Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 19 (2000), No. 1, 23–34

On B-Bounded Semigroups as a Generalization of C_0-Semigroups

L. Arlotti

Abstract. In this paper we consider the one-parameter family of linear operators that A. Belleni Morante recently introduced and called B-bounded semigroups. Such a family was studied by A. Belleni Morante himself and by J. Banasiak. Here we give a necessary and sufficient condition that a pair (A, B) of linear operators be the generator of a B-bounded semigroup. Our procedure is constructive and is equivalent to the Yosida procedure for the construction of a C_0-semigroup when $B = I$. We also show that our result represents a generalization of Banasiak's result.

Keywords: *B-bounded semigroups, C_0-semigroups*

AMS subject classification: 47 D 06

1. Introduction

In a recent paper Belleni Morante [5] introduced a new class of evolution operators which he called B-bounded semigroups. The definition was suggested by an evolution problem in particle transport theory with multiplying boundary conditions [13]. Such a problem was studied in a Banach space X and its evolution was described by an operator $A \in G(M, \omega, X)$, that is an operator which is the generator of a strongly continuous semigroup $(\exp(tA))_{t\geq 0}$, such that $\|\exp(tA)\| \leq M e^{\omega t}$ for all $t \geq 0$. To prove this property the authors used Arendt's method for resolvent positive operators [1] and therefore they could not evaluate the positive constant M, directly from the evolution system. Nevertheless they noticed that another operator B with norm not larger than 1 exists, such that the inequality $\|B \exp(tA) f\| \leq \|Bf\|$ is satisfied for all $t \geq 0$ and $f \in X$. In such a way they could describe the asymptotic behaviour of the physical system by means of the one-parameter family of linear operators $(Y(t) = B \exp(tA))_{t\geq 0}$. Such a family was called a B-bounded semigroup. The original definition of this new class of evolution operators was introduced by Belleni Morante in [5] and generalized by himself in [6]. A further generalization has been given by Banasiak in [2]. Banasiak's definition reads as follows.

Definition 1.1. Let X and Z be Banach spaces and suppose the following:

L. Arlotti: Dipartimento di Ingegneria Civile, via delle Scienze 208, 33100 Udine, Italy
e-mail: Arlotti@dic.uniud.it

ISSN 0232-2064 / \$ 2.50

(i) $A : D(A) \to X$ and $B : D(B) \to Z$ are two linear operators such that $D(A) \subset D(B) \subset X$.

(ii) For some $\omega \in \mathbb{R}$ the resolvent set of A satisfies $\rho(A) \supset (\omega, \infty)$.

A one-parameter family of linear operators $(Y(t))_{t\geq 0}$ which satisfies

1. $Y(t) : \Omega \to Z$, with $X \supset \Omega \supseteq D(B)$, and $\|Y(t)f\| \leq M \exp(\omega t)\|Bf\|$ for any $t \geq 0$ and $f \in D(B)$
2. $t \to Y(t)f \in C([0,\infty), Z)$ for any $f \in \Omega$
3. $Y(t)f = Bf + \int_0^t Y(s)Af\, ds \ \ (t \geq 0)$ for any $f \in \Omega_0 = \{f \in D(A) : \ Af \in \Omega\} \subset D(A) \subset D(B)$

is called a *B-quasi bounded semigroup* generated by A and B.

Here and in the sequel we denote by $\|\cdot\|$ the norm in the Banach space Z. In the case B bounded, $D(B) = X = Z$, $M = 1$, $\omega = 0$ one obtains the original definition, while the case $X = Z$ gives Belleni's generalization. In any case one can verify that Definition 1.1 is correct. Indeed, if all the above conditions are satisfied, then for any $f \in \Omega, \lambda > \omega$ and $n \in \mathbb{N}$ the relation

$$B(\lambda I - A)^{-n} f = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} \exp(-\lambda t) Y(t) f\, dt \tag{1.1}$$

holds which gives for $f \in D(B)$

$$\|B(\lambda I - A)^{-n} f\| \leq \frac{M}{(\lambda - \omega)^n} \|Bf\|. \tag{1.2}$$

Thanks to these relations one can prove that a given pair (A, B) of operators generates at most one family satisfying conditions 1 - 3. For such a pair we use the notation $A \in B\text{-}G(M, \omega, X, Z)$ (or $A \in B\text{-}G(M, \omega, X)$ if $X = Z$). A constructive procedure of the family $(Y(t))_{t\geq 0}$ is presented in [5] under very restrictive assumptions: B bounded, X weakly compact, and $D(A)$ dense in X. Indeed, in [5] sufficient conditions for the existence of the family $(Y(t))_{t\geq 0}$ are given, but a full characterization of generators of B-quasi bounded semigroups is not obtained. In the general case, that is without any restrictive assumption, a full characterization has been obtained in [2], where the following theorem is proved.

