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Oscillations of Certain Delay Integro-Differential Equations

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Abstract. In this paper, the oscillation of solutions of certain delay integro-differential systems are considered. Sufficient conditions for the non-existence of non-oscillatory solutions for these systems are obtained. Comparison results are obtained also.

Keywords: *Integro-differential systems, oscillation, non-oscillation, comparison theorems*

AMS subject classification: 34 A 15

1. Introduction

Recently the oscillation in systems of functional-differential equations has received attention in the literature [1 - 6]. In this paper we consider the delay integro-differential system

$$x_i'(t) + \sum_{j=1}^{n} \int_0^t D_{i,j}(t-s)x_j(s-\tau)\,ds = 0 \qquad (i = 1, 2, \ldots, n) \tag{1}$$

and

$$x_i'(t) + \sum_{j=1}^{n} \int_0^t D_{i,j}(t-s)x_j(s-\tau)\,ds = f_i(t) \qquad (i = 1, 2, \ldots, n) \tag{2}$$

with initial condition

$$x(s) = \phi(s) \qquad (s \in [-\tau, 0]) \tag{3}$$

where $\tau > 0$, $D_{i,j} \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\phi \in C([-\tau, 0])$. System (1) with $n = 1$ has been considered in [7].

By a *solution* of (1) - (3) we mean a vector $x = (x_1, \ldots, x_n)^T$, which is continuous on $[-\tau, \infty)$ and continuously differentiable and satisfies (1) for $t \geq 0$ and such that (3) holds. By a *solution* of (1) on $[T, \infty)$ we mean a vector $x \in \mathbb{R}^n$ which is continuous on $[-\tau, \infty)$ and continuously differentiable and satisfies (1) for $t \geq T$. A vector $x \in \mathbb{R}^n$ is said to be *non-oscillatory*, if $x_i(t) \neq 0$ on $[-\tau, \infty)$ $(i = 1, 2, \ldots, n)$, and to be *positive* if every component of x is positive on $[-\tau, \infty)$.

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We will reduce the non-existence of non-oscillatory solutions of (1) to the same for the scalar integro-differential inequality

$$y'(t) + \int_0^t D(t-s)y(s-\tau)\,ds \le 0 \tag{4}$$

where

$$D(t) = \begin{cases} D_{11}(t) & \text{for } n=1 \\ \min_{1\le j\le n}[D_{jj}(t) - \sum_{i=1,i\ne j}^n |D_{i,j}(t)|] & \text{for } n>1 \end{cases} \qquad (t \ge 0).$$

By a *solution* of (4) on $[T,\infty)$ we mean a function $y \in C([-\tau,\infty),\mathbb{R}) \cap C^1([T,\infty),\mathbb{R})$ which satisfies (4) on $[T,\infty)$ where $T \ge 0$. A solution y of (4) is called *positive*, if $y > 0$ on $[-\tau,+\infty)$.

2. Main results

Let us start with the following theorem.

Theorem 1. *If* (4) *has no positive solutions, then* (1) *has no non-oscillatory solutions.*

Proof. Suppose the contrary, let x be a non-oscillatory solution of (1) with $x_i(t) \ne 0$ $(t \ge -\tau; i = 1,2,\ldots,n)$ and set $\delta_i = \frac{x_i(t)}{|x_i(t)|}$. Then $w = (w_1,w_2,\ldots,w_n)^T = (\delta_1 x_1,\ldots,\delta_n x_n)^T$ satisfies $w_i(t) > 0$ $(t \ge -\tau)$ and

$$w_i'(t) + \sum_{j=1}^n \int_0^t \bar{D}_{i,j}(t-s)w_j(s-\tau)\,ds = 0 \qquad (t \ge 0) \tag{5}$$

where $\bar{D}_{i,j} = \frac{\delta_i}{\delta_j} D_{i,j}$. Clearly, $\bar{D}_{i,i} = D_{i,i}$ and $|\bar{D}_{i,j}| = |D_{i,j}|$ $(i \ne j)$. Define $y(t) = \sum_{i=1}^n w_i(t) > 0$ $(t \ge -\tau)$. Summing (5), we have

