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Initial-Boundary Value Problem for Some Coupled Nonlinear Parabolic System of Partial Differential Equations Appearing in Thermodiffusion in Solid Body

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Abstract. We prove a theorem about existence, uniqueness and regularity of the solution to an initial-boundary value problem for a nonlinear coupled parabolic system consisting of two equations. Such a system appears in the thermodiffusion in solid body. In our proof we use an energy method, methods of Sobolev spaces, semigroup theory and the Banach fixed point theorem.

Keywords: *Linear and nonlinear parabolic systems of partial differential equations, initial-boundary value problems, energy estimates, Sobolev spaces, semigroup theory, Banach fixed point theorem*

AMS subject classification: 35 E 15, 35 K 15, 35 K 25, 35 K 50, 35 K 30

1. Introduction

We consider the initial-boundary value problem for the nonlinear coupled parabolic system of partial differential equations

$$c(\theta_1,\theta_2)\partial_t\theta_1 - a^1_{\alpha\beta}(\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2\theta_1}{\partial x_\alpha\partial x_\beta} + d(\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial\theta_2}{\partial t} = Q_1 \quad (1.1)$$

$$n(\theta_1,\theta_2)\partial_t\theta_2 - a^2_{\alpha\beta}(\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2\theta_2}{\partial x_\alpha\partial x_\beta} + d(\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial\theta_1}{\partial t} = Q_2 \quad (1.2)$$

with initial conditions

$$\left.\begin{aligned}\theta_1(0,x) &= \theta_1^0(x)\\ \theta_2(0,x) &= \theta_2^0(x)\end{aligned}\right\} \quad (1.3)$$

and boundary conditions (Dirichlet Type)

$$\left.\begin{aligned}\theta_1(t,\cdot)|_{\partial\Omega} &= 0\\ \theta_2(t,\cdot)|_{\partial\Omega} &= 0\end{aligned}\right\} \quad (1.4)$$

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where $\theta_1 = \theta_1(t,x)$ and $\theta_2 = \theta_2(t,x)$ are unknown scalar functions denoting the temperature and the chemical potential of the body, respectively, both depending on $t \in \mathbb{R}_+$ and $x \in \Omega$, $\Omega \subset \mathbb{R}^3$ being a bounded domain with smooth boundary $\partial\Omega$, $\nabla\theta_1 = (\partial_1\theta_1, \partial_2\theta_1, \partial_3\theta_1)$ and $\nabla\theta_2 = (\partial_1\theta_2, \partial_2\theta_2, \partial_3\theta_2)$ are the gradients of the functions θ_1 and θ_2, respectively, with $\partial_j = \frac{\partial}{\partial x_j}$ $(j = 1,2,3)$ (analogously, $\partial_t = \frac{\partial}{\partial t}$). Further, $Q_1 = Q_1(t,x)$ and $Q_2 = Q_2(t,x)$ denote scalar functions depending on $t \in \mathbb{R}_+$ and $x \in \Omega$, which describe the source intensities of the heat and of the diffusing mass, respectively. At last, c and n are nonlinear coefficients depending on the unknown functions θ_1 and θ_2, d and $a^1_{\alpha\beta}, a^2_{\alpha\beta}$ are nonlinear coefficients depending additionally on the gradients $\nabla\theta_1$ and $\nabla\theta_2$.

For $0 < m < \infty$ we denote by $H^m(\Omega)$ and $H^m_0(\Omega)$ the usual Sobolev spaces with norm $\|\cdot\|_m$ [1]. For $1 \le p \le \infty$ we denote by $L^p(\Omega)$ the Lebesgue function space on Ω with norm $\|\cdot\|_{L^p}$; the norm and inner product in $L^2(\Omega)$ are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively.

We shall use the notation $\partial^\alpha_x = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ $(|\alpha| = \alpha_1 + \alpha_2 + \alpha_3)$ and denote for any integer $N \ge 0$

$\mathcal{D}^N u = (\partial_t^j \partial_x^\alpha u : j + \alpha = N)$
$\bar{\mathcal{D}}^N u = (\partial_t^j \partial_x^\alpha u : j + \alpha \le N)$
$\mathcal{D}_x^N u = (\partial_x^\alpha u : |\alpha| = N)$
$\bar{\mathcal{D}}_x^N u = (\partial_x^\alpha u : |\alpha| < N)$.