Theorem 1.1 (Banasiak characterization theorem). *Let operators A and B satisfy conditions* (i) *and* (ii) *of Definition* 1.1. *Then $A \in B\text{-}G(M, \omega, X, Z)$ if and only if the following conditions hold:*

1. *$B(D_B(A))$ is dense in Z_B.*
2. *There exists $M > 0$ such that $\|B(\lambda - A)^{-n} f\| \leq \frac{M}{(\lambda-\omega)^n} \|Bf\|$ for any $f \in D(B)$, $\lambda > \omega$ and $n \in \mathbb{N}$.*
3. *$A(N(B) \cap D(A)) \subset N(B)$.*

Here $D_B(A)$ represents the part of A in $D(B)$, that is $D_B(A) = \{f \in D(A) : \ Af \in D(B)\}$, $N(B)$ denotes the null space of B, and Z_B denotes the closure of the range of B in Z, that is $Z_B = \overline{R(B)}^Z$. To obtain this result the extrapolation space X_B, which represents the completion of X with respect to the (semi)norm $\|\cdot\|_B = \|B\cdot\|$, is introduced and the extension of the operator A to X_B is studied. It is shown (main generation theorem) that $A \in B\text{-}G(M,\omega,X,Z)$ if and only if there is an extension of the operator A which is the generator of a strongly continuous semigroup in X_B. The two cases of invertible and non-invertible B are separately examined.

The main aim of this paper is to prove the characterization theorem of B-quasi bounded semigroups directly, using a constructive procedure which can be seen as a generalization of Yosida's method for the construction of strongly continuous semigroups. Indeed, our procedure essentially coincides with Yosida's procedure when $B = I$ (see [8] and [12: p. 60]). In our proof we do not require any assumption on the operator B and we neither need to distinguish the two cases of invertible and non-invertible B, nor need to introduce the extrapolation space X_B, as in [2]. Moreover, considering that only the part of A in $D(B)$, that is $(A, D_B(A))$, appears in the construction of the family $(Y(t))_{t\geq 0}$, we suggest a further generalization of the definition of B-quasi bounded semigroups. Namely, instead of (ii) we can assume that

(ii)′ There exists $\omega \in \mathbb{R}$ such that for every real number $\lambda > \omega$ the map $(\lambda I - A) : D_B(A) \to D(B)$ is bijective

which, as we shall see, defines a more general framework.

In Section 2 we give a new version of the characterization theorem. More precisely, by supposing that operators A and B satisfy conditions (i) and (ii)′, we are able to prove that conditions 1 and 2 of Theorem 1.1 are sufficient to assure that $A \in B\text{-}G(M,\omega,X,Z)$. As for the necessity we have to observe the following: if there exists a B-quasi bounded semigroup $(Y(t))_{t\geq 0}$ generated by the pair (A,B), then formulas (1.1) and (1.2) hold [5], and therefore condition 2 is necessary; but only if we suppose that the family $(Y(t))_{t\geq 0}$ has the further property

4. $Y(0)f = Bf$ for all $f \in D(B)$

we can prove that condition 1 is necessary, too. It is easy to see that property 4 is generally independent from properties 1 - 3 of Definition 1.1, which only imply $Y(0)f = Bf$ for all $f \in D_B(A)$. We also note that property 4 has been implicitely assumed in the proof of [2: Lemma 3.1]. In Section 3 we obtain a new version of the generation theorem of [2]. Indeed, we prove that the pair (A,B) generates the B-quasi bounded semigroup $(Y(t))_{t\geq 0}$, with $Y(0) = B$, if and only if by putting for each $g \in B(D_B(A))$

$$\mathcal{A}(g) = \Big\{ h \in R(B) : \; h = BAf \text{ for } f \in D_B(A) \text{ with } Bf = g \Big\} \tag{1.3}$$

we obtain a single-valued mapping $\mathcal{A} : B(D_B(A)) \to R(B)$; such an operator is linear and closeable in Z_B and its closure generates a C_0-semigroup. Finally, we show that this result is equivalent to Banasiak's generation theorem. In such a way we obtain the same representation of $Y(t)$ as Banasiak. This representation is very important for applications because, as shown in [3], it allows to relate B-bounded semigroups to B-evolutions and the empathy theory introduced and developed in [9 - 11].

2. The characterization theorem

The aim of this section is to obtain a full characterization of B-quasi bounded semigroups directly, using a constructive procedure which generalizes Yosida's method for C_0-semigroups. Here we suppose that the operators A and B satisfy conditions (i) and (ii)$'$, which are less restrictive than conditions (i) and (ii) of Definition 1.1. Indeed, suppose that (i) and (ii) hold. Then clearly for any $\lambda > \omega$ and $f \in D_B(A)$ the function $\lambda f - Af$ belongs to $D(B)$ and the mapping $(\lambda I - A) : D_B(A) \to D(B)$ is injective. But such a mapping is surjective too, because for any $g \in D(B)$ the unique solution $f \in D(A)$ of the equation $(\lambda I - A)f = g$ is such that $Af = \lambda f - g \in D(B)$. This implies $f \in D_B(A)$ and shows that if conditions (i) and (ii) hold, then (ii)$'$ holds too. Furthermore, we have to note that the operator

$$(\lambda I - A)^{-1} : D(B) \to D_B(A)$$

is bounded if condition (ii) is satisfied, but it could be unbounded if (ii)$'$ holds. But, even in the case of the weaker condition (ii)$'$, the resolvent formula

$$(\lambda I - A)^{-1} f - (\mu I - A)^{-1} f = (\mu - \lambda)(\mu I - A)^{-1}(\lambda I - A)^{-1} f$$

is satisfied for any $f \in D(B)$ and $\lambda, \mu > \omega$, and therefore the operators $(\lambda I - A)^{-1}$ and $(\mu I - A)^{-1}$ commute.