$$\begin{aligned}
0 &= \sum_{i=1}^n w_i'(t) + \sum_{i=1}^n \sum_{j=1}^n \int_0^t \bar{D}_{i,j}(t-s)w_j(s-\tau)\,ds \\
&= y'(t) + \sum_{j=1}^n \int_0^t \bar{D}_{j,j}(t-s)w_j(s-\tau)\,ds + \sum_{i,j=1,i\ne j}^n \int_0^t \bar{D}_{i,j}(t-s)w_j(s-\tau)\,ds \\
&\ge y'(t) + \sum_{j=1}^n \int_0^t \Big(\bar{D}_{j,j}(t-s) - \sum_{i,j=1,i\ne j}^n |\bar{D}_{i,j}(t-s)|\Big) w_j(s-\tau)\,ds \\
&\ge y'(t) + \int_0^t D(t-s)y(s-\tau)\,ds
\end{aligned}$$

which is a contradiction ∎

Theorem 2. *Assume that $D_{ii}(t) \not\equiv 0$ on $[0,\tau]$. If*

$$z_i'(t) + \sum_{j=1}^{n} \int_0^t D_{i,j}(t-s) z_j(s-\tau)\,ds \le 0 \qquad (i = 1,2,\dots,n) \tag{6}$$

has a positive solution $z(t)$ on $[-\tau,\infty)$, then (1) has a positive solution $x(t)$ with $0 < x(t) \le z(t)$ $(t \ge 0)$ and $\lim_{t\to\infty} x(t) = 0$.

Proof. Define the set

$$X = \Big\{ x \in C([-\tau,\infty),\mathbb{R}^n) \Big| \, 0 \le x(t) \le z(t) \ \ (t \ge -\tau),\ x(t) > 0 \ \text{ on } [-\tau,0] \Big\}$$

and an operator S on X by

$$(Sx_i)(t) = \begin{cases} \displaystyle\sum_{j=1}^{n} \int_t^{\infty}\int_0^{u} D_{i,j}(u-s)x_j(s-\tau)dsdu & \text{if } t \ge 0 \\ \displaystyle\Big(\sum_{j=1}^{n} \int_0^{\infty}\int_0^{u} D_{i,j}(u-s)x_j(s-\tau)dsdu\Big)\Big(1-\tfrac{|t|}{\tau}\Big)\tfrac{z_i(t)}{z_i(0)} + z_i(t)\tfrac{|t|}{\tau} & \text{if } -\tau \le t \le 0 \end{cases} \tag{7}$$

for $i = 1,2,\dots,n$. Clearly, $(Sx_i)(t) \le z_i(t)$ $(t \ge -\tau)$ and $(Sx_i)(t) > 0$ on $[-\tau,0]$, i.e. $SX \subset X$. Define the sequences $\{y_m(t)\}, y_m \in \mathbb{R}^n$ by

$$\left.\begin{aligned} y_0 &= z \\ y_m &= Sy_{m-1} \quad (m \in \mathbb{N}). \end{aligned}\right\}$$

Then

$$y_0 \ge y_1 \ge \dots \ge y_m(t) \ge \dots \qquad (t \ge -\tau)$$

and hence $\lim_{m\to\infty} y_m(t) = y(t)$ $(t \ge -\tau)$ exists. It is easy to see that $y(t)$ is a solution of (1) for $t \ge 0$ and $y(t) \le z(t)$. We claim that $y(t) > 0$ $(t \ge 0)$. From (7) and the condition $D_{i,i}(t) \not\equiv 0$ on $[0,\tau]$ we have $y(t) > 0$ on $[-\tau,0]$. If ξ is the first zero of y_i on $[0,\infty)$, i.e. $y_i(t) > 0$ $(0 \le t < \xi)$ and $y_i(\xi) = 0$, then

$$\begin{aligned} 0 &= y_i(\xi) \\ &= \sum_{j=1}^{n} \int_{\xi}^{\infty}\int_0^{u} D_{i,j}(u-s)y_j(s-\tau)\,dsdu \\ &\ge \int_{\xi}^{\infty}\int_0^{u} D_{ii}(u-s)y_i(s-\tau)\,dsdu \end{aligned}$$

which implies that $\int_0^u D_{i,i}(u-s)y_i(s-\tau)\,ds = 0$, which contradicts the fact that $D_{i,i} \not\equiv 0$ on $[0,\tau]$ and $y_i(s) > 0$ $(s \in [-\tau,0])$. This contradiction proves the Theorem ∎

From Theorem 2 we can derive a comparison theorem.

