Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 19 (2000), No. 1, 203–225

Relaxation for Dirichlet Problems Involving a Dirichlet Form

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Abstract. For a fixed Dirichlet form, we study the space of positive Borel measures (possibly infinite) which do not charge polar sets. We prove the density in this space of the set of the measures which represent varying domains. Our method is constructive. For the Laplace operator, the proof was based on a pavage of the space. Here, we substitute this notion by that of *homogeneous covering* in the sense of Coiffman and Weiss.

Keywords: *Dirichlet spaces, Asymptotic behavior, Variational methods*

AMS subject classification: 31 C 25, 35 B 40, 35 A 15

1. Introduction

Let X be a connected, locally compact, separable Hausdorff space, X_0 a relatively compact open connected subset of X and Ω an open subset of X_0. Let m be a positive Radon measure on X. In this paper, we consider a strongly local regular Dirichlet-Poincaré form a on X whose domain relative to Ω is denoted by $D_o(a,\Omega) \subset L^2(X,m)$ (see [2],[3]). Let $\mathcal{M}_o^a$ be the space of the non-negative Borel measures on Ω, which do not charge the polar sets with respect to a (i.e. sets whose a-capacity is 0). The space $\mathcal{M}_0^a$ is compact with respect to the γ^a-convergence: see [17] for strongly eliptic operators, [29] for symmetric Dirichlet forms and [28] for non-symmetric Dirichlet forms.

Consider a sequence (C_h) of closed subsets of Ω and denote $\Omega_h = \Omega \backslash C_h$; we shall define

$$\mu_h = \infty_{C_h}$$

in a suitable way (see Lemma 3.1) such that $\mu_h \in \mathcal{M}_o^a$. The γ^a-convergence of the sequence (μ_h) means that there exists a limit measure μ in $\mathcal{M}_o^a$ such that the solution u_h of the variational problem:

Find $u_h \in D_o(a,\Omega_h)$ such that for any $v \in D_o(a,\Omega_h)$,

$$a(u_h, v) = \int_{\Omega\backslash C_h} f v \, dm. \tag{1.1}$$

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This research has been supported by Progetto di ricerca MURST 9801262841

ISSN 0232-2064 / \$ 2.50

or, equivalently, find $u_h \in D_o(a,\Omega) \cap L^2(\Omega,\mu_h)$ such that for any $v \in D_o(a,\Omega) \cap L^2(\Omega,\mu_h)$,

$$a(u_h, v) + \int_\Omega u_h v \, d\mu_h = \int_\Omega f v \, dm, \tag{1.2}$$

converges strongly in $L^2(\Omega, m)$ to the solution u of the problem: $u \in D_o(a,\Omega) \cap L^2(\Omega,\mu)$ and, for any $v \in D_o(a,\Omega) \cap L^2(\Omega,\mu)$,

$$a(u, v) + \int_\Omega uv \, d\mu = \int_\Omega f v \, dm. \tag{1.3}$$

Dal Maso and Mosco in [17] proved that in the strongly elliptic case for $X = \mathbb{R}^n$ $(n \geq 2)$ the space $\mathcal{M}_o^a$ is the closure of the Dirichlet problems of type (1.1) with respect to the γ^a-convergence, i.e. for all $\mu \in \mathcal{M}_o^a$, there exists a sequence of open sets Ω_h (and $C_h = \Omega \backslash \Omega_h$) such that the solutions u_h of (1.1) converge strongly in $L^2(\Omega, m)$ to the solution u of problem (1.3). For this reason, (1.3) has been called a relaxed Dirichlet problem.

Other density results have been proved:

- In the case where a corresponds to a strongly elliptic symmetric operator, Dal Maso and Mosco [17] have proved the density in $\mathcal{M}_o^a$ of the measures with very regular density: $\mu_h = q_h \mathcal{L}$ where $\mathcal{L}$ is the Lebesgue measure, and $q_h \in C_0^\infty(\Omega)$.
- Dal Maso and Garroni [14] have proved the density in $\mathcal{M}_0^a$ of the Radon measures, when a corresponds to a strongly elliptic non-symmetric operator.
- The same density result holds when a is a possibly non-symmetric strongly local Dirichlet-Poincaré form (see [28]).

The aim of this paper is on the one hand to prove that, for a Dirichlet-Poincaré form, the problems of type (1.3) are the relaxed Dirichlet problems (at least under suitable assumption on the form a), i.e. the space $\mathcal{M}_o^a$ is the closure with respect to the γ^a-convergence of the set of measures of type ∞_C , and on the other hand, for a given a measure $\mu \in \mathcal{M}_o^a$, to provide an explicit construction of the subsets Ω_h for which the solutions of (1.2) with $\mu_h = \infty_{C_h}$ converge to the solution of (1.3). The last question has been solved by Dal Maso and Malusa [15] in the case of strongly elliptic symmetric operators.

Before defining more precisely the framework for our study, we wish to describe the method used in [15]. We take $X = \mathbb{R}^n$ $(n \geq 2)$ (this hypothesis on the dimension yields that the points in $\mathbb{R}^n$ have zero a-capacity). Let μ be a Radon measure. For any $h \in \mathbb{N}$, we fix a pavage of $\mathbb{R}^n$ made of the cubes Q_h^i of sides $\frac{1}{h}$ and centers x_i. We denote by I_h the sets of integers such that $Q_h^i \subset \Omega$. For $i \in I_h$ let B_h^i be the Euclidean ball with center x^i and radius $\frac{1}{2h}$. Let E_h^i be a concentric ball whose radius is chosen so that

$$cap^a(E_h^i, B_h^i) = \mu(Q_h^i).$$

Observe that such an E_h^i always exists by virtue of the continuity and the monotonicity of the a-capacity, and because

$$cap^a(\{x_i\}, B_h^i) = 0 \qquad \text{and} \qquad cap^a(B_h^i, B_h^i) = +\infty. \tag{1.4}$$

Let C_h be the set $\cup_{i\in I_h} E_h^i$. Dal Maso and Malusa [15], generalizing the method of Cioranescu and Murat [9], and using a special Poincaré inequality, proved that the sequence $\Omega_h = \Omega\backslash C_h$ has the desired property.

This method can be used for some degenerate operators, for instance for the Heisenberg operator

$$\Delta_H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (4y^2+4x^2)\frac{\partial^2}{\partial z^2} + (4y)\frac{\partial^2}{\partial z\partial x} - (4x)\frac{\partial^2}{\partial z\partial y} \tag{1.5}$$

naturally associated with the homogeneous group of the translations of Heisenberg. The ingredients for generalizing the previous technique are

1. a special pavage (using the above mentioned translations) which has been studied in [6].
2. the notion of balls in a metric associated with the operator, which is different from the Euclidean metric because the operator (1.5) is degenerate.

However, it is not clearly possible to construct a suitable pavage for general degenerate elliptic operators, even if they are associated with homogeneous groups of translations.

In this paper, we shall investigate the possibility of replacing the disjoint covering (pavage) by a covering with a small overlapping. The method that we shall use consists of taking an homogeneous covering by balls $B(x_i, r)$ $(i = 1, \ldots, q)$, in the metric associated with the operator (see [10]). We shall see that this covering has the property $d(x_i, x_j) \geq \frac{r}{2}$ for $i \neq j$. This implies that the balls $B(x_i, \frac{r}{2})$ are disjoint and that any point x belongs at most to M balls $B(x_i, r)$ (where M is an intrinsic constant). Moreover, a property similar to (1.4) (connected with the behavior of the Green functions for the Dirichlet form a (see [3]) allows us to define the sets E_h^i. More precisely, if $\mu \in \mathcal{M}_o^a$ is a Radon measure and if we suppose that $\mathcal{N} = \{x_0 : a - \text{cap}(\{x_0\}) > 0\} \cap \Omega = \emptyset$, we can construct a finite covering of Ω: $B(x_i, r)$ $(i \in I)$ such that $B(x_i, \frac{r}{2})$ are pairwise disjoint. Let $I^* \subset I$ such that for all $i \in I^*$, $B(x_i, \frac{r}{2}) \subset \Omega$. Defining

$$A(x_1, r) = B(x_1, r), \ldots, A(x_i, r) = B(x_i, r)\backslash \cup_{j<i} B(x_j, r), \ldots,$$

let $E_r^i = E(x_i, r) \subseteq A(x_i, r)$ be a ball centered in x_i, contained in $B(x_i, \frac{r}{4})$ with radius such that

$$\text{cap}^a\left(E(x_i, r), B\left(x_i, \frac{r}{4}\right)\right) = \mu(A(x_i, r)),$$

then if $E_r = \cup_{i\in I^*} E(x_i, r)$, the sequence ∞_{E_r} γ-converges to μ.

This result is then generalized for measures in $\mathcal{M}_o^a$. Note that the assumptions $\mathcal{N} = \emptyset$ can be weakened (see [7]), but not suppressed, because in the case of the dimension 1 (the points have non-zero capacity) this density result is false.

The γ^a-convergence of the measures is proved thanks to a generalized Poincaré inequality for positive Kato measures (see Theorem 4.1 which has an interest for itself.

