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Geometry of Numerical Ranges in Locally m-Convex $*$-Algebras

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Abstract. In this paper we examine the symmetry of numerical ranges in a unital locally m-convex C^*- algebra of a given element and its adjoint, with respect to a rotated real-axis, where the rotation angle depends on the value of the positive linear forms of the algebra (states) at the unit element of the algebra.

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0. Introduction

Given a unital locally m-convex $*$-algebra E the union of the (Bonsall) numerical ranges of an element $a \in E$ and its adjoint a^*, $V(E,a) \cup V(E,a^*)$, contains a subset, symmetric with respect to the real axis $\mathbb{R}$. In the particular case that E is a locally m-convex C^*-algebra (more specifically, locally C^*-algebra in the terminology of [8]), the previous symmetric subset is the union itself, as before. As a consequence, one thus infers the symmetry with respect to $\mathbb{R}$ of $V(E,a)$, for any self-adjoint element $a \in E$, with E a locally m-convex C^*-algebra (in this connection see, for instance, [7: Proposition 3.2]). In this symmetry of the numerical range, a crucial role is assigned to the "normalized states" of the algebra (i.e. continuous positive linear forms f of E, with $f(1_E) = 1$). Thus, by changing to generalized normalized states, that is, to similar forms, as before, where now $f(1_E) = k \in \mathbb{C}$, this implies the appearance of a rotation angle equal to $\arg k$. So the previous symmetry properties of the numerical range are still in force, with respect to the new rotated axis. The justification of the latter symmetry properties constitute thus our main objective of this paper.

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1. Preliminaries

Throughout this paper all algebras are complex and the topological spaces are always assumed to be Hausdorff.

A locally m-convex algebra $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ is a complex algebra E which is also a topological vector space, the topology of which is defined by an upper directed family of submultiplicative semi-norms $\Gamma \equiv \{p_\alpha\}_{\alpha \in I}$. If a locally m-convex algebra E has a continuous involution "*", then E is said to be a locally m-convex $*$-algebra, and if the involution has the C^*-property (i.e. $p_\alpha(x^*x) = p_\alpha(x)^2$ for all $x \in E$ and $\alpha \in I$), then we speak about a locally m-convex C^*-algebra. At the end, a locally m-convex Q-algebra is a unital locally m-convex algebra E whose set of invertible elements is an open subset of E (cf. [10; p. 43/Definition 6.2]).

On the other hand, if E is a $*$-algebra (i.e. an involutive algebra), then by a $*$-representation of E, $\phi : E \to \mathcal{L}(H_\phi)$ we shall always mean a $*$-morphism of E into the C^*-algebra $\mathcal{L}(H_\phi)$ of all bounded linear operators on a Hilbert space H_ϕ. If E is furthermore a locally m-convex $*$-algebra, the continuity of ϕ will be always considered with respect to the norm-operator topology of $\mathcal{L}(H_\phi)$. Moreover, we denote by $\mathcal{P}(E)$ the set of all continuous positive linear forms on E, that is

$$\mathcal{P}(E) = \big\{f \in E' : f(x^*x) \geq 0 \ \text{ for all } x \in E\big\}.$$

On the other hand, if E has a unit 1_E and for an $f \in \mathcal{P}(E)$ we have $f(1_E) = 1$, then we speak of a normalized continuous positive linear form on E, and if there is a vector $\xi \in H_\phi$ such that $\{\phi(x)\xi : x \in E\}^- = H_\phi$ (the "bar" here means "topological closure"), then ϕ is called cyclic and the vector ξ is called a cyclic vector for ϕ.

Now, let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be a locally m-convex algebra with a unit 1_E and $k \in \mathbb{C}$ such that $0 < |k| \leq 1$. The family of semi-norms $\Gamma_1 = \big\{p_\alpha, \frac{1}{|k|}p_\alpha\big\}_{\alpha \in I}$ defines on E the same topology as that of the family Γ. In fact, if T and T_1 are the topologies which are defined on E by the families Γ and Γ_1, respectively, then obviously $T \subseteq T_1$ (T_1 is finer than T). Conversely, let

$$V \equiv V_{\alpha_1, \ldots, \alpha_n, \varepsilon_1} = \big\{x \in E : q_{\alpha_i}(x) \leq \varepsilon_1 \ \ (1 \leq i \leq n)\big\}$$

be a neighborhood of $0 \in E$ for the topology T_1 where $q_{\alpha_i} = p_{\alpha_i}$ or $q_{\alpha_i} = \frac{1}{|k|}p_{\alpha_i}$. Then the neighborhood of $0 \in E$ for the topology T,

