is filled up by the joined thin domains in the limit as $\varepsilon \to 0$.

Extension operators play an important role in the proofs of convergence theorems for boundary-value problems in domains depending on a small parameter. They give us the possibility to pass from a domain depending on a small parameter to a fixed domain that does not depend on one. For many problems, these operators have to be uniformly bounded with respect to a small parameter in the Sobolev space H^1. The uniformly boundedness of extension operators is the necessary condition in the statement of some problems (see [30]). For other problems, such extension operators exist, for example, for domains that are $\varepsilon-$periodically perforated by holes with the diameter of order ε [6, 25, 30]. But for thick periodic junctions, there exist no extension operators that are bounded uniformly in ε. This is one more difficulty in the research.

It should be emphasized here the difference between the asymptotic investigation of boundary-value problems in thick periodic junctions and domains with rapidly oscillating boundaries [4] (those results were also referred in [26: Subsection 3.3]). The main difference is the following: the function h, which defines the oscillating boundary, must be a continuously differentiable periodic function, i.e., the boundary is smooth, and there must exist the reciprocal functions of h to construct an extension operator. These conditions do not hold for periodic thick junction: there does not exist any function, which would define the boundary of the junction Ω_ε. This junction has only the Lipschitz boundary and the periodical structure of joining of thin domains only (in our case thin cylinders, the cylinder can have various length). Therefore, the scheme of the construction of the extension operator in [4] is not applicable for thick periodic junctions. Furthermore, if the right-hand side of a boundary-value problem depends on the small parameter ε, then we need the special condition on this right-hand side (see (1.3)) to construct the bounded extension of the solution, and to obtain the convergence results. Also in this paper, using the method of matched asymptotic expansions, we construct the leading terms for the asymptotics of the solution and obtain the asymptotic estimates of the difference between the solution to the initial problem and the solution to the limiting problem.

In Section 1 we formulate the boundary-value problem (the initial problem) and prove auxiliary inequalities. Section 2 deals with the construction of an extension operator provided that the right-hand side of the initial problem satisfies a special condition. This operator is uniformly bounded in ε for the solution to the initial problem. The construction of the extension operator for junctions of type $3:2:1$ is both most complex with respect to the constructions of extensions for other types of thick periodic junctions and most general for ones in conception. In Section 3 we prove the convergence theorem. If the right-hand side has a special form, then we can construct an approximation function and obtain asymptotic estimates. This is done in Section 4. In this section we use some symmetry in the structure of the thick periodic junction. It helps us to define more exactly asymptotic relations for junctions-layer solutions, to detect other properties of ones, and to obtain better estimates for residuals of the approximation function.

1. The initial problem

1.1 Statement of the problem. Let a thick periodic junction Ω_ε of type $3:2:1$ consist of the "body"

$$\Omega_0 = \Big\{ x \in \mathbb{R}^3 : x' \in K \text{ and } 0 < x_3 < \gamma(x') \Big\}$$

and a large number of thin cylinders

$$G_\varepsilon = \bigcup_{i,j=0}^{N-1} G_\varepsilon(i,j),$$

where

$$G_\varepsilon(i,j) = \Big\{ x \in \mathbb{R}^3 : \Big(\frac{x_1}{\varepsilon} - i, \frac{x_2}{\varepsilon} - j\Big) \in \omega \text{ and } -l < x_3 \le 0 \Big\},$$

i.e. $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon$. Here $x' = (x_1, x_2)$, $K = (0,a)^2$, γ is a smooth positive function on $\overline{K}$ and $\gamma > \gamma_0 = \text{const} > 0$, N is a large positive integer and, therefore, the value $\varepsilon = \frac{a}{N}$ is a small parameter which characterizes the distance between thin cylinders and their thickness; the plane domain ω with smooth boundary belongs together with its closure to the disk $\big\{ x' \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 < \rho_0^2 < \frac14 \big\}$.

Define $\Gamma_\varepsilon = \partial\Omega_\varepsilon \cap \{x_3 = -l\}$ and consider the mixed boundary-value problem

$$\left.\begin{array}{rl} \Delta_x u_\varepsilon(x) = f_\varepsilon(x) & (x \in \Omega_\varepsilon) \\ \partial_\nu u_\varepsilon(x) = 0 & (x \in \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon) \\ u_\varepsilon(x) = 0 & (x \in \Gamma_\varepsilon) \end{array}\right\} \tag{1.1}$$

in the junction Ω_ε with given Dirichlet data on the bases Γ_ε of the joined thin cylinders and the Neumann boundary condition on the remaining part of the surface $\partial\Omega_\varepsilon$. Here $\partial_\nu = \frac{\partial}{\partial\nu}$ is the outward normal derivative.

We can regard without loss of generality that the right-hand side f_ε belongs to $L_2(\Omega)$ where Ω is the interior of the union $\overline{\Omega}_0 \cup \overline{D}$, $D = K \times (-l, 0)$ being a parallelepiped that is filled up by the thin cylinders $G_\varepsilon(i,j)$ $(i,j = 0,\dots,N-1)$ in the limit as $\varepsilon \to 0$ $(N \to \infty)$. We also assume that the function f_ε satisfies the following two conditions:

$$f_\varepsilon \to f_0 \qquad \text{in } L_2(\Omega) \text{ as } \varepsilon \to 0 \tag{1.2}$$

and there exist positive constants C_i, r_0, ε_0 such that for all $\varepsilon \in (0,\varepsilon_0)$

$$\int_{\widehat{\Omega}^{(i)}_{\varepsilon,r_0}} \big(\mathbf{F}^{(i)}_\varepsilon(x)\big)^2 dx \le C_i \qquad (i = 1,2) \tag{1.3}$$

where

$$\mathbf{F}^{(i)}_\varepsilon(x) = \varepsilon^{-1}\big(\widehat{f_\varepsilon}^{(i)}(x + \varepsilon\bar{e}_i) - \widehat{f_\varepsilon}^{(i)}(x)\big) \qquad \big(\bar{e}_1 = (1,0,0), \bar{e}_2 = (0,1,0)\big).$$

Here and further we interpret the symbol $\widehat{Y}^{(i)}$ as follows: if Y is a set, then $\widehat{Y}^{(i)}$ is the union of Y and of its image, symmetric with respect to the plane $\{x_i = 0\}$, and if Y is a function, then $\widehat{Y}^{(i)}$ is its even extension into the relevant domain with respect to the plane $\{x_i = 0\}$. Further, $\Omega_{0,r_0} = K \times (0, r_0)$ and Ω_{ε,r_0} is the interior of the union $\overline{\Omega}_{0,r_0} \cup \overline{G_\varepsilon}$. Condition (1.3) means that the function f_ε has not strong scattering of the values on the neighboring cylinders.

Consider some example: The function

$$\phi_0(x) = \begin{cases} 0 & \text{if } x \in \Omega_0 \\ (-1)^{i+j} x_3 & \text{if } x \in G_\varepsilon(i,j) \quad (i,j = 0,\dots,N-1) \end{cases}$$

does not satisfy condition (1.3). For this function, the variation of its values on the neighboring thin cylinders is very big, i.e., if we construct any extension of this function in the Sobolev space H^1, then the gradient of the extension will be of order $\frac{1}{\varepsilon}$.

Let for a function $g_\varepsilon \in L_2(\Omega)$ there exist constants $C_0 > 0$ and $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0,\varepsilon_0)$ and $x', x'' \in \Omega$, $|g_\varepsilon(x') - g_\varepsilon(x'')| \le C_0|x' - x''|$. Then condition (1.3) holds for g_ε.

So, our aim is to describe the asymptotic behaviour of the solution u_ε to problem (1.1) as $\varepsilon \to 0$ $(N \to +\infty)$.

1.2 Auxiliary inequalities. Consider a space $H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ formed by functions of the Sobolev space $H^1(\Omega_\varepsilon)$ whose traces vanish on Γ_ε. In this subspace we introduce along with the usual norm $\|u\|_1 = \big(\int_{\Omega_\varepsilon}(|\nabla u|^2 + u^2)\,dx\big)^{\frac{1}{2}}$ a new norm $\|\cdot\|_\varepsilon$ that is generated by the scalar product

$$\langle u, v\rangle_\varepsilon = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v\, dx. \tag{1.4}$$

Denote the space $H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ with scalar product (1.4) by H_ε.

Lemma 1.1. *For ε small enough, the norms $\|\cdot\|_1$ and $\|\cdot\|_\varepsilon$ are equivalent, i.e. there exist constants $c_1 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the inequalities*

$$\|u\|_\varepsilon \le \|u\|_1 \le c_1\|u\|_\varepsilon \qquad (u \in H_\varepsilon) \tag{1.5}$$

hold.

Proof. In (1.5), it is not obvious only that the second inequality holds. Suppose the contrary. Then there exist sequences $\{\varepsilon_m\}_{m\ge1}, \{v_m\}_{m\ge1} \in H_{\varepsilon_m}$ such that

$$\lim_{m\to0} \varepsilon_m = 0$$

$$\|v_m\|_1 = 1 \tag{1.6}$$

$$\|v_m\|^2_{\varepsilon_m} = \int_{\Omega_{\varepsilon_m}} |\nabla v_m|^2 dx < \frac{1}{m}. \tag{1.7}$$

Since the sequence $\{v_m\}_{m\ge1}$ is bounded in $H^1(\Omega_0)$, we may assume without loss of generality that it is a Cauchy sequence in $L_2(\Omega_0)$. From inequality (1.7) it follows that $\{v_m\}_{m\ge1}$ is a Cauchy sequence also in $H^1(\Omega_0)$:

$$\big\|v_m - v_n; H^1(\Omega_0)\big\|^2 \le \big\|v_m - v_n; L_2(\Omega_0)\big\|^2 + \frac{1}{m} + \frac{1}{n}.$$

Hence, $\{v_m\}_{m\ge1}$ converges in this space to some element $v_0 \in H^1(\Omega_0)$. By virtue of the Friedrich inequality and inequality (1.7) we have

$$\int_{G_{\varepsilon_m}} v_m^2 dx \le \int_{\Omega_{\varepsilon_m}} \Big(\frac{\partial v_m}{\partial x_3}\Big)^2 dx < \frac{1}{m}.$$

Granting this, from (1.6) and (1.7) we obtain that

$$1 = \|v_m\|_1^2 \;\to\; \int_{\Omega_0} v_0^2 dx \;\text{ as }\; m\to\infty \qquad \text{and} \qquad \int_{\Omega_0} |\nabla v_0|^2 dx = 0.$$

This means that $v_0 = \frac{1}{|\Omega_0|_3^{\frac{1}{2}}}$ in Ω_0, where $|\Upsilon|_n$ is the measure of a domain Υ in $\mathbb{R}^n$.

