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On the Hilbert Inequality

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Abstract. It is shown that the Hilbert inequality for double series can be improved by introducing the positive real number $\frac{1}{\pi^2}\left(\frac{s^2(a)}{\|a\|^2}+\frac{s^2(b)}{\|b\|^2}\right)$ where $s(x)=\sum_{n=1}^{\infty}\frac{x_n}{n}$ and $\|x\|^2=\sum_{n=1}^{\infty}x_n^2$ $(x=a,b)$. The coefficient π of the classical Hilbert inequality is proved not to be the best possible if $\|a\|$ or $\|b\|$ is finite. A similar result for the Hilbert integral inequality is also proved.

Keywords: *Hilbert inequality, binary quadratic form, exponential integral, inner product*

AMS subject classification: 26 D, 46 C

1. Introduction

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be arbitrary real sequences. Then the *Hilbert inequality* for double series can be written as

$$\left(\sum_{m,n=1}^{\infty}\frac{a_m b_n}{m+n}\right)^2 \leq \pi^2\left(\sum_{n=1}^{\infty}a_n^2\right)\left(\sum_{n=1}^{\infty}b_n^2\right). \tag{1}$$

Additionally,

$$\left(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{a_m b_n}{m-n}\right)^2 \leq \pi^2\left(\sum_{n=1}^{\infty}a_n^2\right)\left(\sum_{n=1}^{\infty}b_n^2\right) \tag{2}$$

is also called *Hilbert inequality*. Furthermore, if $f,g\in L^2(\mathbb{R}_+)$ where $\mathbb{R}_+=(0,\infty)$, then the inequality analogous to (1)

$$\left(\iint_{\mathbb{R}_+^2}\frac{f(s)g(t)}{s+t}\,dsdt\right)^2 \leq \pi^2\left(\int_{\mathbb{R}_+}f^2(t)\,dt\right)\left(\int_{\mathbb{R}_+}g^2(t)\,dt\right) \tag{3}$$

is called the *Hilbert integral inequality*. The constant π contained in these inequalities, especially in (1), was proved to be the best possible (see [3]). However, if $0<\sum_{n=1}^{\infty}a_n^2<\infty$ or $0<\sum_{n=1}^{\infty}b_n^2<\infty$, then we can select a number $r>0$ such that the right-hand side of (1) can be replaced by

$$\pi^2(1-r)\left(\sum_{n=1}^{\infty}a_n^2\right)\left(\sum_{n=1}^{\infty}b_n^2\right),$$

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i.e. an improvement of (1) will be obtained. Similarly, an improvement of (3) will be established. Namely, the right-hand side of (3) can be written as

$$\pi^2(1-R)\Big(\int_{\mathbb{R}_+} f^2(t)\,dt\Big)\Big(\int_{\mathbb{R}_+} g^2(t)\,dt\Big)$$

with a number $R > 0$. The main purpose of the present paper is to prove the existence of such numbers r and R and to find expressions for them.

We first introduce some notations and functions.

If α and β are elements of an inner product space E, then its inner product is denoted by (α, β) and the norm of α is given by $\|\alpha\| = \sqrt{(\alpha, \alpha)}$. Further, if $a = (a_n)_{n\geq 1}$ and $b = (b_n)_{n\geq 1}$ are two real sequences, then its inner product (a, b) and the norm $\|a\|$ of a are defined by

$$(a,b) = \sum_{n=1}^{\infty} a_n b_n \qquad \text{and} \qquad \|a\| = \sqrt{(a,a)}. \tag{4}$$

Analogously, for functions $f, g \in L^2(a, b)$ its inner product (f, g) and the norm $\|f\|$ of f are defined by

$$(f,g) = \int_a^b f(t)g(t)\,dt \qquad \text{and} \qquad \|f\| = \Big(\int_a^b f^2(t)\,dt\Big)^{\frac{1}{2}}. \tag{5}$$