Corollary 1. *If* $D_{i,j}(t) \le E_{i,j}(t) \quad (t \ge 0)$ *and*

$$y_i'(t) + \sum_{j=1}^{n} \int_0^t E_{i,j}(t-s) y_j(s-\tau)\, ds = 0 \qquad (i = 1, 2, \ldots, n)$$

has a positive solution $y(t)$*, then* (1) *has a positive solution* $x(t)$ *with* $x(t) \le y(t)$*.*

Proof. In fact, we have

$$y_i'(t) + \sum_{j=1}^{n} \int_0^t D_{i,j}(t-s) y_j(t-\tau)\, ds \le y_i'(t) + \sum_{j=1}^{n} \int_0^t E_{i,j}(t-s) y_j(t-\tau)\, ds = 0.$$

By Theorem 2, (1) has a positive solution ∎

Corollary 2. *Let* $d(t) = \max_{1 \le i \le n} \sum_{j=1}^{n} D_{i,j}(t)$*. If there exists* $\lambda > 0$ *such that*

$$-\lambda + e^{\lambda\tau} \int_0^\infty d(s) e^{\lambda s} ds \le 0,$$

then system (1) *has a positive solution* x *with*

$$0 < x_i(t) \le e^{-\lambda t} \qquad (t \ge 0;\ i = 1, 2, \ldots, n). \tag{8}$$

Proof. In fact, let $z_i(t) = e^{-\lambda t} \quad (i = 1, 2, \ldots, n)$. Then

$$\begin{aligned}
&z_i'(t) + \sum_{j=1}^{n} \int_0^t D_{i,j}(t-s) z_j(s-\tau)\, ds \\
&= z_i'(t) + \sum_{j=1}^{n} \int_0^t D_{i,j}(s) z_j(t-s-\tau)\, ds \\
&= e^{-\lambda t}\Big[-\lambda + \sum_{j=1}^{n} e^{\lambda\tau} \int_0^t D_{i,j}(s) e^{\lambda s}\, ds \Big] \\
&\le e^{-\lambda t}\Big[-\lambda + e^{\lambda\tau} \int_0^t d(s) e^{\lambda s} ds \Big] \\
&\le 0.
\end{aligned}$$

By Theorem 2, (1) has a solution x, which satisfies (8) ∎

Now we want to show a sufficient condition for the non-existence of positive solutions of (4). Let

$$D_1(t) = \int_{t-\tau}^{t} \int_{0}^{u} D(u-s)\,dsdu$$

$$D_n(t) = \int_{t-\tau}^{t} \int_{0}^{u} D(u-s)D_{n-1}(s)\,dsdu \quad (n \geq 2)$$

and

$$p_0 = \min_{0 \leq t \leq n\tau} D_n(t)$$

$$p_k = \min_{kn\tau \leq t \leq (k+1)n\tau} D_n(t) \quad (k \geq 1).$$

Lemma 1. *Assume that*

$$D_n(t) \geq \frac{1}{e^n}. \tag{9}$$

Let $y(t)$ be a positive solution of (4)*. Then*

$$\frac{y(t-\tau)}{y(t)} \leq 4e^{2n}. \tag{10}$$

Proof. Integrating (4), we have

$$y(t) - y(t-\tau) + \int_{t-\tau}^{t} \int_{0}^{u} D(u-s)y(s-\tau)\,dsdu \leq 0. \tag{11}$$

Thus

$$\begin{aligned} y(t-\tau) &> \int_{t-\tau}^{t} \int_{0}^{u} D(u-s)y(s-\tau)\,dsdu \\ &\geq y(t-\tau) \int_{t-\tau}^{t} \int_{0}^{u} D(u-s)\,dsdu \\ &= D_1(t)y(t-\tau). \end{aligned} \tag{12}$$