On the one hand, from the trace theorem for functions in Sobolev spaces and [24: Corollary 1.7] it follows that

$$\int_{Q_{\varepsilon_m}} v_m^2 dx \to |\omega|_2\, |\Omega_0|_3^{-1} a^2 \qquad \text{as } m\to\infty$$

where $Q_\varepsilon = G_\varepsilon \cap \{x_3 = 0\}$. On the other hand, we have

$$\int_{Q_{\varepsilon_m}} v_m^2 dx \le \int_{G_{\varepsilon_m}} \Big(\frac{\partial v_m}{\partial x_3}\Big)^2 dx < \frac{1}{m} \to 0 \qquad \text{as } m\to\infty.$$

The lemma is proved ∎

Remark 1.1. It should be noted that here and further all constants c_i and C_i in asymptotic inequalities are independent of the parameter ε.

Using Lemma 1.1 we can state that the right-hand side of the integral identity

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon(x) \cdot \nabla \varphi(x)\, dx = - \int_{\Omega_\varepsilon} f_\varepsilon(x) \varphi(x)\, dx \qquad \big(\varphi \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)\big) \tag{1.8}$$

for problem (1.1) defines a linear continuous functional in $H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ and, therefore, there exists a weak solution u_ε that is unique and satisfies the inequality

$$\|u_\varepsilon; H^1(\Omega_\varepsilon, \Gamma_\varepsilon)\| \le c_2 \|f_\varepsilon; L_2(\Omega)\| \le c_3. \tag{1.9}$$

Show that u_ε has not strong scattering of values on the neighboring cylinders. Let χ_{r_0} be a smooth cut-off function on $\mathbb{R}^n$,

$$\chi_{r_0}(x_3) = \begin{cases} 0 & \text{if } x_3 \ge r_0 \\ 1 & \text{if } x_3 \le \frac{r_0}{2} \end{cases} \qquad (r_0 \text{ taken from (1.3)}).$$

Define $\Gamma_{r_0} = \{x : x' \in K \text{ and } x_3 = r_0\}$. Then the function $v_\varepsilon = \chi_{r_0} u_\varepsilon$ is a weak solution to the problem

$$\left.\begin{array}{ll} \Delta_x v_\varepsilon(x) = \Phi_\varepsilon(x) & (x \in \Omega_{0,r_0}) \\ \Delta_x v_\varepsilon(x) = f_\varepsilon(x) & (x \in G_\varepsilon) \\ v_\varepsilon(x', r_0) = 0 & ((x', r_0) \in \Gamma_{r_0}) \\ v_\varepsilon(x) = 0 & (x \in \Gamma_\varepsilon) \\ \partial_\nu v_\varepsilon(x) = 0 & (x \in \partial\Omega_{\varepsilon,r_0} \setminus (\Gamma_\varepsilon \cup \Gamma_{r_0})) \end{array}\right\} \tag{1.10}$$

where $\Phi_\varepsilon = \chi_{r_0} f_\varepsilon + 2(\chi_{r_0})' \partial_{x_3} u_\varepsilon + (\chi_{r_0})'' u_\varepsilon$.

We extend problem (1.10) in the even way through the plane $\{x_i = 0\}$ $(i = 1, 2)$ and set $2a$−periodicity conditions on the corresponding side of the parallelepiped $\widehat{\Omega}^{(i)}_{0,r_0}$. Since the extended problem is invariant with respect to shifts by ε along the axis x_i, the function

$$\mathbf{V}^{(i)}_\varepsilon(x) = \frac{\widehat{v}^{(i)}_\varepsilon(x + \varepsilon \bar{e}_i) - \widehat{v}^{(i)}_\varepsilon(x)}{\varepsilon} \qquad \big(\bar{e}_1 = (1,0,0), \bar{e}_2 = (0,1,0)\big) \tag{1.11}$$

that is $2a$-periodic in x_i satisfies the relations

$$\begin{array}{ll} -\Delta_x \mathbf{V}^{(i)}_\varepsilon(x) = \dfrac{\widehat{\Phi}^{(i)}(x + \varepsilon\bar{e}_i) - \widehat{\Phi}^{(i)}(x)}{\varepsilon} & (x \in \widehat{\Omega}^{(i)}_{0,r_0}) \\ -\Delta_x \mathbf{V}^{(i)}_\varepsilon(x) = \dfrac{\widehat{f}^{(i)}(x + \varepsilon\bar{e}_i) - \widehat{f}^{(i)}(x)}{\varepsilon} & (x \in \widehat{G}^{(i)}_\varepsilon) \\ \mathbf{V}^{(i)}_\varepsilon(x', r_0) = 0 & (x' \in \widehat{K}^{(i)}) \\ \mathbf{V}^{(i)}_\varepsilon(x) = 0 & (x \in \widehat{\Gamma}^{(i)}_\varepsilon) \\ \partial_\nu \mathbf{V}^{(i)}_\varepsilon(x) = 0 & (x \in \partial\widehat{\Omega}^{(i)}_{\varepsilon,r_0} \setminus (\widehat{\Gamma}^{(i)}_\varepsilon \cup \widehat{\Gamma}^{(i)}_{r_0} \cup \{x_i = \pm a\}) \end{array}$$

whence, using condition (1.3), Lemma 1.1, (1.9) and the second energy inequality [13], we deduce the inequalities (i = 1,2)

$$\begin{aligned}
&\|\mathbf{V}_\varepsilon^{(i)}; H^1(\widehat{\Omega}_{\varepsilon,r_0}^{(i)}, \widehat{\Gamma}_\varepsilon^{(i)})\| \\
&\quad \le c_i \Big(\|\mathbf{F}_\varepsilon^{(i)}; L_2(\widehat{\Omega}_{\varepsilon,r_0}^{(i)})\| + \|\partial_{x_i} u_\varepsilon; L_2(\Omega_\varepsilon)\| + \|(\chi_{r_0})' \partial_{x_i x_3} u_\varepsilon; L_2(\Omega_\varepsilon)\| \Big) \\
&\quad \le c_i \Big(\|\mathbf{F}_\varepsilon^{(i)}; L_2(\widehat{\Omega}_{\varepsilon,r_0}^{(i)})\| + \|f_\varepsilon; L_2(\Omega_\varepsilon)\| + \|u_\varepsilon; H^1(\Omega_\varepsilon, \Gamma_\varepsilon)\| \Big) \\
&\quad \le C_i.
\end{aligned} \tag{1.12}$$

2. The extension operator

Now our aim is to pass to the limit in identity (1.8). To this end we construct an extension operator. As we mentioned above, there exists no extension operator bounded uniformly in ε. In order to verify this assertion, it suffices to make use of the function ϕ_0 from Subsection 1.1. But we can prove the following

Theorem 2.1. *For the solution to problem* (1.1) *there exists an extension operator*

$$\mathbf{P}_\varepsilon : H^1(\Omega_\varepsilon, \Gamma_\varepsilon) \to H^1(\Omega, \Gamma_{-l})$$

such that

$$\big\|\mathbf{P}_\varepsilon u_\varepsilon; H^1(\Omega, \Gamma_{-l})\big\| \le C_0 \left(\sum_{k=1}^{2} \big\|\mathbf{F}_\varepsilon^{(k)}; L_2(\widehat{\Omega}_{\varepsilon,r_0}^{(k)})\big\| + \big\|f_\varepsilon; L_2(\Omega_\varepsilon)\big\| \right) \tag{2.1}$$

where the space $H^1(\Omega, \Gamma_{-l})$ is formed by functions of the Sobolev space $H^1(\Omega)$ whose traces vanish on $\Gamma_{-l} = \{x : x' \in K$ and $x_3 = -l\}$.

To prove this theorem, just as in [20] for thin junctions, first we use an extension from a thin cylinder into a thin beam. Since the proof of this result was not given in the mentioned paper, we give it here.

