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Estimates for Quasiconformal Mappings onto Canonical Domains

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Abstract. In this paper we establish estimates for K-quasiconformal mappings $z = g(w)$ of a domain bounded by two circles $|w| = 1, |w| = q$ and n continua situated in $q < |w| < 1$ onto a circular ring $Q(g) < |z| < 1$ that has been slit along n arcs on the circles $|z| = R_j(g)$ $(j = 1, \ldots, n)$ such that $|z| = 1$ and $|z| = Q$ correspond to $|w| = 1$ and $|w| = q$, respectively. The bounds in the estimates for Q, R_j and $|g(w)|$ are explicitly given, most of them are optimal. They are deduced mainly from [17].

Keywords: *K-quasiconformal mappings, Riemann moduli of a multiply-connected domain, monotony of the modulus of a doubly-connected domain*

AMS subject classification: 30 C 62, 30 C 75, 30 C 80, 30 C 30

1. Introduction and notations

The generalization of Carleman's (see [1: p. 212], [2: p. 177], [12: p. 15]) area inequality for doubly-connected domains to multiply-connected domains in [15] improves many Grötzsch's [4, 6, 8] and Rengel's [13] significant circular slits theorems for conformal mappings. In [16] we establish further area inequalities for *K-quasiconformal mappings* (see the definition in [10: p. 16]). Combining this with Grötzch's [4, 5, 7] inequalities yields in [17] sharp or asymptotic sharp estimates for K-quasiconformal mappings of the circular ring $Q < |z| < 1$ with concentric circular slits onto domains lying in $q < |w| < 1$. In this paper we establish estimates for the inverse mappings of those studied in [17]. Here the consideration is partly similar to the case of conformal mappings ($K = 1$, see [14: pp. 121 - 124]) using two auxiliary functions introduced in Section 4.

Let now B be any domain given in the w-plane, bounded by two circles $|w| = 1$, $|w| = q$ and pn $(p, n \in \mathbb{N})$ boundary components $\sigma_1, \ldots, \sigma_{pn}$ lying in $(0 <)$ $q < |w| < 1$, and transformed into itself by the rotation $t = e^{i\frac{2\pi}{p}} w$. We shall write $B = B_0$ when all σ_j are circular arcs concentric with the circular ring. Let G be the family of all K-quasiconformal mappings $z = g(w)$ each of which maps B onto a circular ring $Q(g) < |z| < 1$ that has been slit along pn circular arcs $L_1(g), \ldots, L_{pn}(g)$ concentric with the

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circular ring such that $|z| = 1$, $|z| = Q$ and L_j correspond to $|w| = 1$, $|w| = q$ and σ_j $(j = 1, \ldots, pn)$, respectively, $g(1) = 1$ and satisfies the p times rotation symmetry

$$g(e^{i\frac{2\pi}{p}} w) = e^{i\frac{2\pi}{p}} g(w) \qquad (w \in B). \tag{1.1}$$

It is clear that the symmetry condition is trivial for $p = 1$. Therefore we see that the inverse mapping f of g, $g \in G$, belongs to the family F studied in [17], because from (1.1) for all $z \in A$, $A = g(B)$, and $f = g^{-1}$, $g \in G$, the relation

$$e^{i\frac{2\pi}{p}} f(z) = f(e^{i\frac{2\pi}{p}} z)$$

follows. Put

$R_j(g) = |z|$ with $z \in L_j(g)$ and $g \in G$
$c_j = \min |w|$ and $d_j = \max |w|$ with $w \in \sigma_j$ $(j = 1, \ldots, pn)$
$c = \min\{c_1, \ldots, c_{pn}\}$ and $d = \max\{d_1, \ldots, d_{pn}\}$
$\mu = \sup \prod_{j=1}^{J} \frac{r'_j}{r_j}$, where $r_j < |w| < r'_j$ are pairwise disjoint sets in B.

Furthermore, denote

by S the inner area of B
by s_j the external area of the compact set bounded by σ_j.

Clearly,

$$\frac{c}{qd} \le \mu \le \frac{1}{q} \tag{1.2}$$

and

$$S + s = \pi(1 - q^2) \qquad \text{with} \;\; s = \sum_{j=1}^{pn} s_j. \tag{1.3}$$

Our task is to estimate the radii $Q(g)$ and $R_j(g)$ $(g \in G;\, j = 1, \ldots, pn)$ that are nothing but the *Riemann moduli* of the domain B in the case $K = 1$ (see [11: p. 334]); as well as $|g(w)|$ $(g \in G, w \in B)$, by at most the quantities $K, p, q, c_j, d_j, s_j, \mu, S$ and $|w|$. The bounds in the estimates will be explicitly calculated by simple functions or the auxiliary function $R(p, t, s)$ introduced in Section 4. Most of them are the best among the bounds that depend on the same quantities.

2. Estimates of Q

The estimate of Q plays an important role in establishing estimates for other quantities. Therefore we begin with this estimate.

