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Topological Structure of Solution Sets to Multi-Valued Asymptotic Problems

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Abstract. Acyclicity of solution sets to asymptotic problems, when the value is prescribed either at the origin or at infinity, is proved for differential inclusions and discontinuous autonomous differential inclusions. Existence criteria showing that such sets are non-empty are obtained as well.

Keywords: *Solution sets, acyclicity, asymptotic problems, differential inclusions, discontinuous autonomous inclusions, inverse systems, essential operators*

AMS subject classification: 34 A 60, 34 C 11, 54 C 60, 58 C 06

1. Introduction

The motivation for the considerations below is taken from studying differential inclusions on non-compact intervals in [1 - 3]. A certain structure of solution sets for differential problems is needed for the existence results presented therein. It is known that solution sets of differential problems often correspond with fixed point sets of some multi-valued operators in functional spaces. Therefore, it is important to study the topological structure of such sets. In the single-valued case of Cauchy problems on infinite intervals, this has been done, e.g., in [9, 28, 29, 34 - 36]. For a survey of results in this field see, e.g., [18] and the references therein.

In [21, 22] the authors have proved that the fixed point set of a multi-valued contraction from a complete absolute retract into itself with convex closed values is an absolute retract, too. A topology of the Fréchet space brings however some troubles in checking the contractivity of operators. Even for an operator which is a contraction in every seminorm (with the same constant of contractivity), it seems to be impossible to prove a contractivity with respect to a metric in this space.

This note gives, at first, a technique which allows us to overcome the troubles mentioned above and to characterize the topological structure of a fixed point set of

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limit maps induced by maps of inverse systems. Applications to a Cauchy problem on a halfline are then given.

Furthermore, the topological structure of a solution set to a target problem (i.e. when the value is prescribed at infinity) is studied, but this time by means of the contractibility argument in [17], of course, under a suitable modification. More concretely, a homotopy is built between this set and a unique solution of a target problem, where the right-hand side of a given system is a Carathéodory selector.

Then, using acyclicity of solution sets of parametrized systems, existence results for both problems under consideration are given, in the absence of Lipschitzianity. In particular, an entirely bounded solution on the whole line with prescribed value at infinity is proved in this way.

The notion of topological essentiality (introduced by A. Granas in [24]) is also applied to higher-order differential systems to get existence results for initial value problems on non-compact intervals. The topological structure of a related solution set is clarified as well.

A non-empty, compact and acyclic set of solutions is finally verified for Cauchy problems to discontinuous autonomous differential inclusions. This rather general type of inclusions has been considered in [7] (see also the references therein), but only in the single-valued case of associated operators and only on compact intervals. Hence, for these inclusions, the topological structure of solution sets is investigated here for the first time.

2. Topological structure of fixed point sets of limit maps

Let us recall that an *inverse system* of topological spaces is a family $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$, where Σ is a set ordered by the relation $\leq$, X_α is a topological space for every $\alpha \in \Sigma$ (we assume that all topological spaces are Hausdorff) and $\pi_\alpha^\beta : X_\beta \to X_\alpha$ is a continuous mapping for each two elements $\alpha, \beta \in \Sigma$ such that $\alpha \leq \beta$. Moreover, for each $\alpha \leq \beta \leq \gamma$, the conditions $\pi_\alpha^\alpha = id_{X_\alpha}$ and $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$ should hold.

A subspace of the product $\Pi_{\alpha\in\Sigma} X_\alpha$ is called a *limit of the inverse system* S and it is denoted by $\lim_\leftarrow S$ or $\lim_\leftarrow \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ if

$$\lim_{\leftarrow} S = \left\{ (x_\alpha) \in \Pi_{\alpha\in\Sigma} X_\alpha \,\middle|\, \pi_\alpha^\beta(x_\beta) = x_\alpha \text{ for all } \alpha \leq \beta \right\}.$$

An element of $\lim_\leftarrow S$ is called a *thread* or a *fibre* of the system S. One can see that if we denote by $\pi_\alpha : \lim_\leftarrow S \to X_\alpha$ a restriction of the projection $p_\alpha : \Pi_{\alpha\in\Sigma} X_\alpha \to X_\alpha$ onto the α-th axis, then we obtain $\pi_\alpha = \pi_\alpha^\beta \pi_\beta$ for each $\alpha \leq \beta$.

Now we summarize some useful properties of limits of inverse systems.

Proposition 2.1. (see [11]). *Let $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ be an inverse system. Then:*

2.1.1. *The limit $\lim_\leftarrow S$ is a closed subset of $\Pi_{\alpha\in\Sigma} X_\alpha$.*

2.1.2. *If, for every $\alpha \in \Sigma$, X_α is*

(i) *compact, then $\lim_\leftarrow S$ is compact*

(ii) *compact and non-empty, then* $\lim_{\leftarrow} S$ *is compact and non-empty*

(iii) *a continuum, then* $\lim_{\leftarrow} S$ *is a continuum*

(iv) *compact and acyclic,* [1] *then* $\lim_{\leftarrow} S$ *is compact and acyclic*

(v) *metrizable and* Σ *is countable, then* $\lim_{\leftarrow} S$ *is metrizable.*

Remark 2.2. Note that if we drop the compactness, then Theorem 2.1.2/(ii) is no longer true.

The following example shows that a limit of an inverse system of AR-spaces [2] need not be an AR-space.

Example 2.3. Consider a family $\{X_n\}_{n=1}^{\infty}$ of subsets of $\mathbb{R}^2$ defined by

$$X_n = \big([0, \tfrac{1}{n\pi}] \times [-1,1]\big) \cup \big\{(x,y)\,|\, y = \sin\tfrac{1}{x} \text{ and } \tfrac{1}{n\pi} < x \le 1\big\}.$$

One can see that, for each $m, n \ge 1$ such that $m \ge n$, we have $X_m \subset X_n$. Define the maps $\pi_n^m : X_m \to X_n$ by $\pi_n^m(x) = x$. Therefore, $S = \{X_n, \pi_n^m, \mathbb{N}\}$ is an inverse system of compact AR-spaces. It is evident that $\lim_{\leftarrow} S$ is homeomorphic to the intersection of all X_n. On the other hand,

$$X = \bigcap_{n=1}^{\infty} X_n = \big\{(0,y)\,|\, y \in [-1,1]\big\} \cup \big\{(x,y)\,|\, y = \sin\tfrac{1}{x} \text{ and } 0 < x \le 1\big\}$$

and $X \notin AR$ since, for instance, X is not locally connected.

Let us give two important examples of inverse systems.

Example 2.4. Let, for every $m \in \mathbb{N}$, $C_m = C([0,m], \mathbb{R}^n)$ be the Banach space of all continuous functions on a closed interval $[0,m]$ into $\mathbb{R}^n$ and $C = C([0,\infty), \mathbb{R}^n)$ be the analogous Fréchet space of continuous functions. Consider the maps $\pi_m^p : C_p \to C_m$ for $p \ge m$ defined by $\pi_m^p(x) = x|_{[0,m]}$. It is easy to see that C is isometrically homeomorphic to the limit of the inverse system $\{C_m, \pi_m^p, \mathbb{N}\}$. The maps $\pi_m : C \to C_m$ defined by $\pi_m(x) = x|_{[0,m]}$ correspond with the suitable projections.

Remark 2.5. In the same manner as above, we can show that the Fréchet spaces $C(J, \mathbb{R}^n)$, where J is an arbitrary interval, $L^1_{loc}(J, \mathbb{R}^n)$ of all locally integrable functions, $AC_{loc}(J, \mathbb{R}^n)$ of all locally absolutely continuous functions and $C^k(J, \mathbb{R}^n)$ of all continuously differentiable functions up to the order k can be considered as limits of suitable inverse systems. More generally, every Fréchet space is a limit of some inverse system of Banach spaces.

Now we introduce the notion of multi-valued maps of inverse systems. Suppose that two systems $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ and $S' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, \Sigma'\}$ are given.

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[1] Acyclic with respect to any continuous theory of cohomology, i.e. a space is *acyclic* if it is homologically the same as a one point space (for more details see, e.g., [16]).

[2] A metric space X is called an *absolute retract* (with respect to the class of metric spaces) (we denote it $X \in AR$ and say that X is an AR-space) if, for any metric space Y and a homeomorphism $h : X \to h(X) \subset Y$ such that $h(X)$ is closed in Y, $h(X)$ is a retract of Y.

Definition 2.6. By a *multi-valued map* of the system $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ into the system $S' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, \Sigma'\}$ we mean a family $\{\sigma, \varphi_{\sigma(\alpha')}\}$ consisting of a monotone function $\sigma : \Sigma' \to \Sigma$, i.e. $\sigma(\alpha') \leq \sigma(\beta')$ for $\alpha' \leq \beta'$, and of multi-valued maps $\varphi_{\sigma(\alpha')} : X_{\sigma(\alpha')} \multimap Y_{\alpha'}$ with non-empty values, defined for every $\alpha' \in \Sigma'$ and such that

$$\pi_{\alpha'}^{\beta'} \varphi_{\sigma(\beta')} = \varphi_{\sigma(\alpha')} \pi_{\sigma(\alpha')}^{\sigma(\beta')} \tag{1}$$

for each $\alpha' \leq \beta'$. A map of systems $\{\sigma, \varphi_{\sigma(\alpha')}\}$ induces a *limit map* $\varphi : \lim_\leftarrow S \multimap \lim_\leftarrow S'$ defined by

$$\varphi(x) = \prod_{\alpha' \in \Sigma} \varphi_{\sigma(\alpha')}(x_{\sigma(\alpha')}) \cap \lim_{\leftarrow} S'.$$

In other words, a limit map is the one such that

$$\pi_{\alpha'} \varphi = \varphi_{\sigma(\alpha')} \pi_{\sigma(\alpha')} \tag{2}$$

for every $\alpha' \in \Sigma'$.

Since a topology of a limit of an inverse system is the one generated by the base consisting of all sets of the form $\pi_\alpha^{-1}(U_\alpha)$, where α runs over an arbitrary set cofinal in Σ and U_α are open subsets of the space X_α, it is easy to prove the following continuity property for limit map.