Lemma 2.1. *Let*

$$T_\varepsilon = \Big\{ x : x' \in \Sigma_\varepsilon = \varepsilon\Sigma \quad \text{and} \quad x_3 \in (-l, 0) \Big\}$$
$$B_\varepsilon = \Big\{ x : x' \in S_\varepsilon = \varepsilon S_1 \quad \text{and} \quad x_3 \in (-l, 0) \Big\}$$

where Σ is a plane domain with Lipschitz boundary belonging with its closure to the square $S_1 = \{x' : |x_1| < 1$ and $|x_2| < 1\}$. Then for all $u \in H^1(T_\varepsilon)$ there exists a linear extension operator $\mathcal{P}_\varepsilon : H^1(T_\varepsilon) \mapsto H^1(B_\varepsilon)$ such that

$$\int_{B_\varepsilon} (\mathcal{P}_\varepsilon u)^2 dx \le c_0 \int_{T_\varepsilon} u^2 dx \tag{2.2}$$

$$\int_{B_\varepsilon} |\nabla_x \mathcal{P}_\varepsilon u|^2 dx \le c_0 \int_{T_\varepsilon} |\nabla_x u|^2 dx. \tag{2.3}$$

Proof. Obviously, it is sufficient to prove inequalities (2.2) and (2.3) for smooth functions. Expand the thin domains T_ε and B_ε with respect to x_1 and x_2 in $\frac{1}{\varepsilon}$ times. By $\widetilde{u}(\eta', x_3)$ we denote the function $u(\varepsilon\eta', x_3)$ with $\eta' = \frac{x'}{\varepsilon}$. There exists a linear extension operator $\mathcal{E}_1 : H^1(\Sigma) \mapsto H^1(S_1)$ (see [6, 14, 21, 25, 28, 30]) such that

$$\left\|\mathcal{E}_1(\widetilde{u}(\cdot, x_3)); H^i(S_1)\right\| \le c_1 \left\|\widetilde{u}(\cdot, x_3); H^i(\Sigma)\right\| \qquad (i = 0, 1) \tag{2.5}$$

where the constant c_1 is independent of both the function u and the variable x_3. By repeating the construction of the operator $\mathcal{E}_1$ in [21, 28] one can see that

$$\partial_{x_3}\big(\mathcal{E}_1(\widetilde{u}(\eta', x_3))\big) = \mathcal{E}_1\big(\partial_{x_3}\widetilde{u}(\eta', x_3)\big). \tag{2.6}$$

Define the desired extension by the formula

$$(\mathcal{P}_\varepsilon u)(x) = t(x_3) + \mathcal{E}_1(\widetilde{u}(\eta', x_3) - t(x_3))\big|_{\eta' = \frac{x'}{\varepsilon}}$$

where

$$t(x_3) = \tfrac{1}{|\Sigma|_2} \int_\Sigma \widetilde{u}(\eta', x_3)\, d\eta'.$$

Using (2.5) and (2.6), the Poincare inequality and the elementary estimate

$$|t(x_3)| \le \frac{\left\|\widetilde{u}(\cdot, x_3); L_2(\Sigma)\right\|}{|\Sigma|_2^{\frac{1}{2}}}$$

we can prove inequality (2.3) :

$$\begin{aligned}
&\int_{B_\varepsilon} |\nabla_{x'} \mathcal{P}_\varepsilon u|^2 dx \\
&\quad = \int_{-l}^{0} \int_{S_1} \left|\nabla_{\eta'} \mathcal{E}_1\big(\widetilde{u}(\eta', x_3) - t(x_3)\big)\right|^2 d\eta' dx_3 \\
&\quad \le c_1 \int_{-l}^{0} \left(\int_\Sigma \big(\widetilde{u}(\eta', x_3) - t(x_3)\big)^2 d\eta' + \int_\Sigma \left|\nabla_{\eta'} \widetilde{u}(\eta', x_3)\right|^2 d\eta' \right) dx_3 \\
&\quad \le c_2 \int_{-l}^{0} \int_\Sigma |\nabla_{\eta'} \widetilde{u}(\eta', x_3)|^2 d\eta' dx_3 \\
&\quad = c_2 \int_{T_\varepsilon} |\nabla_{x'} u|^2 dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_\varepsilon} \big(\partial_{x_3}(\mathcal{P}_\varepsilon u)\big)^2 dx \\
&\quad = \varepsilon^2 \int_{-l}^{0} \int_{S_1} \Big(t'(x_3) + \mathcal{E}_1\big(\partial_{x_3}\widetilde{u}(\eta', x_3) - t'(x_3)\big)\Big)^2 d\eta' dx_3 \\
&\quad \le \varepsilon^2 \int_{-l}^{0} \left(\tfrac{1}{|\Sigma|} \int_\Sigma \big(\partial_{x_3}\widehat{u}(\eta', x_3)\big)^2 d\eta' + c_1 \int_\Sigma \big(\partial_{x_3}\widetilde{u}(\eta', x_3) - t'(x_3)\big)^2 d\eta' \right) dx_3 \\
&\quad \le c_3 \int_{T_\varepsilon} (\partial_{x_3} u)^2 dx.
\end{aligned}$$

Inequality (2.2) can be proved by analogy with the previous inequality ∎

Remark 2.1. In just the same way one can prove the existence of an extension $\mathcal{P}_\varepsilon^{(1)} : H^1(B_\varepsilon \setminus T_\varepsilon) \mapsto H^1(B_\varepsilon)$ with the same properties as the operator $\mathcal{P}_\varepsilon$.

Proof of Theorem 2.1. This construction will be carried out in several steps. At first we prove that there exists an extension $P_\varepsilon^{(1)}$ from each thin cylinder $G_\varepsilon(i,j)$ into the thin beam

$$B_\varepsilon(i,j) = \Big\{x : \Big(\frac{x_1}{\varepsilon} - i, \frac{x_2}{\varepsilon} - j\Big) \in B^{(h)} \text{ and } -l < x_3 \le 0\Big\}$$

such that

$$\big\|P_\varepsilon^{(1)} u; H^1\big(\Omega_0 \cup \big(\cup_{i,j=0}^{N-1} B_\varepsilon(i,j)\big)\big)\big\| \le c_1 \big\|u; H^1(\Omega_\varepsilon)\big\| \qquad \big(u \in H^1(\Omega_\varepsilon)\big). \tag{2.7}$$

Here $B^{(h)} = \big\{x' : |x_1 - \frac{1}{2}| < h \text{ and } |x_2 - \frac{1}{2}| < h\big\}$ is a square containing the domain ω, $\rho_0 < h < \frac{1}{2}$. To this end, we use Lemma 2.1 to extend a function u to each cut beam $B_\varepsilon(i,j) \cap \{x : -l < x_3 < -\frac{\varepsilon}{2}\}$.

Next, we use the operator $\mathfrak{P}_\varepsilon$ from [14] to extend the function u from the domain

$$\Big\{x : \Big|\frac{x_1}{\varepsilon} - i - \frac{1}{2}\Big| < h, \text{ and } \Big|\frac{x_2}{\varepsilon} - i - \frac{1}{2}\Big| < h \ (-\varepsilon < x_3 < \varepsilon\Big\}$$
$$\Big\backslash \Big(\big(B_\varepsilon(i,j) \setminus G_\varepsilon(i,j)\big) \cap \Big\{x : -\frac{\varepsilon}{2} < x_3 < 0\Big\}\Big)$$

to the domain

$$\Big\{x : \Big|\frac{x_1}{\varepsilon} - i - \frac{1}{2}\Big| < h \text{ and } \Big|\frac{x_2}{\varepsilon} - i - \frac{1}{2}\Big| < h \quad -\varepsilon < x_3 < \varepsilon\Big\}$$

$(i,j = 0,1,\ldots,N-1)$. In [14], there are given two-side estimates for the norm of the extension operator $\mathfrak{P}_\varepsilon$ acting in Sobolev spaces on the exterior or interior of a domain with small diameter of order ε. In our case, as follows from this paper, the H^1-norm of $\mathfrak{P}_\varepsilon$ is uniformly bounded in ε, and estimates of the type as in Lemma 2.1 are valid.

For the next steps it is important that we construct the extension for the solution u_ε to problem (1.1). By u_ε we denote again the extended function $P_\varepsilon^{(1)} u_\varepsilon$ for which estimate (2.7) is valid and which vanishes when $x_3 = -l$. For each $i \in \{0,1,\ldots,N-1\}$ we extend the function u_ε from the system of thin beams $B_\varepsilon(i,j)$ $(j = 0,1,\ldots,N-1)$ to the thin plate

$$B_\varepsilon(i,*) = \Big\{x : \Big|\frac{x_1}{\varepsilon} - i - \frac{1}{2}\Big| < h, \quad 0 < x_2 < a, \quad -l < x_3 \le 0\Big\}.$$

At first we extend the function u_ε to each domain

$$\widetilde{D}_\varepsilon(i,j) = \Big\{x : \Big|\frac{x_1}{\varepsilon} - i - \frac{1}{2}\Big| < h, \ h < \frac{x_2}{\varepsilon} - j - \frac{1}{2} < 1 - h, \ -l < x_3 < -\varepsilon\Big\}$$

$(j = -1,0,1,\ldots,N)$, that is situated between two cut beams

$$\widetilde{B}_\varepsilon(i,j) = B_\varepsilon(i,j) \cap \{x : -l < x_3 < -\varepsilon\} \qquad \text{and} \qquad \widetilde{B}_\varepsilon(i,j+1)$$

using the “linear matching”

$$\breve{u}_\varepsilon(x) := a_{ij}(\varepsilon, x_1, x_3) + b_{ij}(\varepsilon, x_1, x_3)\Big(x_2 - \varepsilon\Big(h + j + \frac{1}{2}\Big)\Big)$$

where

$$\begin{aligned} a_{ij}(\varepsilon, x_1, x_3) &= u_\varepsilon\Big(x_1, \varepsilon\Big(j + h + \frac{1}{2}\Big), x_3\Big) \\ b_{ij}(\varepsilon, x_1, x_3) &= \frac{1}{\varepsilon(1-2h)}\left(u_\varepsilon\Big(x_1, \varepsilon\Big(j - h + \frac{3}{2}\Big), x_3\Big) - a_{ij}(\varepsilon, x_1, x_3)\right). \end{aligned}$$

In the case of extreme beams, we perform the even extension of problem (1.1) through the plane $\{x_2 = 0\}$ and $\{x_2 = a\}$, respectively. Estimating the norm of the function $\breve{u}_\varepsilon$, we get

$$\begin{aligned} &\big\|\breve{u}_\varepsilon; H^1(\widetilde{D}_\varepsilon(i,j))\big\|^2 \le \\ &c_1 \int_{\varepsilon(i-h+\frac{1}{2})}^{\varepsilon(i+h+\frac{1}{2})} \int_{-l}^{-\varepsilon} \Big(\varepsilon\big(a_{ij}^2 + |\nabla_{x_1x_3} a_{ij}|^2\big) + \varepsilon^3\big(b_{ij}^2 + |\nabla_{x_1x_3} b_{ij}|^2\big) + \varepsilon b_{ij}^2\Big) dx_3 dx_1. \end{aligned} \tag{2.8}$$