The inclusion $f \in X$ for a space X with norm $\|\cdot\|_X$ means that each component $f_1, \dots, f_n$ of f is in X and $\|f\|_X = \|f_1\|_X + \ldots + \|f_n\|_X$. For any $0 \le m < \infty$ and $T > 0$ we also use the notation $|u|_{m,T} = \sup_{0 \le t \le T} \|u(t)\|_m$ where $\|\cdot\|_0$ denotes $\|\cdot\|$.

The aim of our paper is to prove existence and uniqueness (local in time) of the solution to the initial-boundary value problem (1.1) - (1.4) using methods of Sobolev spaces (cf. [6 - 8, 11, 12]). In Section 2 we present the related main theorem. In Section 3 we present a theorem about existence, uniqueness and regularity of the solution to the linearized problem associated with problem (1.1) - (1.4). Section 4 is devoted to the proof of an energy estimate for that linearized system. Finally, in Section 5 we prove the main theorem using the Banach fixed point theorem.

2. The main theorem

In this section we formulate the theorem about existence and uniqueness (local in time) of the solution to initial-boundary value problems to the nonlinear system (1.1) - (1.2). Before starting to formulate the main theorem we notice that under the condition

$$cn - d^2 > 0 \tag{2.1}$$

we can convert system (1.1) - (1.2) into the form

$$\begin{aligned} &\partial_t\theta_1 - \overline{a}^{11}_{\alpha\beta}(\theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2)\frac{\partial^2\theta_1}{\partial x_\alpha \partial x_\beta} - \overline{a}^{12}_{\alpha\beta}(\theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2)\frac{\partial^2\theta_2}{\partial x_\alpha \partial x_\beta} \\ &\quad = \overline{g}_1(\theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2) \end{aligned} \tag{2.2}$$

$$\begin{aligned}\partial_t\theta_2 - \overline{a}^{21}_{\alpha\beta}(\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2\theta_1}{\partial x_\alpha \partial x_\beta} - \overline{a}^{22}_{\alpha\beta}(\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2\theta_2}{\partial x_\alpha \partial x_\beta}\\ = \overline{g}_2(\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2) \qquad (2.3)\end{aligned}$$

where $\overline{a}^{11}_{\alpha\beta} = \frac{n}{\delta}a^1_{\alpha\beta}$, $\overline{a}^{12}_{\alpha\beta} = -\frac{d}{\delta}a^2_{\alpha\beta}$ and

$$\overline{g}_1 = \tfrac{Q_1 n - dQ_2}{\delta},\ \overline{a}^{21}_{\alpha\beta} = -\tfrac{d}{\delta}a^2_{\alpha\beta},\ \overline{a}^{22}_{\alpha\beta} = \tfrac{c}{\delta}a^2_{\alpha\beta},\ \overline{g}_2 = \tfrac{cQ_2 - dQ_1}{\delta},\ \delta = cn - d^2. \qquad (2.4)$$

With system (2.2) - (2.3) we associate initial conditions

$$\left.\begin{aligned}\theta_1(0,x) = \theta_1^0(x)\\ \theta_2(0,x) = \theta_2^0(x)\end{aligned}\right\} \qquad (2.5)$$

and boundary conditions (Dirichlet type)

$$\left.\begin{aligned}\theta_1(t,\cdot)|_{\partial\Omega} = 0\\ \theta_2(t,\cdot)|_{\partial\Omega} = 0\end{aligned}\right\} \qquad (2.6)$$

Now, we formulate the main theorem.

Theorem 2.1 (Local existence in time). *Let the following assumptions be satisfied:*

1° *$s \geq [\frac{3}{2}] + 4 = 5$ is an arbitrary but fixed integer.*

2° *$\partial_t^k Q_1, \partial_t^k Q_2 \in C^0\big([0,T], H^{s-2-k}(\Omega)\big)$ for $k = 0,1,...,s-2$ and $\partial_t^{s-1}Q_1, \partial_t^{s-1}Q_2 \in L^2\big([0,T], L^2(\Omega)\big)$.*

3° *There exists a constant $\gamma > 0$ such that $(a_{\alpha\beta}\xi_\alpha\xi_\beta\eta,\eta) \geq \gamma|\xi|^2|\eta|^2 \quad (\eta\in\mathbb{R}^2, \xi\in\mathbb{R}^3)$ where $a_{\alpha\beta} = [a^{ij}_{\alpha\beta}] \quad (i,j = 1,2)$, $a^{ij}_{\alpha\beta} = a^{ji}_{\beta\alpha}$, with $a^{ij}_{\alpha\beta}, d \in C^{s-1}(\mathbb{R}^6)$, $c, n \in C^{s-1}(\mathbb{R}^2)$ and $cn - d^2 > 0$.*