$$U \equiv U_{\alpha_1, \ldots, \alpha_n, \varepsilon} = \big\{x \in E : p_{\alpha_i}(x) \leq \varepsilon \equiv |k|\varepsilon_1 \ \ (1 \leq i \leq n)\big\}$$

is obviously contained to V, that is $T_1 \subseteq T$ and hence $T = T_1$. So without loss of generality we may assume that the family of semi-norms $\Gamma \equiv \{p_\alpha\}_{\alpha \in I}$ which defines the locally m-convex topology on E, with every semi-norm $p_\alpha \in \Gamma$ contains the semi-norm $\frac{1}{|k|}p_\alpha$ ($\alpha \in I$), too, where k is an arbitrary but constant complex number with $0 < |k| \leq 1$.

2. The k-numerical range

Let $(E, \{p_\alpha\}_{\alpha \in I})$ be a unital locally m-convex algebra. Moreover, let E' be the weak topological dual space of E and $k \in \mathbb{C}$ with $0 < |k| \leq 1$. We consider the sets

$$(U_\alpha(1))^0 = \left\{ f \in E' : |f(x)| \leq 1 \quad (x \in U_\alpha(1)) \right\} \tag{2.1}$$

where $U_\alpha(1) = \{x \in E : p_\alpha(x) \leq 1\}$,

$$D_k^\alpha(E, 1_E) = \left\{ f \in (U_\alpha(1))^0 : f(1_E) = k \right\} \tag{2.2}$$

and

$$D_k(E, 1_E) = \bigcup_{\alpha \in I} D_k^\alpha(E, 1_E) \tag{2.3}$$

where 1_E is the unit of E. The elements of the set (2.3) are called k-states of E. For $k = 1$ we set $D_1^\alpha(E, 1_E) = D_\alpha(E, 1_E)$ and $D_1(E, 1_E) = D(E, 1_E)$.

We call *k-numerical range* of an element $a \in E$ the set

$$V_k(E, a) = \hat{a}(D_k(E, 1_E)) = \{f(a) : f \in D_k(E, 1_E)\} \tag{2.4}$$

where

$$\hat{a} : D_k(E, 1_E) \subseteq E' \to \mathbb{C}, \qquad f \to \hat{a}(f) := f(a)$$

is the generalization of the Gel'fand transform of a. The numbers,

$$\nu_k(E, a) \equiv \nu_k(a) := \sup\{|\lambda| : \lambda \in V_k(E, a)\} \tag{2.5}$$

is called the *k-numerical radius* of a. For $k = 1$ we set $V_1(E, a) = V(E, a)$ and $\nu_1(E, a) = \nu(E, a)$.

Now, if $\lambda \in V(E, a) \quad (a \in E)$, there is an $f \in E'$ with $f(1_E) = 1$ and $|f(x)| \leq p_\alpha(x) \quad (x \in E)$, for some $\alpha \in I$, such that $\lambda = f(a)$. So the linear form $g = kf$ belongs to the set $D_k^\alpha(E, 1_E) \subset D_k(E, 1_E)$ and hence $k\lambda = g(a) \in V_k(E, a)$, that is $k \cdot V(E, a) \subset V_k(E, a)$.

On the other hand, if $\rho \in V_k(E, a)$, then $\rho = g(a)$ for some $g \in E'$ with $g(1_E) = k$ and $|g(x)| \leq p_\alpha(x) \;\}(x \in E)$ for some $\alpha \in I$. So $\left|\left(\frac{1}{k}g\right)(x)\right| \leq \frac{1}{|k|}p_\alpha(x) = p_\beta(x) \quad (x \in E)$ for some other $\beta \in I$ (see the comments before Section 2) which means that the linear form $f = \frac{1}{k}g \in E'$ belongs to the set $D_1^\beta(E, 1_E) \subseteq D(E, 1_E)$, hence $f(a) \in V(E, a)$. So $\rho = g(a) \in k \cdot V(E, a)$, that is $V_k(E, a) \subseteq k \cdot V(E, a)$.