From Lemma 2.1 and the inequalities

$$\begin{aligned} u^2(0) &\le \tfrac{2}{\varepsilon}\int_0^\varepsilon u^2(t)\,dt + 2\varepsilon\int_0^\varepsilon (u')^2(t)\,dt \\ \big(u(\varepsilon) - u(0)\big)^2 &\le \varepsilon\int_0^\varepsilon (u')^2(t)\,dt \quad \big(u \in H^1([0,\varepsilon])\big) \end{aligned}$$

we obtain

$$\begin{aligned} &\varepsilon\int_{\varepsilon(i-h+\frac{1}{2})}^{\varepsilon(i+h+\frac{1}{2})} \int_{-l}^{-\varepsilon} a_{ij}^2\,dx_3 dx_1 \\ &\le c_2\left(\big\|u_\varepsilon; L_2(\widetilde{B}_\varepsilon(i,j))\big\|^2 + \varepsilon^2\big\|\partial_{x_2}u_\varepsilon; L_2(\widetilde{B}_\varepsilon(i,j))\big\|^2\right) \\ &\le c_3\big\|u_\varepsilon; H^1(\widetilde{G}_\varepsilon(i,j))\big\|^2, \end{aligned}$$

and

$$\begin{aligned} &\varepsilon\int_{\varepsilon(i-h+\frac{1}{2})}^{\varepsilon(i+h+\frac{1}{2})} \int_{-l}^{-\varepsilon} b_{ij}^2\,dx_3 dx_1 \\ &\le c_4\left(\big\|\mathbf{V}_\varepsilon^{(2)}; L_2(\widetilde{B}_\varepsilon(i,j))\big\|^2 + \varepsilon^2\big\|\partial_{x_2}\mathbf{V}_\varepsilon^{(2)}; L_2(\widetilde{B}_\varepsilon(i,j))\big\|^2 + \big\|\partial_{x_2}u_\varepsilon; L_2(\widetilde{B}_\varepsilon(i,j))\big\|^2\right) \\ &\le c_5\left(\big\|\mathbf{V}_\varepsilon^{(2)}; H^1(\widetilde{G}_\varepsilon(i,j))\big\|^2 + \big\|u_\varepsilon; H^1(\widetilde{G}_\varepsilon(i,j))\big\|^2\right) \end{aligned}$$

where the function $\mathbf{V}_\varepsilon^{(2)}$ is defined by (1.11) and $\widetilde{G}_\varepsilon(i,j) = G_\varepsilon(i,j) \cap \{x : -l < x_3 < -\varepsilon\}$.

Using the second energy inequality [12] with the smooth cut-off function χ satisfying

$$\chi_\varepsilon(x_3) = \begin{cases} 0 & \text{if } x_3 \ge -\frac{\varepsilon}{2} \\ 1 & \text{if } x_3 \le -\varepsilon \end{cases}$$

we analogously estimate the other values in (2.8) as

$$\begin{aligned}
&\varepsilon \int_{\varepsilon(i-h+\frac{1}{2})}^{\varepsilon(i+h+\frac{1}{2})} \int_{-1}^{-\varepsilon} |\nabla_{x_1x_3} a_{ij}|^2 dx_3 dx_1 \\
&\leq c_1 \Big(\big\|\nabla u_\varepsilon; L_2(\widetilde{G}_\varepsilon(i,j))\big\|^2 + \varepsilon^2 \big\|u_\varepsilon; H^2(\widetilde{G}_\varepsilon(i,j))\big\|^2\Big) \\
&\leq c_2 \Big(\big\|u_\varepsilon; H^1(G_\varepsilon(i,j))\big\|^2 + \big\|f_\varepsilon; L_2(G_\varepsilon(i,j))\big\|^2\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon^3 \int_{\varepsilon(i-h+\frac{1}{2})}^{\varepsilon(i+h+\frac{1}{2})} \int_{-1}^{-\varepsilon} |\nabla_{x_1x_3} b_{ij}|^2 dx_3 dx_1 \\
&\leq c_3 \varepsilon^2 \Big(\big\|\nabla \mathbf{V}_\varepsilon^{(2)}; L_2(\widetilde{G}_\varepsilon(i,j))\big\|^2 + \varepsilon^2 \big\|\mathbf{V}_\varepsilon^{(2)}; H^2(\widetilde{G}_\varepsilon(i,j))\big\|^2 + \big\|u_\varepsilon; H^2(\widetilde{G}_\varepsilon(i,j))\big\|^2\Big) \\
&\leq c_4 \Big(\big\|\mathbf{V}_\varepsilon^{(2)}; H^1(G_\varepsilon(i,j))\big\|^2 + \big\|u_\varepsilon; H^1(G_\varepsilon(i,j))\big\|^2 \\
&\quad + \big\|\mathbf{F}_\varepsilon^{(2)}; L_2(G_\varepsilon(i,j))\big\|^2 + \big\|f_\varepsilon; L_2(G_\varepsilon(i,j))\big\|^2\Big).
\end{aligned}$$

Thus, the right-hand side of (2.8) is estimated by the sum of the terms

$$\big\|\mathbf{V}_\varepsilon^{(2)}; H^1(G_\varepsilon(i,j))\big\|^2,\ \big\|u_\varepsilon; H^1(G_\varepsilon(i,j))\big\|^2,\ \big\|\mathbf{F}_\varepsilon^{(2)}; L_2(G_\varepsilon(i,j))\big\|^2\ \big\|f_\varepsilon; L_2(G_\varepsilon(i,j))\big\|^2.$$

To extend the function u_ε to the whole domain

$$D_\varepsilon(i,j) = \Big\{x : \Big|\frac{x_1}{\varepsilon} - i - \frac{1}{2}\Big| < h,\quad h < \frac{x_2}{\varepsilon} - j - \frac{1}{2} < 1-h,\quad -l < x_3 \leq 0\Big\}$$

it is sufficient to use the periodicity of the domains $D_\varepsilon(i,j)$ $(j = -1, 0, 1, \ldots, N)$ and Remark 2.1 with the domains

$$\begin{aligned}
T_\varepsilon &= \Big\{x : \Big|\frac{x_1}{\varepsilon} - \frac{1}{2}\Big| < h,\quad h < \frac{x_2}{\varepsilon} - \frac{1}{2} < 1-h,\quad -\varepsilon < x_3 < 0\Big\} \\
B_\varepsilon &= \Big\{x : \Big|\frac{x_1}{\varepsilon} - \frac{1}{2}\Big| < h,\quad \frac{1}{2}\varepsilon < x_2 < \frac{3}{2}\varepsilon,\quad -\frac{3}{2}\varepsilon < x_3 < \frac{1}{2}\varepsilon\Big\}.
\end{aligned}$$

As a result, we obtain the extension to the thin plate $B_\varepsilon(i,*)$ $(i = 0, 1, \ldots, N-1)$. Similarly we extend the function u_ε to each thin plate

$$B_\varepsilon(*,j) = \Big\{x : 0 < x_1 < a,\quad \Big|\frac{x_2}{\varepsilon} - j - \frac{1}{2}\Big| < h,\quad -l < x_3 \leq 0\Big\}$$

along the direction of the x_1-axis $(j = 0, 1, \ldots, N-1)$.

Thus, on the second step we have constructed the extension $P_\varepsilon^{(2)} u_\varepsilon$ $(P_\varepsilon^{(2)} u_\varepsilon = 0$ when $x_3 = -l)$ for which on the basis of conditions (1.2), (1.3) and inequalities (1.9), (1.12), (2.8) the estimate

$$\begin{aligned}
&\Big\|P_\varepsilon^{(2)} u_\varepsilon; H^1\Big(\Omega_0 \cup \big(\cup_{i,j=0}^{N-1} \big(B_\varepsilon(i,*) \cup B_\varepsilon(*,j)\big)\big)\Big)\Big\| \\
&\qquad \leq c_9 \left(\sum_{i=k}^{2} \big\|\mathbf{F}_\varepsilon^{(k)}; L_2(\widehat{\Omega}_{\varepsilon,r_0}^{(k)})\big\| + \big\|f_\varepsilon; L_2(\Omega_\varepsilon)\big\| \right)
\end{aligned} \tag{2.9}$$

holds.

Now it remains to construct an extension $P_\varepsilon^{(3)}$ into the thin beams

$$T_\varepsilon(i,j) = \Big\{x : h < \frac{x_1}{\varepsilon} - i - \frac{1}{2} < 1-h, \quad h < \frac{x_2}{\varepsilon} - j - \frac{1}{2} < 1-h, \quad -l < x_3 \le 0\Big\}$$

$(i,j = -1,0,1,\dots,N)$ just as previously, we perform the even extension of problem (1.1) through the planes $\{x_i = 0\}$ and $\{x_i = a\}$ $(i = 1,2)$ for the extreme beams. To this end, we use at first Remark 2.1 with the domains

$$T_\varepsilon = T_\varepsilon(0,0) \cap \{x:\ x_3 \le -\varepsilon\}$$

$$B_\varepsilon = \Big\{x:\ \frac{1}{2}\varepsilon < x_1 < \frac{3}{2}\varepsilon, \quad \frac{1}{2}\varepsilon < x_2 < \frac{3}{2}\varepsilon, \quad -l < x_3 \le -\varepsilon\Big\}$$

to extend $P_\varepsilon^{(2)} u_\varepsilon$ into the cut beams $T_\varepsilon(i,j) \cap \{x_3 \le -\varepsilon\}$ $(i,j = -1,0,1,\dots,N)$. Next, the extension $P_\varepsilon^{(3)}$ of the function $P_\varepsilon^{(2)} u_\varepsilon$ to parallelepipeds with the diameter of order ε is constructed in the same way as for perforated domains [6, 25, 30]. As we mentioned in the introduction, this extension is bounded uniformly with respect to the parameter ε in the Sobolev space H^1. Thus, according to what has been said and estimate (2.9), the extension operator $\mathbf{P}_\varepsilon := P_\varepsilon^{(3)} \circ P_\varepsilon^{(2)} \circ P_\varepsilon^{(1)}$ was constructed, and it satisfies estimate (2.1) ∎

3. The convergence theorem

Its proof consists of several steps. First, using the extension operator $\mathbf{P}_\varepsilon$, we pass to the limit in the integral identity (1.8). Next, selecting test-functions, we find the weak limit of $\mathbf{P}_\varepsilon u_\varepsilon$, and conclude that this limit is the solution of a homogenized problem.

