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ON THE LOCAL BEHAVIOR OF CERTAIN HOMEOMORPHISMS. II

Dedicated to the 90th anniversary
of G. M. Goluzin

1. In our first paper [1] on this topic we showed, among other results, that a sense-preserving homeomorphism, absolutely continuous on lines (ACL), under certain integral conditions on the complex dilation is conformal at a point. This result includes the Teichmüller–Wittich–Belinski theorem. In the present paper we show that under weakened conditions the local behavior will be similar to that of a quasiconformal mapping.

Our basic assumptions, maintained throughout the paper except where otherwise stipulated are as follows:

$w(z)$ is an ACL homeomorphism of the unit disc U_0 onto itself with $w(0) = 0$;

the complex dilation

$$k(z) = \frac{w_{\overline{z}}}{w_z}$$

is defined almost everywhere in U_0;

with $z = re^{i\phi}$

$$h_1(z) = \frac{|1 + e^{-2i\phi}\,k|^2}{1 - |k|^2}, \quad h_2(z) = \frac{|1 - e^{-2i\phi}\,k|^2}{1 - |k|^2}$$

are defined almost everywhere in U_0;

$$h_1^*(\phi) = \int\limits_{r_1}^{r_2} h_1(te^{i\phi})\frac{dt}{t}, \quad h_2^*(r) = \int\limits_0^{2\pi} h_2(re^{i\phi})d\phi$$

exist for almost all $\phi \in [0, 2\pi]$, $0 < r_1 < r_2 < 1$, $0 < r < 1$, and $h_1^*(\phi) \in L^1(0, 2\pi)$, $h_2^*(r) \in L^1(r_1, r_2)$, $r_1 < r < r_2$.

2. Our first result is

Theorem 1. *If there exists a positive constant T such that*

$$\iint_{U_0} \left| \frac{1+|k|^2}{1-|k|^2} - \frac{1}{2}(T+T^{-1}) \right| \frac{dA}{|z|^2} < \infty \tag{1}$$

and

$$\iint_{U_0} \left| \frac{2\Re(e^{-2i\phi}k)}{1-|k|^2} - \frac{1}{2}(T-T^{-1}) \right| \frac{dA}{|z|^2} < \infty \tag{2}$$

then there exists a positive constant A such that

$$|w(z)| \sim A|z|^T, \quad z \to 0,$$

uniformly in $\arg z$.

We observe first that inequalities (1) and (2) imply that

$$\iint_{U_0} \left| \frac{|1-e^{-2i\phi}k|^2}{1-|k|^2} - T^{-1} \right| \frac{dA}{|z|^2} < \infty \tag{3}$$

and

$$\iint_{U_0} \left| \frac{|1+e^{-2i\phi}k|^2}{1-|k|^2} - T \right| \frac{dA}{|z|^2} < \infty \tag{4}$$

which could have been used alternatively in the statement of Theorem 1.

Let for $0 < r < 1$

$$m_2(r) = \max_{|z|=r} |w(z)|, \quad m_1(r) = \min_{|z|=r} |w(z)|.$$

The quantity $\log(m_2(r)/m_1(r))$ is called the logarithmic oscillation of $w(z)$ on $|z| = r$. Let $B(r_1, r_2)$, $0 < r_1 < r_2 < 1$, be the circular ring $r_1 < |z| < r_2$ of module $M(r_1, r_2) = \frac{1}{2\pi} \log \frac{r_2}{r_1}$ and $B^*(r_1, r_2)$ its image under $w(z)$ of module $M^*(r_1, r_2)$.

We define

$$I_T(r_1, r_2) = \iint_{B(r_1,r_2)} \left| h_1(z) - T \right| \frac{dA}{|z|^2},$$

$$J_T(r_1, r_2) = \iint_{B(r_1,r_2)} \left| h_2(z) - T^{-1} \right| \frac{dA}{|z|^2}.$$

We begin by proving

$$\left| M^*(r_1, r_2) - \frac{T}{2\pi} \log \frac{r_2}{r_1} \right| \leqslant \frac{1}{4\pi^2} \max\left(T^2 J_T(r_1, r_2), I_T(r_1, r_2)\right). \tag{5}$$