Theorem 1. *Under the above hypothesis and notations we have, for every $g \in G$, the estimate*

$$\left(1 + \frac{S}{\pi q^2}\right)^{-\frac{K}{2}} \le Q(g) \le \mu^{-\frac{1}{K}}, \tag{2.1}$$

where the equality on the left-hand side holds if and only if $B = B_0$ and $g(w) = w|w|^{K-1}$ $(w \in B)$, and the equality on the right-hand side holds if and only if $B = B_0$ and $g(w) = w|w|^{\frac{1}{K}-1}$ $(w \in B)$.

Proof. Applying [17: Theorem 3.1] to the mapping $f = g^{-1}$, $g \in G$, we have

$$S \geq \pi q^2 (Q^{-\frac{2}{K}} - 1),$$

hence the lower bound of Q in (2.1) follows. Here the equality holds if and only if $f(z) = z|z|^{\frac{1}{K}-1}$ $(z \in A)$, i.e. $B = B_0$ and $g(w) = w|w|^{K-1}$ $(w \in B)$. On the other hand, applying [16: Theorem 1] to the mapping $g \in G$, we obtain

$$\pi \geq \pi Q^2 \mu^{\frac{2}{K}},$$

hence the upper bound of Q in (2.1) follows. Here with the help of [17: Formula (2.5)] we have the assertion on the occurence of the equality ∎

Corollary 1. *By* (1.3), *the lower bound of* Q *in* (2.1) *may be written in the form*

$$Q(g) \geq q^K \left(1 - \frac{s}{\pi}\right)^{-\frac{K}{2}} \qquad (g \in G) \tag{2.2}$$

hence

$$Q(g) \geq q^K \qquad (g \in G). \tag{2.3}$$

Equality in (2.2) *and* (2.3) *can only occur if* $B = B_0$ *and* $g(w) = w|w|^{K-1}$ $(w \in B)$.

Corollary 2. *From* (2.1) *and* (1.2) *we obtain the estimate*

$$Q(g) \leq \left(\frac{dq}{c}\right)^{\frac{1}{K}} (< 1) \qquad (g \in G)$$

where the equality can only occur if $c = d$ *and* $g(w) = w|w|^{\frac{1}{K}-1}$ $(w \in B)$.

3. Lower bounds of R_j

Since $R_j(g) > Q(g)$ $(g \in G;\ j = 1, \ldots, pn)$, with the help of (2.2) or (2.3) we can get lower bounds of R_j. However, we want to establish other relations that, in certain situations, may give sharper estimates.

Theorem 2. *Under the hypothesis and notations given in Section* 1, *for every* $g \in G$ *with* $Q(g) = Q > q^K$ *we have the estimates*

$$R_j(g) > Q\left[\frac{ps_j}{\pi(Q^{\frac{2}{K}} - q^2)}\right]^{\frac{K}{2}} \qquad (j = 1, \ldots, pn) \tag{3.1}$$

and

$$\max_{1 \leq j \leq pn} R_j(g) > Q\left[\frac{s}{\pi(Q^{\frac{2}{K}} - q^2)}\right]^{\frac{K}{2}}. \tag{3.2}$$

Proof. First, we notice that by (1.1), each circle $|z| = R_j$ contains at least p slits belonging to the boundary of $A = g(B), g \in G$. By [17: Formula (3.1)], we therefore have

$$ps_j \leq \pi R_j^{\frac{2}{K}} (1 - q^2 Q^{-\frac{2}{K}}),$$

hence for every $g \in G$ with $q < Q^{\frac{1}{K}}$, i.e., $Q(g) > q^K$, estimate (3.1) follows. Similarly we obtain, with the help of [17: formula (3.2)], estimate (3.2) ∎

4. The auxiliary functions $R(p, t, s)$ and $T(p, r, s)$

In order to establish other estimates of R_j, Q and $|g(w)|$ we will introduce the following two real functions.

Definition. The real functions

$$\left.\begin{array}{ll} r = R(p,t,s) & (0 \le s < t < 1) \\ t = T(p,r,s) & (0 \le s < r < 1) \end{array}\right\} \qquad (p \in \mathbb{N})$$

are defined in such a way that the circular ring $s < |w| < 1$ with p radial slits

$$P_j = \left\{ w \middle| \, s \le |w| \le t \text{ and } \arg w = j\frac{2\pi}{p} \right\} \qquad (j = 0, \dots, p-1)$$

and the circular ring $r < |z| < 1$ can be schlicht conformal mapped onto each other.