Proposition 2.7. *Let $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ and $S' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, \Sigma'\}$ be two inverse systems and $\varphi : \lim_\leftarrow S \multimap \lim_\leftarrow S'$ be the limit map induced by the map $\{\sigma, \varphi_{\sigma(\alpha')}\}$. If, for every $\alpha' \in \Sigma'$, $\varphi_{\sigma(\alpha')}$ is*

- **(i)** *upper semicontinuous* [3] *and compact-valued, then φ is upper semicontinuous*
- **(ii)** *lower semicontinuous* [4] *, then φ is lower semicontinuous*
- **(iii)** *continuous* [5] *and compact-valued, then φ is continuous.*

Proof. For the proof of statement (i) it is sufficient to show that preimages of sets of the form $\pi_{\alpha'}^{-1}(U_{\alpha'})$ are open in $\lim_\leftarrow S$. Indeed, if $U = \bigcup \pi_{\alpha'}^{-1}(U_{\alpha'})$ is an arbitrary open neighbourhood of a compact set $\varphi(x)$, then we can choose a finite subcovering $\bigcup_{i=1}^n \pi_{\alpha_i'}^{-1}(U_{\alpha_i'})$. By the definition of a limit map, it follows that there is i, $1 \leq i \leq n$, such that $\varphi(x) \subset \pi_{\alpha_i'}^{-1}(U_{\alpha_i'})$.

Moreover, for every α' we have

$$\begin{aligned} \varphi^{-1}(\pi_{\alpha'}^{-1}(U_{\alpha'})) &= \big\{x \in \lim\nolimits_\leftarrow S \,|\, \varphi(x) \subset \pi_{\alpha'}^{-1}(U_{\alpha'})\big\} \\ &= \big\{x \in \lim\nolimits_\leftarrow S \,|\, \pi_{\alpha'}\varphi(x) \subset U_{\alpha'}\big\} \\ &= \big\{x \in \lim\nolimits_\leftarrow S \,|\, \varphi_{\sigma(\alpha')}\pi_{\sigma(\alpha')}(x) \subset U_{\alpha'}\big\}. \end{aligned}$$

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[3] A multi-valued map $\varphi : X \multimap Y$ is *upper semicontinuous* if $\varphi^{-1}(U) = \{x \in X \,|\, \varphi(x) \subset U\}$ is open in X for every open subset U of Y.

[4] A multi-valued map $\varphi : X \multimap Y$ is *lower semicontinuous* if $\varphi_+^{-1}(U) = \{x \in X \,|\, \varphi(x) \cap U \neq \emptyset\}$ is open in X for every open subset U of Y.

[5] A multi-valued map $\varphi : X \multimap Y$ is *continuous* if φ is upper and lower semicontinuous.

Now, if $\varphi_{\sigma(\alpha')}$ is upper semicontinuous, then the composition $\varphi_{\sigma(\alpha')}\pi_{\sigma(\alpha')}$ is upper semicontinuous and the above set is open in $\lim_{\leftarrow} S$. It follows that φ is upper semicontinuous and the proof of statement (i) is complete.

Similarly, notice that

$$\begin{aligned}
\varphi_+^{-1}(\pi_{\alpha'}^{-1}(U_{\alpha'})) &= \big\{x \in \lim_{\leftarrow} S \,|\, \varphi(x) \cap \pi_{\alpha'}^{-1}(U_{\alpha'}) \neq \emptyset\big\} \\
&= \big\{x \in \lim_{\leftarrow} S \,|\, \exists y \in \lim_{\leftarrow} S' \ y \in \varphi(x) \wedge \pi_{\alpha'}(y) \in U_{\alpha'}\big\} \\
&= \big\{x \in \lim_{\leftarrow} S \,|\, \pi_{\alpha'}\varphi(x) \cap U_{\alpha'} \neq \emptyset\big\} \\
&= \big\{x \in \lim_{\leftarrow} S \,|\, \varphi_{\sigma(\alpha')}\pi_{\sigma(\alpha')}(x) \cap U_{\alpha'} \neq \emptyset\big\}.
\end{aligned}$$

Therefore, lower semicontinuity of $\varphi_{\sigma(\alpha')}$ implies lower semicontinuity of φ.

Statement (iii) is an immediate consequence of statements (i) and (ii) ∎

Now, we are able to formulate the main result of this section.

Theorem 2.8. *Let $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ be an inverse system and $\varphi : \lim_{\leftarrow} S \multimap \lim_{\leftarrow} S$ be a limit map induced by the map $\{id, \varphi_\alpha\}$, where $\varphi_\alpha : X_\alpha \multimap X_\alpha$. If the fixed point sets of φ_α are compact acyclic, then the fixed point set of φ is compact acyclic, too.*

Proof. Denote by $\mathcal{F}_\alpha$ the fixed point set of φ_α, for every $\alpha \in \Sigma$, and by $\mathcal{F}$ the fixed point set of φ. We will show that $\pi_\alpha^\beta(\mathcal{F}_\beta) \subset \mathcal{F}_\alpha$. Let $x_\beta \in \mathcal{F}_\beta$. Then $x_\beta \in \varphi_\beta(x_\beta)$ and $\pi_\alpha^\beta(x_\beta) \in \pi_\alpha^\beta\varphi_\beta(x_\beta) \subset \varphi_\alpha\pi_\alpha^\beta(x_\beta)$, which implies that $\pi_\alpha^\beta(x_\beta) \in \mathcal{F}_\alpha$. Similarly, we show that $\pi_\alpha(\mathcal{F}) \subset \mathcal{F}_\alpha$. Denote by $\bar{\pi}_\alpha^\beta : \mathcal{F}_\beta \to \mathcal{F}_\alpha$ the restriction of π_α^β. One can see that $\bar{S} = \{\mathcal{F}_\alpha, \bar{\pi}_\alpha^\beta, \Sigma\}$ is an inverse system. By Proposition 2.1, the set $\mathcal{F}$ is acyclic and the proof is complete ∎

Corollary 2.9. *Let $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ be an inverse system and $\varphi : \lim_{\leftarrow} S \multimap \lim_{\leftarrow} S$ be a limit map induced by the map $\{id, \varphi_\alpha\}$, where $\varphi_\alpha : X_\alpha \multimap X_\alpha$ is compact-valued for every $\alpha \in \Sigma$. Assume that all the sets X_α are complete AR-spaces and all φ_α are contractions, i.e. they are Lipschitz* [6] *with constants $0 \leq k_\alpha < 1$, and have the selection property.* [7] *Then the fixed point set of φ is compact and acyclic.*

Proof. By [21: Theorem 3.1], all the fixed point sets $\mathcal{F}_\alpha$ of φ_α are AR-spaces, and so acyclic. Since all φ_α have compact values, then [33: Theorem 1] implies the compactness of $\mathcal{F}_\alpha$. By Proposition 2.1.2, we get the statement ∎

Using [21: Remark 3.1], we immediately get

Corollary 2.10. *If all X_α are Fréchet spaces and all φ_α are contractions with convex and compact values, then the fixed point set of the limit map φ is non-empty, compact and acyclic.*

Let us still add one more information on the structure of the fixed point set of a limit map in a special case of functional spaces.

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[6] A multi-valued map $\varphi : X \multimap Y$ is a *Lipschitz map*, if there exists a constant $k \geq 0$ such that $d_H(\varphi(x), \varphi(y)) \leq kd(x, y)$ for every $x, y \in X$, where d_H stands for the Hausdorff distance.

[7] For the definition and some properties see, e.g., [5, 21, 22].

Proposition 2.11. *Let $\{id, \varphi_m\}$ be a map of the inverse system $\{C_m, \pi_m^p, \mathbb{N}\}$ considered in Example* 2.4*. If all the fixed point sets $\mathcal{F}_m$ of φ_m are convex or convex and compact, then the fixed point set $\mathcal{F}$ of the induced limit map is convex* (*possibly, empty*) *or non-empty, convex and compact, respectively.*

Proof. The statement is a consequence of the linearity of the maps π_m^p. In fact, let $x, y \in \mathcal{F}$. We want to show that $tx + (1-t)y \in \mathcal{F}$ for every $t \in [0,1]$. But $(tx + (1-t)y)|_{[0,m]} = tx|_{[0,m]} + (1-t)y|_{[0,m]}$ and, by the linearity of π_m^p, we have $\pi_m^p(tx|_{[0,p]} + (1-t)y|_{[0,p]}) = tx|_{[0,m]} + (1-t)y|_{[0,m]}$, which completes the proof ∎

3. Application to a Cauchy problem without convexity

Consider the Cauchy problem

$$\left.\begin{aligned} &\dot{x}(t) \in F(t, x(t)) \quad \text{for a.a. } t \in [0,\infty) \\ &x(0) = x_0 \end{aligned}\right\} \tag{3}$$

where F satisfies the following conditions:

(A) $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$, F has non-empty, compact values, where J denotes the halfline $[0,\infty)$, and $F(\,\cdot\,, x)$ is measurable for all $x \in \mathbb{R}^n$.

(B) There exists a locally integrable function $\eta : J \to J$ such that, for every $t \in J$ and all $x, y \in \mathbb{R}^n$, $d_H(F(t,x), F(t,y)) \le \eta(t)|x-y|$.

Observe that condition (B) implies

(C) There exists a locally integrable function $\alpha : J \to J$ and a positive constant B such that, for every $x \in \mathbb{R}^n$ and for a.a. $t \in J$, $|F(t,x)| \le \alpha(t)(B + |x|)$ where $|F(t,x)| = \sup\{|y| : y \in F(t,x)\}$.

