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THE INTEGRAL MEANS SPECTRUM OF UNIVALENT FUNCTIONS

Dedicated to the memory of Professor G. M. Goluzin

1. The integral means

The works of G. M. Goluzin and in particular his book [8, 9] has had a major impact on the development of geometric function theory and has greatly influenced the present author.

This is a survey article about the integral means of the derivative of univalent functions. Recently great progress has been made by L. Carleson, P. W. Jones and in particular N. G. Makarov on this question which is of central interest for the boundary behaviour of conformal maps.

Let the function f be analytic and univalent in the unit disk $\mathbb{D}$. For $p \in \mathbb{R}$, the integral means of the derivative is

$$I_p(r) = I_p(r, f') = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{\mathrm{it}})|^p \, dt \quad (0 \leqslant r < 1); \tag{1.1}$$

it is convenient not to take the power $1/p$ of this expression. If $p > 0$ then $I_p(r)$ measures how large $|f'(z)|$ is on average and thus measures the mean expansion of the conformal map f. If $p < 0$ then $I_p(r)$ measures how small $|f'(z)|$ is on average and thus the mean compression of f.

We now define

$$\beta_f(p) = \limsup_{r \to 1} \left[\log I_p(r, f') / \log \frac{1}{1-r} \right] \quad (p \in \mathbb{R}). \tag{1.2}$$

Thus $\beta_f(p)$ is the smallest number such that

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{\mathrm{it}})|^p \, dt = O((1-r)^{-\beta_f(p)-\varepsilon}) \quad (r \to 1) \tag{1.3}$$

for every $\varepsilon > 0$. For example, if $f(\mathbb{D})$ is heart-shaped domain with an inward pointing (zero angle) cusp and an outward pointing cusp then

$$\beta_f(p) = |p| - 1 \text{ for } |p| \geqslant 1, = 0 \text{ for } |p| \leqslant 1. \tag{1.4}$$

It easily follows from the Hölder inequality that $\beta_f(p)$ is always convex in $p \in \mathbb{R}$.

We now assume that f is bounded so that $G = f(\mathbb{D})$ is a bounded domain. The Koebe distortion theorem shows that

$$\frac{1}{|f'(z)|} = O\left(\frac{1}{1-r}\right), |f'(z)| \leqslant \frac{4\,\mathrm{dist}(f(z), \partial G)}{1-r^2} = o\left(\frac{1}{1-r}\right)$$

as $|z| = r \to 1$. Hence, by (1.3),

$$\beta_f(p \pm q) \leqslant \beta_f(p) + q \text{ for } p \gtrless 0,\ \ q \geqslant 0. \tag{1.5}$$

Furthermore, if $r \geqslant 1/2$, then

$$(1-r)\int_0^{2\pi} |f'(re^{\mathrm{it}})|^2 dt \leqslant 2\int_r^1 \left(\int_0^{2\pi} |f'(\rho e^{it})|^2 dt\right) \rho\, d\rho$$
$$\leqslant 2\pi \text{ area } G < \infty$$

so that $I_2(r) = o((1-r)^{-1})$. Together with (1.5) this shows that

$$\beta_f(p) \leqslant p - 1 \text{ for } p \geqslant 2. \tag{1.6}$$

2. The universal integral means spectrum

The *universal integral means spectrum* of bounded univalent functions is defined [17] by

$$B(p) = \sup\{\beta_f(p) : f \text{ bounded univalent in } \mathbb{D}\}$$

for $-\infty < p < \infty$. This is a convex function.

Its determination would solve many interesting problem in geometric function theory. We shall later see that $B(p)$ also determines e.g. what happens for unbounded univalent functions. We now discuss what is known for the different ranges of p.

Theorem 1. *We have*

$$B(p) = p - 1 \textit{ for } p \geqslant 2, \tag{2.1}$$

$$B(p) = p - 1 + O((p-2)^2) \textit{ as } p \to 2. \tag{2.2}$$

The first result follows from example (1.4) and from (1.6). The second result, due to Jones and Makarov [11], is much deeper. In particular it implies that the derivative $B'(2) = 1$ exists.