Now by the above we have

$$k \cdot V(E, a) = V_k(E, a) \qquad \text{and} \qquad |k| \cdot \nu(E, a) = \nu_k(E, a). \tag{2.6}$$

On the other hand, let $(E_\alpha = E/\ker p_\alpha)_{\alpha \in I}$ be the Arens-Michael decomposition of $E \quad (E = \lim_{\leftarrow} E_\alpha$, see [10: p. 88/Theorem 3.1]). Then, $D_k^\alpha(E, 1_E)$ is isomorphic to $D_k(E_\alpha, 1_\alpha)$ where $1_\alpha = 1_E + \ker p_\alpha$ is the unit of the normed algebra $E_\alpha = E/\ker p_\alpha \quad (\alpha \in I)$, so that $D_k(E, 1_E) = \cup_{\alpha \in I} D_k(E_\alpha, 1_\alpha)$ by (2.3) and hence

$$V_k(E, a) = \bigcup_{\alpha \in I} V_k(E_\alpha, a_\alpha) \tag{2.7}$$

and

$$\nu_k(E, a) = \sup_{\alpha \in I} \nu_k(E_\alpha, a_\alpha) \tag{2.8}$$

where $a_\alpha = a + \ker\, p_\alpha \in E_\alpha \quad (\alpha \in I)$. In this respect we have the following

Proposition 2.1. *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha\in I})$ be a unital complete locally m-convex algebra. Then:*

(i) *For each $\alpha \in I$, the set $D_k^\alpha(E, 1_E) \subseteq E'$ is convex and compact.*

(ii) *For each $a \in E$, the set $V_k(E, a) \subseteq \mathbb{C}$ is convex.*

(iii) *For each $a \in E$, the set $V_k(E, a) \subseteq \mathbb{C}$ is bounded if and only if $\sup_{\alpha\in I} p_\alpha(a) < +\infty$.*

Moreover, if the family $\{p_\alpha\}_{\alpha\in I}$ is finite, then $V_k(E, a)$ is a compact subset of $\mathbb{C}$.

Proof. (i) The set $D_k^\alpha(E, 1_E) \subseteq E'$ is convex and closed since the sets $(U_\alpha(1))^0$ and $\hat{1}_E^{-1}(\{k\})$ are such. Moreover, the set $D_k^\alpha(E, 1_E)$ is equicontinuous since it is contained in the polar of a neighborhood of $0 \in E$. So by the Alaoglu-Bourbaki theorem it is relatively compact. Hence $D_k^\alpha(E, 1_E)$ $(\alpha \in I)$ is a compact subset of E'.

(ii) For $f, g \in D_k(E, 1_E)$ there are $\alpha, \beta \in I$ such that $|f(x)| \leq p_\alpha(x)$ and $|g(x) \leq p_\beta(x)$ $(x \in E)$. Since the family of semi-norms $\Gamma \equiv \{p_\alpha\}_{\alpha\in I}$ is upper directed, there is $\gamma \in I$ such that $f, g \in D_k^\gamma(E, 1_E) \subseteq (U_\gamma(1))^0$. Now, if h is a convex combination of f and g, then by the convexity of $D_k^\gamma(E, 1_E)$ we have $h \in D_k^\gamma(E, 1_E) \subseteq D_k(E, 1_E)$, that is $D_k(E, 1_E) \subseteq E'$ is a convex subset, and by (2.4) $V_k(E, a) \subseteq \mathbb{C}$ is also a convex subset.

(iii) Let $(E_\alpha \equiv E/\ker p_\alpha)_{\alpha\in I}$ be the Arens-Michael decomposition of E. Then for the normed algebras E_α $(\alpha \in I)$ we have

$$\nu(E_\alpha, a_\alpha) \geq \tfrac{1}{e}||a_\alpha||_\alpha = \tfrac{1}{e}p_\alpha(a) \tag{2.9}$$

(cf. [1: p. 34/Theorem 1]) and

$$\nu_k(E_\alpha, a_\alpha) \leq ||a_\alpha||_\alpha = p_\alpha(a). \tag{2.10}$$

By (2.9), (2.6) and (2.10) we get

$$\tfrac{|k|}{e}p_\alpha(a) \leq \nu_k(E_\alpha, a_\alpha) \leq p_\alpha(a)$$

and taking the suprema we get

$$\tfrac{|k|}{e}\sup_\alpha p_\alpha(a) \leq \sup_\alpha \nu_k(E_\alpha, a_\alpha) \leq \sup_\alpha p_\alpha(a)$$

so that, by (2.8),

$$\tfrac{|k|}{e}\sup_\alpha p_\alpha(a) \leq \nu_k(E, a) \leq \sup_\alpha p_\alpha(a)$$

which means that $V_k(E, a) \subseteq \mathbb{C}$ is bounded if and only if $\sup p_\alpha(a) < +\infty$. On the other hand, if the family of semi-norms $\Gamma \equiv \{p_\alpha\}_{a\in I}$ is finite, then the semi-norm $q(x) = \max_{\alpha\in I} p_\alpha(x)$ $(x \in E)$ defines on E the same topology as the family Γ, and $D_k(E, 1_E) = D_k^q(E, 1_E)$. Since $D_k^q(E, 1_E)$ is convex and compact one has that $V_k(E, a)$ $(a \in E)$ has the same property ∎

Now, let E be a unital locally m-convex algebra and F be a subalgebra of E. Then, for every $a \in E$,

$$V_k(E, a) = V_k(F, a). \tag{2.11}$$

In fact, since the map $f \mapsto f|_F : D_k(E, 1_E) \to D_k(F, 1_E)$ is "onto", by the Hahn-Banach theorem, (2.4) implies (2.11).