Theorem 3.1. *Under assumptions* (A) *and* (B), *the set of solutions to problem* (3) *is non-empty and can be obtained as a limit of an inverse system of AR-spaces, for every $x_0 \in \mathbb{R}^n$.*

Proof. Fix $x_0 \in \mathbb{R}^n$ and denote

$$\mathcal{S} = \Big\{x \in AC_{loc} \Big| \, \dot{x}(t) \in F(t, x(t)) \text{ for a.a. } t \in J \text{ and } x(0) = x_0\Big\},$$

i.e. $\mathcal{S}$ is the set of all solutions to problem (3). A standard application of the well-known Gronwall inequality (see [3: Theorem 4.7]) allows us to find a map $G : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ which satisfies conditions(A), (B) and

(C′) There exists a locally Lebesgue integrable function $\beta : J \to J$ such that, for every $x \in \mathbb{R}^n$ and for a.a. $t \in J$, $|G(t,x)| \le \beta(t)$

and, moreover, the set of solutions to the Cauchy problem (3) with F replaced by G is equal to $\mathcal{S}$. Thus,

$$\mathcal{S} = \Big\{x \in AC_{loc} \Big| \, \dot{x}(t) \in G(t, x(t)) \text{ for a.a. } t \in J \text{ and } x(0) = x_0\Big\}.$$

At first, we prove that $\mathcal{S}$ is non-empty. Define

$$AC^0 = \{x \in AC_{loc} \,|\, x(0) = x_0\}$$
$$S = cl_{C(J,\mathbb{R}^n)}\{x \in AC^0 \,|\, |\dot{x}(t)| \leq \beta(t)\}$$

and denote, for simplicity, $L^1_m = L^1([0,m],\mathbb{R}^n)$. One can see that AC^0 is a closed, convex subset of the Fréchet space AC_{loc} and, by the well-known Ascoli theorem, S (considered as a subset of $C(J,\mathbb{R}^n)$) is compact. Moreover, S is convex.

We shall define, by induction, multi-valued lower semicontinuous maps $K_m : S \multimap L^1_m$ with closed, decomposable values, for every $m \geq 1$. At first, by properties of the map G, for every $m \geq 1$ and $s \in S$, there exists a measurable selection of $G(\cdot, s(\cdot))$, restricted to $[0,m] \times \mathbb{R}^n$. Define $K_1 : S \multimap L^1_1$ by

$$K_1(s) = \left\{u \in L^1_1 \,\middle|\, u(t) \in G(t, s(t)) \text{ for a.a. } t \in [0,1]\right\}.$$

It is easy to see that K_1 has closed, decomposable values. Moreover, one can show (see, e.g., [14]) that K_1 is lower semicontinuous. By the well-known selection theorem (see [13]), there exists a continuous selection k_1 of K_1, i.e. a map $k_1 : S \to L^1_1$ such that $k_1(s)(t) \in G(t, s(t))$ for all $s \in S$ and for a.a. $t \in [0,1]$.

Now, suppose that there is defined $K_{m-1} : S \multimap L^1_{m-1}$ which is a lower semicontinuous map with closed, decomposable values, for some $m \geq 1$. Denote by k_{m-1} a continuous selection of K_{m-1} and define

$$K_m(s) = \left\{u \in L^1_m \,\middle|\, u(t) \in G(t, s(t)) \text{ for a.a. } t \in [0,m] \text{ and } u|_{[0,m-1]} = k_{m-1}(s)\right\}.$$

We see again that K_m is a lower semicontinuous map with closed, decomposable values and, therefore, there exists a continuous selection of K_m. We denote it by k_m.

Define a continuous map $k : S \to L^1_{loc}(J,\mathbb{R}^n)$ by

$$k(s)(t) = k_m(s)(t) \qquad (t \in [0,m])$$

for every $s \in S$. It is obvious that $k(s)(t) \in G(t, s(t))$, for all $s \in S$ and for a.a. $t \in J$. The above consideration allows us to define a map $l : S \to S$ by

$$l(s)(t) = x_0 + \int_0^t k(s)(\tau)\, d\tau.$$

By continuity of k we see that l is continuous as well. By the well-known Schauder-Tikhonov fixed point theorem, there exists a fixed point of l, i.e. a locally absolutely continuous function $x : J \to \mathbb{R}^n$ such that

$$x(t) = x_0 + \int_0^t k(x)(\tau)\, d\tau.$$

It means that x is a solution of problem (3).

In the second part of the proof, we study the structure of the solution set $\mathcal{S}$. Let us make some further notations:

$$AC_m = \{x \in AC([0,m],\mathbb{R}^n) \mid x(0) = x_0\}$$
$$\varphi_m : AC_m \multimap L^1_m,\ \varphi_m(x) = \{u \in L^1_m \mid u(t) \in G(t,x(t)) \text{ for a.a. } t \in [0,m]\}$$
$$T_m : L^1_m \to AC_m,\ T_m(u)(t) = x_0 + \int_0^t u(s)\,ds$$
$$\Phi_m : AC_m \multimap AC_m,\ \Phi_m = T_m \circ \varphi_m.$$

Let $\mathcal{S}_m \subset AC_m$ denote the set of solutions to the Cauchy problem

$$\left.\begin{aligned} &\dot{x}(t) \in G(t,x(t)) \quad \text{for a.a. } t \in [0,m] \\ &x(0) = x_0 \end{aligned}\right\} \tag{4}$$

and

$$\mathcal{F}_m = \{\dot{x} \mid x \in \mathcal{S}_m\}.$$

Observe that, for every $m \geq 1$, the set AC_m is a closed, convex subset of the Banach space $AC([0,m],\mathbb{R}^n)$ and T_m is a homeomorphism. One can see that $\mathcal{S}_m = Fix\Phi_m$. Moreover, assumption (B) implies that φ_m is a Lipschitz map, by which φ_m is continuous. By [6: Theorem 2] the set $\mathcal{F}_m$ is an absolute retract. Since T_m is a homeomorphism, the set $\mathcal{S}_m \in AR$ is an absolute retract, too. Hence, it is acyclic.

We show that $\{\Phi_m\}$ is a map of the inverse system $\{AC_m, \pi^p_m, \mathbb{N}\}$, where $\pi^p_m(x) = x|_{[0,m]}$, for every $x \in AC_p$ and $p \geq m$. This easily follows from the equalities ($t \in [0,m]$)

$$\Phi_m \pi^p_m(x)(t) = \left\{ x_0 + \int_0^t u(s)\,ds \,\middle|\, u \in L^1_m \text{ and } u(t) \in G(t,x(t)) \text{ for a.a. } t \in [0,m] \right\}$$
$$\pi^p_m \Phi_p(x)(t) = \left\{ x_0 + \int_0^t u(s)\,ds \,\middle|\, u \in L^1_p \text{ and } u(t) \in G(t,x(t)) \text{ for a.a. } t \in [0,p] \right\}$$

and from the observation that

$$\begin{aligned} &\{u \in L^1_m \mid u(t) \in G(t,x(t)) \text{ for a.a. } t \in [0,m]\} \\ &\quad = \{u|_{[0,m]} \mid u \in L^1_p \text{ and } u(t) \in G(t,x(t)) \text{ for a.a. } t \in [0,p]\}. \end{aligned}$$

So the map $\{\Phi_m\}$ induces the limit one $\Phi : AC^0 \multimap AC^0$ such that $\Phi(x)|_{[0,m]} = \Phi_m(x|_{[0,m]})$. It means that the fixed point set of Φ is equal to $\mathcal{S}$. By the proof of Theorem 2.8 and the first part of the present one, it follows that $\mathcal{S}$ is as required ∎

4. Structure for a target problem

In this part, we study the problem when, instead of the origin, the value of solutions is prescribed at infinity, namely

$$\left.\begin{array}{l}\dot{x}(t) \in F(t, x(t)) \text{ for a.a. } t \in [0, \infty) \\ \lim\limits_{t \to \infty} x(t) = x_\infty \in \mathbb{R}^n\end{array}\right\} \tag{5}$$

where $F : [0, \infty) \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a *Carathéodory map*, i.e. (note that (i) - (iii) $\Rightarrow$ (A))

(i) values of F are non-empty, compact and convex for all $(t, x) \in [0, \infty) \times \mathbb{R}^n$

(ii) $F(t, \cdot)$ is upper semicontinuous for a.a. $t \in [0, \infty)$

(iii) $F(\cdot, x)$ is measurable for all $x \in \mathbb{R}^n$.

Modifying an idea in [17], where an initial value problem has been studied, we will prove acyclicity of the solution set of problem (5).

Definition 4.1. We say that a metric space X is *contractible* if there exists a homotopy $h : X \times [0, 1] \to X$ such that $h(x, 0) = x$ and $h(x, 1) = y$ for each $x \in X$, where y is a given point in X.

It is well-known that any contractible set is acyclic.

Theorem 4.2. *Consider the target problem* (5)*, where* $F : [0, \infty) \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is a Carathéodory map and* $x_\infty \in \mathbb{R}^n$ *is arbitrary. Assume that condition* (B) *in Section* 3 *is satisfied, but this time with a globally integrable function* $\eta : [0, \infty) \to [0, \infty)$ *such that* $\int_0^\infty \eta(t)\, dt = E < 1$*. Moreover, assume that* $d_H(F(\cdot, 0), 0)$ *can be absolutely estimated by some globally integrable function. If* E *is a sufficiently small constant, then the set of solutions to problem* (5) *is compact and acyclic, for every* $x_\infty \in \mathbb{R}^n$*.*

Proof. Observe that since $\alpha(t)$ in condition (C) ($\Leftarrow$ (B)) becomes globally integrable as well, by the same reason as in Section 3, problem (5) can be equivalently replaced by the problem

$$\left.\begin{array}{l}\dot{x} \in G(t, x(t)) \text{ for a.a. } t \in [0, \infty) \\ \lim\limits_{t \to \infty} x(t) = x_\infty \in \mathbb{R}^n\end{array}\right\} \tag{6}$$

where G is a suitable Carathéodory map which can be estimated by a sufficiently large positive constant M, i.e.

$$|G(t, x)| \le M \qquad \text{for every } x \in \mathbb{R}^n \text{ and a.a. } t \in [0, \infty),$$

and which satisfies condition (B) as well. In other words, the solution set $\mathcal{S}$ for problem (5) is the same as for problem (6), where

$$\mathcal{S} = \left\{ x \in C([0, \infty), \mathbb{R}^n) \,\middle|\, \dot{x}(t) \in F(t, x(t)) \text{ for a.a. } t \in [0, \infty) \text{ and } x(\infty) = x_\infty \right\}.$$

For the structure of $\mathcal{S}$, we will modify an approach in [17]. Observe that, under the above assumptions, F as well G are well-known (see, e.g., [5]) to be product-measurable, and subsequently having a Carathéodory selector $g \subset G$ which is Lipschitzian with a not necessarily same, but again sufficiently small constant (see, e.g., [4]). By the sufficiency we mean that, besides others,

$$|g(t,x) - g(t,y)| \leq \gamma(t)\,|x-y|$$

holds for all $x, y \in \mathbb{R}^n$ and a.a. $t \in [0,\infty)$, with a Lebesgue integrable function $\gamma : [0,\infty) \to [0,\infty)$ such that $\int_0^\infty \gamma(t)\,dt < 1$.