Theorem 3.1. *Let conditions* (1.2) *and* (1.3) *hold. Then for the extension* $\mathbf{P}_\varepsilon u_\varepsilon$ *of the solution* u_ε *to problem* (1.1) *we have*

$$\mathbf{P}_\varepsilon u_\varepsilon \to v_0 \qquad \textit{weakly in } \ H^1(\Omega, \Gamma_{-l}) \ \ \textit{as} \ \ \varepsilon \to 0$$

where the function

$$v_0(x) = \begin{cases} v_0^+(x) & \textit{for } x \in \Omega_0 \\ v_0^-(x) & \textit{for } x \in D \end{cases} \tag{3.1}$$

is a weak solution to the problem

$$\left.\begin{aligned} \Delta v_0^+(x) &= f_0(x) && (x \in \Omega_0) \\ \partial_{x_3}^2 v_0^-(x) &= f_0(x) && (x \in D) \\ \partial_\nu v_0^+(x) &= 0 && (x \in \partial\Omega_0 \setminus K) \\ v_0^-(x', -l) &= 0 && ((x',-l) \in \Gamma_{-l}) \\ v_0^+(x', 0) &= v_0^-(x', 0) && (x' \in K) \\ \partial_3 v_0^+(x', 0) &= |\omega|_2\, \partial_3 v_0^-(x', 0) && (x' \in K). \end{aligned}\right\} \tag{3.2}$$

Proof. Using the extension operator constructed in Theorem 2.1, we can rewrite the integral identity (1.8) as

$$\begin{aligned}\int_{\Omega_0} \nabla u_\varepsilon \cdot \nabla \varphi \, dx + \int_D \chi_\omega\Big(\frac{x'}{\varepsilon}\Big) \nabla \mathbf{P}_\varepsilon u_\varepsilon \cdot \nabla \varphi \, dx \\ = - \int_{\Omega_0} f_\varepsilon \varphi \, dx + \int_D \chi_\omega \Big(\frac{x'}{\varepsilon}\Big) f_\varepsilon \varphi \, dx\end{aligned} \qquad \big(\varphi \in H^1(\Omega_\varepsilon, \Gamma_{-l})\big) \tag{3.3}$$

where $\chi_\omega(\eta')$ $(\eta' = (\eta_1, \eta_2) \in \mathbb{R}^2)$ is a 1-periodic function in η' satisfying

$$\eta_\omega(\eta') = \begin{cases} 1 & \text{on } \omega \\ 0 & \text{on } [0,1]^2 \setminus \omega. \end{cases}$$

Due to conditions (1.2), (1.3) and inequality (2.1), the sequences

$$\Big\{ \chi_\omega \Big(\frac{x'}{\varepsilon}\Big) \partial_{x_i} (\mathbf{P}_\varepsilon u_\varepsilon) \Big\} \qquad (i = 1, 2, 3) \tag{3.4}$$

are bounded in $L_2(D)$. Therefore, we can choose a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) and pass to the limit $\varepsilon \to 0$ in (3.3). We obtain

$$\int_{\Omega_0} \nabla v_0^+(x) \cdot \nabla \varphi(x) \, dx + \int_D \gamma_i(x) \partial_{x_i} \varphi(x) \, dx = - \int_{\Omega_0} f_0 \varphi \, dx - |\omega|_2 \int_D f_0 \varphi \, dx \tag{3.5}$$

for all $\varphi \in H^1(\Omega, \Gamma_{-l})$ where γ_i $(i = 1, 2, 3)$ are weak limits of sequences (3.4) in $L_2(D)$, respectively, and $v_0 = v_0^\pm$ (see (3.1)) is a weak limit of the sequence $\{\mathbf{P}_\varepsilon u_\varepsilon\}$ in $H^1(\Omega, \Gamma_{-l})$.

Next, we select test functions to find these values. Since

$$\int_D \chi_\omega \Big(\frac{x'}{\varepsilon}\Big) \partial_{x_3} \mathbf{P}_\varepsilon (u_\varepsilon(x)) \phi \, dx = - \int_D \chi_\omega \Big(\frac{x'}{\varepsilon}\Big) \mathbf{P}_\varepsilon (u_\varepsilon(x)) \partial_{x_3} \phi \, dx$$

for all $\phi \in C_0^\infty(D)$, we have

$$\gamma_3(x) = |\omega|_2 \; \partial_{x_3} v_0^-(x) \qquad (x \in D).$$

In order to determine γ_1 and γ_2, we consider the integral identity (1.8) for problem (1.1) with the test function ψ_i defined by

$$\psi_i(x) = \begin{cases} 0 & \text{for } x \in \Omega_0 \\ \varepsilon Y_i\big(\frac{x_i}{\varepsilon}\big) \phi(x) & \text{for } x \in G_\varepsilon \end{cases}$$

where $\phi \in C_0^\infty(D)$ and $Y_i(\eta_i) = -\eta_i + [\eta_i] + \frac{1}{2}$ $(i = 1, 2)$ ($[x]$ is the entire part of x); it is obvious that $\psi_i \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$. As a result, we get

$$\int_D \chi_\omega \Big(\frac{x'}{\varepsilon}\Big) \partial_{x_i} \mathbf{P}_\varepsilon (u_\varepsilon(x)) \phi \, dx = O(\varepsilon) \qquad (\varepsilon \to 0)$$

whence $\gamma_1 \equiv \gamma_2 \equiv 0$. Thus we have

$$\int_{\Omega_0} \nabla v_0^+(x) \cdot \nabla \varphi(x)\, dx + |\omega|_2 \int_D \partial_{x_3} v_0^-(x) \partial_{x_3} \varphi(x)\, dx = -\int_{\Omega_0} f_0 \varphi\, dx - |\omega|_2 \int_D f_0 \varphi\, dx$$

for all $\varphi \in H^1(\Omega, \Gamma_{-l})$. This integral identity means that the function v_0 is a weak solution to problem (3.2). This problem is called *resulting* for problem (1.1). The operator that corresponds to problem (3.2) is non-compact as the differentiation in D is taken only with respect to x_3. It should be noted there are no boundary conditions on the vertical sides of D.

By solving the ordinary equation of problem (3.2) in the parallelepiped D with regard to the boundary condition on Γ_{-l} and the first transmission condition in the contact zone K, we find that

$$v_0^-(x) = \int_{-l}^{x_3} (x_3 - t) f_0(x', t)\, dt + \frac{x_3 + l}{l} \left(v_0^+(x', 0) + \int_{-l}^{0} t f_0(x', t)\, dt \right). \tag{3.6}$$

Now, according to the second transmission condition in problem (3.2), we obtain the problem

$$\left.\begin{aligned} \Delta v_0^+(x) &= f_0(x) && (x \in \Omega_0) \\ \partial_\nu v_0^+(x) &= 0 && (x \in \partial\Omega_0 \setminus K) \\ \partial_3 v^+(x', 0) - |\omega|_2 l^{-1} v^+(x', 0) &= \widehat{f}_0(x') && (x' \in K) \end{aligned}\right\} \tag{3.7}$$

where

$$\widehat{f}_0(x') = \frac{|\omega|_2}{l} \int_{-l}^{0} (l + x_3) f_0(x', x_3)\, dx_3.$$

Obviously, problem (3.7) has a unique solution. It means that there is also a unique solution to problem (3.2).

Due to the uniqueness of the solution to problem (3.2), the above reasoning holds for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof. Therefore, the theorem is proved ∎

4. Asymptotic estimates

In this section we suppose that the function f_ε satisfies only the condition

$$f_\varepsilon(x) = f_0(x) + \varepsilon f_1(\varepsilon, x) \qquad (x \in \Omega) \tag{4.1}$$

where

$$f_0, f_1 \in L_2(\Omega) \qquad \text{and} \qquad \big\| f_1(\varepsilon, \cdot); L_2(\Omega) \big\| = O(1) \quad (\varepsilon \to 0)$$

and the plane domain ω is symmetric with respect to the straight lines $\{x_i = \frac{1}{2}\}$ $(i = 1, 2)$.

4.1 Formal derivation of the resulting problem using asymptotic expansions. Combining the algorithm of constructing asymptotics in thin domains with methods of the homogenization theory (see [3] and others), we seek the main terms of the asymptotics for the solution u_ε restricted to Ω_0 in the form

$$u_\varepsilon \approx v_0^+(x) + \sum_{k=1}^{\infty} \varepsilon^k v_k^+(x,\varepsilon) \tag{4.2}$$

and, restricted to $G_\varepsilon(i,j)$, in the form

$$u_\varepsilon \approx v_0^-(x) + \sum_{k=1}^{\infty} \varepsilon^k v_k^-\big(x, \eta_1^{(i)}, \eta_2^{(j)}\big) \qquad \Big(\eta_1^{(i)} = \frac{x_1}{\varepsilon} - i, \eta_2^{(j)} = \frac{x_2}{\varepsilon} - j\Big). \tag{4.3}$$

