N. Kehayopulu, J. S. Ponizovskii, M. Tsingelis

THE EMBEDDING OF AN ORDERED SEMIGROUP INTO AN ORDERED GROUP BY MEANS OF PSEUDOORDER

A commutative semigroup can be embedded in a group if and only if it is cancellative (cf. for example [1; p. 34]). We prove here the analogous result for ordered semigroups by means of pseodoorder. Pseudoorder plays an important role in the structure of orderd semigroups. We give the necessary definitions, and sketch first the proof in the usual sense.

Let $(S, \cdot, \leq)$ be an ordered semigroup. An equivalence relation σ on S is called congruence is $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$, $(ca, cb) \in \sigma$ for every $c \in S$ [2]. We call S cancellative if $ca \leq cb$ implies $a \leq b$ and $ac \leq bc$ implies $a \leq b$. A relation ρ on S is called pseudoorder if

1) $\leq \subseteq \rho$.
2) $(a, b) \in \rho$, $(b, c) \in \rho$ imply $(a, c) \in \rho$.
3) $(a, b) \in \rho \Rightarrow (ac, bc) \in \rho$, $(ca, cb) \in \rho \ \forall \ c \in S$ [3, 4].

Let $(S, \cdot, \leq_s)$, $(T, {}^*, \leq_T)$ be ordered semigroup. Let $f : S \to T$ be a mapping from S into T. f is called isotone if x, $y \in S$, $x \leq_s y$ implies $f(x) \leq_T f(y)$. f is called reverse isotone if x, $y \in S$, $f(x) \leq_T f(y)$ implies $x \leq_s y$. [Each reverse isotone mapping is (1-1)]. f is called a homomorphism if it is isotone and satisfies $f(xy) = f(x) * f(y)$ for all x, $y \in S$. f is called an isomorphism if it is onto, homomorphism and reverse isotone [3].

Two ordered semigroups are called isomorphic if there exists an isomorphism between them. S *is embedded* in T if, by definition, S is isomorphic to a subset of T, i.e., if there exists a mapping $f : S \to T$ which is homomorphism and reverse isotone [5].

Let $(S, \cdot, \leqq)$ be a commutative, cancellative ordered semigroup. The set $S \times S$ with the multiplication "$\circ$" on $S \times S$ defined by

$$\circ : (S \times S) \times (S \times S) \to S \times S \mid ((a, b), (c, d)) \to (ac, bd)$$

is a semigroup. Let σ be the congruence on $S \times S$ defined by:

$$\sigma := \{((a, b), (c, d)) \mid ad = bc\}.$$

Then the set $(S\times S)/\sigma$ with the multiplication

$$* : (S\times S)/\sigma\times(S\times S)/\sigma \to (S\times S)/\sigma \mid ((a,b))_\sigma,\, (c,d)_\sigma) \to$$
$$\to ((a,b))_\sigma * ((c,d))_\sigma := ((a,b)\circ(c,d))_\sigma = ((ac,bd))_\sigma$$

is a semigroup (cf. e.g. [6; p. 13]).

We define a relation "$\preceq$" on $(S\times S)/\sigma$ as follows:

$$\preceq := \big\{(((x,y))_\sigma,\, ((z,t))_\sigma) \mid \exists\,(a,b)\in((x,y))_\sigma,$$
$$(c,d)\in((z,t))_\sigma : ad\le bc\big\}.$$

The following are equivalent:

1) $\exists\,(a,b)\in((x,y))_\sigma,\,(c,d)\in((z,t))_\sigma : ad\le bc.$
2) $\forall\,(a,b)\in((x,y))_\sigma\ \forall\,(c,d)\in((z,t))_\sigma,\ ad\le bc.$
3) $xt\le yz.$

In fact: 1)$\Rightarrow$ 2). Let $(g,k)\in((x,y))_\sigma$, $(h,s)\in((z,t))_\sigma{:}gs\le kh$. Let $(a,b)\in((x,y))_\sigma$, $(c,d)\in((z,t))_\sigma$ ($\Rightarrow ad\le bc$?) We have:

$$((g,k),(x,y))\in\sigma,\ \ ((h,s),(z,t))\in\sigma,\ \ gs\le kh$$
$$((a,b),(x,y))\in\sigma,\quad ((c,d),(z,t))\in\sigma\ \ (\Rightarrow\ ad\le bc?)$$

We have:

$$((a,b),(g,k))\in\sigma,\quad ((c,d),(h,s))\in\sigma.$$

Then $ak=bg$, $cs=dh$, $gs\le kh$.
Then

$$adks = d(ak)s = d(bg)s = (db)(gs)\le(db)(kh) = (bk)(dh)$$
$$= (bk)(cs) = bc(ks).$$

Since S is cancellative, $ad\le bc$.
2)$\Rightarrow$3), 3)$\Rightarrow$1). Since $(x,y)\in((x,y))_\sigma$, $(z,t)\in((z,t))_\sigma$.
Hence we have

$$\preceq := \big\{(((x,y))_\sigma,\, ((z,t))_\sigma) \mid xt\le yz\big\}.$$

The relation "$\preceq$" is an order on $(S\times S)/\sigma$ and $((S\times S)/\sigma,\,*\,,\preceq)$ is an ordered semigroup.