Before giving the precise hypothesis on the Dirichlet Poincaré form we want to point out that our results apply for instance to the following cases:

- forms connected with strongly elliptic operators for $X = \mathbb{R}^n$ and $n \geq 2$,

- forms connected with degenerate elliptic operators with a weight w in the A_2 Muckenhoupt class such that $\mathcal{N} = \emptyset$; let us remember that in the model case $X = \mathbb{R}^n$ $(n \geq 2)$ and $w(x) = |x|^\alpha$ the requirement $w \in A_2$ means that $-n < \alpha < n$, but if we want that $\operatorname{cap}^a(\{0\}, \mathbb{R}^n) = 0$, we have to add the hypothesis $-n+2 \leq \alpha$ (see [20]).
- forms connected with vector fields satisfying a Hörmander condition $X = \mathbb{R}^n$ and $\nu \geq 2$ where ν is the *intrinsic* dimension (in X_0) (see [3, 21, 23, 27]).

2. Hypothesis on the space, the measure and the Dirichlet form

Let X be a *connected, locally compact, separable Hausdorff space.* We fix a positive Radon measure m on X, with $\operatorname{supp} m = X$, which is called the "volume" measure on X. For the general theory of Dirichlet forms we refer to [22]. Let us consider a *strongly local regular symmetric Dirichlet form* $a(u, v)$ on the Hilbert space $L^2(X, m)$ with domain $D(a)$; we recall that $D(a)$ is a Hilbert space with the *intrinsic norm* $\sqrt{a(u,v) + \int_X uv\, dm}$. It is possible to associate with $a(u, v)$ a Radon-measure-valued non-negative definite symmetric bilinear form $\alpha(u, v)(dx)$, called the *energy measure* of a, such that

$$a(u,v) = \int_X \alpha(u,v)(dx) \tag{2.1}$$

for $u, v \in D(a)$. We refer to [3] for the definition and for the main properties of the energy measure, and to [26] for the proofs in a more general context. Since the form $a(u, v)$ is regular, there exists a *core* $C \subset C_c(X) \cap D(a)$ which is dense in $C_c(X)$ with the uniform norm, and in $D(a)$ with the intrinsic norm. We assume that C is an *m-separating core*, that is, for every $x, y \in X$, with $x \neq y$, there exists $\phi \in C$ such that $\phi(x) \neq \phi(y)$ and $\alpha(\phi, \phi) \leq m$, where the last inequality is taken in the sense of Radon measures on X. Let $X_0 \neq X$ be a connected relatively compact open subset of X and Ω an open subset of X_0, the closure of $C_c(\Omega) \cap D(a)$ in $D(a)$ for the intrinsic norm is denoted by $D_o(a, \Omega)$. We define $D(a, \Omega)$ as the set of all restrictions $u|_\Omega$ to Ω of the functions $u \in D(a)$. By the strong local property of the form $a(u, v)$, the restriction to Ω of the energy measure $\alpha(u, v)$ depends only on the restrictions of u and v to Ω. For the properties of the space $D(a, \Omega)$ we refer to [3,8]. By using the energy measure of the Dirichlet form $a(u, v)$ we can introduce a metric on X, called the *intrinsic metric*, defined by

$$d(x,y) = \sup\Big\{\varphi(x) - \varphi(y) : \varphi \in C,\ \alpha(\varphi,\varphi) \leq m \text{ in } X\Big\}. \tag{2.2}$$

By $B(x, r)$ we denote the intrinsic ball centered at x with radius r, i.e.

$$B(x,r) = \big\{y \in X : d(x,y) < r\big\}. \tag{2.3}$$

We assume that the topology induced by this metric coincides with the given topology of X.

We also assume that the measure m on X satisfies the following doubling condition with respect to the intrinsic metric: there exist two constants $R_o > 0$ and $C_o > 1$ such that

$$0 < m(B(x,2r)) \leq C_o\, m(B(x,r)) < +\infty \tag{2.4}$$

for every $x \in X_0$ and for every $0 < r \leq R_o$. Moreover, we suppose that

$$B(x, 2r) \neq B(x, r) \tag{2.5}$$

for every $x \in X_0$ and for every $0 < r \leq \frac{R_o}{2}$. Let us suppose that the metric space (X, d) is complete.

Let us recall that (X, d) together with this doubling measure m is a space of homogeneous type or, for brevity, a *homogeneous space* in the sense of Coifmman and Weiss [10].

We remark that if X is the union of a sequence of balls of radius R_o, then the separability of X is a consequence of the homogeneity.

Remark 2.1. We recall the following property of homogeneous spaces which is a fundamental tool for our method (see [10: pp. 66 – 71] and [8: Proof of Lemma 2.5]): there exist two constants $R_o > 0$ and M such that for every $0 < r \leq \frac{R_o}{4}$ there exists a covering of Ω: $B(x_i, r)$ $(i = 1, \dots, q)$ such that $d(x_i, x_j) \geq r$. This implies that the balls $B(x_i, \frac{r}{2})$ are disjoint and that any point x belongs at most to M balls $B(x_i, r)$ (M is an intrinsic constant) and at most to $K(k)M$ balls $B(x_i, kr)$ where $k \geq 1$.

Let us state here an easy lemma on the inverse doubling property of measure:

Lemma 2.1. *There exist a constant $C_* > 1$ such that for every ball $B(x, 2r) \subset\subset X_0$, $0 < r \leq \frac{R_o}{2}$, we have*

$$m(B(x, 2r)) \geq C_* m(B(x, r)). \tag{2.6}$$

Proof. Using the connexity of X and the continuity of the distance, it is easy to see that there exists $y_0 \in B(x, 2r)$ such that $d(x, y_0) = \frac{3}{2}r$. Let us consider a ball centered in y_0 with radius $\frac{1}{2}r$, that is $B(y_0, \frac{1}{2}r)$. Easily,

$$B(y_0, \tfrac{1}{2}r) \subset B(x, 2r) \tag{2.7}$$

and

$$B(y_0, \tfrac{1}{2}r) \subset B(x, r)^c. \tag{2.8}$$

On the other hand,

$$B(x, 2r) \subset B(y_0, 4r). \tag{2.9}$$

Indeed, for $z \in B(x, 2r)$,

$$d(z, y_0) \leq d(z, x) + d(x, y_0) \leq 2r + \frac{3}{2}r < 4r, \tag{2.10}$$

so we have

$$m(B(y_0, \tfrac{1}{2}r)) \geq \frac{1}{C_o^3} m(B(y_0, 4r)) \geq \frac{1}{C_o^3} m(B(x, 2r)) \geq \frac{1}{C_o^3} m(B(x, r)) \tag{2.11}$$

and

$$m(B(x, 2r)) - m(B(x, r)) = m\big(B(x, 2r) \backslash B(x, r)\big) \geq m(B(y_0, \tfrac{1}{2}r)) \geq \frac{1}{C_o^3} m(B(x, r)), \tag{2.12}$$

and the statement follows for $C_* = 1 + \frac{1}{C_o^3}$ ∎

We assume that the following Poincaré inequality holds: there exist two constants $C_1 > 0$ and $k \geq 1$ such that for every $0 < r \leq R_o$, $y \in X_0$ and for every $u \in D(a)$ we have

$$\int_{B(y,r)} |u - u_{y,r}|^2 m(dx) \leq C_1 r^2 \int_{B(y,kr)} \alpha(u,u)(dx), \tag{2.13}$$

where $u_{y,r}$ is the average of u on $B(y,r)$ with respect to the measure m. Let us remember that the embedding of $D_o[a,\Omega]$ into $L^2(\Omega,m)$ is compact

$$D_o(a,\Omega) \subset\subset L^2(\Omega,m) \tag{2.14}$$

when $\Omega \subset\subset X_0$ (this fact can be proved as in Lemma 2.5 of [8]).

3. Review on the definitions of measure space

We recall here some definitions and properties on measures not charging polar sets and on Kato measures (see [3, 5, 11, 12, 13, 16, 22]; all these definitions and properties are introduced and studied in previous papers, but for the sake of completeness, we describe them hereafter.

The *capacity* of a set $E \subset X$ associated with the Dirichlet form $a(u,v)$ is denoted by $\mathrm{cap}^a(E)$ (see [22: Section 3.1]). The capacity $\mathrm{cap}^a(E,\Omega)$ of a set E with respect to an open set $\Omega \subset X$ is defined in a similar way, replacing X by Ω and $D(a)$ by $D_o(a,\Omega)$.

A function u defined on an open set $\Omega \subset X$ is said to be *quasi-continuous* if for every $\varepsilon > 0$ there exists an open set G_ε with capacity less than ε ($\mathrm{cap}^a(G_\varepsilon) \leq \varepsilon$) such that the restriction of u to $\Omega \setminus G_\varepsilon$ is continuous.

Every $u \in D(a,\Omega)$ admits a quasi-continuous representative, that is, there exists a quasi-continuous function $\tilde{u}$ on Ω, unique up to modifications on a set of capacity zero (with respect to X), such that $u = \tilde{u}$ m-almost everywhere on Ω (see [22: Chapter 3]). In this paper, when considering pointwise values of functions in $D(a,\Omega)$, we shall tacitly consider their quasi-continuous representatives.