Because of the monotony of the modulus of a doubly-connected domain (see [2: p. 176]) we have the following monotonies of the auxiliary function $R(p,t,s)$ with $p \in \mathbb{N}$:

$$s < R(p,t,s) < t \quad (0 \le s < t < 1) \tag{4.1}$$
$$R(p,t_1,s) < R(p,t_2,s) \quad (0 \le s < t_1 < t_2 < 1) \tag{4.2}$$
$$R(p,t,s_1) < R(p,t,s_2) \quad (0 \le s_1 < s_2 < t < 1) \tag{4.3}$$
$$R(p,t,s) > R(1,t,s) \quad (0 \le s < t < 1; p \ge 2)$$

With the help of Hersch's [9: p. 316] and Nehari's [11: p. 295] formulae, I myself found in [14: pp. 101 - 104] the following expression for $R(p,t,s)$:

$$R(p,t,0) = \exp\left\{ \frac{-\pi K'(t^p)}{2pK(t^p)} \right\} \qquad (0 < t < 1;\ p \in \mathbb{N}) \tag{4.4}$$

with

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$
$$K'(k) = K(\sqrt{1-k^2})$$

and for $0 < s < t < 1$

$$R(p,t,s) = \exp\left\{ \frac{-\pi K'(u)}{2pK(u)} \right\}$$

with

$$u = 1 + h - \sqrt{h(2+h)}$$

where

$$h = \frac{(1-k)(1-ak)}{k(1+a)}, \quad k = 4s^p \prod_{n=1}^{\infty} \left[\frac{1+s^{4pn}}{1+s^{2p(2n-1)}} \right]^4$$

$$a = sn\left(b + \frac{i2pb}{\pi} \log\frac{t}{s}, k \right), \ b = K(k).$$

Here $sn(z,k)$ means the Jacobian elliptic sinus with the parameter k. Another expression for $R(p,t,s)$ was shown by Graeser in [3: pp. 77 - 78].

In view of [14: Formula (1.21)] we have the estimate

$$4^{-\frac{1}{p}}t < R(p,t,s) < t \qquad (0 \le s < t < 1;\, p \in \mathbb{N}), \tag{4.5}$$

hence $R(p,t,s) \approx t$ when $p \to \infty$.

The evaluation of $K(t^p)$ and $K'(t^p)$ (see [18: p. 177]) yields the asymptotic behaviour of $R(p,t,0)$:

$$R(p,t,0) \approx 4^{-\frac{1}{p}}t \qquad \text{when } t \to 0 \tag{4.6}$$

and

$$1 - R(p,t,0) \approx \frac{\pi^2}{2p \log \frac{8}{p(1-t)}} \qquad \text{when } t \to 1.$$

Successive approximations for $R(1,t,0)$ are given by Lehto [10: p. 64]. The expression for $T(p,r,s)$, that is not needed here, was shown by Thao ([14: pp. 102 - 105] or [17: p. 61]).

5. Other estimates of R_j, Q and $|g(w)|$

Using the auxiliary functions studied in Section 4, other estimates for R_j, Q and $|g(w)|$ will be given. In particular, when $s_j = 0$ or $s = 0$, they may be sharper than ones of (3.1) and (2.2).

Theorem 3. *Under the hypothesis and notations given in Sections* 1 *and* 4, *for every* $g \in G$, $w \in B$ *and* $j = 1, \ldots, pn$, *we have the estimates*

$$R^K(p,d_j,q) < R_j(g) < Q(g) R^{-K}\Big(p, \frac{q}{c_j}, q\Big) \tag{5.1}$$

$$Q(g) > R^K(p,d_j,q)\, R^K\Big(p, \frac{q}{c_j}, q\Big) \tag{5.2}$$

$$R^K(p,|w|,q) < |g(w)| < Q(g) R^{-K}\Big(p, \frac{q}{|w|}, q\Big). \tag{5.3}$$

Proof. Considering the mapping $f = g^{-1}$, $g \in G$, with the help of [17: Theorem 6.1] we obtain

$$d_j < T(p, R_j^{\frac{1}{K}}, q) = t \qquad \text{and} \qquad \frac{q}{c_j} < T\left[p, \Big(\frac{Q}{R_j}\Big)^{\frac{1}{K}}, q\right] = t'.$$

Hence the definition of the auxiliary functions and the monotony (4.2) yield the relations

$$R_j^{\frac{1}{K}} = R(p,t,q) > R(p,d_j,q) \qquad \text{and} \qquad \Big(\frac{Q}{R_j}\Big)^{\frac{1}{K}} = R(p,t',q) > R\Big(p, \frac{q}{c_j}, q\Big).$$

Thus we have estimate (5.1). Estimate (5.2) is just a consequence of (5.1). Using [17: Formula (6.10)], we obtain similary estimate (5.3) ∎

Corollary 3. *From Theorem* 3 *and* (4.5), *for every* $g \in G$, $w \in B$ *and* $j = 1, \dots, pn$ *we obtain the simple estimates*

$$4^{-\frac{K}{p}} d_j^K < R_j(g) < 4^{\frac{K}{p}} Q(g) \Big(\frac{c_j}{q}\Big)^K \tag{5.4}$$

$$Q(g) > 4^{-2\frac{K}{p}} \Big(\frac{q d_j}{c_j}\Big)^K \tag{5.5}$$

$$4^{-\frac{K}{p}} |w|^K < |g(w)| < 4^{\frac{K}{p}} Q(g) \Big(\frac{|w|}{q}\Big)^K. \tag{5.6}$$

In view of (4.3) and (4.6) we see that the coefficients in (5.4) - (5.6) are the best possible.

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