Decomposing formally the function v_k^- in the Taylor series with respect to the variables x_1 and x_2 in a neighborhood of the pont $x_1^{(i)} = \varepsilon(i+\frac{1}{2})$ and $x_2^{(j)} = \varepsilon(j+\frac{1}{2})$, we rewrite (4.3) in the form

$$u_\varepsilon = v_0^-(i,j,x_3) + \sum_{k=1}^{2} \varepsilon^k V_k^{i,j}\big(x_3, \eta_1^{(i)}, \eta_2^{(j)}\big) + O(\varepsilon^3) \qquad (x \in G_\varepsilon(i,j)) \tag{4.4}$$

where

$$V_k^{i,j} = \sum_{m=0}^{k} \frac{1}{m!} \left(\Big(\eta_1^{(i)} - \frac{1}{2}\Big)\frac{\partial}{\partial x_1} + \Big(\eta_2^{(j)} - \frac{1}{2}\Big)\frac{\partial}{\partial x_2} \right)^m v_{k-m}^-\big(i,j,x_3,\eta_1^{(i)},\eta_2^{(j)}\big) \tag{4.5}$$

and $v_k^-(i,j,\ldots) = v_k^-(x_1^{(i)}, x_2^{(j)}, \ldots)$. Let us substitute (4.4) into (1.1) instead of u_ε. Collecting and equating to 0 the coefficients by equal degrees of ε, we obtain

$$\left.\begin{aligned} &\Delta_{\eta'} V_1^{i,j}(x_3,\eta') = 0 && (\eta' \in \omega) \\ &\partial_{\nu_{\eta'}} V_1^{i,j}(x_3,\eta') = 0 && (\partial_{\nu_{\eta'}} V_1^{i,j}(x_3,\eta') = 0 \end{aligned}\right\} \tag{4.6}$$

and

$$\left.\begin{aligned} \Delta_{\eta'} V_2^{i,j}(x_3,\eta') + \partial_{x_3}^2 v_0^-(i,j,x_3) &= f_0(i,j,x_3) && (\eta' \in \omega) \\ \partial_{\nu_{\eta'}} V_2^{i,j}(x_3,\eta') &= 0 && (\eta' \in \partial\omega). \end{aligned}\right\} \tag{4.7}$$

Here $\eta' = (\eta_1^{(i)}, \eta_2^{(j)})$, the variable $x_3 \in (-l, 0)$ is regarded as a parameter in these problems. From (4.6), it follows that the function $V_1^{i,j}$ does not depend on η'. We restrict ourselves to the leading term of the asymptotics, and thus set $V_1^{i,j} = 0$. Then, by virtue of (4.5), we have

$$v_1^-(i,j,x_3,\eta') = -\partial_{x_1} v_0^-(i,j,x_3)\Big(\eta_1^{(i)} - \frac{1}{2}\Big) - \partial_{x_2} v_0^-(i,j,x_3)\Big(\eta_2^{(j)} - \frac{1}{2}\Big). \tag{4.8}$$

The solvability condition for (4.7) is given by the ordinary differential equation with respect to x_3 that is present in problem (3.2), and the points $(x_1^{(i)}, x_2^{(j)})$ $(i,j = 0,1,\ldots,N-1)$ are regarded as parameters in this equation. Since these points make up the ε-net in K, we can spread this equation in all points of the square K.

It is understandable what equation we will obtain for the function v_0^+ from (4.2) if we substitute (4.2) into (1.1). It remains to provide the continuity of the asymptotic approximation and their gradients in the contact zone K. It is doubtless that

$$v_0^+(x',0) = v_0^-(x',0) \qquad (x' \in K).$$

To get the second transmission condition in (3.2) we can use the method of matched asymptotic expansion: the outer expansions are given by (4.2) and (4.3); the main terms of the inner one takes the form

$$u_\varepsilon \approx v_0^+(x',0) + \varepsilon \sum_{i=1}^{3} Z_i(\eta)\partial_{x_i} v_0^+(x',0) + \ldots \tag{4.9}$$

where $\eta = \frac{x}{\varepsilon}$ and $\{Z_i\}$ are junction-layer solutions which we will consider in the following section. Thus, the first term $v_0^\pm$ in asymptotics (4.2) and (4.3) is a solution to the resulting problem (3.2).

4.2 Junction-layer problems. Let us introduce the "rapid" coordinates $\eta = \frac{x}{\varepsilon}$ in problem (1.1). Passing to $\varepsilon = 0$, we see that the cylinder $G_\varepsilon(0,0)$ transforms into the semi-infinite cylinder

$$\Pi^- = \omega \times (-\infty, 0]$$

and the set Ω_0 transforms into the first octant $\{\eta_i > 0 \quad (i = 1,2,3)\}$. Taking into account the periodicity of the cylinders $G_\varepsilon(i,j)$ $(i,j = 0,\ldots,N-1)$ we can regard that the union Π of the semi-cylinders Π^- and

$$\Pi^+ = (0,1) \times (0,1) \times (0,+\infty)$$

is a base domain in which the junction-layer problems have to be considered. Obviously, solutions of these junction-layer problems must be 1-periodic in η_1 and η_2, i.e.

$$\partial^k_{\eta_\tau} Z(\eta)\big|_{\eta_\tau=0} = \partial^k_{\eta_\tau} Z(\eta)\big|_{\eta_\tau=1} \qquad (\eta \in \partial\Pi^+, \eta_3 > 0;\ k = 0,1;\ \tau = 1,2). \tag{4.10}$$

Let us investigate some properties of solutions to the following junction-layer problem:

$$\left.\begin{aligned}
-\Delta_{\eta\eta} Z(\eta) &= F(\eta) && (\eta \in \Pi)\\
\partial_{\nu_\eta} Z(\eta) &= B(\eta) && (\eta \in \partial\Pi^- \setminus \omega\\
\partial_{\eta_3} Z(\eta',0) &= 0 && \big(\eta',0) \in \partial\Pi^+ \setminus \omega\big)\\
\partial^k_{\eta_\tau} Z(\eta)\big|_{\eta_\tau=0} &= \partial^k_{\eta_\tau} Z(\eta)\big|_{\eta_\tau=1} && (\eta \in \partial\Pi^+, \eta_3 > 0; k = 0,1;\ \tau = 1,2).
\end{aligned}\right\} \tag{4.11}$$

At first we study the solvability of this problem. In this connection we use the scheme given in [22]. Let $\widehat{C}_0^\infty(\overline{\Pi})$ be a space of infinitely differentiable functions in $\overline{\Pi}$ that satisfy the periodical conditions (4.10) and are finite in η_3, i.e.

$$\forall\ v \in \widehat{C}_0^\infty(\overline{\Pi})\ \exists\ R > 0\ \forall \eta \in \overline{\Pi}\ |\eta_3| \geq R: \quad v(\eta) = 0.$$

Let $\mathcal{H}$ be the completion of the space $\widehat{C}_0^\infty(\overline{\Pi})$ by the norm

$$\|u\|_{\mathcal{H}} = \left(\|\nabla_\eta u\|^2_{L_2(\Pi)} + \|\rho u\|^2_{L_2(\Pi)}\right)^{\frac{1}{2}}$$

where $\rho(\eta_3) = \frac{1}{1+|\eta_3|}$ $(\eta_3 \in \mathbb{R})$. We will call a function Z a *generalized solution* to problem (4.11) if for all functions $v \in \mathcal{H}$ the integral identity

$$\int_\Pi \nabla_\eta Z \cdot \nabla_\eta v \, d\eta = \int_\Pi F v \, d\eta + \int_{\partial\Pi^- \setminus \omega} B v \, d\eta \tag{4.12}$$

holds.

Lemma 4.1. *Let $\frac{1}{\rho} F \in L_2(\Pi)$ and $\frac{1}{\rho} B \in L_2(\partial\Pi^- \setminus \omega)$, and let*

$$\int_\Pi F(\eta) \, d\eta + \int_{\partial\Pi^- \setminus \omega} B(\eta) \, d\sigma_\eta = 0. \tag{4.13}$$

Then there exists a solution $Z \in \mathcal{H}$ to problem (4.11) *that is defined up to an additive constant.*

Proof. We rewrite identity (4.12) in the form

$$\langle Z, v\rangle - \int_{\Pi_{-2,2}} Z v \, d\eta = \int_\Pi F v \, d\eta + \int_{\partial\Pi^- \setminus \omega} B v \, d\eta \tag{4.14}$$

where

$$\Pi_{\alpha,\beta} = \{\eta \in \Pi : \alpha < \eta_3 < \beta\}$$

and

$$\langle u, v\rangle = \int_\Pi \nabla_\eta u \cdot \nabla_\eta v \, d\eta + \int_{\Pi_{-2,2}} u v \, d\eta. \tag{4.15}$$

Then the new scalar product (4.15) generates an equivalent norm in $\mathcal{H}$. It is obvious that $\langle u, u\rangle \le c_1 \|u\|^2_{\mathcal{H}}$ $(u \in \mathcal{H})$. The inverse inequality with another constant follows from the Hardy inequality

$$\int_0^{+\infty} \frac{\phi^2(\eta_3)}{(1+\eta_3)^2} \, d\eta_3 \le 4 \int_0^{+\infty} |\partial_{\eta_3}\phi|^2 \, d\eta_3 \qquad \big(\phi \in C^1([0,+\infty)) \text{ with } \phi(0) = 0\big)$$

and the inequality

$$\begin{aligned}
&\int_\Pi \rho^2(\eta_3) u^2(\eta) \, d\eta \\
&\quad \le \int_{\Pi_{-2,2}} \rho^2 u^2 \, d\eta + \int_\Pi \rho^2 \big((1-\chi(\eta_3))u\big)^2 d\eta \\
&\quad \le \int_{\Pi_{-2,2}} \rho^2 u^2 \, d\eta + c_\chi \left(\int_\Pi (\partial_{\eta_3} u)^2 \, d\eta + \int_{\Pi_{-2,2}} (\chi'(\eta_3) u)^2 \, d\eta\right) \\
&\quad \le c_2 \langle u, u\rangle
\end{aligned} \tag{4.16}$$

where $\chi \in C^\infty(\mathbb{R}), 0 \le \chi \le 1$ and

$$\chi(\eta_3) = \begin{cases} 1 & \text{if } |\eta_3| \le 1 \\ 0 & \text{if } |\eta_3| \ge 2. \end{cases} \tag{4.17}$$