Considering the single-valued problem ($g \subset G$)

$$\left.\begin{array}{l} \dot{x}(t) = g(t,x(t)) \quad \text{for a.a. } t \in [0,\infty) \\ \lim\limits_{t\to\infty} x(t) = x_\infty \end{array}\right\} \tag{7}$$

we can easily prove (cf. Theorem 5.4) the existence of a unique solution $\overline{x}(t)$ of problem (7). The uniqueness can be verified in a standard manner by the contradiction, when assuming the existence of another solution $\overline{y}(t)$ of that problem, because so we would arrive at the false inequality

$$\begin{aligned} \sup_{t\in[0,\infty)} |\overline{x}(t) - \overline{y}(t)| &= \sup_{t\in[0,\infty)} \left| \int_\infty^t g(s,\overline{x}(s))\,ds - \int_\infty^t g(s,\overline{y}(s))\,ds \right| \\ &\leq \int_0^\infty \big|g(t,\overline{x}(t)) - g(t,\overline{y}(t))\big|\,dt \\ &\leq \int_0^\infty \gamma(t) \sup_{t\in[0,\infty)} |\overline{x}(t) - \overline{y}(t)|\,dt \\ &\leq \sup_{t\in[0,\infty)} |\overline{x}(t) - \overline{y}(t)| \int_0^\infty \gamma(t)\,dt \\ &< \sup_{t\in[0,\infty)} |\overline{x}(t) - \overline{y}(t)|. \end{aligned}$$

Hence, according to Definition 4.1 of contractibility, it is sufficient to show that the solution set $\mathcal{S}$ of problem (6) is homotopic to a unique solution $\overline{x}(t)$ of problem (7), which is at the same time a solution of problem (6) as well. The desired homotopy reads ($\lambda \in [0,1]$)

$$h(x,\lambda)(t) = \begin{cases} x(t) & \text{for } t \geq \frac{1}{\lambda} - \lambda, \lambda \neq 0 \\ \bar{z}(t) & \text{for } 0 \leq t < \frac{1}{\lambda} - \lambda, \lambda \neq 0 \\ \bar{x}(t) & \text{for } \lambda = 0 \end{cases}$$

where $\bar{z}$ is a unique solution to the reverse Cauchy problem

$$\left.\begin{array}{l} \dot{z}(t) = g(t,z(t)) \quad \text{for a.a. } t \in [0, \frac{1}{\lambda} - \lambda] \\ z(\frac{1}{\lambda} - \lambda) = x(\frac{1}{\lambda} - \lambda) \end{array}\right\}$$

for each $\lambda \in [0,1]$. Since the continuity of h can be verified quite analogously as in [17] and $h(x,0) = \bar{x}$, $h(x,1) = x$, as required, the set $\mathcal{S}$ is acyclic. Using the convexity assumption on values of F, we can prove by the standard manner (Mazur's Theorem) that $\mathcal{S}$ is closed in $C([0,\infty),\mathbb{R}^n)$. By Ascoli's theorem this set is compact and the proof is complete ∎

Remark 4.3 For the Cauchy problem (3) on the half-line, a similar homotopy can be constructed, as an alternative proof of Theorem 3.1, instead of an inverse systems approach.

5. Using the solution sets to existence results

Although in Theorems 3.1 and 4.2 the solvability of given problems is guaranteed as well, for the sole existence criteria, the Lipschitzianity is rather restrictive. Therefore, for obtaining better existence results, the following proposition (representing [3: Corollary 2.35]) will be very useful, when using the acyclicity of solution sets of parametrized systems.

Proposition 5.1. *Consider the problem*

$$\left.\begin{aligned} &\dot{x}(t) \in F(t, x(t)) \quad \text{for a.a. } t \in J \\ &x(t) \in S \end{aligned}\right\} \tag{8}$$

where J is a given (arbitrary) real interval, $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a Carathéodory map (see (i) - (iii) *in Section* 4) *and S is a subset of $C(J, \mathbb{R}^n)$. Let $G : J \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a Carathéodory map such that*

$$G(t, c, c) \subset F(t, c) \qquad \text{for all } (t, c) \in J \times \mathbb{R}^n.$$

Assume the following:

(i) *There exists a convex closed subset Q of $C(J, \mathbb{R}^n)$ such that the associated problem*

$$\left.\begin{aligned} &\dot{x}(t) \in G(t, x(t), q(t)) \quad \text{for a.a. } t \in J \\ &x(t) \in S \cap Q \end{aligned}\right\} \tag{9}$$

has an acyclic set of solutions $T(q)$, for each $q \in Q$.

(ii) *There exists a locally Lebesgue integrable function $\alpha : J \to [0, \infty)$ such that $|G(t, x(t), q(t))| \le \alpha(t)$ a.e. in J for any pair $(q, x) \in \Gamma_T$ (i.e. from the graph of T).*

(iii) *$T(Q)$ is bounded in $C(J, \mathbb{R}^n)$ and $\overline{T(Q)} \subset S$.*

Then problem (8) *admits a solution.*

At first, we deal with a Cauchy problem, in the absence of Lipschitzianity. For a local result of this type see, e.g., [27].

Theorem 5.2. *Consider the Cauchy problem* (3), *where $F : [0, \infty) \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a Carathéodory product-measurable map satisfying condition* (C) *in Section* 3 *with $\int_0^\infty \alpha(t)\, dt < \infty$. Then problem* (3) *admits a bounded solution on the half-line.*

Proof. By the same reason as in the proof of Theorem 3.1, problem (3) can be equivalently replaced by

$$\left.\begin{aligned} &\dot{x}(t) \in G(t, x(t)) \quad \text{for a.a. } t \in [0, \infty) \\ &x(0) = x_0 \end{aligned}\right\} \tag{10}$$

where

$$G(t,x) = \begin{cases} F(t,x) & \text{for } |x| \le D \text{ and } t \in [0,\infty) \\ F(t, D\frac{x}{|x|}) & \text{for } |x| \ge D \text{ and } t \in [0,\infty) \end{cases}$$

$$D \ge (|x_0| + AB)\exp A, \qquad A = \int_0^\infty \alpha(t)\,dt < \infty.$$

Moreover, there certainly exists a positive constant γ such that

$$|x_0| + |G(t,x)| \le |x_0| + A(B+D) \le \gamma \quad \text{for all } x \in \mathbb{R}^n \text{ and a.a. } t \in [0,\infty). \tag{11}$$

Hence, besides problem (10), consider still a one-parameter family of linear problems (notice that a product measurability of F implies the measurability of $G(t,q(t))$; see [5] and cf. (9))

$$\left.\begin{array}{l} \dot{x}(t) \in G(t,q(t)) \quad \text{for a.a. } t \in [0,\infty), q \in Q \\ x(t) \in Q \cap S \end{array}\right\} \tag{12}$$

where

$$S = \{x \in C([0,\infty),\mathbb{R}^n)\,|\, x(0) = x_0\}$$
$$Q = cl_{C([0,\infty),\mathbb{R}^n)}\left\{x(t) \in AC_{loc}([0,\infty),\mathbb{R}^n)\,\middle|\, \sup \operatorname{ess}_{t\in[0,\infty)}|\dot{x}(t)| \le \gamma\right\}.$$

It is well-known (see [26]) that problem (12) is equivalent to

$$x(t) \in x_0 + \int_0^t G(s,q(s))\,ds := T(q)$$

where the integral is understood in the generalized sense of Aumann (see [25]). It follows from Theorem 3.1 that the operator T is, for each $q \in Q$, acyclic and so conditions (i) - (iii) of Proposition 5.1 are satisfied, whenever $T(Q) \subset Q$. This, however, follows immediately from (11):

$$\begin{aligned} \sup_{t\in[0,\infty)} \left| x_0 + \int_0^t G(s,q(s))\,ds \right| &\le |x_0| + \int_0^\infty |G(t,q(t))|\,dt \\ &\le |x_0| + (B+D)\int_0^\infty \alpha(t)\,dt \\ &= |x_0| + A(B+D) \\ &< \infty. \end{aligned}$$

This also implies the boundedness of solutions ∎

Remark 5.3. By the substitution $t := -\tau$, the conclusion of Theorem 5.2 is also true for the problem

$$\left.\begin{array}{l} \dot{x}(t) \in F(t,x(t)) \quad \text{for a.a. } t \in (-\infty,0] \\ x(0) = x_0 \in \mathbb{R}^n \end{array}\right\} \tag{13}$$

but provided $F : (-\infty,0] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a Carathéodory product-measurable map satisfying condition (C) on $J = (-\infty,0]$ with $\int_{-\infty}^0 \alpha(t)\,dt < \infty$.

Now, we proceed to a target problem, again in the absence of Lipschitzianity.