The value $p = 1$ is closely connected with the coefficient problem. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be bounded and univalent in $\mathbb{D}$. Then

$$n|a_n| \leqslant \frac{1}{2\pi r^{n-1}} \int_0^{2\pi} |f'(re^{it})| dt \quad (0 < r < 1). \tag{2.3}$$

Choosing $r = 1 - 1/n$ we deduce that, for every $\varepsilon > 0$,

$$n\, a_n = O(n^{\beta_f(1)+\varepsilon}) = O(n^{B(1)+\varepsilon}) \quad (n \to \infty). \tag{2.4}$$

A remarkable result of Carleson and Jones [4] shows that one does not lose much by taking the absolute value inside the integral in (2.3):

Theorem 2. *If*

$$\gamma = \sup \left\{ \limsup_{n\to\infty} \frac{\log(n|a_n|)}{\log n} \;:\; f \text{ bounded univalent} \right\} \tag{2.5}$$

then $\gamma = B(1)$.

Thus the exponent in the second estimate in (2.4) is best possible. By convexity it is trivial that $B(1) \leqslant 0.5$. Much work has been done on determining $\gamma = B(1)$; see e.g. [6, 18] and [19, Sect. 5.2; 20, Sect. 8.4].

Theorem 3. $0.17 < B(1) < 0.4886$.

Now we turn to small p; see [6, 15, 21] and [20, Sect. 8.4].

Theorem 4. *If* $|p|$ *is small then*

$$0.117p^2 \leqslant B(p) \leqslant 3p^2 + 7|p|^3, \tag{2.6}$$

furthermore $B(-1) < 0.601$.

The proof of the upper estimates uses differential inequalities starting from $|(1 - |z|^2)f''(z)/f'(z) - 2\bar{z}| \leqslant 4$ for (2.6) and from the Schwarzian derivative for $p = -1$.

Finally we come to $p \leqslant -2$. Motivated by a problem in approximation theory, J. E. Brennan [3] asked whether $B(-2) = 1$. Since

$$|p| - 1 \leqslant B(p) \leqslant B(-2) + |p| - 2 \text{ for } p \leqslant -2 \tag{2.7}$$

by example (1.4) and by (1.5), this would imply

$$B(p) = |p| - 1 \text{ for } p \leqslant -2. \tag{2.8}$$

This *Brennan conjecture* was proved by Carleson and Makarov [5] for large $|p|$.

Theorem 5. *There exists $q_0 \geqslant 2$ such that*

$$B(p) = |p| - 1 \text{ for } p \leqslant -q_0. \tag{2.9}$$

Makarov [17, Th. 5.2] has given a complete description of the functions that may occur as $\beta_f(p)$.

Theorem 6. *Let $\beta : \mathbb{R} \to [0, +\infty)$ be convex. Then there exists a bounded univalent function f such that $\beta = \beta_f$, if and only if*

(i) *$\beta(p) \leqslant B(p)$ for $-\infty < p < +\infty$,*

(ii) *$|\beta'(p \pm 0)| \leqslant (\beta(p) + 1)/|p|$ for $p \neq 0$.*

The last condition states that the tangents to the graph $y = \beta(p)$ intersect the y-axis in the interval $[-1, 0]$.

3. Numerical evidence and conjectures

In [4] the domain of attraction of ∞ (compare Theorem 7 below) for the iterates of quadratic polynomials was used to obtain examples with large $\beta_f(1)$. The results suggested that $B(1) > 0.23\ldots$ which led Carleson and Jones to conjecture that

$$\gamma = B(1) = \frac{1}{4}. \tag{3.1}$$

This would vastly improve Theorem 3. Note that the bound $B(1) > 0.17$ is analytic and not experimental.

The experiments were repeated in [13, 14] using a different algorithm with a more secure theoretical basis. Note however that one always makes the plausible assumption that if the computation finally gives stable numerical values then the mathematical limit has been reached. Therefore computations of this sort do not prove anything and thus remain experimental.

p	-1.4	-1.2	-1.0	-0.8	-0.6	-0.4	-0.2
$B(p) >$	0.467	0.340	0.231	0.149	0.085	0.037	0.0095
$p^2/4 =$	0.490	0.360	0.250	0.160	0.090	0.040	0.0100

p	0.2	0.4	0.6	0.8	1.0	1.2	1.4
$B(p) >$	0.0094	0.037	0.086	0.154	0.242	0.346	0.467
$p^2/4 =$	0.0100	0.040	0.090	0.160	0.250	0.360	0.490

The lower estimates that were obtained are listed in the table above. They led Ph. Kraetzer [14] to propose the

BCJK conjecture. $B(p) = p^2/4$ *for* $|p| \leqslant 2$.

