Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 4, 1101–1106

A Kneser-Type Theorem for the Equation $x^{(m)} = f(t,x)$ in Locally Convex Spaces

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Abstract. We shall give sufficient conditions for the existence of solutions of the Cauchy problem for the equation $x^{(m)} = f(t,x)$. We also prove that the set of these solutions is a continuum.

Keywords: *Differential equations, set of solutions, measures of non-compactness*

AMS subject classification: 34 G 20

Let E be a quasicomplete locally convex topological vector space, and let P be a family of continuous seminorms generating the topology of E. Assume that $I = [0,a]$ and $B = \{x \in E : p_i(x) \le b \ \ (i = 1,\ldots,k)\}$, where $p_1,\ldots,p_k \in P$.

In this paper we investigate the existence of solutions and the structure of the set of solutions of the Cauchy problem

$$\left.\begin{aligned} x^{(m)} &= f(t,x) \\ x(0) &= 0 \\ x'(0) &= \eta_1 \\ &\vdots \\ x^{(m-1)}(0) &= \eta_{m-1} \end{aligned}\right\} \tag{1}$$

where m is a positive integer, $\eta_1,\eta_2,\ldots,\eta_{m-1} \in E$ and f is a bounded continuous function from $I \times B$ into E. Our considerations are a continuation of Szufla's paper [8]. For other results concerning differential equations in locally convex spaces see [4].

Put

$$M = \sup\Big\{p_i(f(t,x)) : t \in I, x \in B, i = 1,\ldots,k\Big\}.$$

Choose a positive number d such that $d \le a$ and

$$\sum_{j=1}^{m-1} p_i(\eta_j)\frac{d^j}{j!} + M\frac{d^m}{m!} \le b \qquad (i = 1,\ldots,k). \tag{2}$$

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ISSN 0232-2064 / \$ 2.50

Let $J = [0, d]$. Denote by $C = C(J, E)$ the space of all continuous functions from J into E endowed with the topology of uniform convergence.

For any bounded subset A of E and $p \in P$ we denote by $\beta_p(A)$ the infimum of all $\varepsilon > 0$ for which there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of E such that $A \subset \{x_1, x_2, \ldots, x_n\} + B_p(\varepsilon)$, where $B_p(\varepsilon) = \{x \in E : p(x) \leq \varepsilon\}$. The family $(\beta_p(A))_{p \in P}$ is called the *measure of non-compactness* of A. It is known [6] that:

1° X is relatively compact in $E \iff \beta_p(X) = 0$ for every $p \in P$.

2° $X \subset Y \implies \beta_p(X) \leq \beta_p(Y)$.

3° $\beta_p(X \cup Y) = \max\{\beta_p(X), \beta_p(Y)\}$.

4° $\beta_p(X + Y) \leq \beta_p(X) + \beta_p(Y)$.

5° $\beta_p(\lambda X) = |\lambda| \beta_p(X) \quad (\lambda \in \mathbb{R})$.

6° $\beta_p(\bar{X}) = \beta_p(X)$.

7° $\beta_p(\operatorname{conv} X) = \beta_p(X)$.

8° $\beta_p(\cup_{0 \leq \lambda \leq h} \lambda X) = h \beta_p(X)$.

The following lemma is given in [8].

Lemma 1. *Let H be a bounded countable subset of C. For each $t \in J$ put $H(t) = \{u(t) : u \in H\}$. If the space E is separable, then for each $p \in P$ the function $t \mapsto \beta_p(H(t))$ is integrable and*

$$\beta_p\left(\left\{\int_J u(s)\,ds : u \in H\right\}\right) \leq \int_J \beta_p(H(s))ds.$$

Moreover, let us recall the following lemma from [9].

Lemma 2. *Let $w : [0, 2b] \mapsto \mathbb{R}_+$ be a continuous non-decreasing function and let $g : [0, c) \mapsto [0, 2b]$ be a C^m-function satisfying the inequalities*

$$\begin{aligned} & g^{(j)}(t) \geq 0 && (j = 0, 1, \ldots, m) \\ & g^{(j)}(0) = 0 && (j = 0, 1, \ldots, m-1) \\ & g^{(m)}(t) \leq w(g(t)) && (t \in [0, c)). \end{aligned}$$

If $w(0) = 0$, $w(r) > 0$ for $r > 0$ and $\int_{0+} \left(r^{m-1} w(r)\right)^{-\frac{1}{m}} dr = \infty$, then $g = 0$.

We can now formulate our main result.