We say that a set $U \subset X$ is *quasi-open* if for every $\varepsilon > 0$ there exists an open set U_ε such that $U \subset U_\varepsilon$ and $\mathrm{cap}^a(U_\varepsilon \setminus U) < \varepsilon$.

Definition 3.1. For every open set $\Omega \subset X$, let $\mathcal{M}_o^a(\Omega)$ be the set of all non-negative Borel measures μ on Ω which are absolutely continuous with respect to cap^a, i.e. $\mu(E) = 0$ for every Borel set $E \subset \Omega$ with $\mathrm{cap}^a(E) = 0$.

We introduce now an equivalence relation on the class $\mathcal{M}_o^a(\Omega)$.

Definition 3.2. We say that two non-negative measures μ and λ belonging to $\mathcal{M}_o^a(\Omega)$ are equivalent (and we write $\mu \simeq \lambda$) if $\int_\Omega u^2 d\mu = \int_\Omega u^2 d\lambda$ for every $u \in D_o(a,\Omega)$.

Definition 3.3. Let Ω be an open subset of X, let $\mu \in \mathcal{M}_o^a(\Omega)$, and let $E \subset\subset \Omega$ be a μ-measurable set. The set E is *μ-admissible* if there exsits a function $u \in D_o(a,\Omega)$ with $(u-1) \in L^2(E,\mu)$. If E is $\mu -$ *admissible*, the *variational μ-capacity* of E in Ω relative to the Dirichlet form $a(u,v)$ is defined by

$$\mathrm{cap}_\mu^a(E,\Omega) = \min\left\{ a(u,u) + \int_E (u-1)^2 d\mu : \, u \in D_o(a,\Omega) \right\}. \tag{3.1}$$

The minimum above is attained taking into account the lower semi-continuity of the functional in the weak topology of $D_o(a,\Omega)$; if $E \subset\subset \Omega$, then the unique minimum point u_E of (3.1) is called the *μ-capacitary potential* of E in Ω. If E is not $\mu-admissible$, we define $\mathrm{cap}^a_\mu(E,\Omega)=+\infty$.

Remark 3.1. Arguing as in [11: Theorem 2.9] we obtain that $\mathrm{cap}^a_\mu(\cdot,\Omega)$ is an increasing countably sub-additive set function, $\mathrm{cap}^a_\mu(\emptyset,\Omega)=0$, $\mathrm{cap}^a_\mu(E,\Omega)\le\mu(E)$, and $\mathrm{cap}^a_\mu(E,\Omega)\le\mathrm{cap}^a(E,\Omega)$. Moreover, $\mathrm{cap}^a_\mu(E_h,\Omega)\nearrow\mathrm{cap}^a_\mu(E,\Omega)$ whenever E is the union of an increasing sequence (E_h) of μ-measurable sets contained in Ω. In particular, for every open set $U\subset\Omega$ we have

$$\mathrm{cap}^a_\mu(U,\Omega)=\sup\Big\{\mathrm{cap}^a_\mu(K,\Omega):\ K \ \text{ compact, } \ K\subset U\Big\}. \tag{3.2}$$

Example 3.1 (see [13: Example 1.6]). Let $E\subset\Omega$ and let ∞_E be the Borel measure on Ω defined by

$$\infty_E(B)=\begin{cases}0 & \text{if } \mathrm{cap}^a(B\cap E)=0\\ +\infty & \text{otherwise.}\end{cases}$$

Then ∞_E belongs to $\mathcal{M}^a_o(\Omega)$. We notice that a Borel function defined on Ω belongs to $L^2(\Omega,\infty_E)$ if and only if $u=0$ on E, except on a subset of capacity zero. Therefore, if E is closed in Ω, we have $D_o(a,\Omega\setminus E)=D_o(a,\Omega)\cap L^2(\Omega,\infty_E)$ (see [22: Theorem 4.4.2/(i)]). It is not difficult to see that $\mathrm{cap}^a_{\infty_E}(B,\Omega)=\mathrm{cap}^a(E\cap B,\Omega)$ for every Borel set $B\subset\Omega$.

The Green function for the form $a(u,v)$ in an open set $\Omega\subset\subset X_0$ is denoted by $G_\Omega(x,y)$. We refer to [2,3] for its definition and main properties. In particular, we shall use the following estimate for balls whose radius is sufficiently small with respect to the constant R_o which appears in the doubling condition (2.4).

Proposition 3.1. *Assume that $20R<R_o$ and that $B(x,40R)\subseteq X_0$ is different from X. If $0<d(x,y)<qR$, $q\in(0,1)$ then*

$$\frac{1}{c}\int_{d(x,y)}^R\frac{s^2}{m(B(x,s))}\frac{ds}{s}\le G_{B(x,R)}(x,y)\le c\int_{d(x,y)}^R\frac{s^2}{m(B(x,s))}\frac{ds}{s}. \tag{3.3}$$

The constant c depends only q and on the constants occurring in the doubling condition and in the Poincaré inequality.

Let us remark that this implies (see [3: Remark 6.1]) that if $0\le r\le qR$, then

$$\mathrm{cap}^a(B(r),B(R))\sim\left(\int_r^R\frac{s^2}{m(B(s))}\frac{ds}{s}\right)^{-1},$$

where the constants depends on q; this implies that if $0\le q\le 1$, then

$$C_1(q)\frac{m(B(r))}{r^2}\le\mathrm{cap}^a(B(qr),B(r))\le C_2(q)\frac{m(B(r))}{r^2}. \tag{3.4}$$

From the definition of capacity we have also

$$\lim_{\rho\to r}\mathrm{cap}^a(B(\rho),B(r))=+\infty.$$

Let us also remark that the hypothesis $\mathcal{N} = \{x_0 : a - \mathrm{cap}(x_0) > 0\} \cap \Omega = \emptyset$ means that

$$\lim_{r\to 0} \int_r^R \frac{s^2}{m(B(s))} \frac{ds}{s} = \infty. \tag{3.5}$$

Following [5], we recall the notion of Kato measure associated with a regular Dirichlet form:

Definition 3.4. Let Ω be a relatively compact open subset of X with $2\,\mathrm{diam}(\Omega) = R < R_o$, where R_o is the constant which appears in the doubling condition (2.4). Assume that there exists $x_o \in \Omega$ with $B(x_o, 4R) \subset\subset X$ and $B(x_o, 4R) \neq X$. We say that μ is a *Kato measure* on Ω if μ is a Radon measure on Ω such that

$$\lim_{r\downarrow 0} \sup_{x\in\Omega} \int_{\Omega\cap B(x,r)} \left(\int_{d(x,y)}^R \frac{s^2}{m(B(x,s))} \frac{ds}{s} \right) |\mu|(dy) = 0, \tag{3.6}$$

where $|\mu|$ denotes the total variation of the measure μ. The space of Kato measures is denoted by $K(\Omega)$, while $K_+(\Omega)$ indicates the cone of non-negative elements of $K(\Omega)$.

In the following we assume that Ω is a relatively compact open subset of X_0 with $\Omega \subset B(R) \subset B(40R) \subset X_0$, where $B(R)$ denotes a ball of radius R andR_o is the constant which appears in the doubling condition (2.4), if it is not the case we decompose Ω into open sets satisfying the previous conditions.

We remark that a Kato measure μ is *diffuse*, i.e. $\mu(\{x\}) = 0$ for every $x \in \Omega$; hence we have $\lim_{r\downarrow 0} |\mu|(B(x,r)) = 0$ for every $x \in \Omega$. From (3.6) we have that for every compact set $K \subset \Omega$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mu|(E) < \varepsilon$ for every Borel set $E \subset K$ with $\mathrm{diam}(E) < \delta$. Moreover, if $\mu \in K(\Omega)$ and g is a bounded Borel function, then it is not difficult to see that $g\mu \in K(\Omega)$. We recall that if μ is a Kato measure and if $u \in D_o(a, \Omega)$ is the solution of $a(u, v) = \langle \mu, v\rangle$ for any $v \in D_o(a, \Omega)$, then u has the representation

$$u(x) = \int_\Omega G_\Omega(x, y)\, d\mu(y) \qquad \text{q.e. in } \Omega \tag{3.7}$$

(see [19: Proposition 37] and [5: Proposition 3.2]). Moreover, u is continuous in Ω (see [5: Proposition 4.1]).

Finally, we recall a decomposition result for diffuse measures in $\mathcal{M}_o^a(\Omega)$ proven in [12] by using an extension to Dirichlet spaces of the Evans-Vasilesco Theorem and a result on reconstruction of a measure using capacities proven in [13].