Now we can state our characterization theorem, which represents the main result of this section.

Theorem 2.1 (New characterization theorem). *Let the operators A and B satisfy conditions* (i) *and* (ii)$'$. *Then the pair (A, B) generates the B-quasi bounded semigroup $(Y(t))_{t\geq 0}$, with $Y(0) = B$, if and only if the following conditions hold:*

1. *$B(D_B(A))$ is dense in Z_B.*

2. *There exists $M \geq 1$ such that $\|B(\lambda I - A)^{-n} f\| \leq \frac{M}{(\lambda-\omega)^n} \|Bf\|$ for any $f \in D(B)$, $\lambda > \omega$ and $n \in \mathbb{N}$.*

Remark 2.1. According to the statement of Theorem 2.1 conditions (i), (ii)$'$, 1 and 2 are not only necessary but also sufficient for the existence of the one-parameter family $(Y(t))_{t\geq 0}$. Thus our method allows to construct the B-quasi bounded semigroup under assumptions which are less strong than those in [2].

Proof of Theorem 2.1. *Necessity*: Suppose that the operators A and B satisfy conditions (i) and (ii)$'$ and that they generate the B-quasi bounded semigroup $(Y(t))_{t\geq 0}$, with $Y(0) = B$. Then condition 2 holds, as shown in [5]. In order to see that condition 1 holds, too, consider $\lambda > \omega_1 = \max\{0, \omega\}$ and $g \in D(B)$ and put $f = \lambda(\lambda I - A)^{-1} g \in D_B(A)$. Clearly,

$$Bg - Bf = Bg - \lambda B(\lambda I - A)^{-1} g = \int_0^\infty \lambda \mathrm{e}^{-\lambda t}(B - Y(t))g \, dt.$$

Because the function $t \to Y(t)g$ is continuous at $t = 0$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|(B - Y(t))g\| < \varepsilon$ for any $0 \le t < \delta$. Therefore,

$$\begin{aligned}\|Bg - Bf\| &\le \int_0^\infty \lambda e^{-\lambda t}\|(B - Y(t))g\|\, dt \\ &= \int_0^\delta \lambda e^{-\lambda t}\|(B - Y(t))g\|\, dt + \int_\delta^\infty \lambda e^{-\lambda t}\|(B - Y(t))g\|\, dt \\ &< \varepsilon + 2M\int_\delta^\infty \lambda e^{-(\lambda-\omega_1)t}\|Bg\|\, dt \\ &< 2\varepsilon\end{aligned}$$

for λ sufficiently large. This shows that $B(D_B(A))$ is dense in $R(B)$, and condition 1 is proved.

Sufficiency. Now we have to prove that if the operators A and B satisfy, together with (i) and (ii)$'$, conditions 1 and 2, then they generate a B-quasi bounded semigroup $(Y(t))_{t\ge 0}$ with $Y(0) = B$. For convenience, following [8] we break up the proof into a series of claims.

Claim 1: $\lim_{\lambda\to\infty} \lambda B(\lambda I - A)^{-1}f = Bf$ *for all* $f \in D(B)$. Indeed, note that $\|\lambda B(\lambda I - A)^{-1}f\| \le M\frac{\lambda}{(\lambda-\omega)}\|Bf\|$ for any $\lambda > \omega$ and $f \in D(B)$. Since $\frac{\lambda}{\lambda-\omega} \to 1$ as $\lambda \to \infty$ it follows that, for a fixed $M_1 > M$, $\|\lambda B(\lambda I - A)^{-1}f\| \le M_1\|Bf\|$ for λ sufficiently large. Thanks to this property and condition 1 we can state that claim 1 is proved if it is proved for every $f \in D_B(A)$. But for $f \in D_B(A)$

$$\|\lambda B(\lambda I - A)^{-1}f - Bf\| = \|B(\lambda I - A)^{-1}Af\| \le \frac{M}{(\lambda-\omega)}\|BAf\| \to 0$$

as $\lambda \to \infty$.

Claim 2: If $A_\lambda f := \lambda(\lambda(\lambda I - A)^{-1} - I)f$ *for any* $\lambda > \omega$ *and* $f \in D(B)$, *then* $\lim_{\lambda\to\infty} BA_\lambda f = BAf$ *for all* $f \in D_B(A)$. This follows from claim 1 because $BA_\lambda f = \lambda B(\lambda I - A)^{-1}Af \to BAf$ for $f \in D_B(A)$ as $\lambda \to \infty$.

Claim 3: For any $f \in D(B)$, $s, t \ge 0$ *and* $\lambda, \mu > \omega$ *the series*

$$\sum_{n=0}^\infty \sum_{k=0}^n \frac{(\lambda^2 t)^k}{k!}\frac{(\mu^2 s)^{n-k}}{(n-k)!}B(\lambda I - A)^{-k}(\mu I - A)^{-n+k}f \tag{2.1}$$

converges in Z *and its sum is a function* $g \in Z_B$. Indeed, observe that if $f \in D(B)$, $n, k \in \mathbb{N}$ and $\lambda, \mu > \omega$, then

$$\|B(\lambda I - A)^{-k}(\mu I - A)^{-n+k}f\| \le \frac{M^2}{(\lambda-\omega)^k(\mu-\omega)^{n-k}}\|Bf\| \tag{2.2}$$

and that the series

$$\sum_{n=0}^\infty \sum_{k=0}^n \frac{(\lambda^2 t)^k}{k!}\frac{(\mu^2 s)^{n-k}}{(n-k)!}\frac{1}{(\lambda-\omega)^k(\mu-\omega)^{n-k}} \tag{2.3}$$

is absolutely convergent because it is the Cauchy product of two series $\sum_{n=0}^\infty \frac{1}{n!}\frac{(\lambda^2 t)^n}{(\lambda-\omega)^n}$ and $\sum_{n=0}^\infty \frac{1}{n!}\frac{(\mu^2 s)^n}{(\mu-\omega)^n}$. Thus the claim is proved.