Scholium. From the above it follows that the numerical range of an element $a \in E$ does not change replacing E by its completion $\hat{E}$. Thus, without loss of generality we may assume that the initial locally m-convex algebra E is complete. Moreover, $V_k(E,a)$ may be computed from the subalgebra F generated by a and 1_E, or from the closure of F.

Proposition 2.2. *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha\in I})$ be a unital locally m-convex algebra. Then the set $H_k(E)$ of strongly k-Hermitian elements (i.e. elements with real k-numerical range) is closed.*

Proof. Let $(x_\delta)_{\delta\in J} \subseteq H_k(E)$ be a net of strongly k-Hermitian elements of E with $x_\delta \underset{\delta}{\longrightarrow} x$. For $\lambda \in V_k(E,x)$ there is an $\alpha \in I$ and an $f \in (U_\alpha(1))^0$ with $f(1_E) = k$ such that $\lambda = f(x)$. On the other hand, the numbers $\lambda_\delta = f(x_\delta)$ $(\delta \in J)$ are reals and

$$|\lambda_\delta - \lambda| = |f(x_\delta - x)| \leq p_\alpha(x_\delta - x) \qquad \text{for } f \in (U_\alpha(1))^0.$$

Since $p_\alpha(x_\delta - x) \underset{\delta}{\longrightarrow} 0$ we have $\lambda \in \mathbb{R}$, that is $x \in H_k(E)$ ∎

Theorem 2.3. *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha\in I})$ be a unital locally m-convex algebra and J a closed two-sided ideal of E. For the locally m-convex quotient algebra $(E/J, \dot{\Gamma} \equiv \{\dot{p}_\alpha\}_{\alpha\in I})$ we have*

$$V_k(E/J, \pi(a)) = \bigcap_{b\in J} V_k(E, a+b) \qquad (a \in E) \tag{2.12}$$

where $\pi : E \to E/J$ is the canonical quotient map.

For the proof of Theorem 2.3 we need the following

Lemma 2.4. *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha\in I})$ be a unital locally m-convex algebra and $a \in E$. Then*

$$V_k(E,a) = \bigcup_{\alpha\in I} \Big(\bigcap_{z\in\mathbb{C}} \{\lambda \in \mathbb{C} : |\lambda - kz| \leq p_\alpha(a - z\,1_E)\} \Big). \tag{2.13}$$

Proof. Let $\lambda \in \mathbb{C}$ satisfy the inequality

$$|\lambda - kz| \leq p_\alpha(a - z\,1_E) \qquad (z \in \mathbb{C}) \tag{2.14}$$

for some $\alpha \in I$. If $a = z_0 1_E$ for some $z_0 \in \mathbb{C}$, then from (2.14) we get $\lambda = kz_0$. So for every $f \in D_k^\alpha(E, 1_E)$ we have $\lambda = z_0 f(1_E) = f(a) \in V_k(E,a)$.

On the other hand, if the elements a and 1_E are linearly independent, we define the function

$$f_0 : M \equiv \mathcal{L}(a, 1_E) \to \mathbb{C}, \qquad \mu a + \nu 1_E \mapsto f_0(\mu a + \nu 1_E) := \mu\lambda + \nu k$$

where M is the subspace of E generated by a and 1_E. Then

$$\begin{aligned} |f_0(\mu a + \nu 1_E)| &= |\mu\lambda + \nu k| \\ &= |\mu| \left|\lambda - k\Big(-\frac{\nu}{\mu}\Big)\right| \\ &\leq |\mu|\, p_\alpha\Big(a - \Big(-\frac{\nu}{\mu}\Big)1_E\Big) \\ &= p_\alpha(\mu a + \nu 1_E) \end{aligned} \qquad (\mu, \nu \in \mathbb{C}, \mu \neq 0).$$

Now, the Hahn-Banach theorem guarantees that there is an $f \in E'$ with $|f(x)| \leq p_\alpha(x)$ $(x \in E)$ and $f|_M = f_0$, so that $f \in D_k^\alpha(E, 1_E)$ and $\lambda = f_0(a) = f(a) \in V_k(E, a)$.