Theorem 3.1. *Let Ω be an open relatively compact subset of X and let $\mu \in \mathcal{M}_o^a(\Omega)$ be a diffuse measure. Then there exist a Borel function $g\colon \Omega \to [0, +\infty]$ and a measure $\lambda \in K_+(\Omega)$ such that $\mu \simeq g\lambda$.*

Theorem 3.2. *Let Ω an open relatively compact subset of X, let μ be a measure in $\mathcal{M}_o^a(\Omega)$, and let*

$$E_\mu = \big\{x \in \Omega :\ \mathrm{cap}_\mu^a(\{x\}, \Omega) > 0\big\}. \tag{3.8}$$

Then for every Borel set $B \subset \Omega$ we have

$$\mu(B) = \sup \sum_{i \in I} \operatorname{cap}_\mu^a(B_i, \Omega) + \sum_{x \in B \cap E_\mu} \frac{\left(\operatorname{cap}_\mu^a(\{x\}, \Omega)\right)^2}{\operatorname{cap}^a(\{x\}, \Omega) - \operatorname{cap}_\mu^a(\{x\}, \Omega)} \tag{3.9}$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i \in I}$ of B (as usual if E_μ is not countable the sum for $x \in B \cap E_\mu$ denotes the supremum of the sums on all finite subfamilies of $B \cap E_\mu$).

For every $\mu \in \mathcal{M}_o^a(\Omega)$ we consider the functional $F_\mu : L^2(\Omega) \to [0, +\infty]$ defined by

$$F_\mu(u) = a(u, u) + \int_\Omega u^2 d\mu. \tag{3.10}$$

We recall a variational convergence in the class $\mathcal{M}_o^a(\Omega)$ (see also [8, 12, 29]).

Definition 3.5. We say that a sequence (μ_h) of measures in $\mathcal{M}_o^a(\Omega)$ γ^a-*converges* to a measure $\mu \in \mathcal{M}_o^a(\Omega)$ if the sequence of functionals F_{μ_h} Γ-converges to the functional F_μ in $L^2(\Omega, m)$ in the sense of [18].

Let us remember that this convergence is equivalent to the L^2-strong convergence of the minima, more precisely:

A sequence (μ_h) of measures in $\mathcal{M}_o^a(\Omega)$ γ^a-converges to a measure $\mu \in \mathcal{M}_o^a(\Omega)$ if and only if for every $f \in D_o'(a, \Omega)$ ($D_o'(a, \Omega)$ is the dual space of $D_o(a, \Omega)$) the sequence of weak solutions $u_h \in D_o(a, \Omega) \cap L^2(\Omega, \mu_h)$ of the problems

$$a(u_h, v) + \int_\Omega u_h v \, d\mu_h = \langle f, v \rangle$$

for any $v \in D_o(a, \Omega) \cap L^2(\Omega, \mu_h)$, converges weakly in $D_o(a, \Omega)$ and strongly in $L^2(\Omega, m)$ to the weak solution $u \in D_o(a, \Omega) \cap L^2(\Omega, \mu)$ of the limit problem

$$a(u, v) + \int_\Omega uv \, d\mu = \langle f, v \rangle$$

for any $v \in D_o(a, \Omega) \cap L^2(\Omega, \mu)$.

The following result (see [28, 29]) shows that the space $\mathcal{M}_o^a(\Omega)$ is sequentially compact under the γ^a-convergence.

Proposition 3.2. *Every sequence of measures of $\mathcal{M}_o^a(\Omega)$ has a subsequence which γ^a-converges to a measure of $\mathcal{M}_o^a(\Omega)$.*

4. Poincaré inequality

In this section, when dealing with balls, we shall omit the dependence on the center. So we shall denote by $B(t)$ any ball with radius t centered in a given point $x_o \in X_0$. We shall also omit the dependence on the form, which will be fixed. For instance, $\mathrm{cap}(B(qr), B(r)) = cap^a(B(qr), B(r))$.

Our main result of this section is a Poincaré inequality on an intrinsic ball $B(r)$. It consists of estimating the $L^2(\mu)$-norm of a function u with zero outer capacitary mean (where μ is a Kato measure, possibly different from m) by the energy of the form times a term tending to 0 as the radius tends to 0. This method can be easily extended to other *bounded* projections. See also [1, 15, 31].

First we define the outer capacitary measure:

Definition 4.1. Let U and V two open sets such that $V \subset\subset U \subset\subset \Omega$, and w the capacitary potential of V with respect to U (see also (3.1) with $\mu = \infty_V$ and $\Omega = U$), i.e. the minimum point :

$$\begin{aligned}\min\left\{\int_U \alpha(v,v)(dx) : v \in D_o(a,U), v \geq 1 \text{ q.e. on } V\right\}& \\ = \mathrm{cap}^a(V,U) = \int_U \alpha(w,w)(dx).&\end{aligned} \tag{4.1}$$

Let us extend w by 0 on $\Omega \backslash U$. Then there exist two positive measures $\nu, \lambda \in D_o'(a)$ ($D_o'(a)$ is the dual space of $D_o(a) = D_o(a, X)$) such that

$$a(w,v) = \langle \nu - \lambda, v\rangle \qquad \text{for any } v \in D_o(a,\Omega), \tag{4.2}$$

$\mathrm{supp}(\nu) \subset \partial V$, $\mathrm{supp}(\lambda) \subset \partial U$ and $\nu(\Omega) = \lambda(\Omega) = \mathrm{cap}^a(V,U)$. We call ν and λ the *inner* and *outer a-capacitary distribution*, respectively, of V with respect to U.

The proof of the existence of such measures is similar to that of [15: Lemma 3.1], where the essential tools are Stampacchia's techniques (see [30: Theorems 2.1 and 3.8] and [25: Theorem 2.6.4]) which can be easily extended in our framework.

Theorem 4.1. *Let $\mu \in K_+(\Omega)$ be a Kato measure and let P be defined by*

$$P(u) = \frac{1}{\lambda_r^\rho(B(2r))}\int_{B(2r)} u\, d\lambda_r^\rho = \frac{1}{\lambda_r^\rho(B(r))}\int_{B(r)} u\, d\lambda_r^\rho \tag{4.3}$$

where λ_r^ρ is the outer a-capacitary distribution of $B(\rho)$ with respect to $B(r)$ for $\rho < r$. There exists $r_o > 0$ such that for any $r \leq r_o$, if $B(r) \subset 40B(r) \subset\subset \Omega$ and $u \in D(a, B(2kr))$, then

$$\int_{B(r)} (u - P(u))^2 d\mu \leq C\eta(r)\int_{B(2kr)} \alpha(u,u)(dx) \tag{4.4}$$

where C is a constant independent of r and η is a function independent of the center of the ball $B(r)$ with $\eta(r) \to 0$ as $r \to 0$.

We begin with an abstract result estimating the L^2 norm with respect to a Kato measure in $B(s)$ by an energy norm in $B(t)$ with $t > s$.

Proposition 4.1. *Let $u \in D(a, B(t))$ and μ be a Kato measure, $\mu \in K_+(B)$, where $B(t) \subset B \subset 40B \subset \Omega$. There exists $t_0 > 0$ such that, for any $s < t \le t_0$,*

$$\int_{B(s)} u^2 d\mu \le C\eta(t) \left(\int_{B(t)} \alpha(u,u)(dx) + \frac{1}{(t-s)^2} \int_{B(t)\setminus B(s)} u^2 dm \right) \tag{4.5}$$

where C is a constant independent of t and $\eta(t) \to 0$ when $t \to 0$.

Proof. We assume, without loss of generality, that $u \in C(B(t))$. Let W be the solution of the problem to find $W \in D_o(a, B(t))$ such that

$$\int_{B(t)} \alpha(W, v)(dx) = \int_{B(t)} v\, d\mu \qquad \text{for any } v \in D_o(a, B(t)). \tag{4.6}$$

Let us denote by $G^x_{B(t)}$ the Green function with singularity in x with respect to $B(t)$. Then we have

$$W(x) = \int_{B(t)} G^x_{B(t)}(y)\, d\mu \le \int_{B(t)} G^x_B(y)\, d\mu \le \eta(t) = \sup_{x\in B} \int_{B(t)} G^x_B(y)\, d\mu. \tag{4.7}$$

Using the definition of Kato measure and (3.3) we obtain that $\eta(t) \to 0$ when $t \to 0$ and $W \in D_o(a, B(t)) \cap C(B(t))$ (see [5]). Let ϕ be the cut-off function between $B(s)$ and $B(\frac{t+s}{2})$. Then $u^2\phi^2 \in C_0(B(t))$ and we obtain using (4.6)

$$\begin{aligned}
\int_{B(t)} u^2\phi^2 d\mu &= \int_{B(t)} \alpha(W, u^2\phi^2)(dx) \\
&= 2\int_{B(t)} u\phi^2 \alpha(W,u)(dx) + 2\int_{B(t)} u^2\phi\alpha(W,\phi)(dx) \\
&\le \frac{\varepsilon}{2}\int_{B(t)} \phi^2\alpha(u,u)(dx) + \frac{2}{\varepsilon}\int_{B(t)} u^2\phi^2\alpha(W,W)(dx) \\
&\quad + \frac{\varepsilon}{2}\int_{B(t)} u^2\alpha(\phi,\phi)(dx) + \frac{2}{\varepsilon}\int_{B(t)} u^2\phi^2\alpha(W,W)(dx) \\
&= \frac{\varepsilon}{2}\int_{B(t)} (\phi^2\alpha(u,u)(dx) + u^2\alpha(\phi,\phi)(dx)) + \frac{4}{\varepsilon}\int_{B(t)} u^2\phi^2\alpha(W,W)(dx).
\end{aligned}$$