Theorem. *Suppose that for each $p \in P$ there exists a continuous non-decreasing function $w_p : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $w_p(0) = 0$, $w_p(r) > 0$ for $r > 0$ and*

$$\int_{0+} \frac{dr}{\sqrt[m]{r^{m-1} w_p(r)}} = \infty. \tag{3}$$

If

$$\beta_p(f(t, X)) \leq w_p(\beta_p(X)) \tag{4}$$

for $p \in P$, $t \in I$ and bounded subsets X of E, then the set S of all solutions of problem (1) *defined on J is non-empty, compact and connected in $C(J,E)$.*

Proof. 1° Put

$$r(x) = \begin{cases} x & \text{for } x \in B \\ \frac{x}{K(x)} & \text{for } x \in E \setminus B \end{cases}$$

and $g(t,x) = f(t, r(x))$ for $(t,x) \in J \times E$, where K is the Minkowski functional of B. As B is a closed, balanced and convex neighbourhood of 0, from known properties of the Minkowski functional it follows that r is a continuous function from E into B and

$$r(X) \subset \bigcup_{0 \le \lambda \le 1} \lambda X \qquad \text{for any subset } X \text{ of } E.$$

Thus $\beta_p(r(X)) \le \beta_p(X)$ for any $p \in P$ and any bounded subset X of E. Consequently, g is a bounded continuous function from $J \times E$ into E such that

$$\beta_p(g(t,X)) \le w_p(\beta_p(X)) \tag{4$'$}$$

for $p \in P$, $t \in J$ and bounded subsets X of E and

$$p_i(g(t,x)) \le M \qquad (i = 1, \dots, k;\ t \in J, x \in E). \tag{5}$$

We introduce a mapping F defined by

$$F(x)(t) = q(t) + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} g(s, x(s))\, ds \qquad (t \in J, x \in C)$$

where $q(t) = \sum_{j=1}^{m-1} \eta_j \frac{t^j}{j!}$. It is known (cf. [2]) that F is a continuous mapping $C \mapsto C$ and the set $F(C)$ is bounded and equicontinuous. It is clear from (1) and (5) that if $x = F(x)$, then

$$\begin{aligned} p_i(x(t)) &\le \sum_{j=1}^{m-1} p_i(\eta_j) \frac{d^j}{j!} + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} M\, ds \\ &\le \sum_{j=1}^{m-1} p_i(\eta_j) \frac{d^j}{j!} + M \frac{d^m}{m!} \qquad\qquad (i = 1, \dots, k) \\ &\le b \end{aligned}$$

so $x(t) \in B$ for $t \in J$. Therefore, a function $x \in C$ is a solution of problem (1) if and only if $x = F(x)$.

2° For any $n \in \mathbb{N}$ put

$$u_n(t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{d}{n} \\ q(t - \frac{d}{n}) + \frac{1}{(m-1)!} \int_0^{t-\frac{d}{n}} (t-s)^{m-1} g(s, u_n(s))\, ds & \text{if } \frac{d}{n} \le t \le d. \end{cases}$$

Then u_n is a continuous function $J \mapsto B$ and

$$\lim_{n\to\infty} \big(u_n(t) - F(u_n)(t)\big) = 0 \tag{6}$$

uniformly for $t \in J$. Let $V = \{u_n : n \in \mathbb{N}\}$. From (6) it follows that the set $\{u_n - F(u_n) : n \in \mathbb{N}\}$ is relatively compact in C. Since

$$V \subset \{u_n - F(u_n) : n \in \mathbb{N}\} + F(V) \tag{7}$$

and the set $F(V)$ is bounded and equicontinuous, we conclude that the set V is also bounded and equicontinuous. Hence for each $p \in P$ the function $t \mapsto \beta_p(V(t))$ is continuous on J. Denote by H a closed separable subspace of E such that

$$g(s, u_n(s)) \in H \qquad (s \in J, n \in \mathbb{N}).$$

Let $(\beta_p^H)_{p\in P}$ be the measure of non-compactness in H. Fix $t \in J$ and $p \in P$. From $(4)'$ we have

$$\beta_p^H\big(g(s, V(s))\big) \le 2\beta_p\big(g(s, V(s))\big) \le 2w_p\big(\beta_p(V(s))\big) \qquad (s \in [0, t]).$$

By Lemma 1, we get

$$\begin{aligned}
\beta_p(F(V)(t)) &= \beta_p\left(\left\{\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} g(s, u_n(s))\, ds : n \in \mathbb{N}\right\}\right)\\
&\le \beta_p^H\left(\left\{\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} g(s, u_n(s))\, ds : n \in \mathbb{N}\right\}\right)\\
&\le \frac{1}{(m-1)!}\int_0^t \beta_p^H\left(\left\{(t-s)^{m-1} g(s, u_n(s)) : n \in \mathbb{N}\right\}\right) ds\\
&= \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} \beta_p^H(g(s, V(s))\, ds\\
&\le \frac{2}{(m-1)!}\int_0^t (t-s)^{m-1} w_p(\beta_p(V(s))\, ds.
\end{aligned}$$