Theorem 5.4. *Consider the target problem* (5) *and assume that* $F : [0,\infty) \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is a Carathéodory product-measurable map satisfying condition* (C) *in Section* 3 *with* $\int_0^\infty \alpha(t)\,dt < \infty$. *Then problem* (5) *admits a* (*bounded*) *solution.*

Proof. Again, it is convenient to consider, instead of problem (5), the equivalent problem (6), where

$$G(t,x) = \begin{cases} F(t,x) & \text{for } |x| \le D \text{ and } t \in [0,\infty) \\ F(t, D\frac{x}{|x|}) & \text{for } |x| \ge D \text{ and } t \in [0,\infty) \end{cases}$$

$$D \ge (|x_\infty| + AB)\exp A, \qquad A = \int_0^\infty \alpha(t)\,dt < \infty \ \text{ and (11) holds.}$$

Besides problem (6), consider still a one-parameter family of linear problems (12) (cf. (9)), where

$$\begin{aligned} S &= \big\{x \in C([0,\infty),\mathbb{R}^n) \mid \lim_{t\to\infty} x(t) = x_\infty\big\} \\ Q &= \big\{q \in C([0,\infty),\mathbb{R}^n) \mid |q(t)| \le |x_\infty| + A(B+D) \text{ for } t \ge 0\big\}. \end{aligned}$$

Consider the set

$$S_1 = \left\{ x \in Q \,\middle|\, |x(t) - x_\infty| \le (B+D)\int_t^\infty \alpha(s)\,ds \text{ for } t \ge 0 \right\} \subset S.$$

It is evident that S_1 is a closed subset of S and all solutions to problem (12) belong to S_1.

At first, we assume that $G = g$ is single-valued. Then we have a single-valued continuous operator

$$T(q) = x_\infty + \int_\infty^t g(s,q(s))\,ds$$

for every $q \in Q$. Thus, to apply Proposition 5.1, only the condition $T(Q) \subset Q$ should be verified. But this follows immediately from (11), because

$$\begin{aligned} \sup_{t\in[0,\infty)} \left| x_\infty + \int_\infty^t G(s,q(s))\,ds \right| &\le |x_\infty| + \int_0^\infty |G(t,q(t))|\,dt \\ &\le |x_\infty| + (B+D)\int_0^\infty \alpha(t)\,dt \\ &= |x_\infty| + A(B+D) \\ &< \infty. \end{aligned}$$

By Proposition 5.1, we obtain a solution to the problem with g as right-hand side. This existence result can be applied to prove Theorem 4.2 which is needed to prove our statement in a general case. In fact, in view of Theorem 4.2, the map T which assigns to every $q \in Q$ the set of solutions to the linear problem (12), is acyclic. Once more, we use Proposition 5.1 obtaining a solution to problem (5), and the proof is complete ∎

Remark 5.5. By the substitution $t := -\tau$, the conclusion of Theorem 5.4 is also true for the problem

$$\left.\begin{aligned} &\dot{x}(t) \in F(t, x(t)) \quad \text{for a.a. } t \in (-\infty, 0] \\ &x(-\infty) = x_{-\infty} \in \mathbb{R}^n \end{aligned}\right\}$$

but provided $F : (-\infty, 0] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a Carathéodory product-measurable map satisfying condition (C) on $J = (-\infty, 0]$ with $\int_{-\infty}^{0} \alpha(t)\, dt < \infty$.

Summarizing Theorems 5.2 and 5.4, we can conclude, in view of Remarks 5.3 and 5.5, by the following

Corollary 5.6. *Let $F : \mathbb{R}^{n+1} \multimap \mathbb{R}^n$ be a Carathéodory product-measurable map satisfying condition* (C) *in Section* 3*, on $J = (-\infty, \infty)$, with $\int_{-\infty}^{\infty} \alpha(t)\, dt < \infty$. Then the inclusion*

$$\dot{x}(t) \in F(t, x(t)) \qquad \text{for a.a. } t \in (-\infty, \infty)$$

admits an entirely bounded solution x_- on $(-\infty, \infty)$ with $x_-(-\infty) = x_{-\infty} \in \mathbb{R}^n$ and an entirely bounded solution x_+ on $(-\infty, \infty)$ with $x_+(\infty) = x_\infty \in \mathbb{R}^n$. If, additionally, $F(t, x) \equiv -F(-t, x)$ holds, then at least one solution x exists with $x(-\infty) = x(\infty) = x_\infty$ for each $x_\infty \in \mathbb{R}^n$.

6. Topological essentiality approach

In the present section, we are interested in some applications of topological essentiality of multi-valued maps to differential problems on non-compact intervals. The notion of topological essentiality for single-valued maps was introduced by Granas in [24] and elaborated in [15]. In the multi-valued case, it has been considered by several authors (see, e.g., [12, 20, 23]). The approach presented in [23] is the most general one, but still it must be modified if we would like to apply it to problems on non-compact intervals. Indeed, the appropriate spaces are usually not normed and there are no open bounded subsets of them. On the other hand, it seems to be inappropriate to consider a topological essentiality for bounded (hence with an empty interior) subsets. An abstract definition and usual consequences are possible, but the class of such essential maps would not contain many important and natural examples of maps. Therefore, one should consider a topological essentiality in linear topological spaces for arbitrary open subsets. This approach has been described in [31] and we only recall here basic definitions and some consequences which will be needed below.

For metric spaces, remark that a map is *admissible* if and only if it is a composition of acyclic maps (for more details see [16]). Assume that E and F are Fréchet spaces and $U \subset E$ is an open subset. Let

$$\mathcal{A}_{\partial U}(U, F) = \Big\{ \varphi : \overline{U} \multimap F \,\Big|\, \varphi \text{ is admissible and } 0 \notin \varphi(\partial U) \Big\}$$

$$\mathcal{A}^c(U, F) = \Big\{ \varphi : \overline{U} \multimap F \,\Big|\, \varphi \text{ is admissible and compact} \Big\}$$

$$\mathcal{A}^0(U, F) = \Big\{ \varphi \in \mathcal{A}^c(U, F) \,\Big|\, \varphi(x) = \{0\} \text{ for all } x \in \partial U \Big\}.$$

Notice that if $\partial U = \emptyset$ $(U = E)$, then $\mathcal{A}_{\partial U}(U, F)$ is the class of all admissible maps $\varphi : U \multimap F$ and $\mathcal{A}^0(U, F) = \mathcal{A}^c(U, F)$.

Definition 6.1 (comp. [23, 31]). A map $\varphi \in \mathcal{A}_{\partial U}(U, F)$ is *essential* if, for every $\Psi \in \mathcal{A}^0(U, F)$, there exists a point $x \in U$ such that $\varphi(x) \cap \Psi(x) \neq \emptyset$.

The most useful properties for us are summarized in the following two propositions.

Proposition 6.2 (Existence). *If $\varphi \in \mathcal{A}_{\partial U}(U, F)$ is essential, then there exists $x \in U$ such that $0 \in \varphi(x)$.*

Proposition 6.3. *Let $\varphi \in \mathcal{A}_{\partial U}(U, F)$ be an essential map. If $\chi : \overline{U} \times [0,1] \multimap F$ is a compact admissible map such that*

(i) $\chi(x, 0) = \{0\}$ *for every* $x \in \partial U$

(ii) $\{x \in \overline{U} | \, \varphi(x) \cap \chi(x,t) \neq \emptyset$ *for some* $t \in [0,1]\} \subset U$,

then $(\varphi - \chi(\cdot, 1))$ *is essential.*

The proofs of both propositions are quite analogous to those of Propositions 2.2 and 2.7 in [23]. For other properties see [31] (comp. also [23]).

Among many important examples of essential maps one has the following

Example 6.4. Let $L : E \to F$ be a continuous linear isomorphism. Then, for any open neighbourhood U of the origin in E, the restriction $L|_{\overline{U}} : \overline{U} \to F$ is essential.

Now, we show how the topological essentiality can be applied to differential inclusions. At first, let us give some preliminary assumptions.

Put $J_1 = [0, \infty)$ and let J_2 be some subinterval of $(-\infty, 0]$ such that $0 \in J_2$. Denote $J = J_2 \cup J_1$. Let $A_i : J_1 \to C(C(J, \mathbb{R}^n), C(J_2, \mathbb{R}^n))$ be continuous for $i = 0, 1, \ldots, k-1$ (e.g. $[A_i(t)(x)](s) = x(t+s)$). Assume still that

$$\begin{aligned} p_i &: C^{k-1}(J, \mathbb{R}^n) \to C(J_2, \mathbb{R}^n) \quad (i = 0, 1, \ldots, k-2) \text{ are continuous} \\ p_{k-1} &: C^{k-1}(J, \mathbb{R}^n) \to C(J_2, \mathbb{R}^n), \;\; p_{k-1}(x) = x^{(k-1)}|_{J_2} \\ \psi_i &: C^{k-1}(J, \mathbb{R}^n) \multimap C(J_2, \mathbb{R}^n) \quad (i = 0, 1, \ldots, k-1) \text{ are admissible and compact.} \end{aligned}$$

Denote

$$\begin{aligned} E_1 &= C(J_2, \mathbb{R}^n) \times \ldots \times C(J_2, \mathbb{R}^n) \qquad ((k-1) - \text{times}) \\ E_2 &= C(J_2, \mathbb{R}^n) \times \ldots \times C(J_2, \mathbb{R}^n) \qquad (k - \text{times}) \end{aligned}$$

and take some multi-valued map

$$\varphi : J_1 \times E_2 \multimap \mathbb{R}^n.$$

Consider the family of problems

$$\left.\begin{aligned} x^{(k)}(t) &\in \rho\varphi\big(t, A_0(t)x, A_1(t)\dot{x}, \ldots, A_{k-1}(t)x^{(k-1)}\big) \quad \text{for a.a.} t \in J_1 \\ p_0(x) &\in \rho\psi_0(x) \\ &\;\vdots \\ p_{k-1}(x) &\in \rho\psi_{k-1}(x) \end{aligned}\right\} \tag{14}$$

where $\rho \in [0,1]$. For every ρ, the set of solutions to problem (14) will be denoted by $S^A_\rho(\varphi)$.

In the degenerate case, when $J_2 = \{0\}$, we can put $[A_i(t)x](0) = x(t)$ and identify $C(J_2, \mathbb{R}^n) \equiv \mathbb{R}^n$, hence we have $p_i : C^{k-1}(J_1, \mathbb{R}^n) \to \mathbb{R}^n$, $\psi_i : C^{k-1}(J_1, \mathbb{R}^n) \multimap \mathbb{R}^n$ for $i = 0, 1, \dots, k-1$, and $p_{k-1}(x) = x^{(k-1)}(0)$. Thus, we get

$$\left.\begin{aligned} x^{(k)}(t) &\in \rho\varphi(t, x(t), \dot{x}(t), \dots, x^{(k-1)}(t)) \quad \text{for a.a. } t \in J_1 \\ p_0(x) &\in \rho\psi_0(x) \\ &\vdots \\ p_{k-1}(x) &\in \rho\psi_{k-1}(x) \end{aligned}\right\} \tag{15}$$

where $\rho \in [0,1]$. We denote by $S^k_\rho(\varphi)$ the set of solutions to problems (15).