Remark 2.2. For $t>0$ and $s=0$ series (2.2) becomes $\sum_{n=0}^{\infty}\frac{(\lambda^2 t)^n}{n!}B(\lambda I-A)^{-n}f$, and for $s=t=0$ it reduces to the only term Bf.

Claim 4: By putting for all $s,t\geq 0$, $\lambda,\mu>\omega$ and $f\in D(B)$

$$Y_{\lambda,\mu}(t,s)f=\mathrm{e}^{-(\lambda t+\mu s)}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\lambda^2 t)^k}{k!}\frac{(\mu^2 s)^{n-k}}{(n-k)!}B(\lambda I-A)^{-k}(\mu I-A)^{-n+k}f, \tag{2.4}$$

for any $\lambda,\mu>\omega$ a two-parameters family $(Y_{\lambda,\mu}(t,s))_{t,s\geq 0}$ of linear operators mapping $D(B)$ into Z_B is defined such that

$$Y_{\lambda,\mu}(t,s)=Y_{\mu,\lambda}(s,t) \tag{2.5}$$

$$\|Y_{\lambda,\mu}(t,s)f\|\leq M^2\exp\Big(\frac{\lambda\omega t}{\lambda-\omega}+\frac{\mu\omega s}{\mu-\omega}\Big)\|Bf\|. \tag{2.6}$$

Moreover, for any $\lambda,\mu>\omega$ and $f\in D(B)$, $Y_{\lambda,\mu}(t,s)f$ is (strongly) continuous in the variable $(t,s)\in[0,\infty)\times[0,\infty)$, and continuously differentiable in both t and s with

$$\frac{\partial Y_{\lambda,\mu}(t,s)f}{\partial t}=Y_{\lambda,\mu}(t,s)A_\lambda f \tag{2.7}$$

$$\frac{\partial Y_{\lambda,\mu}(t,s)f}{\partial s}=Y_{\lambda,\mu}(t,s)A_\mu f. \tag{2.7$'$}$$

Indeed, claim 3 allows to define the two-parameters family $(Y_{\lambda,\mu}(t,s))_{t,s\geq 0}$ and gives estimate (2.6), while the resolvent formula implies symmetry. Concerning the continuity and the continuous differentiability of the function $Y_{\lambda,\mu}(t,s)f$, thanks to inequality (2.2) these properties can be shown in the same way as if $\exp(\lambda t+\mu s)Y_{\lambda,\mu}(t,s)f$ was a complex-valued function defined by a power series (see [4: p. 49]). Finally, formulas (2.7) and (2.7)$'$ can be easily verified because the series appearing in (2.4) can be differentiated termwise.

Claim 5: For any $\lambda,\mu>\omega$, $f\in D(B)$, $t>0$ and $0\leq s\leq t$, the function $Y_{\lambda,\mu}(t-s,s)f$ is continuously differentiable in s with

$$\frac{\partial Y_{\lambda,\mu}(t-s,s)f}{\partial s}=Y_{\lambda,\mu}(t-s,s)(A_\mu f-A_\lambda f). \tag{2.8}$$

Moreover, for a fixed $\omega_1>\omega$,

$$\|Y_{\lambda,\mu}(t-s,s)f\|\leq M^2\exp(\omega_1 t)\|Bf\| \tag{2.9}$$

for λ and μ sufficiently large. Clearly, formula (2.8) is an immediate consequence of (2.7) and (2.7)$'$. Inequality (2.9) follows from (2.6), since $\lambda\frac{\omega(t-s)}{\lambda-\omega}+\frac{\mu\omega s}{\mu-\omega}\to\omega t$ as both $\lambda,\mu\to\infty$.

Claim 6: If we put

$$Y_\lambda(t)f=\exp(-\lambda t)\sum_{n=0}^{\infty}\frac{(\lambda^2 t)^n}{n!}B(\lambda I-A)^{-n}f \tag{2.10}$$

for $\lambda > \omega, f \in D(B)$ and $t \geq 0$, then for any $f \in D(B)$ and $t \geq 0$ we have that $\lim_{\lambda\to\infty} Y_\lambda(t)f$ exists and defines a linear operator $Y(t) : D(B) \to Z_B$; moreover, $Y(0) = B$. Clearly, by virtue of definition (2.4), Remark 2.2 and symmetry property (2.5) we have for any $\lambda, \mu > \omega, f \in D(B)$, and $t \geq 0$

$$Y_\lambda(t)f = Y_{\lambda,\mu}(t,0)f = Y_{\mu,\lambda}(0,t)f. \tag{2.11}$$

This and claim 5 immediately give

$$\begin{aligned} Y_\mu(t)f - Y_\lambda(t)f &= Y_{\lambda,\mu}(0,t)f - Y_{\lambda,\mu}(t,0)f \\ &= \int_0^t \frac{\partial Y_{\lambda,\mu}(t-s,s)f}{\partial s}\,ds \\ &= \int_0^t Y_{\lambda,\mu}(t-s,s)(A_\mu f - A_\lambda f)ds. \end{aligned}$$