4° *$\theta_1^0, \theta_2^0 \in H^s(\Omega)\cap H_0^1(\Omega)$, $\theta_1^k, \theta_2^k \in H^{s-k}(\Omega)\cap H_0^1(\Omega) \quad (1 \leq k \leq s-2)$ and $\theta_1^{s-1}, \theta_2^{s-1} \in L^2(\Omega)$ where $\theta_i^k = \frac{\partial^k\theta_i(0,\cdot)}{\partial t^k}$ $(i = 1,2)$ are calculated formally from system $(1.1) - (1.2)$ and expressed with the initial data θ_1^0 and θ_2^0.*

Then there exists a unique solution (θ_1,θ_2) to problem $(2.2)-(2.6)$ with the properties

$$\left.\begin{aligned}\theta_i \in \cap_{k=0}^{s-2} C^k\big([0,T], H^{s-k}(\Omega)\cap H_0^1(\Omega)\big)\\ \partial_t^{s-1}\theta_i \in C^0\big([0,T], L^2(\Omega)\big)\\ \partial_t^{s-1}\nabla\theta_i \in L^2\big([0,T], L^2(\Omega)\big)\end{aligned}\right\} \qquad (i = 1,2).$$

The proof of Theorem 2.1 is divided into the following three steps:

Step 1°: Proof of existence, uniqueness and regularity of the solution to the initial-boundary value problem for the linearized system of equations associated with system (2.2) - (2.6).

Step 2°: Proof of an energy estimate for the linearized initial-boundary value problem (2.2) - (2.6).

Step 3°: Proof of existence and uniqueness of the solution of the nonlinear initial-boundary value problem (2.2) - (2.6) using the Banach fixed point theorem.

3. Existence, uniqueness and regularity of the solution to the linearized problem associated with problem (2.2) - (2.6)

In this section we consider the linearized problem associated with (2.2) - (2.6)

$$\partial_t\theta_1 - a^{11}_{\alpha\beta}(t,x)\frac{\partial^2\theta_1}{\partial x_\alpha \partial x_\beta} - a^{12}_{\alpha\beta}(t,x)\frac{\partial^2\theta_2}{\partial x_\alpha \partial x_\beta} = g_1(t,x) \tag{3.1}$$

$$\partial_t\theta_2 - a^{21}_{\alpha\beta}(t,x)\frac{\partial^2\theta_1}{\partial x_\alpha \partial x_\beta} - a^{22}_{\alpha\beta}(t,x)\frac{\partial^2\theta_2}{\partial x_\alpha \partial x_\beta} = g_2(t,x) \tag{3.2}$$

with initial conditions

$$\left.\begin{aligned} \theta_1(0,x) &= \theta_1^0(x) \\ \theta_2(0,x) &= \theta_2^0(x) \end{aligned}\right\} \tag{3.3}$$

and boundary conditions

$$\left.\begin{aligned} \theta_1(t,\cdot)|_{\partial\Omega} &= 0 \\ \theta_2(t,\cdot)|_{\partial\Omega} &= 0 \end{aligned}\right\}. \tag{3.4}$$

We consider the solvability of problem (3.1) - (3.4). At first, introducing the vector $V = (\theta_1, \theta_2)^*$, we can write problem (3.1) - (3.4) in the equivalent matrix form

$$\partial_t V - a_{\alpha\beta}(t,x)\frac{\partial^2 V}{\partial x_\alpha \partial x_\beta} = G(t,x) \tag{3.5}$$

$$V(0,x) = V^0(x),\ V(t,\cdot)|_{\partial\Omega} = 0 \tag{3.6}$$

where

$$a_{\alpha\beta}(t,x) = \begin{pmatrix} a^{11}_{\alpha\beta}(t,x) & a^{12}_{\alpha\beta}(t,x) \\ a^{21}_{\alpha\beta}(t,x) & a^{22}_{\alpha\beta}(t,x) \end{pmatrix} \qquad (\alpha,\beta = 1,2,3) \tag{3.7}$$

and $G(t,x) = (g_1(t,x), g_2(t,x))^*$.

Before proving an energy estimate to problem (3.1) - (3.4), we present two theorems.