The following special case of problems (14) will be considered below. We assume that

$$\left.\begin{aligned} [A_i(t)x](s) = [A(t)x](s) &= x(t+s) \\ p_i(x) &= x^{(i)}|_{J_2} \end{aligned}\right\} \qquad (i = 0, 1, \dots, k-1).$$

There is a map $b \in C^{k-1}(J_2, \mathbb{R}^n)$ such that $\psi_i(x) = \{b^{(i)}\}$ $(i = 0, 1, \dots, k-1)$. Under these assumptions, we get the problems

$$\left.\begin{aligned} x^{(k)}(t) &\in \rho\varphi\big(t, A(t)x, A(t)\dot{x}, \dots, A(t)x^{(k-1)}\big) \quad \text{for a.a. } t \in J_1 \\ x|_{J_2} &= \rho b \\ &\vdots \\ x^{(k-1)}|_{J_2} &= \rho b^{(k-1)} \end{aligned}\right\} \tag{16}$$

where $\rho \in [0,1]$. The set of solutions to problems (16) will be denoted by $S_\rho(\varphi)$. One can see that for $J_2 = \{0\}$ we obtain a family of Cauchy problems.

Now, we prove an existence result for problems (14) using a topological essentiality.

Theorem 6.5. *Assume that $\varphi : J_1 \times E_2 \multimap \mathbb{R}^n$ is a locally integrably bounded upper semicontinuous map with compact, convex values and there exists an open subset $\Omega \subset C^{k-1}(J, \mathbb{R}^n)$ such that*

(i) *$S^A_\rho(\varphi) \subset \Omega$ for every $\rho \in [0,1]$*

(ii) *$g : C^{k-1}(J, \mathbb{R}^n) \to C(J, \mathbb{R}^n) \times E_1$ given by $g(x) = \big(x^{(k-1)}, p_0(x), \dots, p_{k-2}(x)\big)$ is essential on $\overline{\Omega}$.*

Then the set $S^A_1(\varphi)$ is non-empty.

Proof. Denote $E = C^{k-1}(J, \mathbb{R}^n)$ and $F = C(J, \mathbb{R}^n)$. For every $x \in E$, $u \in \psi_{k-1}(x)$ and $z_x : J_1 \to \mathbb{R}^n$ such that

$$z_x(t) \in \varphi\big(t, A_0(t)x, A_1(t)\dot{x}, \dots, A_{k-1}(t)x^{(k-1)}\big) \qquad \text{for a.a. } t \in J_1 \tag{17}$$

(such z_x exists since φ is upper semicontinuous) we define the map $y_{x,u,z_x} : J \to \mathbb{R}^n$ by

$$y_{x,u,z_x} = \begin{cases} u(0) + \int_0^t z_x(\tau)\, d\tau & \text{for } t \in J_1 \\ u(t) & \text{for } t \in J_2. \end{cases}$$

Define $T : E \multimap F$ by

$$T(x) = \{y_{x,u,z_x} \,|\, u \in \psi_{k-1}(x),\ z_x \text{ satisfies (17)}\}.$$

This operator is the sum of ψ_{k-1} and an integral operator which is a compact map with compact, convex values (as a consequence of the local integrable boundedness of φ). Therefore, T is admissible and compact.

Consider the map $\Phi : \overline{\Omega} \multimap F \times E_1$ defined by

$$\Phi(x) = g(x) - T(x) \times \psi_0(x) \times \ldots \times \psi_{k-2}.$$

One can see that $0 \notin \Phi(x)$ for every $x \in \partial\Omega$. Indeed, if $0 \in \Phi(x)$, then $x \in S_1^A(\varphi)$. Assumption (i) implies that $S_1^A(\varphi) \subset \Omega$, thus $x \in \Omega$. We want to prove now that $0 \in \Phi(x)$ for some $x \in \Omega$. For this purpose we show that Φ is essential. By assumption (ii) we know that g is essential. Define a homotopy $\chi : \overline{\Omega} \times [0,1] \multimap F \times E_1$ by

$$\chi(x,t) = g(x) - t(T(x) \times \psi_0(x) \times \ldots \times \psi_{k-2})$$

for every $x \in \overline{\Omega}$ and $t \in [0,1]$. Applying assumption (i) again, we conclude that χ is a homotopy appropriate to use Proposition 6.3. This implies that $\Phi = \chi(\cdot,1)$ is essential. By Proposition 6.2, there is a point $x \in \Omega$ such that $0 \in \Phi(x)$, which completes the proof ∎

Corollary 6.6. *Assume that $J = [0,\infty)$ and let $\varphi : J \times \mathbb{R}^{nk} \multimap \mathbb{R}^n$ be a locally integrably bounded upper semicontinuous map with compact, convex values. Suppose that there exists an open subset $\Omega \subset C^{k-1}(J,\mathbb{R}^n)$ such that*

- **(i)** *$S_\rho^k(\varphi) \subset \Omega$ for every $\rho \in [0,1]$*
- **(ii)** *$g : C^{k-1}(J,\mathbb{R}^n) \to C(J,\mathbb{R}^n) \times E_1$ defined by $g(x) = (x^{(k-1)}, p_0(x), \ldots, p_{k-2}(x))$ is essential on $\overline{\Omega}$.*

Then the set $S_1^k(\varphi)$ is non-empty.

Corollary 6.7. *Assume that $\varphi : J_1 \times E_2 \multimap \mathbb{R}^n$ is a locally integrably bounded upper semicontinuous map with compact, convex values. Then the set S_1 is non-empty.*

Proof. Define a closed subspace G of $C(J,\mathbb{R}^n) \times E_1$ (see the above notation) as follows:

$$G = \left\{ (z, u_0, \ldots, u_{k-2}) \in C(J,\mathbb{R}^n) \times E_1 \;\middle|\; \begin{array}{l} \exists\, y \in C^{k-1}(J,\mathbb{R}^n) \text{ with } y^{(k-1)} = z \\ \text{and } y^{(i)}|_{J_2} = u_i \quad (i = 0, \ldots, k-2) \end{array} \right\}.$$

The map $g : F \to G$ defined by

$$g(x) = (x^{(k-1)}, x|J_2, \ldots, x^{(k-2)}|_{J_2})$$

is a linear isomorphism, hence by Example 6.4 g is essential on each open neighbourhood of the origin in F. By Theorem 6.5 we get that $S_1 \neq \emptyset$ ∎

Remark 6.8. Note that all the above results are true if φ is a measurable-continuous map. Moreover, in Corollary 6.7 we can assume only that φ is measurable-upper semicontinuous.

Now, we are interested in the topological structure of the solution set of problem (16). Assume that $J_2 = [r, 0]$ for some $r < 0$.

Theorem 6.9. *Assume that $\varphi : J_1 \times E_2 \multimap \mathbb{R}^n$ is a locally integrably bounded map with compact, convex values satisfying the following conditions:*

- **(A)** *For all $x_0, \ldots, x_{k-1} \in C(J_2, \mathbb{R}^n)$ the map $\varphi(\cdot, x_0, \ldots, x_{k-1})$ is measurable.*
- **(B)** *There exists $L \geq 0$ for all $x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1} \in C(J_2, \mathbb{R}^n)$ and for all $t \in J_1$ such that $d_H\big(\varphi(t, x_0, \ldots, x_{k-1}), \varphi(t, y_0, \ldots, y_{k-1})\big) \leq L \sum_{i=0}^{k-1} ||x_i - y_i||$.*

Then $S_1(\varphi)$ is a compact, acyclic subset of $C^{k-1}(J, \mathbb{R}^n)$.

Proof. We can assume $L \geq 1$. Consider the map $l : L^1_{loc}(J_1, \mathbb{R}^n) \to C^{k-1}(J, \mathbb{R}^n)$ defined by

$$l(z)(t) = \begin{cases} \sum_{j=0}^{k-1} \frac{t^j}{j!} b^{(j)}(0) + \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} z(s)\, ds ds_{k-1} \ldots ds_1 & \text{for } t \in J_1 \\ b(t) & \text{for } t \in J_2 \end{cases}$$

and a sequence of maps $l_m : L^1([0, m], \mathbb{R}^n) \to C^{k-1}([r, m], \mathbb{R}^n)$ defined by

$$l_m(z)(t) = \begin{cases} \sum_{j=0}^{k-1} \frac{t^j}{j!} b^{(j)}(0) + \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} z(s)\, ds ds_{k-1} \ldots ds_1 & \text{for } t \in [0, m] \\ b(t) & \text{for } t \in J_2. \end{cases}$$

Define the operator $\Phi : C^{k-1}(J, \mathbb{R}^n) \multimap C^{k-1}(J, \mathbb{R}^n)$ by

$$\Phi(x) = \left\{ y \in C^{k-1}(J, \mathbb{R}^n) \;\middle|\; \begin{array}{l} y(t) = l(z)(t) \text{ and, a.a. in } J_1 \\ z(t) \in \varphi\big(t, A(t)x, A(t)\dot{x}, \ldots, A(t)x^{(k-1)}\big) \end{array} \right\}.$$

In a similar way, we define a sequence of multi-valued maps $\Phi_m : C^{k-1}([r, m], \mathbb{R}^n) \multimap C^{k-1}([r, m], \mathbb{R}^n)$ by

$$\Phi_m(x) = \left\{ y \in C^{k-1}([r, m], \mathbb{R}^n) \;\middle|\; \begin{array}{l} y(t) = l_m(z)(t) \text{ and, a.a. in } [0, m] \\ z(t) \in \varphi\big(t, A(t)x, A(t)\dot{x}, \ldots, A(t)x^{(k-1)}\big) \end{array} \right\}.$$

Observe that all values of $\Phi_m(x)$ are non-empty (since φ is measurable-continuous) and convex (since φ is convex-valued). Moreover, because of the compactness and convexity of values and by the local integrable boundedness of φ, one can check that the values of Φ_m are compact, too.