We next introduce a binary quadratic form $F(\cdot,\cdot)$ defined by

$$F(x,y) = \|\alpha\|^2 x^2 - 2(\alpha,\beta)xy + \|\beta\|^2 y^2 \tag{6}$$

where $x = (\beta, \gamma)$ and $y = (\alpha, \gamma)$ for $\gamma \in E$. We further denote

$$G(\alpha,\beta,\gamma) = F((\beta,\gamma),(\alpha,\gamma)). \tag{7}$$

The results involve $G(\alpha, \beta, \gamma)$ with α and β specified beforehand, and γ to be chosen for maximum felicity. It is obvious that if γ is orthogonal to both α and β, then $G(\alpha, \beta, \gamma) = 0$. It will turn out that if $(\alpha, \gamma)^2 + (\beta, \gamma)^2 > 0$ (see Lemma 1). Therefore, it is shrewd in every case to choose γ not orthogonal to both α and β.

For convenience, we introduce yet the notations

$$u(a,b) = \sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n}, \quad v(a,b) = \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{a_m b_n}{m-n}, \qquad s(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}.$$

We shall frequently use these notations below.

2. Lemmas

To prove our theorems, we need the following results.

Lemma 1. *Let $G(\alpha,\beta,\gamma)$ be defined as in* (7)*. If $\alpha,\beta \in E$ are linearly independent and $(\alpha,\gamma)^2+(\beta,\gamma)^2>0$, then $G(\alpha,\beta,\gamma)>0$.*

Lemma 2. *Let $G(\alpha,\beta,\gamma)$ be defined as defined in* (7)*. If $\alpha,\beta \in E$ are linearly dependent, then $G(\alpha,\beta,\gamma)=0$.*

Lemma 3. *Let $G(\alpha,\beta,\gamma)$ be defined as defined in* (7)*. If $\alpha,\beta \in E$ are arbitrary and $\gamma \in E$ with $\|\gamma\|=1$, then*

$$(\alpha,\beta)^2 \le \|\alpha\|^2\|\beta\|^2 - G(\alpha,\beta,\gamma), \tag{8}$$

and equality holds in (8) *if and only if α,β,γ are linearly dependent.*

The proofs of Lemmas 1 and 2 have been given in our previous paper [1]. Lemma 3 is actually a sharpening of the Cauchy-Schwarz inequality. This result has been given also in the paper [1], and in [5]. Hence the proofs of all lemmas are omitted.

Using the inner product defined by (5) and Lemma 3, we have the following result.

Corollary 1. *If $f,g \in L^2(a,b)$, then*

$$(f,g)^2 \le \|f\|^2\|g\|^2 - F(x,y) \tag{9}$$

where $F(x,y)=\|f\|^2x^2-2(f,g)xy+\|g\|^2y^2$ with $x=(g,\gamma)$ and $y=(f,\gamma)$, $\gamma \in L^2(a,b)$ with $\|\gamma\|=1$.

3. Main results

In this section we will combine the two forms (1) and (2) of the Hilbert inequality into one similar form, and make inequalities (1) - (3) relaize significant improvements. The following theorems are the main results in this paper.

Theorem 1. *If $a=(a_n)$ and $b=(b_n)$ are real sequences with non-negative terms, with $0<\|a\|<\infty$ or $0<\|b\|<\infty$, then*

$$u^2(a,b)+v^2(a,b) < \pi^2(1-r)\|a\|^2\|b\|^2 \tag{10}$$

where $r=\frac{1}{\pi^2}\big(\frac{s^2(a)}{\|a\|^2}+\frac{s^2(b)}{\|b\|^2}\big)$.