Theorem 3.1. *Let the assumptions*

$$\left.\begin{aligned} \overline{D}^1 a^{ij}_{\alpha\beta} &\in C^0([0,T]\times\overline{\Omega}) \cap L^\infty([0,T], L^\infty(\Omega)) \\ \partial_t \nabla a^{ij}_{\alpha\beta} &\in L^\infty([0,T], L^\infty(\Omega)) \\ G &\in C^0([0,T], L^2(\Omega)) \\ \partial_t G &\in L^2([0,T], H^{-1}(\Omega)) \\ V^0 &\in H^1_0(\Omega) \\ V^1 &= a_{\alpha\beta}(0)\frac{\partial^2 V^0}{\partial x_\alpha \partial x_\beta} + G(0) \in L^2(\Omega) \end{aligned}\right\} \quad (i,j = 1,2) \tag{3.8}$$

and

$$\left.\begin{aligned} a^{ij}_{\alpha\beta}(t,x) &= a^{ji}_{\beta\alpha}(t,x) \quad \text{for } (t,x) \in [0,T]\times\overline{\Omega} \\ (a_{\alpha\beta}\xi_\alpha\xi_\beta\eta,\eta) &\geq \gamma|\xi|^2|\eta|^2 \quad \text{for } \xi\in\mathbb{R}^3, \eta\in\mathbb{R}^2 \end{aligned}\right\} \quad (i,j = 1,2)$$

be satisfied where $\gamma > 0$ is some constant. Then there exists a unique solution $V = (\theta_1, \theta_2)^*$ to problem (3.1) – (3.4) *with*

$$\left.\begin{aligned} \theta_1 &\in C^0([0,T], H^2(\Omega)) \cap H_0^1(\Omega) \\ \partial_t\theta_1 &\in C^0([0,T], L^2(\Omega)) \\ \partial_t\nabla\theta_1 &\in L^2([0,T], L^2(\Omega)) \\ \theta_2 &\in C^0([0,T], H^2(\Omega) \cap H_0^1(\Omega)) \\ \partial_t\theta_2 &\in C^0([0,T], L^2(\Omega)) \\ \partial_t\nabla\theta_2 &\in L^2([0,T], L^2(\Omega)) \end{aligned}\right\}. \tag{3.9}$$

Proof. It can be done by using semigroup theory and it follows directly from considerations in [3] ∎

Now we present a higher regularity theorem connected with the solution to problem (3.1) - (3.4). The existence result is a special case of a classical theorem on local existence for parabolic systems (cf. [9]).

Theorem 3.2 (Existence, Uniqueness and Regularity). *Let the following assumptions be satisfied:*

1° $a_{\alpha\beta}^{ij} \in C^0([0,T]\times\overline{\Omega}) \cap L^\infty([0,T], L^\infty(\Omega))$, $\mathcal{D}_x a_{\alpha\beta}^{ij} \in L^\infty([0,T], H^{s-2}(\Omega))$, $\partial_t^k a_{\alpha\beta}^{ij} \in L^\infty([0,T], H^{s-1-k}(\Omega))$ $(1 \le k \le s-2)$ *and* $\partial_t^{s-1} a_{\alpha\beta}^{ij} \in L^2([0,T],\ L^2(\Omega))$.

2° *For $\theta_1, \theta_2 \in H_0^1(\Omega)$ and all $t \in [0,T]$ the inequality* $\|\theta_1\|_1^2 + \|\theta_2\|_1^2 \le \gamma_2\{(a_{\alpha\beta}^{ij}\frac{\partial\theta_i}{\partial x_\alpha}, \frac{\partial\theta_j}{\partial x_\beta}) + \|\theta_1\|^2 + \|\theta_2\|^2\}$ *is satisfied for a constant $\gamma > 0$.*

3° *For $t \in [0,T]$, $-a_{\alpha\beta}^{ij}(t)\frac{\partial^2\theta_j}{\partial x_\alpha \partial x_\beta} \in H^k(\Omega)$ with $\theta_1, \theta_2 \in H_0^1(\Omega)$ implis that $\theta_1, \theta_2 \in H^{s+2}(\Omega)$ and $\|V\|_{k+2} \le \gamma_3(\|-a_{\alpha\beta}^{ij}(t)\frac{\partial^2 V_j}{\partial x_\alpha \partial x_\beta}\|_k + \|V\|)$ where $V = (\theta_1, \theta_2)$, $0 \le k \le s-2$ and $\gamma_3 > 0$ is some constant.*

4° $\partial_t^k g_i \in C^0([0,T], H^{s-2-k}(\Omega))$ $(0 \le k \le s-2)$ *and* $\partial_t^{s-1} g_i \in L^2([0,T], H^{-1}(\Omega))$ $(i = 1,2)$, *where $s \ge [\frac{3}{2}] + 4 = 5$ is an arbitrary but fixed fixed integer.*