Using this property, inequality (2.9) and claim 2 we obtain for $f \in D_B(A)$

$$\begin{aligned} \|Y_\mu(t)f - Y_\lambda(t)f\| &\leq \int_0^t \|Y_{\lambda,\mu}(t-s,s)(A_\mu f - A_\lambda f)\|\,ds \\ &\leq M^2 t \exp(\omega_1 t)\|B(A_\mu f - A_\lambda f)\| \\ &\to 0 \end{aligned}$$

as $\lambda, \mu \to \infty$. Moreover, the convergence is uniform in every finite interval of t. This together with inequality (2.9) and condition 1 implies that there exists a linear operator $Y(t) : D(B) \to Z_B$ such that for any $f \in D(B)$ and $t \geq 0$

$$\lim_{\lambda\to\infty} Y_\lambda(t)f = Y(t)f. \tag{2.12}$$

Since $Y_\lambda(0)f = Bf$ for all $\lambda > \omega$ and $f \in D(B)$, we obtain $Y(0) = B$.

Claim 7: The family $(Y(t))_{t\geq 0}$ is a B-quasi bounded semigroup generated by A and B such that $Y(0) = B$. Indeed, definition (2.10) and condition 2 imply that for every $t \geq 0, f \in D(B)$ and $\omega_1 > \omega$

$$\|Y_\lambda(t)f\| \leq M \exp(\omega_1 t)\|Bf\| \tag{2.13}$$

for λ sufficiently large. This and (2.12) show that property 1 of Definition 1.1 is satisfied. As the convergence (2.12) is uniform in every finite interval of t, the limit $Y(t)f$ is strongly continuous in t for each $f \in D(B)$, and property 2 of B-quasi bounded semigroups is satisfied. Thanks to (2.7), (2.10) and (2.11) we have for any $t \geq 0$, $f \in D(B)$ and $\lambda > \omega$

$$Y_\lambda(t)f = Bf + \int_0^t Y_\lambda(s)A_\lambda f\,ds. \tag{2.14}$$

Using(2.12), (2.13) and claim 2, we easily obtain $\lim_{\lambda\to\infty} Y_\lambda(s)A_\lambda f = Y(s)Af$ for any $s \geq 0$ and $f \in D_B(A)$ and the convergence is uniform in every finite interval of s. This and (2.14) imply that also property 3 of Definition 1.1 is satisfied. Finally, property 4 has been verified in claim 6 ∎

Remark 2.3. Suppose $X = Z, B = I$ and $D(B) = X$. Then condition (i) is satisfied, while conditions (ii)$'$, 1 and 2 of Theorem 2.1 can be rewritten in the following way: $D(A)$ is dense in X; there exist $\omega \in R$ and $M \geq 1$ such that $\lambda \in \rho(A)$ and $\|R(\lambda; A)^n\| \leq \frac{M}{(\lambda-\omega)^n}$ $(n \in \mathbb{N})$ for every real number $\lambda > \omega$. In other words, we obtain the assumptions of the Hille-Yosida theorem. In such a case put, as in [8],

$$S_\lambda(t) = \exp(-\lambda t) \sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n}{n!} R(\lambda; A)^n.$$

By comparing this definition and formula (2.4) we immediately recognize that $Y_{\lambda,\mu}(t,s) = S_\lambda(t) S_\mu(s)$ and that our claims reduce to the corresponding claims of [8]. Because of this the family $(Y(t))_{t\geq 0}$ can be seen as a generalized C_0-semigroup.

3. The generation theorem

In this section we first prove a theorem which can be seen as a new version of the Banasiak "main generation theorem", then we compare with each other. To obtain our version of the generation theorem we have to consider the mapping defined by formula (1.3) and study its properties. Such a study is carried out in the following theorem.

Theorem 3.1 (New generation theorem). *Suppose that the operators A and B satisfy conditions* (i) *and* (ii)$'$. *Then the pair (A, B) generates a B-quasi bounded semigroup $(Y(t))_{t\geq 0}$ with the property $Y(0)f = Bf$ for any $f \in D(B)$ if and only if formula* (1.3) *defines a linear single-valued mapping $\mathcal{A} : B(D_B(A)) \to Z_B$ which is closeable in Z_B and whose closure $\overline{\mathcal{A}} \in G(M, \omega, Z_B)$. The B-quasi bounded semigroup $(Y(t))_{t\geq 0}$ is given for $f \in D(B)$ by*

$$Y(t)f = \exp(t\overline{\mathcal{A}})\, Bf. \tag{3.1}$$

Proof. *Necessity.* Suppose that the operators A and B satisfy conditions (i) and (ii)$'$ and that they generate a B-quasi bounded semigroup $(Y(t))_{t\geq 0}$ with $Y(0) = B$. We have to prove that $\mathcal{A}$ is single-valued, linear and closeable in Z_B and that its closure $\overline{\mathcal{A}} \in G(M, \omega, Z_B)$.