Proof. Let us define two real functions $f,g:(0,2\pi)\to\mathbb{R}$ by

$$f(t)=\sum_{n=1}^{\infty} a_n\sqrt{t}\sin(nt) \qquad \text{and} \qquad g(t)=\sum_{n=1}^{\infty} b_n\sqrt{t}\cos(nt).$$

It is easily to deduce that, with the notations of the space $L^2(0,2\pi)$,

$$|u(a,b)+v(a,b)| = \tfrac{1}{\pi}|(f,g)|. \tag{11}$$

According to (5) and (6) we have $(f,g)^2 \le \|f\|^2\|g\|^2 - F(x,y)$ where $\|f\|^2 = \pi^2\|a\|^2$, $\|g\|^2 = \pi^2\|b\|^2$ and

$$F(x,y) = \|f\|^2x^2 - 2(f,g)xy + \|g\|^2y^2 \ge (\|f\|x - \|g\|y)^2 = \pi^2(\|a\|x - \|b\|y)^2.$$

Hence

$$(f,g)^2 \le \pi^4\|a\|^2\|b\|^2 - \pi^2(\|a\|x - \|b\|y)^2 \tag{12}$$

where $x = (g,\gamma)$ and $y = (f,\gamma)$, $\gamma \in L^2(0,2\pi)$ with $\|\gamma\| = 1$. We can choose $\gamma = \frac{1}{2\pi}\sqrt{2t}$. Then $x = 0$ and $y = -\sqrt{2}\sum_{n=1}^{\infty}\frac{a_n}{n} = -\sqrt{2}\,s(a)$. Hence

$$(\|a\|x - \|b\|y)^2 = 2\|b\|^2s^2(a). \tag{13}$$

In virtue of (11) - (13) we obtain

$$|u(a,b) + v(a,b)|^2 \le \pi^2\|a\|^2\|b\|^2 - 2\|b\|^2s^2(a). \tag{14}$$

Since the vectors f, g, γ are linearly independent, by Lemma 3, it is impossible to take equality in (14). Hence we have

$$|u(a,b) + v(a,b)|^2 < \pi^2\|a\|^2\|b\|^2 - 2\|b\|^2s^2(a). \tag{15}$$

Notice that $u(b,a) = u(a,b)$ and $v(b,a) = -v(a,b)$. Interchanging a and b in (11), similarly we obtain

$$|u(a,b) - v(a,b)|^2 < \pi^2\|a\|^2\|b\|^2 - 2\|a\|^2s^2(b). \tag{16}$$

Adding (15) and (16), inequality (10) is yielded after some simplifications. Thus the proof of the theorem is completed ∎

Remark. Since $a = (a_n)$ and $b = (b_n)$ are real sequences with non-negative terms, with $0 < \|a\| < \infty$ or $0 < \|b\| < \infty$, it follows that $r > 0$. Hence inequality (10) is a significant refinement of the paper [4].

Corollary 2. *If $a = (a_n)$ is a real sequence with non-negative terms and $0 < \|a\| < \infty$, then*

$$u^2(a,a) + v^2(a,a) < \pi^2(1 - \tilde{r})\|a\|^4 \tag{17}$$

where $\tilde{r} = \frac{2}{\pi^2}\frac{s^2(a)}{\|a\|^2}$.

If $v^2(a,b)$ in (10) is replaced by 0, then we have the following

Corollary 3. *With the assumptions of Theorem 1, then*

$$u^2(a,b) < \pi^2(1 - r)\|a\|^2\|b\|^2 \tag{18}$$

where $r = \frac{1}{\pi^2}\left(\frac{s^2(a)}{\|a\|^2} + \frac{s^2(b)}{\|b\|^2}\right)$.

We see from the above Remark that inequality (18) is a significant improvement of (1). According to Corollary 2 we obtain at once the following

Corollary 4. *If $a = (a_n)$ is a real sequence with non-negative terms and $0 < \|a\| < \infty$, then*

$$u^2(a,a) < \pi^2(1-\tilde{r})\|a\|^4 \tag{19}$$

where $\tilde{r} = \frac{2}{\pi^2}\frac{s^2(a)}{\|a\|^2}$.