Then there exists a unique solution $V = (\theta_1, \theta_2)^*$ to the initial-boundary value problem (3.1) – (3.4) *with the properties*

$$\left.\begin{aligned} \partial_t^k\theta_i &\in C^0([0,T], H^{s-2-k}(\Omega) \cap H_0^1(\Omega)) \quad (0 \le k \le s-2) \\ \partial_t^{s-1}\theta_i &\in C^0([0,T], L^2(\Omega)) \\ \partial_t^{s-1}\nabla\theta_i &\in L^2([0,T], L^2(\Omega)) \end{aligned}\right\} \quad (i = 1,2). \tag{3.10}$$

Proof. It is based on Theorem 2.1, the assumption of Theorem 2.2 and mathematical induction ∎

Remark 3.1. In order to obtain the solution of problem (3.1) - (3.4) with regularity (3.10) the initial data must satisfy the compatibility conditions

$$V^k = (\theta_1^k, \theta_2^k) \in (H^{s-k}(\Omega) \cap H_0^1(\Omega)) \times (H^{s-k}(\Omega) \cap H_0^1(\Omega))$$

where $k = 0, 1, ..., s-2$ and

$$V^{s-1} = (\theta_1^{s-1}, \theta_2^{s-1}) \in L^2(\Omega) \times L^2(\Omega). \tag{3.11}$$

We define V^k successively by

$$V^k = \sum_{j=0}^{k-1} \binom{k-1}{j} \partial_k^j a_{\alpha\beta}(0) \frac{\partial^2 V^{k-1-j}}{\partial x_\alpha \partial x_\beta} + \partial_t^{k-1} G(0) \qquad (k \geq 1).$$

4. An energy estimate for problem (3.1) - (3.4)

We start with the formulation of the following

Theorem 4.1 (Energy estimate). *Let the conditions of Theorem* 2.2 *be fulfilled. Then the solution* $V = (\theta_1, \theta_2)$ *to the initial-boundary value problem* (3.1) – (3.4) *established in Theorem* 3.2 *satisfies the inequality*

$$\begin{aligned}
&\sum_{k=0}^{s-2} |\partial_t^k \theta_1|_{s-k,T}^2 \\
&+ \sum_{k=0}^{s-2} |\partial_t^k \theta_2|_{s-k,T}^2 + |\partial_t^{s-1}\theta_1|_{0,T}^2 + |\partial_t^{s-1}\theta_2|_{0,T}^2 \\
&+ \int_0^t \left[\|\partial_t^{s-1}\nabla\theta_1(\tau)\|^2 + \|\partial_t^{s-1}\nabla\theta_2(\tau)\|^2\right] d\tau \leq K_3 M_0 e^{K_4 \eta(T)}
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
M_0 = (1+T)\Bigg\{ &\sum_{k=0}^{s-2} (\|\theta_1^k\|_{s-k}^2 + \|\theta_2^k\|_{s-k}^2) + \|\theta_1^{s-1}\|^2 \\
&+ \|\theta_2^{s-1}\|^2 + \|\overline{D}^{s-2} g_1\|_{0,T}^2 + \|\overline{D}^{s-2} g_2\|_{0,T}^2 \\
&+ \int_0^T \left[\|\partial_t^{s-1} g_1(\tau)\|_{H^{-1}}^2 + \|\partial_t^{s-1} g_2(\tau)\|_{H^{-1}}^2\right] d\tau \Bigg\}
\end{aligned} \tag{4.2}$$

and $K_3 = K_3(P_0, \gamma_2, \gamma_3)$, $K_4 = K_4(P, \gamma_2, \gamma_3)$ *are positive constants depending continuously on* P_0, P, γ_2, γ_3 *are constants defined in the assumption of Theorem* 3.1,

$$\begin{aligned}
P = &\sup_{0 \leq t \leq T} \sum_{i,j=1}^{3} \|a_{\alpha\beta}^{ij}(t)\|_{L^\infty} + \sum_{i,j=1}^{2} \|D_x a_{\alpha\beta}^{ij}\|_{s-2,T} \\
&+ \sum_{k=1}^{s-2} \sum_{i,j=1}^{2} |\partial_t^k a_{\alpha\beta}^{ij}|_{s-1-k} + \int_0^T \sum_{i,j=1}^{2} |\partial_t^{s-1} a_{\alpha\beta}^{ij}(\tau)|^2 d\tau \\
P_0 = &\sum_{i,j=1}^{2} \|a_{\alpha\beta}^{ij}(0)\|_{L^\infty} + \sum_{i,j=1}^{2} \|D_x a_{\alpha\beta}^{ij}(0)\|_{s-3}
\end{aligned}$$