(α): *The mapping defined by (1.3) is single-valued and linear.* Indeed, consider $f \in D_B(A)$ such that $Bf = 0$. From properties 1 and 3 of Definition 1.1 we obtain for any $t \geq 0$

$$Y(t)f = 0 \qquad \text{and} \qquad \int_0^t Y(s)Af\, ds = Y(t)f - Bf = 0.$$

But then $Y(t)Af = 0$ for any $t \geq 0$. In particular, $BAf = Y(0)Af = 0$. Because the linearity is easily verified, the assertion is completely proved.

(β): *$\mathcal{A}$ is closeable in Z_B.* Indeed, consider a sequence $(g_n)_{n\in\mathbb{N}}$ in $D(\mathcal{A}) = B(D_B(A))$ such that $g_n \to 0$ and $\mathcal{A}g_n \to h$ in Z_B. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $D_B(A)$ such that $Bf_n = g_n \to 0$ and $BAf_n = \mathcal{A}g_n \to h$. From property 1 of Definition 1.1 we have for any $t > 0$ that $Y(t)f_n \to 0$ and

$$\|Y(t)Af_n - Y(t)Af_m\| \leq M \exp(\omega t) \|BAf_n - BAf_m\| \to 0$$

as $n, m \to \infty$. Therefore for each $t \geq 0$ there exists $\lim_{n\to\infty} Y(t)Af_n = F(t)$ and the convergence is uniform in every finite interval of t. Thus the function $t \to F(t)$ is strongly continuous. But from property 3 of Definition 1.1 we have for all $t > 0$ and $n \in \mathbb{N}$

$$Y(t)f_n = Bf_n + \int_0^t Y(s)Af_n ds.$$

If we consider the limit as $n \to \infty$ we obtain $\int_0^t F(s)\,ds = 0$ for any $t > 0$ and therefore $F(t) = 0$ for all $t \geq 0$. In particular, $F(0) = 0$, that is $h = \lim_{n\to\infty} BAf_n = \lim_{n\to\infty} \mathcal{A}g_n = 0$. Thus the operator $\mathcal{A}$ is closeable.

Consider the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ in Z_B. By remembering the definition of $\mathcal{A}$ we have (see [7: p. 166])

$$D(\overline{\mathcal{A}}) = \Big\{ g \in Z_B : \exists\ (f_n)_{n\in\mathbb{N}} \text{ in } D_B(A), h \in Z_B \text{ such that } Bf_n \to g, BAf_n \to h \Big\}$$

and $\overline{\mathcal{A}}g = \lim_{n\to\infty} BAf_n$ for $g = \lim_{n\to\infty} Bf_n \in D(\overline{\mathcal{A}})$. Now we show that $\overline{\mathcal{A}}$ generates a C_0-semigroup in Z_B.

(γ): *$D(\overline{\mathcal{A}})$ is dense in Z_B.* Indeed, $B(D_B(A)) = D(\mathcal{A}) \subset D(\overline{\mathcal{A}}) \subset Z_B$. But we know by Theorem 2.1 that, if the pair (A, B) generates the B-quasi bounded semigroup $(Y(t))_{t\geq 0}$ and if $Y(0) = B$, then $B(D_B(A))$ is dense in Z_B. Thus $D(\overline{\mathcal{A}})$ is dense, too.

(δ): *For any $\lambda > \omega$ the mapping $(\lambda I - \overline{\mathcal{A}}) : D(\overline{\mathcal{A}}) \to Z_B$ is injective.* Indeed, let $g \in D(\overline{\mathcal{A}})$ such that $(\lambda I - \overline{\mathcal{A}})g = 0$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $D_B(A)$ such that $Bf_n \to g$ and $BAf_n \to \overline{\mathcal{A}}g$ and therefore $\lambda Bf_n - BAf_n \to \lambda g - \overline{\mathcal{A}}g = 0$. But by virtue of condition 2 of Theorem 2.1

$$\|Bf_n\| = \|B(\lambda I - A)^{-1}(\lambda I - A)f_n\| \leq \frac{M}{\lambda - \omega} \|B(\lambda I - A)f_n\| \to 0$$

that is $g = 0$. The assertion is thus proved.

(ε): *For any $\lambda > \omega$ the mapping $(\lambda I - \overline{\mathcal{A}}) : D(\overline{\mathcal{A}}) \to Z_B$ is surjective.* Indeed, consider $H \in Z_B$. Then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $D(B)$ such that $Bg_n \to H$. But for any n

$$f_n = (\lambda I - A)^{-1} g_n \in D_B(A). \tag{3.2}$$

Since

$$\|Bf_n - Bf_m\| = \|B(\lambda I - A)^{-1}(g_n - g_m)\| \leq \frac{M}{\lambda - \omega} \|Bg_n - Bg_m\| \to 0$$

as $n, m \to \infty$ we see that there exist

$$\left.\begin{aligned} \lim\nolimits_{n\to\infty} Bf_n &= h \in Z_B \\ \lim\nolimits_{n\to\infty} BAf_n &= \lim_{n\to\infty} (\lambda Bf_n - Bg_n) = \lambda h - H. \end{aligned}\right\} \tag{3.3}$$

Therefore $h \in D(\overline{\mathcal{A}})$ with

$$(\lambda I - \overline{\mathcal{A}})h = \lim_{n\to\infty} (\lambda Bf_n - BAf_n) = H \tag{3.4}$$

and the thesis is proved.