Conversely, let $\lambda \in V_k(E, a)$. Then there are $\alpha \in I$ and $f \in (U_\alpha(1))^0$ with $f(1_E) = k$ such that $\lambda = f(a)$. So we have $|\lambda - kz| = |f(a - z1_E)| \leq p_\alpha(a - z1_E)$ for every $z \in \mathbb{C}$ ∎

Proof of Theorem 2.3. Let $\lambda, z \in \mathbb{C}$ be such that

$$|\lambda - kz| \leq \dot{p}_\alpha(\pi(a) - z\dot{1}_E) \tag{2.15}$$

for some $\alpha \in I$ where $\dot{1}_E = \pi(1_E) = 1_E + J$ is the unit of the quotient algebra E/J. Since the canonical map π is a homomorphism we have

$$\dot{p}_\alpha(\pi(a) - z1_E) = \dot{p}_\alpha(\pi(a - z1_E)) := \inf_{b \in I} p_\alpha((a + b) - z1_E)$$

so that by (2.15) $|\lambda - kz| \leq p_\alpha((a + b) - z1_E)$ $(b \in I)$ for some $\alpha \in I$. Hence, Lemma 2.4 implies (2.12) ∎

3. Geometry of the k-numerical range

In this section we prove some elegant geometrical properties of the k-numerical range (see Introduction). We first have the following

Theorem 3.1. *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be a locally m-convex $*$-algebra with unit 1_E, $a \in E$, and $k \in \mathbb{C}$ with $|k| \leq 1$. Then the set $V_k(E, a) \cup V_k(E, a^*)$ contains a subset $A \subset \mathbb{C}$ symmetric with respect to the line $\varepsilon = \{re^{i\theta_0} : r \in \mathbb{R}\}$, where $\theta_0 \in [0, 2\pi]$ with $\cos\theta_0 = \operatorname{Re} k$ and $\sin\theta_0 = \operatorname{Im} k$.*

Proof. Let $f \in D_1(E, 1_E) \equiv D(E, 1_E)$ be a normalized state of E which is further a positive linear form (i.e. $f \in \mathcal{P}(E)$). Then there is a continuous cyclic $*$-representation of E, $\phi_f : E \to \mathcal{L}(H_\phi)$, and a unital cyclic vector $\xi \in H_{\phi_f}$ such that $f(x) = (\phi_f(x)\xi|\xi)$ $(x \in E)$ (cf. [3, 8]). We have

$$f(a^*) = (\phi_f(a^*)\xi|\xi) = (\phi_f(a)^*\xi|\xi) = (\xi|\phi_f(a)\xi) = \overline{(\phi_f(a)\xi|\xi)} = \overline{f(a)}.$$

So the set

$$B = \Big\{\{f(a)\} \cup \{f(a^*)\} : f \in D(E, 1_E) \cap \mathcal{P}(E)\Big\} \subseteq \mathbb{C}$$

is symmetric with respect to the $\mathbb{R}$-axis, hence the set

$$A = e^{i\theta_0}B \subseteq e^{i\theta_0}\big(V(E, a) \cup V(E, a^*)\big) = V_k(E, a) \cup V_k(E, a^*)$$

(see (2.6)) is symmetric with respect to the line ε ∎

Corollary 3.2. *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be a unital locally m-convex $*$-algebra, $a \in E$ self-adjoint, and $k \in \mathbb{C}$ with $|k| \leq 1$. Then the k-numerical range of a contains a subset of $\mathbb{C}$, symmetric with respect to the line $\varepsilon = \{re^{i\theta_0} : r \in \mathbb{R}\}$, where $\theta_0 \in [0, 2\pi]$ with $\cos\theta_0 = \operatorname{Re} k$ and $\sin\theta_0 = \operatorname{Im} k$.*

Corollary 3.3. *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be a locally m-convex C^*-algebra with unit 1_E, $a \in E$, and $k \in \mathbb{C}$ with $|k| \leq 1$. Then the set $V_k(E,a) \cup V_k(E,a^*) \subseteq \mathbb{C}$ is symmetric with respect to the line $\varepsilon = \{re^{i\theta_0} : r \in \mathbb{R}\}$, where $\theta_0 \in [0, 2\pi]$ with $\cos\theta_0 = \operatorname{Re} k$ and $\sin\theta_0 = \operatorname{Im} k$.*

Proof. In every unital locally m-convex C^*-algebra E we have $D(E, 1_E) = \mathcal{P}(E)$ (cf. [4: Theorem $3.7_{(i)\Leftrightarrow(ii)}$]). So by the above Theorem 3.1 we have what we wanted to prove ∎

By the above Corollary 3.3 we see that the value of the states of a unital locally m-convex C^*-algebra E at the unit element 1_E determines the argument of the line, with respect to which the union of the numerical ranges of the elements a and a^* is symmetric.