Let us estimate the last term:

$$\begin{aligned}
&\int_{B(t)} u^2\phi^2\alpha(W,W)(dx) \\
&\quad = \int_{B(t)} \alpha(W, Wu^2\phi^2)(dx) - 2\int_{B(t)} u\phi W\alpha(W, u\phi)(dx) \\
&\quad = \int_{B(t)} Wu^2\phi^2 d\mu - 2\int_{B(t)} u\phi W\alpha(W,u\phi)(dx) \\
&\quad \le \eta(t)\int_{B(t)} u^2\phi^2 d\mu + \int_{B(t)} u^2 2\phi^2\alpha(W,W)(dx) + 2\int_{B(t)} W^2\alpha(u\phi,u\phi)(dx) \\
&\quad \le \eta(t)\int_{B(t)} u^2\phi^2 d\mu + \tfrac{1}{2}\int_{B(t)} u^2\phi^2\alpha(W,W)(dx) \\
&\qquad + 4\eta^2(t)\int_{B(t)} \left(u^2\alpha(\phi,\phi)(dx) + \phi^2\alpha(u,u)(dx)\right)
\end{aligned}$$

and we obtain

$$\begin{aligned}\int_{B(t)} & u^2\phi^2\alpha(W,W)(dx)\\ &\le 2\eta(t)\int_{B(t)} u^2\phi^2 d\mu + 8\eta^2(t)\int_{B(t)} \big(u^2\alpha(\phi,\phi)(dx)+\phi^2\alpha(u,u)(dx)\big).\end{aligned}$$

But

$$\begin{aligned}\int_{B(t)} u^2\phi^2 d\mu &\le \frac{\varepsilon}{2}\int_{B(t)} \big(\phi^2\alpha(u,u)(dx)+u^2\alpha(\phi,\phi)(dx)\big)+\frac{8}{\varepsilon}\eta(t)\int_{B(t)} u^2\phi^2 d\mu\\ &\quad+\frac{32}{\varepsilon}\eta^2(t)\int_{B(t)} \big(u^2\alpha(\phi,\phi)(dx)+\phi^2\alpha(u,u)(dx)\big)\\ &=\frac{8}{\varepsilon}\eta(t)\int_{B(t)} u^2\phi^2 d\mu\\ &\quad+\Big(\frac{\varepsilon}{2}+\frac{32\eta^2(t)}{\varepsilon}\Big)\int_{B(t)} \big(u^2\alpha(\phi,\phi)(dx)+\phi^2\alpha(u,u)(dx)\big).\end{aligned}\tag{4.8}$$

Let us choose $\varepsilon = 16\,\eta(t)$ in the preceding inequality. Then

$$\begin{aligned}&\int_{B(t)} u^2\phi^2 d\mu \le\\ &\tfrac{1}{2}\int_{B(t)} u^2\phi^2 d\mu + 10\,\eta(t)\int_{B(t)} \big(u^2\alpha(\phi,\phi)(dx)+\phi^2\alpha(u,u)(dx)\big),\end{aligned}\tag{4.9}$$

and then

$$\begin{aligned}\int_{B(s)} u^2 d\mu &\le \int_{B(t)} u^2\phi^2 d\mu\\ &\le 20\,\eta(t)\int_{B(t)} \big(u^2\alpha(\phi,\phi)(dx)+\phi^2\alpha(u,u)(dx)\big)\\ &\le 20\,\eta(t)\left(\int_{B(t)}\alpha(u,u)(dx)+\frac{C}{(t-s)^2}\int_{B(t)\setminus B(s)} u^2 dm\right)\end{aligned}\tag{4.10}$$

for any $u \in C(B(t))$, and we conclude by density ∎

Remark 4.1. Easily, from Proposition 4.1 we have that, if $u \in D(a, B(2r))$ and $c \in \mathbb{R}$,

$$\int_{B(r)} (u-c)^2 d\mu \le C\eta(2r)\left(\int_{B(2r)}\alpha(u,u)(dx)+\frac{1}{r^2}\int_{B(2r)}(u-c)^2 dm\right). \tag{4.11}$$

Let H be the Hilbert space $H = D(a, B(2r))$ endowed with the scalar product

$$\langle u, v\rangle = C\left(\int_{B(2r)}\alpha(u,v)(dx)+\frac{1}{r^2}\int_{B(2r)} uv\,dm\right) \tag{4.12}$$

where C is a suitable constant . Let $H_1 = \mathbb{R}$. The Hilbert projection on H_1 is given by $\Pi(u) = \frac{1}{m(B(2r))}\int_{B(2r)} u\,dm$. On the other hand, we denote by $\mathcal{P}$ a bounded linear operator $\mathcal{P} : H \to H_1$ such that $\mathcal{P}(u_1) = u_1$ for any $u_1 \in H_1$. We can use (4.11) with $c = \mathcal{P}(u)$. Then

$$\begin{aligned}\int_{B(r)} &(u - \mathcal{P}(u))^2 d\mu \\ &\leq C\eta(2r)\left(\int_{B(2r)} \alpha(u,u)(dx) + \frac{1}{r^2}\int_{B(2r)} (u - \mathcal{P}(u))^2 dm\right).\end{aligned} \tag{4.13}$$

In view of obtaining our Poincaré inequality (4.4), we have to prove that

$$\int_{B(2r)} (u - \mathcal{P}(u))^2 dm \leq Cr^2 \int_{B(2kr)} \alpha(u,u)(dx). \tag{4.14}$$

Using the Poincaré inequality (2.13) with $\Pi(u) = \frac{1}{m(B(2r))}\int_{B(2r)} u\,dm$, it is enough to prove that

$$(\Pi(u) - \mathcal{P}(u))^2 \leq \frac{Cr^2}{m(B(2r))}\int_{B(2kr)} \alpha(u,u)(dx), \tag{4.15}$$

but

$$(\Pi(u) - \mathcal{P}(u))^2 = (\mathcal{P}(u - \Pi(u)))^2 \leq \|\mathcal{P}\|^2_{H'}\|u - \Pi(u)\|^2_H. \tag{4.16}$$

Thanks to

$$\|u - \Pi(u)\|^2_H \leq C\int_{B(2kr)} \alpha(u,u)(dx) \tag{4.17}$$

we have Theorem 4.1 below for any $\mathcal{P}$ such that $\|\mathcal{P}\|^2_{H'} \leq \frac{Cr^2}{m(B(2r))}$.

It remains to prove that for P defined as in (4.3)

$$P(u) = \frac{1}{\lambda_r^\rho(B(2r))}\int_{B(2r)} u\,d\lambda_r^\rho = \frac{1}{\lambda_r^\rho(B(r))}\int_{B(r)} u\,d\lambda_r^\rho \tag{4.18}$$

we have

$$\|P\|^2_{H'} \leq \frac{Cr^2}{m(B(2r))}$$

where λ_r^ρ is the outer a-capacitary distribution of $B(\rho)$ with respect to $B(r)$ for $\rho < r$.

Lemma 4.1. *Let $r \leq \frac{R_o}{4}$, $40B(r) \subset\subset \Omega$ and P as defined in (4.3). If λ_r^ρ is the outer capacitary measure of $B(\rho)$ in $B(r)$ concentrated on the boundary $\partial B(r)$ and if there exist $q \in (0,1)$ and a constant C such that for any $\rho \in (0, qr)$ and for any $0 \leq \phi \in H$*

$$\frac{1}{\lambda_r^\rho(\partial B(r))}\int_{\partial B(r)} \phi\,d\lambda_r^\rho \leq C\frac{1}{\lambda_r^{qr}(\partial B(r))}\int_{\partial B(r)} \phi\,d\lambda_r^{qr}, \tag{4.19}$$

then

$$\|P\|^2_{H'} \leq \frac{Cr^2}{m(B(2r))}. \tag{4.20}$$

Proof. Using (4.19), for any $\phi \in H$

$$\left| \frac{1}{\lambda_r^\rho(\partial B(r))} \int_{\partial B(r)} \phi \, d\lambda_r^\rho \right| \leq C \frac{1}{\lambda_r^{qr}(\partial B(r))} \int_{\partial B(r)} |\phi| \, d\lambda_r^{qr} \tag{4.21}$$

and then

$$\|P\|_{H'} = \frac{1}{\lambda_r^\rho(\partial B(r))} \|\lambda_r^\rho\|_{H'} \leq C \frac{1}{\lambda_r^{qr}(\partial B(r))} \|\lambda_r^{qr}\|_{H'}. \tag{4.22}$$

We want to prove that

$$\|\lambda_r^{qr}\|_{H'}^2 \leq \frac{Cr^2}{m(B(2r))} (\lambda_r^{qr}(\partial B(r)))^2.$$