By standard length-area arguments

$$M^*(r_1, r_2) \leqslant \left[\int_0^{2\pi} \left[\int_{r_1}^{r_2} h_1(re^{i\phi}) \frac{dr}{r} \right]^{-1} d\phi \right]^{-1},$$

$$M^*(r_1, r_2) \geqslant \int_{r_1}^{r_2} \left[\int_0^{2\pi} h_2(re^{i\phi}) d\phi \right]^{-1} \frac{dr}{r}.$$

By Schwarz's inequality

$$M^*(r_1, r_2) \leqslant \frac{1}{4\pi^2} \int_0^{2\pi} h_1^*(\phi) d\phi$$

and

$$\left(M^*(r_1, r_2)\right)^{-1} \leqslant \left(\log(r_2/r_1)\right)^{-2} \int_{r_1}^{r_2} h_2^*(r)\, dr.$$

Thus if

$$M^*(r_1, r_2) \leqslant TM(r_1, r_2)$$

we have

$$0 \leqslant TM(r_1, r_2) - M^*(r_1, r_2) \leqslant$$
$$\leqslant \frac{1}{4\pi^2} \frac{TM^*(r_1, r_2)}{M(r_1, r_2)} J_T(r_1, r_2) \leqslant \frac{1}{4\pi^2} T^2 J_T(r_1, r_2),$$

while if

$$TM(r_1, r_2) \leqslant M^*(r_1, r_2)$$

we have

$$0 \leqslant M^*(r_1, r_2) - TM(r_1, r_2) \leqslant \frac{1}{4\pi^2} I_T(r_1, r_2).$$

Now we prove that

$$\lim_{r \to 0} \log \frac{m_2(r)}{m_1(r)} = 0. \tag{6}$$

We use the following lemma (Lemma 2.1 in [1]).

Lemma 1. *Let γ_j, $j = 0, 1, \dots, n$, be a set of Jordan curves separating 0 from ∞. Let*

$$\xi_1^{(j)} = \frac{1}{2\pi} \min_{z \in \gamma_j} \log |z|^{-1}, \quad \xi_2^{(j)} = \frac{1}{2\pi} \max_{z \in \gamma_j} \log |z|^{-1}$$

with $\xi_2^{(j-1)} < \xi_1^{(j)}$, $j = 1, \dots, n$. Let M_j be the module of the doubly-connected domain with boundary components γ_{j-1} and γ_j, $j = 1, \dots, n$. Then

$$\sum_{j=1}^{n} M_j \leqslant \xi_1^{(n)} - \xi_2^{(0)} + 2 - \sum_{j=1}^{n-1} f(\xi_2^{(j)} - \xi_1^{(j)}),$$

where f is a positive monotone increasing function independent of any geometric properties of the configuration.

If (6) did not hold there would exists a sequence of numbers $\{s_j\}$, $j = 1, 2, \dots$, $0 < s_j < 1$, decreasing to zero and a number $\lambda > 0$ with $m_1(s_n) > m_2(s_{n+1})$ and

$$\log \frac{m_2(s_n)}{m_1(s_n)} \geqslant \lambda$$

for all n. For arbitrary $\varepsilon > 0$ we can suppose s_1 sufficiently small so that

$$\frac{1}{4\pi^2} T^2 J_T(s_m, s_n) < \varepsilon, \quad \frac{1}{4\pi^2} I_T(s_m, s_n) < \varepsilon$$

for $m > n$. Using the monotonicity property of the module and (5) we have for any $N > 1$

$$\frac{1}{2\pi} \log \frac{m_1(s_1)}{m_2(s_{N+1})} \leqslant M^*(s_{N+1}, s_1) \leqslant \frac{1}{2\pi} T \log \frac{s_1}{s_{N+1}} + \varepsilon$$

and

$$\frac{1}{2\pi} T \log \frac{s_1}{s_{N+1}} - N\varepsilon \leqslant \sum_{j=1}^{N} M^*(s_{j+1}, s_j).$$

Applying Lemma 1 we have

$$\sum_{j=1}^{N} M^*(s_{j+1}, s_j) \leqslant \frac{1}{2\pi} \log \frac{m_1(s_1)}{m_2(s_{N+1})} + 2 - \sum_{j=2}^{N} f\left(\log \frac{m_2(s_j)}{m_1(s_j)} \right).$$

Combining these inequalities we get

$$\frac{1}{2\pi} T \log \frac{s_1}{s_{N+1}} - N\varepsilon \leqslant \frac{1}{2\pi} T \log \frac{s_1}{s_{N+1}} + \varepsilon + 2 - (N-1) f(\lambda)$$

or

$$(N-1)f(\lambda) \leqslant 2 + (N+1)\varepsilon.$$

Dividing by N and letting N tend to infinity we obtain the contradiction $f(\lambda) \leqslant \varepsilon$ proving (6).