Proposition 3.4. *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be a locally m-convex C^*-algebra with unit 1_E, $a \in E$ self-adjoint, and $k \in \mathbb{C}$ with $|k| \leq 1$. Then the k-numerical range of a is a convex subset of the line $\varepsilon = \{re^{i\theta_0} : r \in \mathbb{R}\}$, where $\theta_0 \in [0, 2\pi]$ with $\cos\theta_0 = \operatorname{Re} k$ and $\sin\theta_0 = \operatorname{Im} k$.*

Proof. Since the locally m-convex $*$-algebra E has the C^*-property, we have $D(E, 1_E) = \mathcal{P}(E)$ (cf. [4: Theorem $1.7_{(i)\Leftrightarrow(ii)}$]). Now, having in mind the proof of Theorem 3.1, we have $f(a) \in \mathbb{R}$ for every $f \in D(E, 1_E) = \mathcal{P}(E)$. So $V(E,a) \subset \mathbb{R}$, hence $V_k(E,a) = e^{i\theta_0} V(E,a)$ is a convex subset of the line ε ∎

Corollary 3.5. *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be a unital locally m-convex Q-$*$-algebra with the C^*-property, $a \in E$ self-adjoint, and $k \in \mathbb{C}$ with $|k| \leq 1$. Then the k-numerical range of a is a line segment of the line $\varepsilon = \{re^{i\theta_0} : r \in \mathbb{R}\}$ where $\theta_0 \in [0, 2\pi]$ with $\cos\theta_0 = \operatorname{Re} k$ and $\sin\theta_0 = \operatorname{Im} k$.*

Proof. The proof is a consequence of Proposition 3.4, Proposition 2.1 and [8: Theorem $7.6_{(1)\Leftrightarrow(3)}$] ∎

Now, let $\mathcal{M}_n(\mathbb{C})$ be the $*$-algebra of all $n \times n$ complex matrices with an involution defined by $A^* = \overline{{}^tA}$ $(A \in \mathcal{M}_n(\mathbb{C}))$, where t means transpose and overline means complex conjugation. If $\mathcal{L}(\mathbb{C}^n)$ is the C^*-algebra of all (bounded) operators on the numerical space $\mathbb{C}^n$, with inner product $(a|b) := \overline{{}^tb}a \equiv b^*a$ $(a, b \in \mathbb{C}^n)$, then by considering each $n \times n$ matrix $A \in \mathcal{M}_n(\mathbb{C}) \cong \mathcal{L}(\mathbb{C}^n)$ as an operator on $\mathbb{C}^n$ (i.e. $A : \mathbb{C}^n \to \mathbb{C}^n$, $x \mapsto A(x) := Ax$, $x = (x_1, \ldots, x_n)^t \in \mathbb{C}^n$) and taking $k = 1$, the classical numerical range of the matrix A is a (convex) subset of the k-numerical range $V_k(\mathcal{M}_n(\mathbb{C}), A) \equiv V_1(\mathcal{M}_n(\mathbb{C}), A) \equiv V(\mathcal{M}_n(\mathbb{C}), A)$ of the element $A \in \mathcal{M}_n(\mathbb{C}) \cong \mathcal{L}(\mathbb{C}^n)$ (cf. [7: Comments after Theorem 2.1] and [6: Theorem 2.1]). So by the above Corollary 3.5 we have that the classical numerical range of a Hermitian matrix A is a line segment of the line $\mathbb{R}$.

On the other hand, if $A \in \mathcal{M}_n(\mathbb{C})$ is an $n \times n$ matrix such that the matrix $e^{-i\theta_0}A$ $(\theta_0 \in [0, 2\pi])$ is Hermitian, then by Corollary 3.5 the classical numerical range $W(A)$ of A is a line segment of the line $\varepsilon = \{r \cdot e^{i\theta_0} : r \in \mathbb{R}\}$, since $W(A)$ is a bounded convex subset of $\mathbb{C}$ and

$$W(A) = e^{i\theta_0} W(e^{-i\theta_0}A) \subseteq e^{i\theta_0} V(\mathcal{M}_n(\mathbb{C}), e^{-i\theta_0}A) = V_k(\mathcal{M}_n(\mathbb{C}), e^{-i\theta_0}A) \subseteq \varepsilon$$

where $k \in \mathbb{C}$ is such that $\operatorname{Re} k = \cos\theta_0$ and $\operatorname{Im} k = \sin\theta_0$.