The name (due to the present author) stands for Breannan–Carlecon–Jones–Kraetzer. By (1.4) and (1.5) it would imply

$$B(p) = |p| - 1 \text{ for } |p| \geqslant 2. \tag{3.2}$$

The BCJK conjecture reduces to the Carleson–Jones conjecture (3.1) for $p = 1$ and reduces to the Brennan conjecture (2.8) for $p \leqslant -2$.

It follows from Theorem 1 that (3.2) is correct for $p \geqslant 2$. Moreover the BCJK conjecture states that $B(p) = p - 1 + (p-2)^2/4$ in accordance with (2.2).

For small p the conjectured $B(p) = p^2/4$ gives the correct quadratic rate of Theorem 4, and $B(p) = |p| - 1$ holds for $p \leqslant -q_0$ by Theorem 5.

Thus the BCJK conjecture is compatible with all that has been proved or conjectured about the universal integral means spectrum. Furthermore $p^2/4$ is the only polynomial of degree at most 3 that satisfies the known values $B(0) = B'(0) = 0$, $B(2) = B'(2) = 1$.

4. Other classes of functions

As usual, let S denote the class of all functions $f(z) = z + a_2 z^2 + \dots$ analytic and univalent in $\mathbb{D}$. Also let Σ denote the class of all univalent functions

$$g(z) = z + b_0 + b_1 z^{-1} + \dots \quad (|z| > 1).$$

We define $\beta_g(p)$ as in (1.2) with r replaced by $1/r$. For the integral means spectrum the essential difference between S and Σ is that the boundary is bounded for $g \in \Sigma$. Thus the functions in Σ behave essentially like bounded functions [4, 13].

Theorem 7.

$$\sup\{\beta_g(p) : g \in \Sigma\} = B(p) \text{ for } p \in \mathbb{R},$$

$$\sup\left\{\limsup_{n\to\infty}[\log(n|b_n|)/\log n] : g \in \Sigma\right\} = \gamma = B(1).$$

The situation is different for the class S. It is however remarkable that the universal integral means spectrum for bounded univalent functions completely describes the behaviour also in the unbounded case [17, Th. 5.4]:

Theorem 8. $\sup\{\beta_f(p) : f \in S\} = \max(B(p), 3p-1)$.

If the BCJK conjecture turns out to be correct the dividing point would be $p = 6 - \sqrt{32}$.

Similar considerations apply to functions with restricted growth [17]. We only describe the coefficient growth which corresponds to $p = 1$ [1; 17, Th. 5.5].

Theorem 9. *If* $0 \leqslant \alpha \leqslant 2$ *then, with* $\gamma = B(1)$,

$$\sup\left\{\limsup_{n\to\infty}\frac{\log n|a_n|}{\log n} : f \in S,\ f(z) = O\left(\frac{1}{(1-|z|)^\alpha}\right)\right\} = \max(\alpha-1, \gamma-1).$$

If $\alpha \geqslant 1/2$ then "univalent" can be replaced by the much weaker assumption "(areally) mean univalent" but not if $\alpha < 1/2$, see [10, Sect. 3.3].

In order put our present knownledge into perspective, we consider k-symmetric functions

$$f(z) = \sum_{\nu=0}^{\infty} a_{k\nu+1} z^{k\nu+1}.$$

The function $(1-z^k)^{-2/k}$ has coefficients of order $n^{-1+2/k}$.

$1 \leqslant k \leqslant 3:$ $a_n = O(n^{2/k-1})$ for f mean univalent [10,Sect. 3.7];

$k = 4:$ $a_n = O(n^{2/k-1})$ for f univalent [1];

$5 \leqslant k \leqslant 8:$ $a_n = O(n^{2/k-1+\varepsilon})$ for f univalent, $\varepsilon > 0$, by Theorem 9 assuming $\gamma \leqslant 0.25$ [4];

$k \geqslant 9:$ $a_n = O(n^{\gamma-1+\varepsilon})$ for f univalent, $\varepsilon > 0$, by Theorem 9 assuming $\gamma > 0.22$.

Finally we consider univalent functions that satisfy a Hölder condition

$$f'(z) = O((1-|z|)^{\eta-1}) \quad (|z| \to 1). \tag{4.1}$$

The following result [17, 18] reduces the problem again to the general case.