Given $\varepsilon > 0$ by the monotonicity property of the module and (5) there exists $\delta > 0$ such that for $0 < r_1 < r_2 < \delta$ we have

$$\left| M^*(r_1, r_2) - \frac{1}{2\pi} \log \frac{m_2(r_2)}{m_2(r_1)} \right| < \varepsilon$$

and

$$\left| M^*(r_1, r_2) - \frac{1}{2\pi} T \log \frac{r_2}{r_1} \right| < \varepsilon,$$

thus

$$\left| \frac{1}{2\pi} \log \frac{m_2(r_2)}{r_2^T} - \frac{1}{2\pi} \log \frac{m_2(r_1)}{r_1^T} \right| < 2\varepsilon.$$

By the Cauchy criterion $\lim\limits_{r\to 0} \log \frac{m_2(r)}{r^T}$ exists and

$$\lim_{r\to 0} \frac{m_2(r)}{r^T} \neq 0$$

proving Theorem 1.

3. By strengthening the conditions on the complex dilation we can also draw conclusion on the behavior of the argument of $w(z)$. In this context we utilize the concept of a homeomorphism being asymptotically a rotation on circles, introduced in [1].

Let $w(z)$ be a sense-preserving homeomorphism of $U_0 - \{0\}$ onto itself with the origins corresponding as boundary components. If the logarithmic oscillation of $w(z)$ on $|z| = r$ tends to zero as r tends to zero we say $w(z)$ is asymptotically a rotation on circles if for any choice of the arguments θ_1 and θ_2

$$\lim_{r\to 0} \bigl(\arg w(re^{i\theta_2}) - \arg w(re^{i\theta_1}) - (\theta_2 - \theta_1)\bigr) = 0$$

uniformly in θ_1 and θ_2.

Our basic result is

Lemma 2. *If $w(z)$ is a sense-preserving homeomorphism of $U_0 - \{0\}$ onto itself with the origins corresponding as boundary components and satisfying the conditions*

(i) $|w(z)| \sim A|z|^T$, T, $A > 0$, *as* $r \to 0$

and

(ii) *if* $Q = Q(r_1, r_2, \theta_1, \theta_2)$ *is the quadrangle with domain* $r_1 < r < r_2$, $\theta_1 < \theta < \theta_2$, *having a pair of opposite sides on* $|z| = r_1$, $|z| = r_2$ *with corresponding module* $M(Q)$ *and if* $\widehat{Q}$ *is the image of* Q *under* $w(z)$ *with corresponding module* $M(\widehat{Q})$ *while for given* $a > 1$ *and* $\varepsilon > 0$ *there exists* $R = R(a, \varepsilon) < 1$ *such that for* $0 < r_1 < r_2 < R$ *and* $\log r_2/r_1 > a$

$$M(\widehat{Q}) \geqslant (\theta_2 - \theta_1)/\left(T \log \frac{r_2}{r_1}\right) - \varepsilon$$

then $w(z)$ *is asymptotically a rotation on circles.*

The proof is essentially identical with that of [1, Lemma 5.1] with the additional observation that there $r_1/r_2 = 4$.

Lemma 3. *If* $w(z)$ *is sense-preserving and satisfies* (1) *and* (2) *then* $w(z)$ *is asymptotically a rotation on circles.*

Condition (i) of Lemma 2 follows from Theorem 1.