On the other hand, from (6) and (7) we obtain

$$\beta_p(V(t)) \le \beta_p(F(V)(t)).$$

Hence

$$\beta_p(V(t)) \le \frac{2}{(m-1)!}\int_0^t (t-s)^{m-1} w_p(\beta_p(V(s))\, ds \qquad (t \in J, p \in P).$$

Putting

$$g(t) = \frac{2}{(m-1)!} \int_0^t (t-s)^{m-1} w_p(\beta_p(V(s))\, ds$$

we see that

$$\begin{aligned}
&g \in C^m \\
&\beta_p(V(t)) \le g(t) \\
&g^{(j)}(t) \ge 0 \text{ for } j = 0,1,\ldots,m \\
&g^{(j)}(0) = 0 \text{ for } j = 0,1,\ldots,m-1 \\
&g^{(m)}(t) = 2w_p(\beta_p(V(t))) \le 2w_p(g(t)) \text{ for } t \in J.
\end{aligned}$$

Moreover, by (3),

$$\int_{0+} \frac{dr}{\sqrt[m]{r^{m-1} 2w_p(r)}} = \infty.$$

By Lemma 2 from this we deduce that $g(t) = 0$ for $t \in J$. Thus $\beta_p(V(t)) = 0$ for $t \in J$ and $p \in P$. Therefore for each $t \in J$ the set $V(t)$ is relatively compact in E. As the set V is equicontinuous, Ascoli's theorem proves that V is relatively compact in C. Hence the sequence (u_n) has a limit point u. As F is continuous from (6) we conclude that $u = F(u)$, i.e. u is a solution of problem (1). This proves that the set S is non-empty.

3° Let us first remark that the set S is compact in C. Indeed, as $(I-F)(S) = \{0\}$, in the same way as in Step 2°, we can prove that S is relatively compact in C. Moreover, from the continuity of F it follows that S is closed in C. Suppose that S is not connected. Thus there exist non-empty closed sets S_0 and S_1 such that $S = S_0 \cup S_1$ and $S_0 \cap S_1 = \emptyset$. As S_0 and S_1 are compact subsets of C and C is a Tichonov space, this implies (see [3: §41, II, Remark 3]) the existence of a continuous function $v : C \mapsto [0,1]$ such that $v(x) = 0$ for $x \in S_0$ and $v(x) = 1$ for $x \in S_1$. Further, for any $n \in \mathbb{N}$ we define a mapping F_n by

$$F_n(x)(t) = F(x)(r_n(t)) \qquad (x \in C, t \in J)$$

where

$$r_n(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{d}{n} \\ t - \frac{d}{n} & \text{for } \frac{d}{n} \le t \le d. \end{cases}$$

It can be easily verified (cf. [10]) that:

(i) F_n is a continuous mapping $C \mapsto C$.

(ii) $\lim_{n\to\infty} F_n(x) = F(x)$ uniformly for $x \in C$.

(iii) $I - F_n$ is a homeomorphism $C \mapsto C$ (I - identity mapping).

Fix $u_0 \in S_0$, $u_1 \in S_1$ and $n \in \mathbb{N}$. Put

$$e_n(\lambda) = \lambda(u_1 - F_n(u_1)) + (1-\lambda)(u_0 - F_n(u_0)) \qquad (0 \le \lambda \le 1).$$

Let $u_{n\lambda} = (I - F_n)^{-1}(e_n(\lambda))$. As $e_n(\lambda)$ depends continuosly on λ and $I - F_n$ is a homeomorphism, we see that the mapping $\lambda \mapsto v(u_{n\lambda})$ is continuous on $[0,1]$. Moreover,

$u_{n0} = u_0$ and $u_{n1} = u_1$, so that $v(u_{n0}) = 0$ and $v(u_{n1}) = 1$. Thus there exists $\lambda_n \in [0,1]$ such that

$$v(u_{n\lambda_n}) = \frac{1}{2}. \tag{8}$$

For simplicity put $v_n = u_{n\lambda_n}$ and $V = \{v_n : n \geq 1\}$. Since $\lim_{n\to\infty} e_n(\lambda) = 0$ uniformly for $\lambda \in [0,1]$, we get

$$\lim_{n\to\infty} (v_n - F(v_n)) = \lim_{n\to\infty} (e_n(\lambda) + F_n(v_n) - F(v_n)) = 0 \tag{9}$$

and therefore the set $(I-F)(V)$ is relatively compact in C. Using now a similar argument as in Step 2°, we can prove that the set V is relatively compact in C. Consequently, the sequence (v_n) has a limit point z. In view of (9) and the continuity of F, we infer that $z \in S$, so $v(z) = 0$ or $v(z) = 1$. On the other hand, from (8) it is clear that $v(z) = \frac{1}{2}$, which yields a contradiction. Thus S is connected ∎

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Received 28.10.1998