For an element $a\in S$, the element $((a,a))_\sigma$ is the identity of $(S\times S)/\sigma$, and for each $((x,y))_\sigma\in(S\times S)/\sigma$, the element $((y,x))_\sigma$ is the inverse of $((x,y))_\sigma$. Thus $(S\times S)/\sigma$ is an ordered group. The mapping:

$$f : S\to(S\times S)/\sigma \mid a\to((a^2,a))_\sigma$$

is well defined, it is homomorphism and reverse isotone. f is inverse isotone: Let $a,\ b \in S,\ f(a) \preceq f(b)$. Since $((a^2,a))_\sigma \preceq ((b^2,b))_\sigma$, we have $a^2b \le ab^2$, $aab \le abb$, and $a \le b$. $\Box$

We prove the Theorem by means of pseodoorder.

Lemma (cf. [3; Lemma 1)]. *Let* $(S,\ .\ ,\le)$ *be an ordered semigroups,* σ *a pseudoorder on* S*. The relation* $\overline{\sigma}$ *on* S *defined by*

$$\overline{\sigma} := \big\{(x,y) \in S\times S \mid (x,y) \in \sigma \ \text{ and } \ (y,x) \in \sigma\big\}$$

is a congruence on S *and the set* $S/\overline{\sigma}$ *with the multiplication "$.$" and the order "$\preceq$" below is an ordered semigroup*

$$.\ : S/\overline{\sigma}\times S/\overline{\sigma} \to S/\overline{\sigma}\ |((x)_{\overline{\sigma}},\ (y)_{\overline{\sigma}}) \to (x)_{\overline{\sigma}}\,.\,(y)_{\overline{\sigma}} := (xy)_{\overline{\sigma}}$$
$$(x)_{\overline{\sigma}} \preceq (y)_{\overline{\sigma}} \Longleftrightarrow (x,y) \in \sigma. \ \Box$$

Theorem. *A commutative ordered semigroup is embedded in an ordered group if and only it is cancellative.*

Proof. $\Rightarrow$. Let $(,\ .\ ,\le)$ be a commutative ordered semigroup, $(G,\ *\ ,\preceq)$ an ordered group, $f : S \to G$ homomorphism and reverse isotone. Let a, $b,\ c \in S,\ ac \le bc$. Then $f(ac) \preceq f(bc)$, and $f(a)*f(c) \preceq f(b)*f(c)$. Since G is a group, $f(a) \preceq f(b)$. Since f is reverse isotone, $a \le b$. Similarly, a, $b,\ c \in S,\ ca \le cb$ imply $a \le b$.

$\Leftarrow$. Let $(s,\ .\ ,\le)$ be a commutative, cancellative ordered semigroup. The set $S\times S$ with the multiplication "$*$" and the order "$\preceq$" below is an ordered semigroup (The proof is easy).

$$*\ : (S\times S)\times(S\times S) \to S\times S \mid ((a,b),(c,d)) \to (a,b) * (c,d) := (ac,bd)$$
$$\preceq := \big\{((a,b),(c,d)) \mid a \le c,\ c \le b\big\}.$$

Let σ be the relation on $S\times S$ defined as follows:

$$\sigma := \big\{((a,b),(c,d)) \mid ad \le bc\big\}.$$

σ is a pseodoorder on $(S\times S,\ *\ ,\preceq)$. In fact: $\preceq\ \subseteq \sigma$: If $((a,b),(c,d)) \in\ \preceq$, then $a \le c,\ d \le b,\ ad \le cb = bc$. Then $((a,b),(c,d)) \in \sigma$. Let $((a,b),(c,d)) \in \sigma$, $((c,d)(e,f)) \in \sigma$. Since $ad \le bc$, $cf \le de$, we have $(ad)(cf) \le (bc)(de)$, $(af)(dc) \le (be)(dc)$. Since S is cancellative, $af \le be$. Then $((a,b)(e,f)) \in \sigma$. Let $((a,b),(c,d)) \in \sigma$, $(e,f) \in S{\times}S$. Since $ad \le bc$, we have $e(ad)f \le e(bc)f$, then

$$(ae)(df) \le (bf)(ce),\ \ ((ae,bf),(ce,df)) \in \sigma,\ \ ((ea,fb),(ec,fd)) \in \sigma$$

(S comm.)