and

$$\eta(T) = T(1+T). \tag{4.3}$$

Proof. It can be found in [7] ∎

5. Proof of Theorem 2.1

The proof of Theorem 2.1 is based on the Banach fixed point theorem. At first, we define $Z(N,T)$ as the set of functions (θ_1, θ_2) which satisfy

$$\left.\begin{aligned} \partial_t^k \theta_i &\in L^\infty([0,T], H^{s-k}(\Omega)) \quad (0 \le k \le s-2) \\ \partial_t^{s-1} \theta_i &\in L^\infty([0,T], L^2(\Omega)) \\ \partial_t^{s-1} \nabla\theta_i &\in L^2([0,T], L^2(\Omega)) \end{aligned}\right\} \qquad (i=1,2) \tag{5.1}$$

$(s \ge [\frac{3}{2}] + 4 = 5)$ with boundary and initial conditions of the form

$$\left.\begin{aligned} \theta_i|_{\partial\Omega} &= 0 \\ \partial_t^k \theta_i(0,x) &= \theta_i^k(x) \end{aligned}\right\} \qquad (i=1,2;\ 0 \le k \le s-2)$$

and the inequality

$$\begin{aligned} &\sum_{l=0}^{s-2} |\partial_t^k \theta_1|^2_{s-k,T} + |\partial_t^{s-2}\theta_1|^2_{0,T} \\ &+ \sum_{l=0}^{s-2} |\partial_t^k \theta_2|^2_{s-k,T} + |\partial_t^{s-2}\theta_2|^2_{0,T} \\ &+ \int_0^T \left[\|\partial_t^{s-1}\nabla\theta_1(\tau)\|^2 + \|\partial_t^{s-1}\nabla\theta_2(\tau)\|^2\right] d\tau \le N \end{aligned} \tag{5.2}$$

for N large enough. Now, we consider the system of equations

$$\partial_t \theta_1 - a^{11}_{\alpha\beta} \frac{\partial^2 \theta_1}{\partial x_\alpha \partial x_\beta} - a^{12}_{\alpha\beta} \frac{\partial^2 \theta_2}{\partial x_\alpha \partial x_\beta} = g_1 \tag{5.3}$$

$$\partial_t \theta_2 - a^{21}_{\alpha\beta} \frac{\partial^2 \theta_1}{\partial x_\alpha \partial x_\beta} - a^{22}_{\alpha\beta} \frac{\partial^2 \theta_2}{\partial x_\alpha \partial x_\beta} = g_2 \tag{5.4}$$

with initial and boundary conditions (2.5) and (2.6) where

$$\left.\begin{aligned} a^{11}_{\alpha\beta} &:= \overline{a}^{11}_{\alpha\beta}(\overline{\theta}_1, \overline{\theta}_2, \nabla\overline{\theta}_1, \nabla\overline{\theta}_2) \qquad & a^{12}_{\alpha\beta} &:= \overline{a}^{12}_{\alpha\beta}(\overline{\theta}_1, \overline{\theta}_2, \nabla\overline{\theta}_1, \nabla\overline{\theta}_2) \\ a^{21}_{\alpha\beta} &:= \overline{a}^{21}_{\alpha\beta}(\overline{\theta}_1, \overline{\theta}_2, \nabla\overline{\theta}_1, \nabla\overline{\theta}_2) \qquad & a^{22}_{\alpha\beta} &:= \overline{a}^{22}_{\alpha\beta}(\overline{\theta}_1, \overline{\theta}_2, \nabla\overline{\theta}_1, \nabla\overline{\theta}_2) \end{aligned}\right\}$$

and

$$\left.\begin{aligned} g_1 &:= \overline{g}_1(\overline{\theta}_1, \overline{\theta}_2, \nabla\overline{\theta}_1, \nabla\overline{\theta}_2, t, x) \\ g_2 &:= \overline{g}_2(\overline{\theta}_1, \overline{\theta}_2, \nabla\overline{\theta}_1, \nabla\overline{\theta}_2, t, x) \end{aligned}\right\}. \tag{5.5}$$

Applying Theorem 3.2 to problem (5.3) - (5.5), (2.3) - (2.4) we can see that there exists a mapping σ such that

$$\sigma:\ Z(N,T) \ni (\overline{\theta}_1, \overline{\theta}_2) \to \sigma(\overline{\theta}_1, \overline{\theta}_2) = (\theta_1, \theta_2).$$

