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RELATIVELY UNSTABLE INVARIANT SETS OF NONLINEAR OPERATORS

ABSTRACT. The note contains accurate proofs of Lipschitz versions of a collection of results from the theory of invariant manifolds of nonlinear mappings which are well known for the case of smooth mappings.

Dedicated to the memory of A. P. Oskolkov

Let $S : X \to X$ be a continuous mapping in a Banach space X, and let 0 be its fixed point, i.e., $S(0) = 0$. Given a positive number λ, denote by $\mathcal{M}_\lambda(S)$ the set of all $u \in X$ for which there exists a sequence $u_k \in X$, $k = 0, 1, \dots$, satisfying

$$u_0 = u, \quad u_k = S(u_{k+1}), \quad \forall k = 0, 1, \dots, \quad \text{and} \quad \lambda^k \|u_k\| \to 0.$$

It follows directly from the definition that $S(\mathcal{M}_\lambda(S)) = \mathcal{M}_\lambda(S)$, $\forall \lambda$, and $\mathcal{M}_\lambda(S) \subset \mathcal{M}_{\lambda'}(S)$ if $\lambda \geq \lambda'$.

The sets $\mathcal{M}_\lambda(S)$ are extensively used in the theory of attractors for quasigradient systems, which has been rapidly developing in recent years. There is no widely accepted term for these sets. The set $\mathcal{M}_1(S)$ is called unstable invariant manifold of the operator S in those cases when it is a manifold. Here it is suggested to call $\mathcal{M}_\lambda(S)$ the λ-*relatively unstable invariant set* of S.

The most full study of the sets $\mathcal{M}_\lambda(S)$ in the case of a C^r operator, $r \geq 1$, was undertaken in [1]. However, for the needs of the theory of attractors the assumption of the operator S being smooth appears to be too strong. The present note contains accurate proofs of Lipschitz versions of some results from [1]. Only minor changes to the proofs of [1] have been made.

Throughout the note L stands for a bounded linear operator in a Banach space X satisfying the following assumptions:

(L1) the exist bounded projections $P, Q : X \to X$ such that $P + Q = I$ and $L(PX) = PX$, $L(QX) \subset QX$;

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(L2) there exist positive numbers λ, δ such that $\|Lu\| \geq (\lambda + \delta)\|u\|$ for all $u \in PX$ and $\|Lu\| \leq (\lambda - \delta)\|u\|$ for all $u \in QX$.

By p we denote the maximum of the norms $\|P\|$ and $\|Q\|$.

Remark. The operator L is not assumed to be invertible on X but the expression $L^{-1}u$ makes sense for $u \in PX$, since by (L1) and (L2) the operator $L\big|_{PX}$ is invertible on PX. Moreover, by (L2) we have $\|L^{-1}u\| \leq (\lambda + \delta)^{-1}\|u\|$, $\forall u \in PX$.

Theorem A. *Let $S : X \to X$ be a Lipschitz continuous mapping and let $S(0) = 0$. Assume there exists a bounded linear operator $L : X \to X$ satisfying* (L1), (L2) *and a positive number σ, such that $2p\sigma < \delta$ and the Lipschitz constant* $\mathrm{Lip}\,(\Phi)$ *of the mapping $\Phi(\cdot) = S(\cdot) - L$ does not exceed σ. Then*

(i) *$\mathcal{M}_\lambda(S) = \{y + g(y),\ y \in PX\}$, where $g : PX \to QX$ is a Lipschitz continuous function such that $g(0) = 0$, and its Lipschitz constant $\gamma_g = \mathrm{Lip}\,(g)$ satisfies*

$$\gamma_g \leq \frac{4\gamma_g}{(1+\gamma_g)^2} \leq \frac{2p\sigma}{\delta} < 1;$$

(ii) *$S(\mathcal{M}_\lambda(S)) = \mathcal{M}_\lambda(S)$ and $S : \mathcal{M}_\lambda(S) \to \mathcal{M}_\lambda(S)$ is a one-to-one map with a Lipschitz continuous inverse;*

(iii) *$\|QS(u) - g(PS(u))\| \leq (\lambda - \delta + 2p\sigma)\|Qu - g(Pu)\|$ for every $u \in X$;*

(iv) *if $u \in \mathcal{M}_\lambda(S)$, then $\|PS(u)\| \geq (\lambda + \delta - 2p\sigma)\|Pu\|$.*

Remark. It is convenient to have closely at hand the following chain of inequalities:

$$0 < \lambda - \delta < \lambda - \delta + 2p\sigma < \lambda < \lambda + \delta - 2p\sigma < \lambda + \delta.$$