Substituting (12) into (11) we obtain $y(t-\tau) > D_2(t)y(t-\tau)$. In general, we have

$$y(t-\tau) > y(t-\tau)D_n(t). \tag{13}$$

Assumption (9) implies that there exists $t^* \in [t-\tau, t]$ such that

$$\int_{t-\tau}^{t^*} \int_{0}^{u} D(u-s)D_{n-1}(s)\,dsdu \geq \frac{1}{2e^n} \quad \text{and} \quad \int_{t^*}^{t} \int_{0}^{u} D(u-s)D_{n-1}(s)\,dsdu \geq \frac{1}{2e^n}.$$

Integrating (4), we have

$$y(t^*) - y(t-\tau) + \int_{t-\tau}^{t^*}\int_0^u D(u-s)y(s-\tau)\,dsdu \le 0. \tag{14}$$

In view of (13) and (14), we obtain

$$\begin{aligned} y(t-\tau) &> \int_{t-\tau}^{t^*}\int_0^u D(u-s)y(s-\tau)\,dsdu \\ &\ge \int_{t-\tau}^{t^*}\int_0^u D(u-s)D_{n-1}(s)y(s-\tau)\,dsdu \\ &\ge \frac{y(t^*-\tau)}{2e^n}. \end{aligned}$$

Similarly, we have $y(t^*) > \frac{y(t-\tau)}{2e^n}$. Combining the above two inequalities we have $\frac{y(t^*-\tau)}{y(t^*)} < (2e^n)^2$ and the proof is complete ∎

Lemma 2. *Assume that* (9) *holds. Let* $y(t)$ *be a positive solution of* (4) *and define*

$$M = \min_{n(m-1)\tau \le t \le mn\tau} \frac{y(t-\tau)}{y(t)} \qquad \text{and} \qquad N = \min_{mn\tau \le t \le (m+1)n\tau} \frac{y(t-\tau)}{y(t)}.$$

Then $M > 1$ *and*

$$N \ge \exp(e^{n-1}Mp_m) \ge \exp\left(\frac{M}{e}\right) \ge M. \tag{15}$$

Proof. $M > 1$ is obvious. Dividing (4) by $y(t)$ and integrating it we have

$$\frac{y(t-\tau)}{y(t)} \ge \exp \int_{t-\tau}^{t} \frac{1}{y(s)} \int_0^s D(s-u)y(u-\tau)\,duds. \tag{16}$$

Let

$$\begin{aligned} d_k &= \min_{(mn+k-1)\tau \le t \le (mn+k)\tau} \{D_n(t)\} \\ N_k &= \min_{(mn+k-1)\tau \le t \le (mn+k)\tau} \frac{y(t-\tau)}{y(t)} \\ M_l &= \min_{((m-1)n+l)\tau \le t \le (mn+l)\tau} \frac{y(t-\tau)}{y(t)} \end{aligned}$$

for $k, l = 1, 2, \dots, n$. By definition, $p_m \le d_k$ $(k = 1, 2, \dots, n)$ and $N = \min_{1 \le k \le n} N_k$.

For $t \in [mn\tau, (mn+1)\tau]$, from (16) and the inequality $e^x \geq ex$ for $x > 0$, we have