Next we prove that σ maps $Z(N,T)$ into itself under the conditions that N is large and T small enough. For this we introduce the notation

$$\begin{aligned} E_0 = &\sum_{k=0}^{s-2} \|\theta_1^k\|_{s-k}^2 + \|\theta_1^{s-1}\|^2 + \sum_{k=0}^{s-2} \|\theta_2^k\|_{s-k}^2 + \|\theta_2^{s-1}\|^2 \\ &+ \sum_{k=0}^{s-2} |\partial_t^k(\theta_1,\theta_2)|_{s-2-k,T}^2 + \sum_{k=0}^{s-2} |\partial_t^k(Q_1,Q_2)|_{s-2-k,T}^2 \\ &+ \int_0^T \|\partial_t^{s-1}(Q_1,Q_2)\|^2 d\tau. \end{aligned} \tag{5.6}$$

After some calculations and taking into account inequality $N(t) = N(0) + \int_0^t \partial_\tau N(\tau)\, d\tau$ we get

$$\begin{aligned} &\sum_{k=0}^{s-2} |\partial_t^k \overline{g}_1|_{s-2-k,T} + \sum_{k=0}^{s-2} |\partial_t^k \overline{g}_2|_{s-2-k,T} + \int_0^T \big(\|\partial_t^{s-1}\overline{g}_1\|_{s-1}^2 + \|\partial_t^{s-1}\overline{g}_2\|_{s-1}^2\big) dt \\ &\quad \le C(E_0) + C(N)T(1+T). \end{aligned} \tag{5.7}$$

Taking into account that

$$K_3, K_4 \le C(E_0) + C(N)T(1+T) \tag{5.8}$$

and putting (5.6) and (5.7) into the energy estimate, we obtain

$$\begin{aligned} &\sum_{k=0}^{s-2} |\partial_t^k\theta_1|_{s-k,T} + \sum_{k=0}^{s-2} |\partial_t^k\theta_2|_{s-k,T} + |\partial_t^{s-1}\theta_1|_{0,T} + |\partial_t^{s-1}\theta_2|_{0,T} \\ &+ \int_0^T \big(\|\partial_\tau^{s-1}\nabla\theta_1\|^2 + \|\partial_\tau^{s-1}\nabla\theta_2\|\big) d\tau \\ &\qquad \le K(E_0,\gamma_2,\gamma_3)\big(1 + C(N)T(1+T)\big)e^{C(N)T(1+T^{\frac{1}{2}}+T^4+T^{\frac{3}{2}})}. \end{aligned} \tag{5.9}$$

Now we choose N such that $K(E_0,\gamma_2,\gamma_3) \le \frac{N}{2}$. Then we can notice that

$$\alpha(T) = \big(1 + C(N)T(1+T)^2\big)e^{C(N)T^{\frac{1}{2}}(1+T^{\frac{1}{2}}+T+T^{\frac{3}{2}})} < 2$$

and for T small enough ($\alpha(0) = 1$) we conclude that

$$\sigma(Z(N,T)) \subset Z(N,T). \tag{5.10}$$

Now we prove that

$$\sigma : Z(N,T) \to Z(N,T) \tag{5.11}$$

is even a contraction mapping. For this we define the matric space (complete) (W,ρ) where

$$W = \Big\{(\theta_1,\theta_2) : \theta_1,\theta_2 \in L^\infty([0,T],L^2(\Omega)),\ \nabla\theta_1,\nabla\theta_2 \in L^2([0,T],L^2(\Omega))\Big\} \tag{5.12}$$

and

$$\rho\big((\overline{\theta}_1,\overline{\theta}_2),(\theta_1,\theta_2)\big) = |\overline{\theta}_1-\theta_1|_{0,T}+|\overline{\theta}_2-\theta_2|_{0,T} + \int_0^T \|\nabla(\overline{\theta}_1-\theta_1)(\tau)\|^2 d\tau + \int_0^T \|\nabla(\overline{\theta}_2-\theta_2)(\tau)\|^2 d\tau. \tag{5.13}$$

The set $Z(N,T)$ is a closed subset in (W,ρ). Let $(\overline{\theta}_1,\overline{\theta}_2),(\overline{\theta}_1^*,\overline{\theta}_2^*)\in Z(N,T)$ and let

$$\left.\begin{aligned}\sigma(\overline{\theta}_1,\overline{\theta}_2) &= (\theta_1,\theta_2)\in Z(N,T)\\ \sigma(\overline{\theta}_1^*,\overline{\theta}_2^*) &= (\theta_1^*,\theta_2^*)\in Z(N,T)\end{aligned}\right\}. \tag{5.14}$$