Let us consider the function $\zeta = 1 - \theta$ where θ is the cut-off function between $B(qr)$ and $B(r)$. Then $\zeta = 1$ on $\partial B(r)$, $\zeta = 0$ in $\overline{B(qr)}$ and $\alpha(\zeta, \zeta)(dx) \leq \frac{c}{r^2} dm$. Using the equality $\lambda_r^{qr}(\partial B(r)) = \operatorname{cap}(B(qr), B(r))$ and the definition of the capacitary potential w_{qr}^r of the ball $B(qr)$ in $B(r)$ (extended to zero in the complement of $B(r)$), we obtain for any $\phi \in H \cap C(B(2r))$

$$\begin{aligned} |\lambda_r^{qr}(\phi)| &= \left| \int_{B(2r)} \phi \, d\lambda_{qr}^r \right| \\ &= \left| \int_{B(2r)} \phi\zeta \, d\lambda_{qr}^r \right| \\ &= \left| \int_{B(2r)} \alpha(w_{qr}^r, \phi\zeta)(dx) \right| \\ &\leq C \sqrt{\operatorname{cap}(B(qr), B(r))} \left(\int_{B(2r)} \alpha(\phi, \phi) dx + \frac{1}{r^2} \int_{B(2r)} \phi^2 dm \right)^{\frac{1}{2}}. \end{aligned} \tag{4.23}$$

So we have

$$\|\lambda_r^{qr}\|_{H'}^2 \leq C \operatorname{cap}(B(qr), B(r)). \tag{4.24}$$

It follows that

$$\operatorname{cap}(B(qr), B(r)) \leq \frac{Cr^2}{m(B(2r))} (\lambda_r^{qr}(\partial B(r)))^2 \tag{4.25}$$

if and only if

$$\operatorname{cap}(B(qr), B(r)) \geq C \frac{m(B(2r))}{r^2}. \tag{4.26}$$

Using (3.4) and the previous inequalities we obtain the desired result ∎

We end the proof of Theorem 4.1 by observing that inequality (4.19) can be proved using the same methods as in [15].

Remark 4.2. Using Lemma 4.1, it is also true that (4.4) holds for $\mathcal{P}(u)$ satisfying an inequality of the type (4.20), for instance for

$$\mathcal{P}(u) = \frac{1}{\lambda_{r4}^\rho B(\frac{r}{4})} \int_{B(\frac{r}{4})} u \, d\lambda_{\frac{r}{4}}^\rho. \tag{4.27}$$

5. Main results

Now we are able to prove the density result announced in the introduction. We shall prove it first for Kato measures in a Dirichlet space without polar points. This result gives a natural meaning to the notion of relaxed Dirichlet problems involving a form, i.e. any weak solution u of the "relaxed Dirichlet problem" in Ω with homogeneous Dirichlet boundary data $a(u,v)+\int_\Omega uv\,d\mu=\int_\Omega fv\,dm$ (with v in a suitable space) is the $L^2(\Omega)$-strong limit of solutions u_h of $a(u_h,v)=\int_{\Omega_h} fv\,dm$ with homogeneous Dirichlet boundary data in varying domains Ω_h.

Different density results have been stated for instance by Mataloni and Tchou [28] proving that the relaxed Dirichlet problems are also the closure of Schrödinger type problems. These results have been already proved in the Laplacian case by Dal Maso and Mosco [16, 17] and in a more constructive way for general strongly elliptic operators by Dal Maso and Malusa [15]. In both articles, the Euclidean pavage associated with the Laplacian plays a fundamental role. The density result in the case of elliptic degenerated Heisenberg operator can be easily proven with the same methods as [16, 17] by using results stated in [6], and in particular the special pavage associated with the operator.

In a more general case, it is not possible to construct a pavage of the space associated to a general degenerated elliptic operator and enjoying the properties needed by the Dal Maso and Mosco theory. The method used here is issued from harmonic analysis on homogeneous spaces (see [10]) and consists of replacing the notion of "pavage" (disjoint covering) associated to the operator, by the notion of "homogeneous" covering associated to the operator. This notion is really weaker than the first one, and more suitable for general degenerated operators. Moreover, also in the classical strongly elliptic case, this method provides new approximating Dirichlet problems associated to homogeneous coverings different from the pavage.

Let us recall that, using the measures defined in Example 3.1 $\mu_h=\infty_{E_h}$ where $\Omega_h=\Omega\backslash E_h$, it is possible to consider the sequence u_h of solutions of the homogeneous Dirichlet problem $a(u_h,v)=\int_{\Omega_h} fv\,dm$ as a special case of a sequence of solutions of the relaxed Dirichlet problem

$$a(u_h,v)+\int_\Omega u_h v\,d\mu_h=\int_\Omega fv\,dm.$$

Let us recall also that for H^{-1} positive measures, Cioranescu, Murat and Kacimi (see [9, 24]) proved that the existence of the Dirichlet approximating problem is a consequence of the existence of two positive measures satisfying special conditions. Indeed, we want also to use a similar approach and we start with the following

Proposition 5.1. *Let $\mu\in D_o^{-1}(a,\Omega)$. If there exist two positive measures $\nu_h,\lambda_h\in D_o^{-1}(a,\Omega)$ and a sequence of functions $w_h\in D(a,\Omega)$ such that*

$$\begin{aligned} w_h &\rightharpoonup 1 && \textit{weakly in } D(a,\Omega)\\ w_h &= 0 && \textit{q.e. in } E_h\\ a(w_h,v) &= \langle \nu_h-\lambda_h, v\rangle && \textit{for any } v\in D_o(a,\Omega)\end{aligned}$$

$$\begin{aligned}
\nu_h \to \mu \qquad & \textit{strongly in } D_o^{-1}(a,\Omega)\\
\lambda_h \rightharpoonup \mu \qquad & \textit{weakly in } D_o^{-1}(a,\Omega)\\
\langle \lambda_h, v\rangle = 0 \qquad & \textit{for any } v \in D_o(a,\Omega) \textit{ with } v = 0 \textit{ in } E_h,
\end{aligned}$$

then the measures ∞_{E_h} γ*-converge to the measure* μ.

Proof. First we remark that $\mu \in D_o^{-1}(a,\Omega)$ is a positive measure (cf. [24: Remarque 2.4]). To prove our result, let us consider for $f \in D_o^{-1}(a,\Omega)$ the sequence u_h of the solutions of the problems

$$\left.\begin{aligned}
a(u_h, v) = \langle f, v\rangle \qquad & \text{for any } v \in D_o(\Omega) \text{ with } v = 0 \text{ in } E_h\\
u_h = 0 \qquad & \text{in } E_h.
\end{aligned}\right\} \tag{5.1}$$

We want to prove its strong $L^2(\Omega)$-convergence to the solution u of the problem

$$a(u,v) + \int_\Omega uv\,d\mu = \langle f, v\rangle \qquad \text{for any } v \in D_o(a,\Omega) \cap L^2(\Omega,\mu). \tag{5.2}$$

Let us take as test function in (5.1) $w_h\phi$ where $\phi \in C$ (the core of the form); the choice is possible since ϕ is continuous and $\alpha(phi, phi)$ Has a bounded density, so $w_h\phi \in D_o(a,\Omega)$. Then

$$\int_\Omega w_h \alpha(u_h,\phi)(dx) + \int_\Omega \alpha(w_h, u_h\phi)(dx) - \int_\Omega u_h \alpha(w_h,\phi)(dx) = \langle f, w_h\phi\rangle \tag{5.3}$$

and, using the definition of w_h,

$$\int_\Omega \alpha(w_h, u_h\phi)(dx) = \langle \nu_h - \lambda_h, u_h\phi\rangle = \langle \nu_h, u_h\phi\rangle. \tag{5.4}$$

By using the techniques used in [8], we can prove that passing to the limit in (5.3) yields

$$\int_\Omega \alpha(u,\phi)(dx) + \langle \mu, u\phi\rangle = \langle f, \phi\rangle \tag{5.5}$$

for any $\phi \in C$ and then

$$\int_\Omega \alpha(u,\phi)(dx) + \int_\Omega u\phi\,d\mu = \langle f, \phi\rangle \tag{5.6}$$

for any $v \in D_o(a,\Omega) \cap L^2(\Omega,\mu)$ ∎

Proposition 5.2. *Let us suppose that* $\mathcal{N} = \{x_0 : a - \mathrm{cap}(\{x_0\}) > 0\} \cap \Omega = \emptyset$, $\mu \in K_+(\Omega)$ *and supp* $\mu \subset\subset \Omega$.

(i) *There exists a positive constant depending only on* R_o *and* $C(R_o)$ *such that if* $0 < r \le min(C(R_o), \frac{1}{40} d(supp\ \mu, \Omega^c))$, *we can construct a finite covering* $B(x_i, r)$ $(i \in I)$ *of* Ω *such that the balls* $B(x_i, \frac{r}{2})$ *are pairwise disjoint.*

(ii) *Let $I^* \subset I$ such that $B(x_i, \frac{r}{2}) \subset \Omega$ for all $i \in I^*$. Let us define*

$$A(x_1, r) = B(x_1, r)$$
$$A(x_i, r) = B(x_i, r) \setminus \cup_{j<i} B(x_j, r) \quad (i \geq 1).$$

Let $E(x_i, r)$ be a ball centered in x_i, contained in $B(x_i, \frac{r}{4})$ and with radius such that

$$\operatorname{cap}^a\big(E(x_i, r), B(x_i, \tfrac{r}{4})\big) = \mu(A(x_i, r)). \tag{5.7}$$

Then if $E_r = \cup_{i \in I^*} E(x_i, r)$ and $r_h \to 0$ the sequence $\infty_{E_{r_h}}$ γ-converges to μ.