To show that condition (ii) is satisfied we note that as in the proof of Theorem 1

$$M(\widehat{Q}) \geqslant \int_{\theta_1}^{\theta_2} \left[\int_{r_1}^{r_2} h_1(re^{i\phi}) \frac{dr}{r} \right] d\phi.$$

Applying Schwarz's inequality we have

$$\left(M(\widehat{Q})\right)^{-1} \leqslant (\theta_2 - \theta_1)^{-2} \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} h_1(re^{i\phi}) \frac{dr}{r}\, d\phi.$$

If $M(\widehat{Q}) \geqslant T^{-1}\, M(Q)$ condition (ii) is immediate.

If not we have

$$T^{-1} M(Q) - M(\widehat{Q}) \leqslant \frac{M(\widehat{Q})}{T M(Q)} \left(\log \frac{r_2}{r_1}\right)^{-2} \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \left|h_1(re^{i\phi}) - T\right| \frac{dr}{r}\, d\phi$$

$$\leqslant \left(T \log \frac{r_2}{r_1}\right)^{-2} I_T(r_1, r_2).$$

Given $a > 1$, $\varepsilon > 0$ by (4) which follows from (1) and (2) there exists $R > 0$ such that for $0 < r_1 < r_2 < R$, $I_T(r_1, r_2) < a^2T^2\varepsilon$. Then for $\log \frac{r_2}{r_1} > a$, $0 < r_1 < r_2 < R$

$$M(\widehat{Q}) \geqslant T^{-1}M(Q) - \varepsilon$$

which is condition (ii).

4. The following theorem deals with the behavior of the argument of $w(z)$ at $z = 0$.

Theorem 2. *If $w(z)$ is sense-preserving and there exists a positive constant T such that*

$$\iint\limits_{U_0} \left| \frac{1+|k|^2}{1-|k|^2} - \frac{1}{2}(T + T^{-1}) \right| \frac{dA}{|z|^2} < \infty \tag{1}$$

and

$$\iint\limits_{U_0} \left| \frac{2e^{-2i\phi}k}{1-|k|^2} - \frac{1}{2}(T - T^{-1}) \right| \frac{dA}{|z|^2} < \infty \tag{7}$$

then there exists a positive constant $C \neq 0$ such that

$$|w(z)| \sim C|z|^{T-1}z, \quad z \to 0,$$

uniformly in $\arg z$.

Let $\widehat{U}_0$ be the disc U_0 slit along the radius from 0 to 1. We map $\widehat{U}_0$ and its image $w(\widehat{U}_0)$ onto semistrips S_1 and S_2 using a branch of $-\log$. Let $f(\sigma) = -\log w(e^{-\sigma})$ be the induced map from S_1 to S_2. We can extend $f(\sigma)$ throughout the half-plane $\Re\sigma > 0$ as a homeomorphism by setting

$$f(\sigma + 2\pi ki) = f(\sigma) + 2\pi ki$$

for all integers k. $f(\sigma) = u(\sigma) + iv(\sigma)$ is ACL and

$$\lim_{s\to\infty} u(s,t) = +\infty, \quad \sigma = s + it.$$

Since (1) and (7) imply (1) and (2) we have by Theorem 1

$$\lim_{s\to\infty} \big(u(s,t) - Ts\big) = B \tag{8}$$

uniformly in t, B real, and by Lemma 3

$$\lim_{s\to\infty} \big(v(s,t_2) - v(s,t_1) - (t_2 - t_1)\big) = 0 \tag{9}$$

uniformly in t_1 and t_2. Moreover from (1) and (7) we have

$$\iint\limits_{U_0} \left| \frac{|1 - ie^{-2i\phi}k|^2}{1-|k|^2} - \frac{1}{2}(T + T^{-1}) \right| \frac{dA}{|z|^2} < \infty. \tag{10}$$

Transforming (10) to the σ-plane we obtain

$$\int_0^{\infty}\int_0^{2\pi} \left| \frac{(u_s + u_t)^2 + (v_s + v_t)^2}{u_s v_t - u_t v_s} - (T + T^{-1}) \right| ds\, dt < \infty.$$

Suppose now that there is a value t_0 such that $\lim\limits_{s\to\infty} (v(s,t) - t)$ does not exist. Then there exist sequences $\{s_n^{(1)}\}^\infty$ and $\{s_n^{(2)}\}^\infty$, $s_n^{(1)} < s_n^{(2)}$, such that for $t = t_0$

$$\delta_1(t) = \lim_{n\to\infty} \left(v(s_n^{(2)}, t) - v(s_n^{(1)}, t) \right) \neq 0.$$