Due to the conditions of Lemma 4.1, inequality (4.16) and the inequality

$$\begin{aligned} \int_{\partial\Pi^- \setminus \omega} \rho^2(\eta_3) v^2(\eta)\, d\sigma_\eta &\le \int_{-\infty}^0 \rho(\eta_3) \int_{\partial\omega} v^2\, d\sigma_{\eta'}\, d\eta_3 \\ &\le c \int_\Pi \left(|\nabla_\eta v|^2 + \rho(\eta_3) v^2 \right) d\eta \end{aligned} \qquad (v \in \mathcal{H})$$

the right-hand side of identity (4.14) defines a linear continuous functional in $\mathcal{H}$. As the embedding $\mathcal{H} \subset L_2(\Pi_{-2,2})$ is compact, there exists a self-adjoint positive compact operator $\mathcal{A} : \mathcal{H} \mapsto \mathcal{H}$ such that

$$\langle \mathcal{A}u, v \rangle = \int_{\Pi_{-2,2}} u(\eta) v(\eta)\, d\eta \qquad (\{u, v\} \in \mathcal{H}).$$

Thus, we can rewrite identity (4.14) as operator equation

$$Z - \mathcal{A}Z = f$$

and apply Fredholm theorems to it. It is obvious that every solution of the homogeneous problem (4.11) in the space $\mathcal{H}$ is constant (its Dirichlet integral is trivial). Therefore, equality (4.13) is the solvability condition for problem (4.11) ∎

Remark 4.1. Let $\exp(\delta_0 |\eta_3|) F \in L_2(\Pi)$ and $\exp(-\delta_0 \eta_3) B \in L_2(\partial\Pi^- \setminus \omega)$ $(\delta_0 > 0)$. Taking into account the properties of solutions to elliptic problems in semi-cylinders, we can select a solution $\widetilde{Z}$ of problem (4.11) such that

$$\exp(-\delta_1 \eta_3) \widetilde{Z} \in H^1(\Pi^-) \tag{4.18}$$

where δ_1 is an arbitrary number that satisfies the inequalities $0 < \delta_1 < \delta_0$ and $\delta_1 < \sqrt{\Lambda(\omega)}$; $\Lambda(\omega)$ is the first positive eigenvalue of the Neumann problem in the plane domain ω. It is clear that the solution $\widetilde{Z}$ has the following asymptotics in the semi-cylinder Π^+:

$$\widetilde{Z}(\eta) = C + O\big(\exp(-\delta_2 \eta_3)\big) \qquad (\eta_3 \to +\infty). \tag{4.19}$$

Remark 4.2. If the functions F and B from Remark 4.1 are even or odd in any of the variables $\{\eta_1, \eta_2\}$ with respect to $\frac{1}{2}$, then the solution $\widetilde{Z}$ has the same symmetry. In fact, let for example F and B be even in η_1 with respect to $\frac{1}{2}$, i.e.

$$F(\eta_1, \eta_2, \eta_3) = F(1 - \eta_1, \eta_2, \eta_3) \quad \text{and} \quad B(\eta_1, \eta_2, \eta_3) = B(1 - \eta_1, \eta_2, \eta_3).$$

Then, due to the symmetry of the domain ω and using the substitution $\eta_1 = 1 - \eta_1'$ in problem (4.11), we obtain that the difference $\widetilde{Z}(\eta_1, \eta_2, \eta_3) - \widetilde{Z}(1 - \eta_1, \eta_2, \eta_3)$ is a solution of the homogeneous problem (4.11), and relation (4.18) is satisfied for it. By virtue of the uniqueness of such a solution, it follows that this difference vanishes.

Corollary 4.1. *The homogeneous problem* (4.11) *has a solution* $\Xi_0 \notin \mathcal{H}$ *with the asymptotics*

$$\Xi_0(\eta) = \begin{cases} C_\omega + \eta_3 + O(\exp(-\delta_3\eta_3)) & \text{as } \eta_3 \to +\infty \\ |\omega|_2^{-1}\eta_3 + O(\exp(\delta_3\eta_3)) & \text{as } \eta_3 \to -\infty \end{cases} \tag{4.20}$$

and this solution is even in η_1, η_2 *with respect to* $1/2$*;* $\delta_3 > 0$.

Proof. The solution Ξ_0 is sought in the form of a sum

$$\Xi_0(\eta) = \chi_+(\eta_3)\eta_3 + \frac{\eta_3}{|\omega|_2}\chi_-(\eta_3) + \widetilde{Z}_0$$

where $\widetilde{Z}_0 \in \mathcal{H}$ and $\widetilde{Z}_0$ is the solution to problem (4.11) with right-hand sides

$$F(\eta) = 2\chi_+'(\eta_3) + \chi_+''(\eta_3)\eta_3 + \frac{2}{|\omega|_2}\chi_-'(\eta_3) + \frac{1}{|\omega|_2}\chi_-''(\eta_3)\eta_3 =: F_\omega(\eta_3)$$

and $B = 0$. Here χ_+ is a smooth cut-off function with

$$\chi_+ = \begin{cases} 1 & \text{when } \eta_3 \geq 2 \\ 0 & \text{when } \eta_3 \leq 1 \end{cases}$$

and $\chi_-(\eta_3) = \chi_+(-\eta_3)$ $(\eta_3 < 0)$. By virtue of Lemma 4.1 and Remarks 4.1 and 4.2, there exists a unique solution $\widetilde{Z}_0 \in \mathcal{H}$ to such problem that is even in η_1 and η_2 with respect to $\frac{1}{2}$ and has the asymptotics

$$\widetilde{Z}_0(\eta) = \begin{cases} C_\omega + O\big(\exp(-\delta_3\eta_3)\big) & \text{as } \eta_3 \to +\infty \\ O\big(\exp(\delta_3\eta_3)\big) & \text{as } \eta_3 \to -\infty. \end{cases} \tag{4.21}$$

In order to find the constant C_ω in (4.21), it is necessary to substitute the function Ξ_0 and $\widetilde{Z}_0$ into the Green formula

$$\int_{\Pi_{-R,R}} \big(\Xi_0\Delta\widetilde{Z}_0 - \widetilde{Z}_0\Delta\Xi_0\big)\,d\eta = \int_{\partial\Pi_{-R,R}} \big(\Xi_0\partial_{\nu_\eta}\widetilde{Z}_0 - \widetilde{Z}_0\partial_{\nu_\eta}\Xi_0\big)\,d\eta$$

and to pass to the limit as $R \to \infty$. As a result, we obtain $C_\omega = \int_\Pi \Xi_0(\eta)F_\omega(\eta_3)\,d\eta$ ∎

Remark 4.3. By analogy we can show that the constant C in (4.19) equals

$$C = \int_\Pi \Xi_0(\eta)F(\eta)\,d\eta + \int_{\partial\Pi^-\setminus\omega} \Xi_0(\eta)B(\eta)\,d\sigma_\eta. \tag{4.22}$$

Let us substitute (4.9) into (1.1) instead of u_ε. Collecting and equating to 0 the coefficients by equal degrees of ε, we get the problems for the functions Z_i. The conditions at infinity for ones follow from the conditions of the matching of the outer and inner expansions (4.2), (4.3) and (4.9). As a result, we obtain that $Z_3 = \Xi_0$ and Z_1, Z_2 are equal to

$$Z_i(\eta) = \chi_-(\eta_3)(-\eta_i + \tfrac{1}{2}) + \widetilde{Z}_i(\eta) \qquad (\eta \in \Pi)$$

where $\widetilde{Z}_i \quad (i=1,2)$ are solutions to problem (4.11) with right-hand sides

$$F_i = \chi''_-(\eta_3)(-\eta_i + \tfrac{1}{2}) \qquad (\eta \in \Pi)$$
$$B_i = -(1-\chi_-(\eta_3))\nu_i(\eta) \qquad (\eta \in \partial\Pi^- \setminus \omega).$$

Remark 4.4. Since the functions F_i and B_i are odd in η_i and even in $\eta_{i+(-1)^{i+1}}$ with respect to $\frac{1}{2}$, then, due to the above-mentioned results, it follows that, in the first place, the solvability condition for these problems is realized. In the second place, the correspondent constants in (4.19) for the solutions $\widetilde{Z}_1$ and $\widetilde{Z}_2$ are equal to zero. And, in the third place, the functions Z_1 and Z_2 have the same symmetry, i.e. with respect to $\frac{1}{2}$ the function Z_1 is odd in η_1 and even in η_2, and the function Z_2 is even in η_1 and odd in η_2.

Remark 4.5. The main asymptotic relations for the functions Z_i can be obtained from general results about the asymptotic behaviour of solutions to elliptic problems in domains with different exits to infinity [9, 24]. In our case, using the symmetry of the domain ω and the existence theorem for the concrete problem, we can define more exactly the asymptotic relations and detect other properties of the junction-layer solutions. These properties help us to find residuals of an approximation function U_ε in the equation and in the boundary conditions of problem (1.1) and to obtain better estimates for ones.