It is easy to see that the fixed point set $Fix(\Phi)$ is equal to the one of all solutions to problem (16). Thus, we are interested in the topological structure of $Fix(\Phi)$. This will be studied, when applying the results from Section 2.

Consider the equivalent norms in $C^{k-1}([r,m],\mathbb{R}^n)$,

$$q_m(x)=\sum_{i=0}^{k-1}\max_{t\in[r,m]}(|x^{(i)}(t)|e^{-Lkt}).$$

For every $x,y\in C^{k-1}([r,m],\mathbb{R}^n)$ and for every $t\in[r,m]$, we have

$$\begin{aligned}
e^{-Lkt}||A(t)x-A(t)y|| &= \max\Big\{e^{-Lkt}|x(t+s)-y(t+s)|\Big|\, s\in[r,0]\Big\}\\
&\le \max\Big\{e^{-Lk(t+s)}|x(t+s)-y(t+s)|\Big|\, s\in[r,0]\Big\}\\
&\le \max\Big\{e^{-Lkv}|x(v)-y(v)|\Big|\, v\in[r,m]\Big\}\\
&= q_m(x-y).
\end{aligned}$$

For any $x_1,x_2\in C^{k-1}([r,m],\mathbb{R}^n)$ and $y_1\in\Phi_m(x_1)$, $y_1(t)=l_m(z_1)(t)$, we can choose $y_2\in\Phi_m(x_2)$ (comp. [30]) such that, for every $t\in[r,m]$, we have $y_2(t)=l_m(z_2)(t)$ and

$$|z_1(t)-z_2(t)|=d_H\Big(z_1(t),\varphi\big(t,A(t)x_2,A(t)\dot{x}_2,\ldots,A(t)x_2^{(k-1)}\big)\Big).$$

Now, for every $t\in[0,m]$ and $i=0,\ldots,k-1$,

$$\begin{aligned}
&\big|y_1^{(i)}(t)-y_2^{(i)}(t)\big|\\
&\quad=\left|\left(\int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-1}}(z_1(s)-z_2(s))\,dsds_{k-1}\cdots ds_1\right)^{(i)}\right|\\
&\quad\le\int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-1-i}}|z_1(s)-z_2(s)|\,dsds_{k-1-i}\cdots ds_1\le\\
&\quad\le L\int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-1-i}}\sum_{j=0}^{k-1}\big\|A(s)x_1^{(j)}-A(s)x_2^{(j)}\big\|dsds_{k-1-i}\cdots ds_1.
\end{aligned}$$

Multiplying it by e^{-Lkt}, we get

$$\begin{aligned}
&e^{-Lkt}\big|y_1^{(i)}(t)-y_2^{(i)}(t)\big|\\
&\quad\le Le^{-Lkt}\int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-1-i}}\sum_{j=0}^{k-1}e^{Lks}e^{-Lks}\big\|A(s)x_1^{(j)}-A(s)x_2^{(j)}\big\|dsds_{k-1-i}\cdots ds_1\\
&\quad\le q_m(x_1-x_2)Le^{-Lkt}\int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-1-i}}e^{Lks}dsds_{k-1-i}\cdots ds_1\\
&\quad\le q_m(x_1-x_2)L\frac{1}{(Lk)^{k-i}}(e^{Lkt}-1)\,e^{-Lkt}\\
&\quad= q_m(x_1-x_2)\frac{1-e^{-Lkt}}{(Lk)^{k-i-1}k}\\
&\quad\le\frac{1-e^{-Lkm}}{k}\,q_m(x_1-x_2).
\end{aligned}$$

Thus,

$$q_m(y_1 - y_2) \le L_m q_m(x_1 - x_2) \qquad \text{where } L_m = 1 - e^{-Lkm} < 1$$

what implies that Φ_m is a contraction.

Now, it is easy to check that, considering $C^{k-1}(J, \mathbb{R}^n)$ as limit of an inverse system of Banach spaces endowed with norms q_m, Φ is a limit map induced by the sequence $\{\Phi_m\}$. Using Corollary 2.10, we get that the set $S_1(\varphi)$ (which is equal to $Fix(\Phi)$) is non-empty, compact and acyclic ∎

Note that the case of an unbounded interval J_2 brings some troubles and the related problem remains open.

7. Discontinuous autonomous differential inclusions

In this section we are interested in existence results as well as in the structure of the solution set to an autonomous Cauchy problem with a multi-valued right-hand side on the halfline $J = [0, \infty)$. The classical Carathéodory problem can be regarded as a special case. Indeed, the problem

$$\left.\begin{aligned} &\dot{x}(t) = g(t, x(t)) \quad \text{for a.a. } t \in J \\ &x(0) = v \end{aligned}\right\}, \tag{18}$$

where $g : J \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function, can be rewritten as follows. Define $f(x_0, x) = (1, g(x_0, x))$, where $x_0 = t$. Let $y = (x_0, x)$ be a new variable. Then we arrive at the autonomous problem

$$\left.\begin{aligned} &\dot{y}(t) = f(y(t)) \quad \text{for a.a. } t \in J \\ &y(0) = (0, v) \end{aligned}\right\} \tag{19}$$

which is equivalent to problem (18).

The results below are generalizations of [7: Theorems 1 and 2] to the case of multi-valued maps and non-compact intervals.

We will consider the Cauchy problem

$$\left.\begin{aligned} &\dot{x}(t) \in \varphi(x(t)) \quad \text{for a.a. } t \in J \\ &x(0) = v. \end{aligned}\right\} \tag{20}$$

Here $\varphi : \mathbb{R}^n \multimap \mathbb{R}^n$ is a multi-valued map of the form $\varphi(x) = \psi(\tau(x), x)$, where $\tau : \mathbb{R}^n \to \mathbb{R}$ is a single-valued map and $\psi : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a multi-valued one.

Theorem 7.1. *Assume the following:*

(i) *The map $\tau : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and $\psi : \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a Carathéodory map.*

(ii) *For some compact, convex set $K \subset \mathbb{R}^n$, at every point x, one has*

$$\varphi(x) \subset K \qquad \textit{and} \qquad \nabla\tau(x) \cdot z > 0 \;\; \textit{for every } z \in K. \tag{21}$$

Then the Cauchy problem (20) *has a solution. If, additionally,*

(iii) *The gradient* $\nabla\tau$ *has bounded directional variation* [8] *with respect to the cone* $\Gamma = \{\lambda z | \, \lambda \geq 0$ *and* $z \in K\}$,

then problem (20) *has a non-empty, compact and acyclic set of solutions.*

The proof of this statement will be given a little later, after some preparations.

Remark 7.2. We can understand assumption (ii) in Theorem 7.1 as a transversality assumption motivated by a classical Carathéodory single-valued problem. Indeed, the map f in (19) can jump across the hyperplanes of the form $x_0 = \text{const}$, which are transversal to f (the inner product of their normal vectors with f is equal to 1). In the above theorem, since $\dot{x} \in K$, for every solution x, transversality assumptions imply that all trajectories must cross transversally any hypersurface of the form $\tau(x) = \text{const}$.

Remark 7.3. The above theorem remains true with weaker assumptions on the regularity of the map τ. Namely, it is sufficient to assume that τ is Lipschitzian and reformulate (21) e.g. in terms of Clarke's generalized gradients (see [8]).

For the proof of Theorem 7.1 we need some results describing possible regularizations of Carathéodory maps (see [10, 19, 25, 32]).

Proposition 7.4. *Let* J *be an interval and* $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ *be a non-empty compact convex-valued Carathéodory map. Suppose that* $\tau : \mathbb{R}^n \to \mathbb{R}$ *satisfies assumptions* (i) - (ii) *of Theorem* 7.1*. Then there exists an almost upper semicontinuous* [9] *map* $G : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ *with non-empty, compact, convex values and such that:*

(i) $G(t,x) \subset F(t,x)$ *for every* $(t,x) \in J \times \mathbb{R}^n$.

(ii) *For every* $T > 0$, *if* $\Delta \subset [0,T]$ *is measurable,* $u : \Delta \to \mathbb{R}^n$ *is a measurable map,* $v \in C_T$ *where*

$$C_T = \left\{ v \in C([0,T], \mathbb{R}^n) \,\middle|\, v(0) = 0 \text{ and } \frac{v(t) - v(s)}{t - s} \in K \text{ for } t > s \right\}$$

and $u(t) \in F\big(\tau(v(t)), v(t)\big)$ *for a.a.* $t \in \Delta$, *then* $u(t) \in G\big(\tau(v(t)), v(t)\big)$ *for a.a.* $t \in \Delta$.

Proof. Due to [25: Theorem 1.5] and [10: Lemma 1], there exists an almost upper semicontinuous map $G : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ with non-empty, compact, convex values satisfying

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[8] A map u has a *bounded directional variation* with respect to a cone Γ if

$$\sup \left\{ \sum_{i=1}^{N} |u(p_i) - u(p_{i-1})| \,\middle|\, N \geq 1 \text{ and } p_i - p_{i-1} \in \Gamma \text{ for every } i \right\} < \infty.$$

[9] For two metric spaces X, Y and an interval J, a multi-valued map $G : J \times X \multimap Y$ is *almost upper semicontinuous* if, for every $\varepsilon > 0$, there exists a measurable set $A_\varepsilon \subset J$ such that $m(J \setminus A_\varepsilon) < \varepsilon$ and the restriction $G|_{A_\varepsilon \times X}$ is upper semicontinuous, where m stands for the Lebesgue measure.