By (9) this is a value δ_1 independent of t. We may assume $\delta_1 > 0$ without loss of generality. Let $\delta = \min(1, \delta_1)$. Let $\varepsilon > 0$ be given and $N = N(\varepsilon)$ so large that for $n > N$

$$\left| u(s_n^{(j)}, t) - T\, s_n^{(j)} \right| < \varepsilon, \quad j = 1, 2, \tag{11}$$

for all t,

$$\left| v(s_n^{(j)}, t_2) - v\,(s_n^{(j)}, t_1) - (t_2 - t_1) \right| < \varepsilon, \quad j = 1, 2, \tag{12}$$

for all t_1, t_2,

$$v(s_n^{(2)}, t) - v\,(s_n^{(1)}, t) - \delta > -\varepsilon \tag{13}$$

for all t and

$$\int_{s_n^{(1)}}^{s_n^{(2)}}\int_0^{2\pi} \left| \frac{(u_s + u_t)^2 + (v_s + v_t)^2}{u_s v_t - u_t v_s} - (T + T^{-1}) \right| dA_\sigma < \varepsilon. \tag{14}$$

For fixed $n > N$ we write $s_1 = s_n^{(1)}$, $s_2 = s_n^{(2)}$. Consider the map

$$\sigma = g(p, q) = (p, p + q).$$

It maps the rectangle $R = \{p, q\}$, $s_1 < p < s_2$, $0 < q < 2\pi$, onto a parallelogram P in the σ-plane. Denote by P^* the image of P under f

and let $h = f \circ g$. The latter maps R onto P^*. Let Γ^* denote the family of curves $\{\gamma_q^*\}$, $0 < q < 2\pi$, where γ_q^* is the image under h of the segment I_q,

$$I_q = \{(p, q), s_1 \leqslant p \leqslant s_2\}.$$

By [14, Theorem 14] the module $M(\Gamma^*)$ of Γ^* satisfies

$$M(\Gamma^*) = \int\limits_0^{2\pi} \int\limits_{s_1}^{s_2} \left[\frac{|dh/dp|^2}{J_h} dp \right]^{-1} dq,$$

where J_h is the Jacobian determinant of the mapping h. By Schwarz's inequality

$$M(\Gamma^*) \geqslant (2\pi)^2 \left[\iint\limits_R \frac{|dh/dp|^2}{J_h} dp\, dq \right]^{-1}.$$

Since

$$\iint\limits_R \frac{|dh/dp|^2}{J_h} dp\, dq = \iint\limits_P \frac{(u_s + u_t)^2 + (v_s + v_t)^2}{u_s v_t - u_t v_s}\, dA_\sigma$$

by (14)

$$M(\Gamma^*) \geqslant (2\pi)^2 \left[2\pi(s_2 - s_1)(T + T^{-1}) + \varepsilon \right]^{-1}. \tag{15}$$

We estimate $M(\Gamma^*)$ from above by the definition of module. By (11), (12), (13)

$$\text{area} \quad P^* \leqslant 2\pi \left[T(s_2 - s_1) + 2\varepsilon \right],$$

$$(\text{length } \gamma_q^*)^2 \geqslant \big(T(s_2 - s_1) - 2\varepsilon \big)^2 + ((s_2 - s_1) - 2\varepsilon + \delta)^2, \; 0 < q < 2\pi,$$

thus

$$M(\Gamma^*) \leqslant \frac{2\pi[T(s_2 - s_1) + 2\varepsilon]}{(T(s_2 - s_1) - 2\varepsilon)^2 + ((s_2 - s_1) - 2\varepsilon + \delta)^2}.$$

Comparing this with (15) we obtain

$$2(\delta - 2\varepsilon) \leqslant 5\varepsilon T + 2\varepsilon(T + T^{-1}),$$

a contradiction for ε sufficiently small. Thus $\lim\limits_{s \to \infty} (v(s, t) - t)$ exists for every t and by Lemma 3 this limit is uniform. This completes the proof of Theorem 2 since

$$w(z) \sim C|z|^{T^{-1}} z, \quad z \to 0, \quad C \neq 0,$$

uniformly in $\arg z$.

References

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Received September 9, 1996

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