4.3 The asymptotic approximation. Let v_0 be a unique solution to the resulting problem (3.2), i.e. v_0^- is defined by (3.6) and v_0^+ is a solution to problem (3.7). Matching the outer and inner expansions for the solution u_ε, we construct the global asymptotic approximation function $U_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$:

$$U_\varepsilon(x) = v_0^+(x) + \varepsilon\chi_0(x_3)\sum_{i=1}^{3}\big(Z_i(\eta) - \delta_{i,3}\eta_3\big)\partial_{x_i}v_0^+(x',0) \qquad (\eta = \tfrac{x}{\varepsilon},\ x \in \Omega_0) \quad (4.23)$$

and

$$\begin{aligned} U_\varepsilon(x) = v_0^-(x) + \varepsilon\Bigg(&Y_1(\eta_1)\partial_{x_1}v_0^-(x) + Y_2(\eta_2)\partial_{x_2}v_0^-(x) \\ &+ \chi_0(x_3)\sum_{i=1}^{3}\Big(Z_i(\eta) - \delta_{i,1}Y_1(\eta_1) - \delta_{i,2}Y_2(\eta_2) \qquad (\eta = \tfrac{x}{\varepsilon},\ x \in G_\varepsilon). \qquad (4.24) \\ &- \delta_{i,3}|\frac{\eta_3}{|\omega|_2}\Big)\partial_{x_i}v_0^+(x',0)\Bigg) \end{aligned}$$

Here $\chi_0(x_3) = \chi(\frac{2}{r_0}x_3) \quad (x_3 \in \mathbb{R})$ (the function χ is defined by (4.17) and $Y_i(\eta_i) = -\eta_i + [\eta_i] + \frac{1}{2}$ (see (4.8)). Substituting U_ε into problem (1.1) in place of u_ε we find that, for any $\psi \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$,

$$\int_{\Omega_\varepsilon}\big(\nabla U_\varepsilon \cdot \nabla\psi + f_\varepsilon\psi\big)\,dx = F_\varepsilon(\psi) \qquad (4.25)$$

and

$$|F_\varepsilon(\psi)| \le c(\delta)\,\varepsilon^{1-\delta}\big\|\psi; H^1(\Omega_\varepsilon, \Gamma_\varepsilon)\big\|, \qquad (4.26)$$

where δ is arbitrary positive number. Using Lemma 1.1 and the integral identity (1.8), we get from (4.25) and (4.26) the following

Theorem 4.3. *The difference between the solution u_ε to problem* (1.1) *with right-hand side* (4.1) *and the approximation function U_ε satisfies the estimate*

$$\left\| u_\varepsilon - U_\varepsilon; H^1(\Omega_\varepsilon, \Gamma_\varepsilon) \right\| \le c_1(\delta)\, \varepsilon^{1-\delta} \qquad (\delta > 0). \tag{4.27}$$

Corollary 4.2. *It follows from* (4.27) *that*

$$\left\| u_\varepsilon - v_0; L_2(\Omega_\varepsilon) \right\| \le c_2(\delta)\, \varepsilon^{1-\delta} \qquad (\delta > 0) \tag{4.28}$$

where v_0 is the solution to the resulting problem.

4.4 Conclusion. The symmetry of the domain ω with respect to the straight lines $\{x_1 = \frac{1}{2}\}$ and $\{x_2 = \frac{1}{2}\}$ is only a technical condition. It helps us to avoid awkward calculation and to obtain better estimates for residuals of the approximation function U_ε. If the right-hand side has the form $f_\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k f_k(x)$, then we can define the other terms in the asymptotic expansions (4.2), (4.3) and (4.9).

From asymptotic estimates (4.27) and (4.28) we come to the following conclusion: For applied problems or for numerical calculation in thick periodic junctions, we can use the resulting problem instead of the initial problem with sufficient probability. The similar boundary-value problem for the elasticity equations in thick periodic junction of type 3 : 2 : 1 is prepared for publication.

I would like to note that the thin cylinders G_ε in the junction Ω_ε have the same length. It was done only for simplification of the presentations. In the case when the cylinders have various length, and their bases describe some surface $\gamma_-(x')$, $x' \in K$; we must perform the odd extension of the solution to problem (1.1); and then construct the extension $\mathbf{P}_\varepsilon u_\varepsilon$ in the domain $\Omega_0 \cap \{x' \in K,\ \gamma_-(x') < x_3 \le 0\}$ as in Section 2. It should be also noted that extension operators for spectral boundary-value problems in thick periodic junction are uniformly bounded in ε only on finite linear combinations of eigenfunctions. For spectral problem we are not need of additional conditions like condition (1.3) (see [16, 20]).

Acknowledgment. The author is grateful to the Alexander von Humboldt Foundation and Prof. W. L. Wendland for the possibility to carry out this research at the University of Stuttgart.

References

[1] Argatov, I. I. and S. A. Nazarov: *Asymptotic solution to the problem of an elastic body lying on several small supports.* J. Appl. Math. Mech. 58 (1994), 303 – 311.

[2] Benkaddour, A. and J. Sanchez-Hubert: *Spectral study of a coupled compact-noncompact problem.* Model. Math. Anal. Numer. 26 (1992), 659 – 672.

[3] Bensoussan, A., Lions, J. L. and G. Papanicolaou: *Asymptotic Analysis of Periodic Structures.* Amsterdam: North-Holland 1978.

[4] Brizzi, R. and J. P. Chalot: *Homogenization and Neumann Boundary Value Problem.* Ric. Mat. 46 (1997), 347 – 387.

[5] Ciarlet, P. G.: *Plates and Junctions in Elastic Multi-Structures.* Paris: Masson 1990.

[6] Cioranescu, D. and J. Saint Jean Paulin: *Homogenization in open sets with holes.* J. Math. Anal. Appl. 71 (1979), 590 – 607.

[7] Fleury, F. and E. Sanchez-Palencia: *Asymptotic and spectral properties of the acoustic vibrations of body, perforated by narrow channels.* Bull. Sci. Math 2 (1986), 149 – 176.

[8] Gryniv, R. O. and T. A. Mel'nyk: *On a singular Rayleigh functional* (in Russian). Matem. zametki 60 (1996), 130 – 134.

[9] Kondrat'ev, V. A. and O. A. Oleinik: *Boundary-value problems for partial differential equations in non-smooth domains* (in Russian). Uspekhi Mat. Nauk 38 (1983)2, 3 – 76.

[10] Kotliarov, V. P. and E. Ya. Khruslov: *On a limit boundary condition of some Neumann problem* (in Russian). Theor. Funkts., Funkts. Anal. Prilozhen, Izd-vo Kharkov Univ. 10 (1970), 83 – 96.

[11] Kozlov, V. A., Maz'ya, V. G. and A. B. Movchan: *Asymptotic analysis of a mixed boundary value problem in a multi-structure.* Asymp. Anal. 8 (1994), 105 – 143.

[12] Kozlov, V. A., Maz'ya, V. G. and A. B. Movchan: *Asymptotic representation of an elastic field in a multi-structure.* Asymp. Anal. 11 (1995), 343 – 415.

[13] Ladyzhenskaya, O. A.: *The Boundary Value Problems of Mathematical Physics.* Berlin: Springer-Verlag 1985.

[14] Maz'ya, V. G. and B. S. Poborchi: *On extension of functions in Sobolev spaces to the exterior or interior of a small domain* (in Russian). Vestnik Leningrdskovo Gos. Univ. 7 (1984), 27 – 32.

[15] Mel'nyk, T. A.: *Asymptotic analysis of the spectral boundary-value problems in the periodic thick singularly degenerate junctions of the different types.* In: Multiple Scale Analyses and Coupled Physical Systems. Proc. Saint-Venant Symposium, Paris, August 28 - 29, 1997. Paris: Presses des Ponts et Chaussees 1997, pp. 453 – 459.

[16] Mel'nyk, T. A.: *Asymptotic analysis of a spectral problem in a periodic thick junction of type 3:2:1.* Preprint. Stuttgart: University, Preprint 98/20 of SFB 404, 1998.

[17] Mel'nyk, T. A. and S. A. Nazarov: *The asymptotic structure of the spectrum in the problem of harmonic oscillations of a hub with heavy spokes* (in Russian). Dokl. Akad. Nauk Russia 333 (1993), 13 – 15; Engl. transl.: Russ. Acad. Sci. Dokl. Math. 48 (1994), 428 – 432.

[18] Mel'nyk, T. A. and S. A. Nazarov: *Asymptotic structure of the spectrum of the Neumann problem in a thin comb-like domain.* C.R. Acad Sci. Paris (Serie 1) 319 (1994), 1343 – 1348.

[19] Mel'nyk, T. A. and S. A. Nazarov: *On the asymptotics of eigenvalues of the Neumann problem in a domain of "tooth-comb" type* (in Russian). Dokl. Akad. Nauk Russia 342 (1995), 23 – 25; Engl. transl.: Doklady Math. 51 (1995), 326 – 328.

[20] Mel'nyk, T. A. and S. A. Nazarov: *Asymptotics of the Neumann spectral problem solution in a domain of "thick comb" type* (in Russian). Trudy Seminara imeni I. G. Petrovskogo 19 (1996), 138 – 173; Engl. transl.: J. Math. Sci. 85 (1997), 2326 – 2346.

[21] Mikhajlov, V. P.: *Partial Differential Equations* (in Russian). Moscow: Nauka 1976.

[22] Nazarov, S. A.: *Junctions of singularly degenerating domains with different limit dimensions* (in Russian). Trudy Seminara imeni I. G. Petrovskogo 18 (1995), 3 – 79.

[23] Nazarov, S. A. and B. A. Plamenevskii: *Asymptotics of the spectrum of the Neumann problem in singularly degenerating thin domains* (in Russian). Algebra and Analiz 2 (1990), 85 – 111.

[24] Nazarov, S. A. and B. A. Plamenevskii: *Elliptic Problems in Domains with Piecewise Smooth Boundaries.* Berlin - New York: Walter de Gruyter 1994.

[25] Oleinik, O. A., Yosifyan, G. A. and A. S. Shamaev: *Mathematical Problems in Elasticity and Homogenization.* Amsterdam: North-Holland 1992.

[26] Pironneau: *Optimal Shape Design for Elliptic Systems.* Berlin - Heidelberg: Springer-Verlag 1984.

[27] Sanchez-Hubert, J. and E. Sanchez-Palencia: *Vibration and Coupling of Continuous Systems.* Berlin - Heidelberg: Springer-Verlag 1989.

[28] Stein, E. M.: *Singular Integrals and Differentiability Properties of Functions.* Princeton: Univ. Press. 1970.

[29] Suzikov, G. V. and E. Ya. Khruslov: *On advancing sound waves through narrow channels in a reflecting layer* (in Russian). Theor. Funkts., Funkts. Anal. Prilozhen, Izd-vo Kharkov Univ. 5 (1967), 35 – 49.

[30] Zhikov, V. V., Kozlov, S. M. and O. A. Oleinik: *Homogenization of Differential Operators and Integral Functionals.* Berlin: Springer 1994.

Received 22.01.1999