Similarly, we can establish an improvement of the Hilbert integral inequality. For this we need the integral

$$e(t) = \int_{\mathbb{R}_+} \frac{e^{-s}}{s+t}\, ds \qquad (t \in \mathbb{R}_+)$$

called *exponential integral with parameter t.*

Theorem 2. *Let $f, g \in L^2(\mathbb{R}_+)$ be positive. Then*

$$\left(\iint_{\mathbb{R}_+^2} \frac{f(s)g(t)}{s+t}\, dsdt \right)^2 < \pi^2(1-R)\|f\|^2\|g\|^2 \tag{20}$$

where $R = \frac{1}{\pi}\left(\frac{x}{\|g\|} - \frac{y}{\|f\|}\right)^2$ with $x = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(g,e)$ and $y = (2\pi)^{\frac{1}{2}}(f, e^{-s})$, e being the exponential integral with parameter.

Proof. Define functions F and G by

$$F(s,t) = \frac{f(s)}{(s+t)^{\frac{1}{2}}}\left(\frac{s}{t}\right)^{\frac{1}{4}} \qquad \text{and} \qquad G(s,t) = \frac{g(t)}{(s+t)^{\frac{1}{2}}}\left(\frac{t}{s}\right)^{\frac{1}{4}}.$$

Using inequality (9) we have in $L^2(\mathbb{R}_+^2)$

$$\begin{aligned} \left(\iint_{\mathbb{R}_+^2} \frac{f(s)g(t)}{s+t}\, dsdt \right)^2 &= (F,G)^2 \\ &\le \|F\|^2\|G\|^2 - F(x,y) \\ &\le \|F\|^2\|G\|^2 - (\|F\|x - \|G\|y)^2 \end{aligned} \tag{21}$$

where $x = (G,\gamma)$ and $y = (F,\gamma)$, $\gamma \in L^2(\mathbb{R}_+^2)$ with $\|\gamma\| = 1$. We can choose

$$\gamma(s,t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{e^{-s}}{(s+t)^{\frac{1}{2}}}\left(\frac{s}{t}\right)^{\frac{1}{4}}.$$

Hence we get

$$x = \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}(g,e) \qquad \text{and} \qquad y = (2\pi)^{\frac{1}{2}}(f, e^{-s}). \tag{22}$$

It is easy to deduce that

$$\|F\|^2 = \pi\|f\|^2 \qquad \text{and} \qquad \|G\|^2 = \pi\|g\|^2. \tag{23}$$

Substituting (22) and (23) into (21) we obtain

$$(F,G)^2 \le \pi^2\|f\|^2\|g\|^2 - \pi(\|f\|x - \|g\|y)^2. \tag{24}$$

Since F, G, γ are linearly independent, it is impossible to have equality in (24). Consequently, inequality (20) is obtained from (24) after some simplifications. Thus the theorem is proved ∎

Corollary 5. *If $f \in L^2(\mathbb{R}_+)$ is positive, then*

$$\left(\iint_{\mathbb{R}_+^2} \frac{f(s)f(t)}{s+t}\, dsdt\right)^2 < \pi^2(1-\tilde{R})\|f\|^4$$

where $\tilde{R} = \frac{1}{\pi}\frac{(x-y)^2}{\|f\|^2}$ with $x = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(f,e)$ and $y = (2\pi)^{\frac{1}{2}}(f,e^{-s})$, e being the exponential integral with parameter.

Obviously, this is an immediate consequence of Theorem 2.

4. Conclusions

Some classical reasults concerning the Hilbert inequality show that the constant π in (1) is the best possible (see, i.e., [1, 2, 5, 6]). We see from (18) that inequality in (1) can be obtained only if $r = 0$. However, to change r into 0, it is necessary to take both $\|a\|$ and $\|b\|$ infinite. Therefore, generally, the constant π in (1) is not the best possible because the constant r contained in (18) is not equal to 0 if $\|a\|$ or $\|b\|$ is finite. In other words, the factor π in (1) can be decreased if $0 < \|a\| < \infty$ or $0 < \|b\| < \infty$.

Similarly, we see from (20) that strong inequality in (3) can be obtained only if $R = 0$. In other words, the factor π in (3) is also not the best possible if $\|f\|$ or $\|g\|$ is finite.

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