(ζ): *For any* $\lambda > \omega$,

$$\lambda \in \rho(\overline{\mathcal{A}}) \qquad \text{and} \qquad \|R(\lambda;\overline{\mathcal{A}})^n\| \leq \frac{M}{(\lambda-\omega)^n} \quad (n \in \mathbb{N}). \tag{3.5}$$

Indeed, the previous points (δ) and (ε) allow to state that, in Z_B, for any $\lambda > \omega$ the domain of the closed operator $(\lambda I - \overline{\mathcal{A}})^{-1}$ is Z_B itself. The closed graph theorem implies that the operator $(\lambda I - \overline{\mathcal{A}})^{-1}$ is bounded and $\lambda \in \rho(\overline{\mathcal{A}})$. Now, in order to prove inequality (3.5) it is sufficient to show that for all $\lambda > \omega$, $n \in \mathbb{N}$ and $g \in D(B)$

$$\|(\lambda I - \overline{\mathcal{A}})^{-n} Bg\| \leq \frac{M}{(\lambda-\omega)^n}\|Bg\| \tag{3.6}$$

since $R(B)$ is dense in Z_B. But it is immediate to recognize that for $\lambda > \omega$, $n \in \mathbb{N}$ and $g \in D(B)$ the formula

$$(\lambda I - \overline{\mathcal{A}})^{-n} Bg = B(\lambda I - A)^{-n} g \tag{3.7}$$

holds. Indeed, formulas (3.2) - (3.4) show that (3.7) holds for $n = 1$. Because of this it is easy to see that, if (3.7) holds for $n = m$, then it holds for $n = m+1$, too. Thus statement (3.7) is proved by induction. From (3.7) and condition 2 of Theorem 2.1 we obtain (3.6). This proves inequality (3.5).

(η): *The operator* $\overline{\mathcal{A}} \in G(M,\omega,Z_B)$ *and formula (3.1) holds.* Indeed, the previous steps (γ) - (ζ) and the Hille-Yosida theorem assure that $\overline{\mathcal{A}} \in G(M,\omega,Z_B)$. Therefore for all $g \in D(\overline{\mathcal{A}})$ and $t \geq 0$ we have

$$\exp(t\overline{\mathcal{A}})g = g + \int_0^t \exp(s\overline{\mathcal{A}})\overline{\mathcal{A}}g\, ds. \tag{3.8}$$

But $B(D_B(A)) \subset D(\overline{\mathcal{A}})$ and

$$\overline{\mathcal{A}}Bf = BAf \qquad \text{for any} \quad f \in D_B(A). \tag{3.9}$$

Thanks to property (3.9) we can rewrite (3.8) for $g = Bf$ with $f \in D_B(A)$. In such way we obtain

$$\exp(t\overline{\mathcal{A}})Bf = Bf + \int_0^t \exp(s\overline{\mathcal{A}})BAf\, ds. \tag{3.10}$$

This together with property 3 of Definition 1.1 shows that formula (3.1) holds.

Sufficiency. Now suppose that the mapping defined by (1.3) is single-valued, that the linear operator $\mathcal{A} : B(D_B(A)) \to Z_B$ is closeable in Z_B and finally that its closure $\overline{\mathcal{A}} \in G(M,\omega,Z_B)$. Then the pair (A,B) of operators generates the B-quasi bounded semigroup given by (3.1). Indeed, consider the one-parameter family $(Y(t))_{t\geq 0}$ given by (3.1). For $t \geq 0$ and $f \in D(B)$ we have

$$\|Y(t)f\| = \|\exp(t\overline{\mathcal{A}})Bf\| \leq M\exp(\omega t)\|Bf\|,$$

that is property 1 of Definition 1.1. Because the function $t \to \exp(t\overline{\mathcal{A}})Bf \in C([0,\infty), Z)$ also property 2 is satisfied. By repeating the considerations contained in step (η) we can see that for any $f \in D_B(A)$ formula (3.10) holds. Thus property 3 is proved, too. Property 4 is immediate, since $\exp(0\overline{\mathcal{A}}) = I$ ∎

Now it is possible to compare our Theorem 3.1 with the Banasiak generation theorem. To this aim we have to introduce the Banach space X_B and to provide some results proved in [2].

Definition 3.1. Let us consider the set X of sequences $(f_n)_{n\in\mathbb{N}}$ such that $f_n \in D(B)$ for $n \in \mathbb{N}$ and $(Bf_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. We define X_B to be the space of all equivalence classes of sequences $(f_n)_{n\in\mathbb{N}} \in$ X with respect to the relation

$$(f_n)_{n\in\mathbb{N}} \equiv (g_n)_{n\in\mathbb{N}} \quad \Longleftrightarrow \quad \lim_{n\to\infty} \|Bf_n - Bg_n\| = 0.$$

The space X_B is a normed space with norm $\|[(f_n)_{n\in\mathbb{N}}]\|_{X_B} = \lim_{n\to\infty} \|Bf_n\|$.

Definition 3.2. Denote by P the operator from $D(B)$ into X_B defined by

$$\mathrm{P}f = [(f, f, ...)]. \tag{3.11}$$

Definition 3.3. Denote by $\mathcal{B}$ the linear operator from X_B into Z_B defined for $\Phi = [(f_n)_{n\in N}] \in X_B$ by $\mathcal{B}\Phi = \lim_{n\to\infty} Bf_n$.