Theorem 10. *For $0 < \eta \leqslant 1$, let T_η be the tangent of slope $1-\eta$ to the graph of B and let $B(p_\eta) = T_\eta(p_\eta)$. Then*

$$\sup\{\beta_f(p) : f'(z) = O((1-|z|)^{\eta-1})\} = \begin{cases} T_\eta(p) & \text{for } p \geqslant p_\eta, \\ B(p) & \text{for } p \leqslant p_\eta. \end{cases}$$

The BJCK conjecture would give that this maximum is

$$= \begin{cases} p^2/4 & \text{for } -2 \leqslant p \leqslant 2-2\eta, \\ (1-\eta)(p-1+\eta) & \text{for } 2-2\eta \leqslant p < \infty. \end{cases}$$

5. The integral means spectrum and Hausdorff dimension

The *Hausdorff dimension* $\dim E$ of the bounded set E is the infimum of all numbers α such that, for every $\varepsilon > 0$, there exists a covering of E by disks D_k with

$$\sum_k (\operatorname{diam} D_k)^\alpha < \varepsilon.$$

The *Minkowski dimension* $m\dim E$ is defined by

$$m\dim E = \limsup_{n\to\infty} [\log N_\delta(E)/\log(1/\delta)]$$

where $N_\delta(E)$ is the minimum number of disks of radius δ that cover E. It satisfies $\dim E \leqslant m\dim E$. See e.g. [7].

Let f be a conformal map of $\mathbb{D}$ into $\mathbb{C}$. Then [2] the angular limit exists except perhaps on a set of zero capacity and therefore of zero Hausdorff dimension. If A is a Borel set on $\mathbb{T} = \partial\mathbb{D}$ then

$$\dim f(A) \geqslant \sup_{q>0} \frac{q \dim A}{\beta_f(-q) + q + 1 - \dim A}$$

as Makarov [16] has proved; see [20, Sect. 10.3].

Hence Theorem 5 [5] shows that

$$\dim f(A) \geqslant (q_0 \dim A)/(2q_0 - \dim A) \text{ for } A \subset \mathbb{T} \tag{5.2}$$

which improves the estimate $\dim f(A) > \frac{1}{2}\dim A$ for $\dim A > 0$ in [16]. If the BCJK conjecture is true then $B(-q) = q^2/4$ for $0 \leqslant q \leqslant 2$, and (5.1) would give

$$\dim f(A) \geqslant \frac{\dim A}{1 + \sqrt{1 - \dim A}} \text{ for } A \subset \mathbb{T} \tag{5.3}$$

which would imply Makarov's result [16]

$$\dim A = 1 \Longrightarrow \dim f(A) \geqslant 1.$$

The connection to the integral means spectrum is even closer for the Minkowski dimension [20, Sect. 10.5; 17, (3.3)].

Theorem 11. *Let f map $\mathbb{D}$ conformally onto the bounded domain G and let $p = m\dim\partial G$. Then*

$$\beta_f(p) \leqslant p - 1. \tag{5.4}$$

Equality holds if f satisfies a Hölder condition but not in general.

It is known [11] that the Hölder condition (4.1) implies $m\dim\partial G \leqslant 2 - c\eta$ for some absolute constant $c > 0$; see [12] for further results. Suppose now that the BCJK conjecture if true. By the remark following Theorem 10 this would give

$$p - 1 = \beta_f(p) = (1-\eta)(p - 1 + \eta)$$

which would imply the beautiful formula

$$m\dim\partial G = p \leqslant 2 - \eta \quad (0 \leqslant \eta \leqslant 1); \tag{5.5}$$

note that this is trivially true in the limit cases $\eta = 0$ and $\eta = 1$.

For $1 \leqslant M \leqslant 2$ we consider [18]

$$\gamma^{(M)} = \sup\left\{\limsup_{n\to\infty}\frac{\log(n|a_n|)}{\log n} : m\dim\partial f(\mathbb{D}) \leqslant M\right\}. \tag{5.6}$$

Thus $\gamma^{(2)} = \gamma = B(1)$ by Theorem 2.

Theorem 12. *Let the critical dimension be defined by*

$$M_c = 1 + B(1)/(1 - B'(1+0)).$$

Then $1.2048 < M_c < 1.9989$ and

$$\gamma^{(M)} = \gamma \text{ for } M_c \leqslant M \leqslant 2,$$
$$\gamma^{(M)} < \gamma \text{ for } 1 \leqslant M < M_c.$$

Thus the additional information that the boundary has Minkowski dimension $\leqslant M$ becomes irrelevant for the coefficient problem if $M \geqslant M_c$. This means that one does not have to consider arbitrarily bad domains in order to determine γ. This stregthens the case for the numerical experiments mentioned in Sec. 3. The BCJK conjecture would give $M_c = 1.5$.

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