Subtracting by side the corespoding system for θ_1,θ_2 and θ_1^*,θ_2^* we get

$$\begin{aligned}&\partial_t(\theta_i-\theta_i^*) - a_{\alpha\beta}^{ij}(\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2)\frac{\partial^2(\theta_j-\theta_j^*)}{\partial x_\alpha \partial x_\beta}\\ &= \Big(a_{\alpha\beta}^{ij}(\overline{\theta}_1^*,\overline{\theta}_2^*,\nabla\overline{\theta}_1^*,\nabla\overline{\theta}_2^*) - \overline{a}_{\alpha\beta}^{ij}(\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2)\Big)\cdot\frac{\partial^2\theta_j^*}{\partial x_\alpha\partial x_\beta}\\ &\quad+\overline{g}_i(\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2)(x,t) - g_i(\overline{\theta}_1^*,\overline{\theta}_2^*,\nabla\overline{\theta}_1^*,\nabla\overline{\theta}_2^*)(x,t)\end{aligned} \tag{5.15}$$

for $i=1,2$. Using the fact that

$$\sup_{0\le t\le T}\big\|\overline{D}^2(\overline{\theta}_1,\overline{\theta}_2,\overline{\theta}_1^*,\overline{\theta}_2^*,\theta_1,\theta_2,\theta_1^*\theta_2^*)\big\|\le CN \quad\text{and}\quad \left.\begin{aligned}(\theta_i-\theta_i^*)|_{\partial\Omega}=0\\ (\theta_i-\theta_i^*)(0,x)=0\end{aligned}\right\}$$

and taking into account the mean value theorem

$$\begin{aligned}C(\theta_1,\theta_2)-C(\theta_1^*,\theta_2^*) &= C\big(\theta_1^*+(\theta_1-\theta_1^*),\theta_2^*+(\theta_2-\theta_2^*)\big)-C(\theta_1^*,\theta_2^*)\\ &= \nabla_\xi C(\xi)\cdot(\theta-\theta^*),\end{aligned}$$

after multiplying equation (5.13) by $\theta_i-\theta_i^*$ and intergrating on $[0,t]\times\Omega$ we get

$$\begin{aligned}&\|\theta_1-\theta_1^*\|^2+\int_0^t\|\nabla(\theta_1-\theta_1^*)\|^2d\tau+\|\theta_2-\theta_2^*\|^2+\int_0^t\|\nabla(\theta_2-\theta_2^*)\|^2d\tau\\ &\le C(N)\big(1+\tfrac{1}{T^{1/2}}\big)\int_0^t\big(\|\theta_1-\theta_1^*\|^2+\|\theta_2-\theta_2^*\|^2\big)d\tau\\ &\quad+(T^{\frac12}(1+T)\big[|\overline{\theta}_1-\overline{\theta}_1^*|_{0,T}^2+|\overline{\theta}_2-\overline{\theta}_2^*|_{0,T}^2\big]\\ &\quad+\int_0^t\big(\|\nabla(\overline{\theta}_1-\overline{\theta}_1^*)\|^2+\|\nabla(\overline{\theta}_2-\overline{\theta}_2^*)\|^2\big)d\tau\\ &\quad+(1+\tfrac{1}{T^{1/2}})\int_0^t\!\!\int_0^s\big(\|\nabla(\theta_1-\theta_1^*)\|^2+\|\nabla(\theta_2-\theta_2^*)\|^2\big)d\tau dt.\end{aligned} \tag{5.16}$$

Applying to (5.14) the Growall inquality we get

$$\begin{aligned}&|\theta_1-\theta_1^*|_{0,T}^2+|\theta_2-\theta_2^*|_{0,T}^2+\int_0^T\big(\|\nabla(\theta_1-\theta_1^*)\|^2+\|\nabla(\theta_2-\theta_2^*)\|^2\big)d\tau\\ &\le\varepsilon\left[|\overline{\theta}_1-\overline{\theta}_1^*|_{0,T}^2+|\overline{\theta}_2-\overline{\theta}_2^*|_{0,T}^2+\int_0^T\big(\|\nabla(\overline{\theta}_1-\overline{\theta}_1^*)\|^2+\|\nabla(\overline{\theta}_2-\overline{\theta}_2^*)\|\big)d\tau\right]\end{aligned} \tag{5.17}$$

where $\varepsilon = C(N)T^{\frac12}(1+T)e^{C(N)(T+T^{\frac12})}$. So choosing T small enough we obtain $\varepsilon<1$. So it means that the mapping σ is a contraction. This ends the proof of Theorem 2.1.

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