$$
\begin{aligned}
\frac{y(t-\tau)}{y(t)} &\geq \exp \int_{t-\tau}^{t} \int_{0}^{s_1} D(s_1 - u) \frac{y(u-\tau)}{y(u)}\, du ds_1 \\
&\geq \exp \int_{t-\tau}^{t} \int_{0}^{s_1} D(s_1 - u) \exp \int_{u-\tau}^{u} \int_{0}^{s_2} D(s_2 - u_1) \frac{y(u_1 - \tau)}{y(u_1)}\, du_1 ds_2 du ds_1 \\
&\geq \exp \left(e \int_{t-\tau}^{t} \int_{0}^{s_1} D(s_1 - u) \int_{u-\tau}^{u} \int_{0}^{s_2} D(s_2 - u_1) \frac{y(u_1-\tau)}{y(u_1)} du_1 ds_2 du ds_1 \right) \\
&\geq \\
&\vdots \\
&\geq \exp \left(e^{n-1} \int_{t-\tau}^{t} \int_{0}^{s_1} D(s_1 - u) \int_{u-\tau}^{u} \int_{0}^{s_2} D(s_2 - u_2) \ldots \right. \\
&\qquad \left. \ldots \int_{u_{n-2}-\tau}^{u_{n-2}} \int_{0}^{s_n} D(s_n - u_{n-1}) \frac{y(u_{n-1} - \tau)}{y(u_{n-1})}\, du_{n-1} ds_n \ldots du ds_1 \right) \\
&\geq \exp \left(e^{n-1} \int_{t-\tau}^{t} \int_{0}^{s_1} D(s_1 - u) \int_{u-\tau}^{u} \int_{0}^{s_2} D(s_2 - u_2) \ldots \right. \\
&\qquad \left. \ldots \int_{u_{n-2}-\tau}^{u_{n-2}} \frac{y(s_n - \tau)}{y(s_n)} \int_{0}^{s_n} D(s_n - u_{n-1})\, du_{n-1} ds_n \ldots du ds_1 \right)
\end{aligned}
$$

where $s_n \in [(m-1)n\tau, (mn+1)\tau]$. In view of the inequality $\exp(\frac{x}{e}) > x$ for $x \neq e$ we have

$$
\begin{aligned}
N_1 &\geq \exp\left(e^{n-1} \min(M, N_1) d_1\right) \\
&\geq \exp\left(e^{n-1} \min(M, N_1) p_m\right) \\
&\geq \exp\left(\frac{\min(M, N_1)}{e}\right) \\
&> \min(M, N_1).
\end{aligned}
$$

Hence $\min(M, N_1) = M$. Therefore

$$N_1 \geq \exp(e^{n-1} M p_m) \geq \exp\left(\frac{M}{e}\right) \geq M$$

and $M_1 \geq \min(M, N_1) \geq M$. Similar to the above, we can prove that

$$N_k \geq \exp(e^{n-1} M p_m) \geq \exp\left(\frac{M}{e}\right) \geq M \qquad (k = 1, 2, \ldots, n).$$

Therefore

$$N = \min_{1 \le k \le n} N_k \ge \exp(e^{n-1} M p_m) \ge \exp\left(\frac{M}{e}\right) \ge M$$

and the proof is complete ∎

Theorem 3. *Assume that* (9) *holds and*

$$\sum_{i=1}^{\infty} \left(p_i - \frac{1}{e^n}\right) = \infty. \tag{17}$$

Then (4) *has no positive solutions.*

Proof. Suppose the contrary, let $y(t)$ be a positive solution of (4). Define the sequence $\{\bar{N}_i\}$ by

$$\bar{N}_i = \min_{(k+i-1)n\tau \le t \le (k+i)n\tau} \frac{y(t-\tau)}{y(t)} \qquad (i \ge 0).$$

By Lemma 2, we have $\bar{N}_i > 1$ and

$$\begin{aligned} \bar{N}_{i+1} &\ge \exp(e^{n-1} \bar{N}_i p_{k+i}) \\ &\ge \exp\left(\frac{\bar{N}_i}{e}\right) \exp\left(\bar{N}_i \left(e^{n-1} p_{k+i} - \frac{1}{e}\right)\right) \\ &\ge \exp\left(\frac{\bar{N}_i}{e}\right) \\ &> \bar{N}_i. \end{aligned} \tag{18}$$

Therefore $\{\bar{N}_i\}$ is increasing. On the other hand, by Lemma 1, $\{\bar{N}_i\}$ is bounded. Hence $\lim_{i\to\infty} \bar{N}_i = \bar{N}$ exists. From the last inequality, we have $\bar{N} \ge \exp(\frac{\bar{N}}{e}) > \bar{N}$ if $\bar{N} \ne e$ which implies that $\bar{N} = e$. From (18),

$$\bar{N}_{i+1} \ge \bar{N}_i \big(1 + e^{n-1} \bar{N}_i (p_{k+i} - e^{-n})\big).$$