It is known in [2: Lemma 2.1 and Proposition 2.1] that $\mathcal{B}$ is an isometric isomorphism of X_B into Z_B. Considering this we can rewrite the Banasiak generation theorem in the following form.

Theorem 3.2. *Suppose that the operators A and B satisfy conditions* (i) *and* (ii)$'$. *Then the pair (A, B) generates a B-quasi bounded semigroup $(Y(t))_{t\geq 0}$ with the property $Y(0)f = Bf$ for all $f \in D(B)$ if and only if the following conditions hold:*

1. *Formula* (1.3) *defines a single-valued mapping $\mathcal{A}: B(D_B(A)) \to Z_B$.*

2. *The operator $\mathcal{T}$ defined by*

$$\mathcal{T} = \mathcal{B}^{-1}\mathcal{A}\mathcal{B} \tag{3.12}$$

is closeable in X_B, and its closure $\overline{\mathcal{T}} \in G(M, \omega, X_B)$. The B-quasi bounded semigroup $(Y(t))_{t\geq 0}$ is given for $f \in D(B)$ by

$$Y(t)f = \mathcal{B}\exp(t\overline{\mathcal{T}})\mathrm{P}f. \tag{3.13}$$

Proof. Suppose that the pair of operators (A, B) generates a B-quasi bounded semigroup $(Y(t))_{t\geq 0}$ with the property $Y(0)f = Bf$ for any $f \in D(B)$. Then, according to Theorem 3.1, condition 1 holds. Moreover, the operator $\mathcal{A}$ is closeable in Z_B and its closure $\overline{\mathcal{A}} \in G(M, \omega, Z_B)$. But definition (3.12) and the properties of the linear operator $\mathcal{B}$ assure that also $\mathcal{T}$ is closeable and its closure $\overline{\mathcal{T}} \in G(M, \omega, X_B)$, with

$$\overline{\mathcal{T}} = \mathcal{B}^{-1}\overline{\mathcal{A}}\mathcal{B} \qquad \text{and} \qquad \exp(t\overline{\mathcal{T}}) = \mathcal{B}^{-1}\exp(t\overline{\mathcal{A}})\mathcal{B}. \tag{3.14}$$

Finally formula (3.13) is an immediate consequence of (3.1), (3.11) and (3.14). Indeed, for $f \in D(B)$ we have

$$Y(t)f = \exp(t\overline{\mathcal{A}})Bf = (\mathcal{B}\exp(t\overline{\mathcal{T}})\mathcal{B}^{-1})Bf = \mathcal{B}\exp(t\overline{\mathcal{T}})\mathrm{P}f,$$

that is (3.13).

Inversely, if conditions 1 and 2 hold, then the operator $\mathcal{A}$ is closeable in Z_B and its closure $\overline{\mathcal{A}} \in G(M, \omega, Z_B)$. We can apply Theorem 3.1 and state that the pair (A, B) generates a B-quasi bounded semigroup $(Y(t))_{t\geq 0}$ with the property $Y(0)f = Bf$ for all $f \in D(B)$ ∎

References

[1] Arendt, W.: *Resolvent positive operators.* Proc. Lond. Math. Soc. 54 (1987), 321 – 349.

[2] Banasiak, J.: *Generation results for B-bounded semigroups.* Ann. Mat. Pura Appl. (Serie IV) 175 (1998), 307 – 326.

[3] Banasiak, J.: *B-bounded semigroups and implicit evolution equations.* Report. Natal: Univ., School Math. & Stat. Sci., Internal Report 1/99.

[4] Beals, R.: *Advanced Mathematical Analysis* (Graduate Texts in Mathematics). New York: Springer-Verlag 1973.

[5] Belleni Morante, A.: *B-bounded semigroups and applications.* Ann. Mat. Pura Appl. (Serie IV) 170 (1996), 359 – 376.

[6] Belleni Morante, A.: *On some properties of B-quasi bounded semigroups and applications.* Firenze: Universitá, Dipartimento di Ingegneria Civile, Quaderno sezione modelli matematici $n^0 1/96$.

[7] Kato, T.: *Perturbation Theory for Linear Operators* (Die Grundlehren der Mathematischen Wissenschaften: Vol. 132). Berlin - Heidelberg - New York: Springer-Verlag 1980.

[8] Ladas, G. A. and V. Lakshmikantham.: *Differential Equations in Abstract Spaces* (Mathematics in Science and Engineering: Vol. 85). London: Academic Press 1972.

[9] Sauer, N.: *Linear evolution equations in two Banach spaces.* Proc. Royal Soc. Edin. 91A (1982), 287 – 303.

[10] Sauer, N.: *Implicit evolution equations and empathy theory.* In: Recent Developments in Evolution Equations (eds.: A. C. McBride and G. F. Roach). Harlow (Essex): Longman Sci. Techn. 1995, pp. 32 – 40.

[11] Sauer, N.: *Empathy theory and the Laplace transform.* Banach Center Publ. 38 (1997), 325 – 338.

[12] Tanabe, T.: *Equations of Evolution* (Monographs and Studies in Mathematics: Vol. 6). London: Pitman 1979.

[13] Totaro, S. and A. Belleni Morante: *The successive reflection method in three dimensional particle transport.* J. Math. Phys. 37 (1996), 2815 – 2823.

Received 16.11.1998