Thus we have

$$((a,b)*(e,f),(c,d)*(e,f))\in\sigma,\quad ((e,f)*(a,b),(e,f)*(c,d))\in\sigma.$$

$(S\times S, *, \preceq)$ is an ordered semigroup and σ is a pseudoorder on $S\times S$.

By the Lemma, the relation $\overline{\sigma}$ jn $S\times S$ defined by:

$$\overline{\sigma}:=\{((a,b),(c,d))\mid ((a,b),(c,d))\in\sigma,\ ((c,d),(a,b))\in\sigma\}$$

is a congruence on $S\times S$ and the set $(S\times S)/\overline{\sigma}$ with the multiplication "$\otimes$" anf the order "$\prec\!\prec$" below is an ordered semigroup.

$$((a,b))_{\overline{\sigma}}\otimes((c,d))_{\overline{\sigma}}:=((a,b))*(c,d))_{\overline{\sigma}}=((ac,bd))_{\overline{\sigma}}$$

$$((a,b))_{\overline{\sigma}}\prec\!\prec((c,d))_{\overline{\sigma}}\Leftrightarrow((a,b),(c,d))\in\sigma.$$

1) $(S\times S)/\overline{\sigma}$ is a commutative ordered semigroup. In fact:

$$((a,b))_{\overline{\sigma}}\otimes((c,d))_{\overline{\sigma}}:=((ac,bd))_{\overline{\sigma}}=((ca,db))_{\overline{\sigma}}=((c,d))_{\overline{\sigma}}\otimes((a,b))_{\overline{\sigma}}.$$

We prove that

$$((a,a))_{\overline{\sigma}}=((b,b))_{\overline{\sigma}}\ \ \forall\ \ a,b\in S\ldots(*)$$

Let a, $b\in S$. Since $ab\le ba$, we have $((a,a),(b,b))\in\sigma$, $((b,b),(a,a))\overline{\sigma}$. Then $((a,a),(b,b))\in\overline{\sigma}$, and $((a,a))_{\overline{\sigma}}=((b,b))_{\overline{\sigma}}$. Let

$$((c,d))_{\overline{\sigma}}\in(S\times S)/\sigma\ \ (\Rightarrow\ \ ((a,a))_{\overline{\sigma}}\otimes((c,d))_{\overline{\sigma}}=((c,d))_{\overline{\sigma}}?).$$

We prove that $((ac,ad),(c,d))\in\overline{\sigma}$. Since $(ac)d\le(ad)c$, $c(ad)\le d(ac)$ (S comm.), we have $((ac,ad),(c,d))\in\sigma$, $((c,d),(ac,ad))\in\sigma$. Thus $((ac,ad),(c,d))\in\overline{\sigma}$. Let

$$((c,d))_{\overline{\sigma}}\in(S\times S)/\overline{\sigma}\ \ (\Rightarrow\ \ ((c,d))_{\overline{\sigma}}\otimes((d,c))_{\overline{\sigma}}=((a,a))_{\overline{\sigma}}?)$$

$$((c,d))_{\overline{\sigma}}\otimes((d,c))_{\overline{\sigma}}=((cd,dc))_{\overline{\sigma}}=((a,a))_{\overline{\sigma}}\ \ \text{(by } (*)).$$

2) S is embeddable in $(S\times S)/\overline{\sigma}$.

We consider the mapping:

$$f:S\to(S\times S)/\overline{\sigma}\mid a\to((a^2.a))_{\overline{\sigma}}.$$

f is well defined. It is clear.
f is an homomorphism. Indeed: let a, $b\in S$. Then

$$f(ab)=((abab,ab))_{\overline{\sigma}}=((a^2b^2,ab))_{\overline{\sigma}}$$

$$= ((a^2, a))_{\overline{\sigma}} \otimes ((b^2, b))_{\overline{\sigma}} = f(a) \otimes f(b).$$

Let a, $b \in S$, $a \leq b$. Since $a^2b = aab \leq abb = ab^2$, we have $((a^2, a), (b^2, b)) \in \sigma$. Then $((a^2, a))_{\overline{\sigma}} \prec\prec ((b^2, b))_{\overline{\sigma}}$.
f is reverse isotone. Indeed: let a, $b \in S$, $((a^2, a)_{\overline{\sigma}} \prec\prec ((b^2, b))_{\overline{\sigma}}$. Since $((a^2, a), (b^2, b)) \in \sigma$, we have $a^2b \leq ab^2$, i.e., $a(ab) \leq b(ab)$. Since S is cancellative, $a \leq b$.

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It is shown that an ordered cancellative commutative semigroup can be embedded into an ordered commutative group.

University of Athens,
Greece

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Ros. Gos. Hydromet. Inst.
St.-Petersburg, Russia

University of Patras,
Greece