Proof. We shall use the previous result to study the asymptotic behavior of the sequence ∞_{E_r}. To this aim, we construct three sequences $w_r \in D(a, \Omega)$, ν_r and λ_r as in the previous remark.

Using Definition 4.1, let us define $\lambda_{i,r}$ as outer capacitary distribution of $E(x_i, r)$ with respect to $B(x_i, \frac{r}{4})$ and $\nu_{i,r}$ as inner capacitary distribution of $E(x_i, r)$ with respect to $B(x_i, \frac{r}{4})$. We introduce

$$\begin{aligned} \lambda_r &= \sum_{i \in I^*} \lambda_{i,r} \\ \nu_r &= \sum_{i \in I^*} \nu_{i,r} \\ w_r &= \begin{cases} 1 & \text{in } \Omega \setminus \cup_{i \in I^*} B(x_i, \frac{r}{4}) \\ \{1 - \psi_{i,r}\} & \text{in } B(x_i, \frac{r}{4}) \setminus E(x_i, r) \quad (i \in I^*) \end{cases} \end{aligned} \tag{5.8}$$

where $\psi_{i,r}$ is the capacitary potential of $E(x_i, r)$ with respect to $B(x_i, \frac{r}{4})$. Thanks to the weak maximum principle we have $0 \leq w_r \leq 1$ and then $\|w_r\|_{L^2(\Omega)} \leq C$. Moreover,

$$\begin{aligned} \int_\Omega \alpha(w_r, w_r)(dx) = &= \sum_{i \in I^*} \int_{B(x_i, \frac{r}{4}) \setminus E(x_i, r)} \alpha(w_{i,r}, w_{i,r})(dx) \\ &= \sum_{i \in I^*} \operatorname{cap}^a\big(E(x_i, r), B(x_i, \tfrac{r}{4}\big) = \\ &= \sum_{i \in I^*} \mu(A(x_i, r)) \\ &\leq \mu(\Omega) \end{aligned}$$

and we obtain

$$\|w_r\|_{D(a,\Omega)} \leq C. \tag{5.9}$$

Using the compact embedding of $D_o(a, \Omega)$ into $L^2(\Omega)$ (see (2.14)) and remarking that $w_r - 1 \in D_o^a(\Omega)$ we have

$$\left.\begin{aligned} w_r &\rightharpoonup w && \text{in } D(a, \Omega) \\ w_r &\to w && \text{in } L^2(\Omega). \end{aligned}\right\} \tag{5.10}$$

We wish to prove that $w = 1$. For this let us consider the characteristic function

$$\chi_r = \chi_{(\cup_{i \in I^*} B(x_i, \frac{r}{2}) \setminus B(x_i, \frac{r}{4}))}. \tag{5.11}$$

We have $\chi_r w_r = \chi_r$. Let us call χ the $w*-L^\infty$-limit of χ_r. Passing to the limit as $r \to 0$,

$$\chi w = \chi. \tag{5.12}$$

We want to prove that q.e. $\chi \neq 0$. Let Ω' be an open subset of Ω. If $I^{*'} \subset I$ is defined by $i \in I^{*'}$ if and only if $B(x_i, \frac{r}{2}) \subset \Omega'$, and if C_* is the constant of the inverse doubling condition (see (2.6)), we have

$$\begin{aligned}\int_{\Omega'} \chi_r &\geq \sum_{i\in I^{*'}} m(B(x_i,\tfrac{r}{2})) - (B(x_i,\tfrac{r}{4}))\\ &\geq \sum_{i\in I^{*'}} (C_*-1)m(B(x_i,\tfrac{r}{4}))\\ &\geq \frac{C_*-1}{C_*^2}\sum_{i\in I^{*'}} m(B(x_i,r))\\ &\geq \frac{C_*-1}{C_*^2}(m(\Omega')-O(r))\end{aligned} \tag{5.13}$$

where $O(r) \to 0$ as $r \to 0$. Letting $r \to 0$ and using a density argument we obtain $\chi \geq \frac{(C_*-1)^2}{C_*}$ and this implies that $w = 1$.

Now we shall study the weak convergence of the sequence $\lambda_r = \sum_{i\in I^*}\lambda_{i,r}$. Let $\phi \in C$, and let us define

$$\phi_r = \sum_{i\in I^*} M_{i,r}(\phi)\chi_{A(x_i,r)}, \quad M_{i,r}(\phi) = \frac{1}{\lambda_{i,r}(B(x_i,\frac{r}{4}))}\int_{B(x_i,\frac{r}{4})}\phi\, d\lambda_{i,r}. \tag{5.14}$$

From now on, let us choose r such that $\frac{3r}{2} < \frac{1}{40} dist(supp\,\mu, \Omega^c)$; then it is easy to see that $\sum_{\Omega\setminus\cup_{i\in I^*}B(x_i,r)}\phi\, d\mu = 0$. We recall that $\lambda_{i,r}$ is the outer capacitary distribution of $E(x_i,r)$ with respect to $B(x_i,\frac{r}{4})$. Then

$$\lambda_{i,r}(B(x_i,\tfrac{r}{4})) = \mathrm{cap}^a\big(E(x_i,r), B(x_i,\tfrac{r}{4})\big) = \mu(A(x_i,r))$$

and

$$\begin{aligned}&|\langle\lambda_r,\phi\rangle - \langle\mu,\phi\rangle|\\ &\quad\leq \left|\int_{\cup_{i\in I^*}B(x_i,r)}\phi\, d\lambda_r - \int_{\cup_{i\in I^*}B(x_i,r)}\phi\, d\mu\right|\\ &\quad= \left|\int_{\cup_{i\in I^*}B(x_i,\frac{r}{4})}\phi\, d\lambda_r - \int_{\cup_{i\in I^*}A(x_i,r)}\phi\, d\mu\right|\\ &\quad= \left|\sum_{i\in I^*}\int_{B(x_i,\frac{r}{4})}\phi\, d\lambda_{i,r} - \sum_{i\in I^*}\int_{A(x_i,r)}\phi\, d\mu\right|\\ &\quad= \left|\sum_{i\in I^*}\frac{\mu(A(x_i,r))}{\lambda_{i,r}(B(x_i,\frac{r}{4}))}\int_{B(x_i,\frac{r}{4})}\phi\, d\lambda_{i,r} - \sum_{i\in I^*}\int_{A(x_i,r)}\phi\, d\mu\right|\end{aligned} \tag{5.15}$$

$$= \left| \sum_{i\in I^*} \left(\frac{\mu(A(x_i,r))}{\lambda_{i,r}(B(x_i,\frac{r}{4}))} \int_{B(x_i,\frac{r}{4})} \phi \, d\lambda_{i,r} - \int_{A(x_i,r)} \phi \, d\mu \right) \right|$$

$$= \left| \sum_{i\in I^*} \int_{A(x_i,r)} (\phi - \phi_r) \, d\mu \right| + \left| \int_{\Omega\setminus\cup_{i\in I^*} B(x_i,r)} \phi \, d\mu \right| .$$

Then

$$\begin{aligned} &\big|\langle \lambda_r, \phi\rangle - \langle \mu, \phi\rangle\big| \\ &\quad \le \sum_{i\in I^*} \int_{A(x_i,r)} |\phi - \phi_r| \, d\mu \\ &\quad \le \mu(\Omega)^{\frac{1}{2}} \left(\sum_{i\in I^*} \int_{A(x_i,r)} (\phi - \phi_r)^2 d\mu \right)^{\frac{1}{2}} \\ &\quad = \mu(\Omega)^{\frac{1}{2}} \left(\sum_{i\in I^*} \int_{A(x_i,r)} (\phi - M_{i,r}(\phi))^2 d\mu \right)^{\frac{1}{2}} \end{aligned} \tag{5.16}$$

and using the Poincaré inequality stated in Theorem 4.1 and Remark 4.2, we obtain that

$$\begin{aligned} \int_{A(x_i,r)} (\phi - M_{i,r}(\phi))^2 d\mu &\le \int_{B(x_i,r)} (\phi - M_{i,r}(\phi))^2 d\mu \\ &= \int_{B(x_i,r)} (\phi - P(\phi))^2 d\mu \\ &\le \eta(r) \int_{B(x_i,2kr)} \alpha(\phi,\phi)(dx). \end{aligned} \tag{5.17}$$

Using (5.15) - (5.17) we deduce that, denoting by N (depending on k and C_o) the maximal number of balls $B(x_i, 2kr)$ such that $x \in B(x_i, 2kr)$,

$$\begin{aligned} \big|\langle \lambda_r, \phi\rangle - \langle \mu, \phi\rangle\big| &\le \eta(r) \left(\sum_{i\in I^*} \int_{B(x_i,2kr)} \alpha(\phi,\phi)(dx) \right) \\ &\le N\eta(r) \left(\int_{\Omega} \alpha(\phi,\phi)(dx) \right) \end{aligned} \tag{5.18}$$

We conclude by density.