(i) and

$$\left.\begin{array}{l}\Delta \subset J \text{ measurable}\\ u, v : \Delta \to \mathbb{R}^n \text{ measurable}\\ u(t) \in F(t, v(t)) \text{ for a.a. } t \in \Delta\end{array}\right\} \quad \Rightarrow \quad u(t) \in G(t, v(t)) \text{ for a.a. } t \in \Delta.$$

Following the proofs of [25: Theorems 1.2, 1.3 and 1.5] one can check that statement (ii) also holds. Note that in this consideration we use the fact that, under the assumptions on τ,

$$m^*((\tau \circ v)^{-1}(B)) = m^*(B) \qquad \text{for every } B \subset \mathbb{R} \text{ with } m^*(B) = 0 \tag{22}$$

where m^* stands for the outer Lebesgue measure on $\mathbb{R}$ and $v \in C_T$. To proof this we define the set

$$X_T = \big\{x \in \mathbb{R}^n : |x| \le T|K|\big\}$$

where $|K| = \sup_{x \in K} |x|$, and notice that, by the continuity of the gradient $\nabla\tau$ and the compactness of X_T and K, there exists $\delta > 0$ such that

$$\nabla\tau(x) \cdot z \ge \delta \qquad \text{for every } x \in X \text{ and } z \in K. \tag{23}$$

Take any $v \in C_T$. By the definition of the set C_T, we get that $v(t) \in X_T$ for every $t \in [0, T]$. Hence, condition (23) implies

$$\liminf_{t \to s^+} \frac{\tau(v(t)) - \tau(v(s))}{t - s} \ge \delta$$

by which the map $\tau \circ v$ is strictly increasing and, moreover,

$$\frac{1}{\delta}\big(\tau(v(b)) - \tau(v(a))\big) \ge b - a \tag{24}$$

for all $a, b \in [0, T]$ with $b > a$. Take a set $B \subset \mathbb{R}$ with $m^*(B) = 0$. We show that $m^*\big((\tau \circ v)^{-1}(B)\big) = 0$ or, equivalently, that for every $\varepsilon > 0$ there is a covering $\alpha = \{P_1, P_2, \ldots\}$ of $(\tau \circ v)^{-1}(B)$ by intervals such that $\mathrm{vol}\,(\alpha) = \sum_{i=1}^{\infty} m(P_i) < \varepsilon$. Let $\varepsilon > 0$ be arbitrary and $\beta = \{D_1, D_2, \ldots\}$ be a covering of B by intervals such that $\mathrm{vol}\,(\beta) < \varepsilon\delta$. Since $\tau \circ v$ is continuous and increasing, all the sets $(\tau \circ v)^{-1}(D_i)$ are intervals and form a covering of $(\tau \circ v)^{-1}(B)$. Putting $b_i - a_i = m(D_i)$, we get

$$\begin{aligned}\mathrm{vol}\,(\tau \circ v)^{-1}(B) &= \sum_{i=1}^{\infty} \mathrm{vol}\,\big((\tau \circ v)^{-1}(D_i)\big)\\ &= \sum_{i=1}^{\infty} \big((\tau \circ v)^{-1}(b_i) - (\tau \circ v)^{-1}(a_i)\big)\\ &\le \frac{1}{\delta}\sum_{i=1}^{\infty}(b_i - a_i)\\ &< \varepsilon\end{aligned}$$

which completes the proof of the proposition ∎

Let us also recall the following

Proposition 7.5 [3, 19]. *Let E, E_1 be two separable Banach spaces, J be an interval and $F : J \times E \multimap E_1$ be an almost upper semicontinuous map with compact convex values. Then F is σ-Carathéodory-selectionable.* [10] *The maps $F_k : J \times E \multimap E_1$ are almost upper semicontinuous and we have $F_k(t,e) \subset \overline{\mathrm{conv}}\,(\bigcup_{x\in E} F(t,x))$ for all $(t,e) \in J \times E$. Moreover, if F is locally integrably bounded, then F is σ-mLL-selectionable.* [11]

Proof of Theorem 7.1. Without loss of generality, we assume that $v = 0$. In the first step, we describe the topological structure of the solution set of our problem considered on compact intervals. Using Proposition 7.4, for every integer $m \geq 1$ we can find an almost upper semicontinuous map $G^m : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ such that the set $S(G^m, \tau, v)$ of all solutions to the problem

$$\left.\begin{array}{ll} \dot{x}(t) \in G^m(\tau(x(t)), x(t)) & \text{for a.a. } t \in [0,m] \\ x(0) = v & \end{array}\right\} \tag{25}$$

is equal to the set of solutions to the problem

$$\left.\begin{array}{ll} \dot{x}(t) \in \psi(\tau(x(t)), x(t)) & \text{for a.a. } t \in [0,m] \\ x(0) = v. & \end{array}\right\} \tag{26}$$

Proposition 7.5 implies the existence of a sequence of Carathéodory convex compact-valued maps $G_k^m : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ $(k \in \mathbb{N})$ such that each G_k^m has a measurable-locally Lipschitz selector and G^m is an intersection of G_k^m. Consider (comp. Proposition 7.4) the compact, convex set

$$C_m = \left\{ v \in C([0,m], \mathbb{R}^n) \,\middle|\, v(0) = 0 \text{ and } \frac{v(t) - v(s)}{t - s} \in K \text{ for } t > s \right\}.$$

Each map from C_m has values in the compact set

$$X_m = \{ x \in \mathbb{R}^n : |x| \leq m|K| \}.$$

Denote by $f_k^m : J \times \mathbb{R}^n \to \mathbb{R}^n$ a measurable-locally Lipschitz selector of G_k^m and by $g_k^m : J \times \mathbb{R}^n \to \mathbb{R}^n$ the map defined by

$$g_k^m(t,x) = \begin{cases} f_k^m(t,x) & \text{for } x \in X_m \\ f_k^m(t, m|K|\frac{x}{|x|}) & \text{for } x \notin X_m. \end{cases}$$

[10] F is *σ-Carathéodory-selectionable*, if there exists a sequence of maps $F_k : J \times E \multimap E_1$ for which there exists a full measure set $A \subset J$ such that, for every $t \in A$, $x \in E$ and $k \in \mathbb{N}$ we have $F(t,x) = \cap_{k\in\mathbb{N}} F_k(t,x)$, $F_{k+1}(t,x) \subset F_k(t,x)$ and each F_k has a Carathéodory single-valued selector.

[11] I.e. F is an intersection (see above) of a decreasing sequence of maps having measurable-locally Lipschitz selectors.

By the compactness of X_m, the map $g_k^m(t,\cdot)$ is globally Lipschitz. [7: Theorem 1] applied for g_k^m implies that the set $S(G_k^m,\tau,v)$ of all solutions to the problem on $[0,m]$ with G_k^m as right-hand side is non-empty. By the well-known Ascoli theorem, this set is also compact.

Now we show that, under assumption (iii), $S(G_k^m,\tau,v)$ is contractible. Following, e.g., [17], we define a required homotopy $h : S(G_k^m,\tau,v)\times[0,1]\to S(G_k^m,\tau,v)$ by

$$h(u,\lambda)(t)=\begin{cases} u(t) & \text{for } 0\le t\le \lambda m\\ S(g_k^m,\tau,\lambda m,u(\lambda m)) & \text{for } \lambda m\le t\le m\end{cases}$$

where $S(g_k^m,\tau,\lambda m,u(\lambda m))$ denotes the unique solution of the problem

$$\left.\begin{aligned} &\dot{x}(t)=g_k^m(\tau(x(t)),x(t)) \quad \text{for a.a. } t\in[0,m]\\ &x(\lambda m)=u(\lambda m).\end{aligned}\right\} \tag{27}$$

The existence and uniqueness result for problem (27) follows from [7: Theorem 2]. Thus, we obtain that $S(G_k^m,\tau,v)$ is compact and contractible. A standard computation shows that

$$S(G^m,\tau,v)=\bigcap_{k=1}^{\infty} S(G_k^m,\tau,v)$$

what implies that $S(G^m,\tau,v)$ is non-empty, compact and, under assumption (iii), also acyclic (more precisely - R_δ).

Consider the following family of maps $\Psi_m : C([0,m],\mathbb{R}^n) \multimap C([0,m],\mathbb{R}^n)$ $(m\ge 1)$:

$$\Psi_m(u)(t)=\left\{\int_0^t v(s)\,ds \;\middle|\; \begin{aligned}&v\in L^1([0,m],\mathbb{R}^n) \text{ and}\\ &v(s)\in\psi(\tau(u(s)),u(s)) \text{ for a.a. } t\in[0,m]\end{aligned}\right\}.$$

It is easy to check (e.g. by means of Propositions 7.4 and 7.5 and properties of measurable maps) that each Ψ_m has non-empty values. Moreover, by the convexity of K, Ψ_m maps C_m into C_m. Denote $\Phi_m : C_m \multimap C_m$, $\Phi_m(u)=\Psi_m(u)$. It is easy to see that $S(G^m,\tau,v)$ is equal to the fixed point set of Φ_m. Notice that the set

$$C=\left\{v\in C(J,\mathbb{R}^n)\;\middle|\; v(0)=0 \text{ and } \frac{v(t)-v(s)}{t-s}\in K \text{ for } t>s\right\}$$

can be considered as limit of the inverse system $\{C_m,\pi_m^p\}$ where $\pi_m^p : C_p\to C_m$ is a bonding map defined by $\pi_m^p(u)=u|_{[0,m]}$. Moreover, the map $\{\Phi_m\}$ of the above system induces the limit map $\Phi : C\multimap C$,

$$\Phi(u)(t)=\left\{\int_0^t v(s)\,ds\;\middle|\; v\in L^1_{loc}(J,\mathbb{R}^n) \text{ and } v(s)\in\psi(\tau(u(s)),u(s)) \text{ for a.a. } t\in J\right\}.$$

Of course, the fixed point set of Φ is equal to the solution set of problem (20). Hence Proposition 2.1 and Theorem 2.8 imply that the set of solutions to problem (20) is non-empty and compact under assumptions (i) - (ii) and, additionally, it is acyclic if we add assumption (iii). The proof of Theorem 7.1 is so complete ∎

Remark 7.6. Note that [7: Theorems 1 and 2] can be reformulated and proved in the case of non-compact intervals (with single-valued right-hand side), when using the same method as in [7], under more restrictive assumptions on τ_i in Theorem 1 and on τ in Theorem 2. Namely, one should assume that $\nabla\tau_i(x) \cdot z \geq \delta$ for some $\delta > 0$ and every $z \in K$.

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