Hence

$$\bar{N}_{i+1} - \bar{N}_i \ge e^{n-1} \bar{N}_i^2 (p_{k+i} - e^{-n})$$

and

$$\bar{N}_{i+2} - \bar{N}_{i+1} \ge e^{n-1} \bar{N}_{i+1}^2 (p_{k+i+1} - e^{-n}) > e^{n-1} \bar{N}_i^2 (p_{k+i+1} - e^{-n}).$$

Summing up the above inequality, we get

$$e - \bar{N}_i > e^{n-1} \bar{N}_i^2 \sum_{j=i}^{\infty} (p_{k+j} - e^{-n})$$

for some k, which contradicts (17) ∎

Corollary 3. *If there exists a positive integer n such that*

$$\liminf_{t\to\infty} D_n(t) > \frac{1}{e^n}, \tag{19}$$

then (4) *has no positive solutions.*

In fact, (19) implies (17). Combining Theorem 1 and Corollary 3 we have the following result.

Corollary 4. *If* (19) *holds, then* (1) *has no non-oscillatory solutions.*

Now we consider the forced system (2).

Theorem 4. *Let $F(t) = \sum_{i=1}^{n} \delta_i f_i(t)$ with $\delta_i = \pm 1$ and $F(t) = h'(t)$, $h_+(t) = \max(h(t), 0) \not\equiv 0$, $h_-(t) = \max(-h(t), 0) \not\equiv 0$, and*

$$\int_0^\infty \int_0^t D(t-s) h_+(s-\tau)\, ds dt = \infty. \tag{20}$$

Then (2) *has no non-oscillatory solutions.*

Proof. Suppose the contrary, let $\{x_i(t)\}$ $(i = 1, 2, \ldots, n)$ be a non-oscillatory solution of (2). Then we have

$$\delta_i x_i'(t) + \sum_{j=1}^{n} \int_0^t D_{i,j}(t-s)\delta_i x_j(s-\tau)\, ds = \delta_i f_i(t)$$

$$w_i'(t) + \sum_{j=1}^{n} \int_0^\infty D_{i,j}(t-s)\delta_i {\delta_j}^{-1} w_j(s-\tau)\, ds = \delta_i f_i(t).$$

That is,

$$w_i'(t) + \sum_{i=1}^{n} \int_0^t \bar{D}_{i,j}(t-s) w_j(s-\tau)\, ds = \delta_i f_i(t).$$

Summing the above equation, we obtain

$$\sum_{i=1}^{n} w_i'(t) + \sum_{j=1}^{n} \int_0^t \bar{D}_{i,j}\bar{D}_{j,j}(t-s) w_j(s-\tau)\, ds$$
$$+ \sum_{i,j=1, i\neq j}^{n} \int_0^t \bar{D}_{i,j}(t-s) w_j(s-\tau)\, ds = \sum_{i=1}^{n} \delta_i f_i(t).$$

Hence

$$y'(t) + \sum_{j=1}^{n} \int_0^t \Big(\bar{D}_{j,j}(t-s) - \sum_{i,j=1, i\neq j} |\bar{D}_{i,j}(t-s)|\Big) w_j(s-\tau)\, ds \le \sum_{i=1}^{n} \delta_i f_i(t)$$

and hence

$$y'(t) + \int_0^t D(t-s)y(s-\tau)\,ds \le F(t).$$

Thus,

$$(y(t) - h(t))' + \int_0^t D(t-s)y(s-\tau)\,ds \le 0. \tag{21}$$

If $y(t) > 0$ eventually, then $y(t) - h(t)$ is non-increasing. There are two possible cases:

(i) $y(t) - h(t) \le 0$ eventually

and

(ii) $y(t) - h(t) \ge 0$ eventually.

For the case (i), $y(t) \le h(t)$ eventually, which contradicts the positivity of y. Therefore, the case (ii) holds. Hence $y(t) \ge h_+(t)$ eventually. From this and (21), we obtain

$$(y(t) - h(t))' + \int_0^t D(t-s)h_+(s-\tau)\,ds \le 0. \tag{22}$$

This together with condition (20) lead to a contradiction and the proof is completed ∎

Remark 1. (2) has no positive solution, if (20) holds, where $F(t) = \sum_{i=1}^n f_i(t)$.

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