Now, if $A \in \mathcal{M}_n(\mathbb{C})$ is an $n \times n$ matrix with elements on the line $\varepsilon = \{r \cdot e^{i\theta_0} : r \in \mathbb{R}\}$, $\theta_0 \in [0, 2\pi]$, then by Corollary 3.3 and the relation $W(A) = W({}^tA)$ (see [12: p. 7]) $W(e^{-i\theta_0}A)$ is symmetric with respect to the R-axis, hence $W(A) = e^{i\theta_0}W(e^{-i\theta_0}A)$ is symmetric with respect to the line ε. That is we take [11: Proposition 2.1 and its Corollary].

4. The k-spatial numerical range

Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be a locally convex space, $T \in \mathcal{L}(E)$ a continuous operator on E, and $k \in \mathbb{C}$ with $|k| \leq 1$. For $p_\alpha \in \Gamma$ and $x \in U_\alpha(1) \equiv \{x \in E : p_\alpha(x) \leq 1\}$ we consider the sets

$$D_k^\alpha(E, x) = \big\{f \in (U_\alpha(1))^0 : f(x) = k\big\} \tag{4.1}$$

$$V_k^\alpha(T, x) = \{f(Tx) : f \in D_k^\alpha(E, x)\} \tag{4.2}$$

$$V_k^\alpha(T) = \bigcup_{x \in S_\alpha(1)} V_k^\alpha(T, x). \tag{4.3}$$

where $S_\alpha(1) = \{x \in E : p_\alpha(x) = 1\}$. Applying an analogous argument as in the proof of Proposition 2.1, we can prove that for each $x \in U_\alpha(1)$ the set $D_k^\alpha(E, x) \subseteq E_s'$ is (weakly) compact and convex.

Definition 4.1. Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be a locally convex space and $T \in \mathcal{L}(E)$. We call *k-spatial numerical range* of T the set

$$V_k(T) = \bigcup_{\alpha \in I} V_k^\alpha(T). \tag{4.4}$$

In this regard, we have the following

Lemma 4.1. *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be a locally convex space, $T \in \mathcal{L}(E)$ and $x \in E$ with $p_\alpha(x) = 1\}$ for some $\alpha \in I$. Then*

$$V_k^\alpha(T, x) = \bigcap_{\zeta \in \mathbb{C}} \big\{\lambda \in \mathbb{C} : |\lambda - k\zeta| \leq p_\alpha((T - \zeta I_E)x)\big\} \tag{4.5}$$

where $k \in \mathbb{C}$ with $|k| \leq 1$.

Proof. We apply an analogous argument as in the proof of Lemma 2.4 ∎

Theorem 4.2. *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be a locally m-convex algebra with unit 1_E, such that $p_\alpha(1_E) = 1$ $(\alpha \in I)$. Moreover, for some $a \in E$ consider the operator*

$$T_a : E \to E, \qquad x \mapsto T_a(x) := ax. \tag{4.6}$$

Then

$$V_k(E, a) = V_k(T_a). \tag{4.7}$$

Proof. The operator T_a is clearly continuous since $p_\alpha(T_a x) = p_\alpha(ax) \le p_\alpha(a)p_\alpha(x)$ $(x \in E)$. Now, for $\lambda \in V_k(T_a)$ there is an $\alpha \in I$ such that $|\lambda - k\zeta| \le p_\alpha((T_a - \zeta I)x)$ for all $\zeta \in \mathbb{C}$. But

$$p_\alpha((T_a - \zeta I)x) = p_\alpha((a - \zeta 1_E)x) \le p_\alpha(a - \zeta 1_E)p_\alpha(x) = p_\alpha(a - \zeta 1_E) \tag{4.8}$$

for all $\zeta \in \mathbb{C}$, so that $|\lambda - k\zeta| \le p_\alpha(a - \zeta 1_E)$ for all $\zeta \in \mathbb{C}$. Thus Lemma 2.4 implies that $\lambda \in V_k(E, a)$.

On the other hand, if $\lambda \in V_k(E, a)$, relation (4.8) is true for some $\alpha \in I$ and for all $\zeta \in \mathbb{C}$. So the equality $p_\alpha(a - \zeta 1_E) = p_\alpha((T_a - \zeta I)1_E)$ and Lemma 4.1 imply that $\lambda \in V_k^\alpha(T_a, 1_E)$, hence $\lambda \in V_k^\alpha(T_a) \subseteq V_k(T_a)$ ∎

For convenience, we recall the definition of an upper semi-continuous map.

Definition 4.2. Let E and F be two topological spaces and let 2^F be the set of all subsets of F. A map $\phi : E \to 2^F$ is said to be *upper semi-continuous* if for each $x \in E$ and each neighborhood U of $\phi(x)$, there exists a neighborhood V of x such that $y \in V$ implies $\phi(y) \subseteq U$.