Now the other conditions stated in Proposition 5.1 are easily verified and we have proved our statement. ∎

Proposition 5.3. *Let μ be a measure in $K_+(\Omega)$ and Ω_h an increasing sequence of open sets with closure contained in Ω ans such that $\cup_h \Omega_h = \Omega$. Denote by mu_h the measure defined by $\mu_h(E) = \mu(E \cap \Omega_h)$ for every μ-measurable set E; then μ_h γ converges to μ as $h \to +\infty$.*

Proof. Consider the problems

$$a(u_h, v) + \int_{\Omega} u_h v \, d\mu_h = \langle f, v\rangle \tag{5.19$_h$}$$

$$\text{for any } v \in D_o(a,\Omega), u_h \in D_o(a,\Omega).$$

and

$$a(u,v) + \int_\Omega uv\,d\mu = \langle f, v\rangle \tag{5.19}$$
$$\text{for any } v \in D_o(a,\Omega), u \in D_o(a,\Omega).$$

where $f \in L^\infty(\Omega,m)$. We may assume without loss of generality that u_h converges weakly to u_o in $D_o(a,\Omega)$, then u_h converges strongly to u_o in $L^2_{loc}(\Omega,m)$, [8]. Moreover the functions u_h are locally equicontinuous in Ω,[5], so we may assume that u_h converges to u_o uniformly locally in Ω. We observe that

$$lim_{h\to+\infty}\int_\Omega u_h v\,d\mu_h = lim_{h\to+\infty}\int_\Omega u_h v\chi_h\,d\mu = \int_\Omega u_o v\,d\mu \tag{5.20}$$

for every $v \in D_o(a,\Omega)\cap C_c(\Omega)$, then for every $v \in D_o(a,\Omega)$, since μ is a Kato measure in Ω, [5]. From (5.20) we obtain easily that u_o is the solution of (5.19). Since $L^\infty(\Omega,m)$ is dense in $D'_o(a,\Omega)$, the above result holds again for $f \in D'_o(a,\Omega)$, so μ_h γ-converges to μ. ∎

We remark that the γ-convergence is metrizable, [16, 17], then by a diagonal argument and by the results in Proposition 5.2 and 5.3 we prove:

Theorem 5.1. *Let us suppose that* $\mathcal{N} = \{x_0 : a - \text{cap}(\{x_0\}) > 0\}\cap\Omega = \emptyset$, $\mu \in K_+(\Omega)$ *and denote* $\Omega_r = \{x \in \Omega; dist(x,\Omega^c) > 3r\}$.

(i) *There exists a positive constant depending only on* R_o *and* $C(R_o)$ *such that if* $0 < r \le min(C(R_o), \frac{1}{40}d(supp\ \mu, \Omega^c))$, *we can construct a finite covering* $B(x_i,r)$ $(i \in I)$ *of* Ω *such that the balls* $B(x_i,\frac{r}{2})$ *are pairwise disjoint.*

(ii) *Let* $I^* \subset I$ *such that* $B(x_i,\frac{r}{2}) \subset \Omega$ *for all* $i \in I^*$. *Let us define*

$$A(x_1,r) = B(x_1,r)$$
$$A(x_i,r) = B(x_i,r)\setminus \cup_{j<i} B(x_j,r) \quad (i \ge 1).$$

Let $E(x_i,r)$ *be a ball centered in* x_i, *contained in* $B(x_i,\frac{r}{4})$ *and with radius such that*

$$\text{cap}^a\big(E(x_i,r), B(x_i,\tfrac{r}{4})\big) = \mu(A(x_i,r)\cap\Omega_r).$$

Then if $E_r = \cup_{i\in I^*}E(x_i,r)$ *and* $r_h \to 0$ *the sequence* $\infty_{E_{r_h}}$ γ*-converges to* μ.

We shall extend our previous result in the case of Radon measures in $\mathcal{M}_o^a$.

Theorem 5.2. *Let us suppose that*

$$\mathcal{N} = \big\{x_0 : a - \text{cap}(x_0) > 0\big\}\cap\Omega = \emptyset \tag{5.21}$$

and $\mu \in \mathcal{M}_o^a$ *is a Radon measure. Then if* E_r *is defined as in Theorem* 5.1, *the sequence* ∞_{E_r} γ*-converges to* μ.

Proof. Using a general result due to Mosco (see [29]) we know that $\mathcal{M}_o^a$ is compact. So it is possible to extract a subsequence of ∞_{E_r} (that we shall call again ∞_{E_r}) which converges to $\pi \in \mathcal{M}_o^a$. We want to prove that $\pi = \mu$. Let us show first that $\pi \leq \mu$. For this, let us prove that $\mathrm{cap}_\pi^a(A, \Omega) \leq \mu(A)$ for any open $A \subset\subset \Omega$. Let A' be an open set, $A' \subset\subset A$ and r be a real number small enough so that $\cup_{E(x_i,r)\cap A' \neq \emptyset} B(x_i, r) \subset A$. We have

$$\begin{aligned}
\mathrm{cap}_\pi^a(A', \Omega) &= \liminf_{r\to 0} \mathrm{cap}_{\infty_{E_r}}^a(A', \Omega) \\
&= \liminf_{r\to 0} \mathrm{cap}^a(A' \cap E_r, \Omega) \\
&\leq \liminf_{r\to 0} \sum_{E(x_i,r)\cap A' \neq \emptyset} \mathrm{cap}^a(A' \cap E(x_i, r), \Omega) \\
&\leq \liminf_{r\to 0} \sum_{E(x_i,r)\cap A' \neq \emptyset} \mathrm{cap}^a(E(x_i, r), B(x_i, \frac{r}{4})) \\
&\leq \liminf_{r\to 0} \sum_{E(x_i,r)\cap A' \neq \emptyset} \mu(A(x_i, r) \cup \Omega_r) \\
&\leq \mu(A).
\end{aligned}$$

In this inequality we have used that $A(x_i, r) \cap A(x_j, r) = \emptyset$ if $i \neq j$. If $A' \to A$ increasingly, we have

$$\mathrm{cap}_\pi^a(A, \Omega) \leq \mu(A). \tag{5.22}$$

If μ is a Radon measure, we can easily extend (5.20) to any Borel set $B \subset \Omega$:

$$\mathrm{cap}_\pi^a(B, \Omega) \leq \mu(B).$$

Using Theorem 3.2, we have

$$\pi(B) = \sup \sum_{i\in I} \mathrm{cap}_\pi^a(B_i, \Omega) \leq \sup \sum_{i\in I} \mu(B_i) \leq \mu(B) \tag{5.23}$$

where the supremum is taken over all finite partitions B_i of B.

Now we want to prove that

$$\pi(B) \geq \mu(B).$$

Thanks to Theorem 3.1 there exist a Borel function $g : \Omega \to [0, +\infty]$ and a measure $\lambda \in K_+(\Omega)$ such that $\mu \simeq g\lambda$. Let us define $g_K(x) = \min\{g(x), K\}$ and $\mu_K = g_K\lambda$ for $K \in \mathbb{N}$ (then $\mu_K \in K_+(\Omega)$), and for any K let us construct the set $E_{r,K}$ as in Theorem 5.2, such that $\infty_{E_{r,K}}$ γ-converges to μ_K. But we have $E_{r,K} \subset E_r$ for any r and K and this implies $\infty_{E_{r,K}} \leq \infty_{E_r}$. Using the monotonicity properties of the γ-convergence we have $\mu_K \leq \pi$ and, also letting $K \to \infty$,

$$\mu \leq \pi. \tag{5.24}$$

We conclude using (5.23) and (5.24). ∎

Let us now give a more abstract result on the density of measures ∞_E in the general case of $\mu \in \mathcal{M}_o^a$.

Theorem 5.3. *Let us suppose that* $\mathcal{N} = \{x_0 : a - \mathrm{cap}(x_0) > 0\} \cap \Omega = \emptyset$ *and* $\mu \in \mathcal{M}_o^a$. *Then there exists a sequence* ∞_{A_r} γ*-converging to* μ.

Proof. We give only a sketch of the proof. We remark that the space $\mathcal{M}_o^a$ with the γ-convergence is metrizable (see [16, 17] for a similar proof in a particular case). Let $\mu \in \mathcal{M}_o^a$. We use the [28: Proposition 4.13] (where no result of type of Theorem 3.1 is used) to find a sequence of positive Radon measures $\mu_h \in \mathcal{M}_o^a$ which γ-converges to μ. Then we use Theorem 5.2 to approximate μ_h and we conclude by a diagonal argument ∎

Remark 5.1. Let us remark that also in the general case when $\mu \in \mathcal{M}_o^a$, the construction of the sequence ∞_{A_r} is explicit. Indeed, the construction of the measures $\mu_h \in \mathcal{M}_o^a$ in [29: Proposition 4.13] is completely explicit and our preceding Theorem 5.2 and the standard diagonal argument are also constructive.

Remark 5.2. No representation theorem (like Theorem 3.1) for the measure $\mu \in \mathcal{M}_o^a$ has been used in the proof of Theorem 5.3 for the approximation of μ by Radon measures.

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Received 08.04.1999