In this regard, we now have the following

Lemma 4.3 (see [5: Lemma 3.6]). *Let E and F be topological spaces with F compact. Let $\phi : E \to 2^F$ be a map such that for every $x \in E$ the set $\phi(x) \subseteq F$ is closed. Then ϕ is upper semi-continuous if and only if for any nets $(x_\delta) \subseteq E$ and $(y_\delta) \subseteq \phi(x_\delta)$ with $\lim_\delta x_\delta = x$ and $\lim_\delta y_\delta = y$ one has $y \in \phi(x)$.*

Theorem 4.4. *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be a locally convex space and suppose that for some $\alpha \in I$ the set $S_\alpha(1) = \{x \in E : p_\alpha(x) = 1\}$ is bounded. If T is a continuous linear operator on E, then the map*

$$x \mapsto V_k^\alpha(T, x), \qquad S_\alpha(1) \to 2^{\mathbb{C}} \tag{4.9}$$

is upper semi-continuous and sends $S_\alpha(1)$ $(\alpha \in I)$ into non-void compact convex subsets of $\mathbb{C}$.

Proof. Let $x \in S_\alpha(1)$ and $\lambda \in V_k^\alpha(T, x)$. There is an $f \in D_k^\alpha(E, x) \subseteq (U_\alpha(1))^0$ such that $\lambda = f(Tx)$, so that $|\lambda| \le p_\alpha(Tx)$ $(x \in E)$ and by the continuity of T there are $\beta \in I$ and $\mu > 0$ such that $p_\alpha(Tx) \le \mu p_\beta(x)$ $(x \in E)$.

On the other hand, since $S_\alpha(1)$ is bounded there is a $\xi > 0$ such that $S_\alpha(1) \subseteq \xi V$, where $V = \{x \in E : p_\beta(x) \le 1\}$ (cf. [9: p. 109]). So for $x \in S_\alpha(1)$ we have $p_\beta(x) \le \xi$, hence $|\lambda| \le \mu\xi \equiv \rho$, that is $V_k^\alpha(T, x) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le \rho\} \equiv S_\rho(\mathbb{C})$ for every $x \in S_\alpha(1)$. Moreover, the sets $V_k^\alpha(T, x)$ $(x \in S_\alpha(1))$ are compact and convex since the sets $D_k^\alpha(E, x)$ $(x \in S_\alpha(1))$ are such (see the comments before Definition 4.1). Now we complete the proof by applying Lemma 4.3: Consider nets $(x_\delta)_{\delta \in J} \subseteq S_\alpha(1)$ and $(\lambda_\delta) \subseteq \mathbb{C}$ with $\lambda_\delta \in V_k^\alpha(T, x) \subseteq S_\rho(\mathbb{C})$. If $\lim_\delta x_\delta = x$ and $\lim_\delta \lambda_\delta = \lambda$, we have to prove that $\lambda \in V_k^\alpha(E, x)$. In fact, there is a net $(f_\delta) \subseteq E'$ with $f_\delta \in D_k^\alpha(E, x_\delta) \subseteq (U_\alpha(1))^0$ such that $\lambda_\delta = f_\delta(Tx)$ $(\delta \in J)$. On the other hand, by compactness of $(U_\alpha(1))^0 \subseteq E'_S$, there is a subnet $(f_{n_\delta})_{\delta \in J}$ with $\lim_\delta f_{n_\delta} = f \in (U_\alpha(1))^0$. Thus, one obtains

$$\begin{aligned} |k - f(x)| &\le |k - f_{n_\delta}(x)| + |f_{n_\delta}(x) - f(x)| \\ &= |f_{n_\delta}(x_{n_\delta} - x)| + |(f_{n_\delta} - f)(x)| \\ &\xrightarrow[\delta]{} 0 \end{aligned}$$

which yields $f(x) = k$. Finally, we have

$$\begin{aligned}|\lambda - f(Tx)| &\leq |\lambda - \lambda_{n_\delta}| + |\lambda_{n_\delta} - f_{n_\delta}(Tx)| + |f_{n_\delta}(Tx) - f(Tx)| \\ &= |\lambda - \lambda_{n_\delta}| + |f_{n_\delta}(Tx_{n_\delta} - Tx)| + |(f_{n_\delta} - f)(Tx)| \\ &\xrightarrow[\delta]{} 0\end{aligned}$$

which gives $\lambda = f(Tx) \in V_k^\alpha(T, x)$. So, by Lemma 4.3, the map (4.9) is upper semi-continuous ∎

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