The proof splits into a few steps. Denote by $\mathcal{G}$ the Banach space of all continuous functions $g : PX \to QX$ satisfying $g(0) = 0$ and having finite norm

$$\|g\|_{\mathcal{G}} = \sup_{u \in PX \setminus \{0\}} \frac{\|g(u)\|}{\|u\|}.$$

For functions $g \in \mathcal{G}$ we shall use the notation $\gamma_g = \mathrm{Lip}\,(g)$. By $\mathcal{G}_0$ we denote the set of all functions $g \in \mathcal{G}$ whose Lipschitz constants do not exceed 1. Clearly, $\mathcal{G}_0$ is a closed bounded subset of $\mathcal{G}$.

For each function $g \in \mathcal{G}_0$ define a Lipschitz operator $T_g : PX \to PX$ by $T_g(u) = PS(u + g(u))$.

Proposition 1. *For each function $g \in \mathcal{G}_0$ the operator T_g is invertible and* $\mathrm{Lip}\,(T_g^{-1}) \le (\lambda + \delta - p\sigma(1+\gamma_g))^{-1} \le (\lambda + \delta - 2p\sigma)^{-1}$.

Proof. Fix some $v \in PX$ and rewrite the equation $T_g(u) = v$ as $u = L^{-1}v + R_g(u)$, where $R_g(u) = -L^{-1}P\Phi(u + g(u))$. One easily checks that $\mathrm{Lip}\,(R_g) \le (\lambda+\delta)^{-1}p\sigma(1+\gamma_g) < 2p\sigma/\delta < 1$ and therefore by the contraction principle there exists a unique solution $u \in PX$ of the equation $T_g(u) = v$, which allows us to set $T_g^{-1}(v) = u$. For two different points v_1, $v_2 \in PX$ from the equations $T_g^{-1}(v_\ell) = L^{-1}v_\ell + R_g(T_g^{-1}(v_\ell))$, $\ell = 1, 2$, we have

$$\|T_g^{-1}(v_1) - T_g^{-1}(v_2)\| \le (\lambda+\delta)^{-1}\|v_1 - v_2\| + \mathrm{Lip}\,(R_g)\|T_g^{-1}(v_1) - T_g^{-1}(v_2)\|,$$

which implies the claimed Lipschitz estimate for T_g^{-1}. $\square$

Define a mapping $G : \mathcal{G}_0 \to \mathcal{G}$ taking each $g \in \mathcal{G}_0$ to $f = G(g)$, where

$$f(u) = QS(v + g(v)) \equiv Lg(v) + Q\Phi(v + g(v)), \quad v = T_g^{-1}(u)$$

for all $u \in PX$. This definition along with Proposition 1 yields

Proposition 2. *If $g \in \mathcal{G}_0$ and $f = G(g)$, then*

$$\gamma_f \le \frac{(\lambda-\delta)\gamma_g + p\sigma(1+\gamma_g)}{\lambda + \delta - p\sigma(1+\gamma_g)} < 1.$$

This, in particular, implies that G maps $\mathcal{G}_0$ into $\mathcal{G}_0$.

The mapping G is called the Hadamard's transform. The definitions of G and T_g imply that if $f = G(g)$, then

$$u + f(u) = S(v + g(v)), \quad \text{for all} \quad v \in PX \quad \text{and} \quad u = T_g(v). \tag{1}$$

For each $g \in \mathcal{G}$ denote by M_g the graph of g, i.e., $M_g = \{u + g(u),\ u \in PX\}$. From (1) we immediately have $S(M_g) \subset M_f$. In fact, a stronger property takes place.

Proposition 3. *If $f = G(g)$ then $S(M_g) = M_f$ and the mapping $S : M_g \to M_f$ is invertible with a Lipschitz continuous inverse.*

Proof. Define the mapping $F_g : PX \to M_g$ by $F_g(u) = u + g(u)$. Clearly, F_g is invertible, the inverse being $P\big|_{M_g}$. By (1) we have $S\big|_{M_g} = F_f \circ T_g \circ F_g^{-1}$, each of the mappings $F_g^{-1} : M_g \to PX$, $T_g : PX \to PX$, and $F_f : PX \to M_f$ having a Lipschitz continuous inverse. $\square$

Proposition 4. *The equation $G(g) = g$ has a unique solution $g \in \mathcal{G}_0$.*

Proof. We again use the contraction principle. Fix some functions g_1, $g_2 \in \mathcal{G}_0$, a point $u \in PX$, and for each of $\ell = 1, 2$ denote $f_\ell = G(g_\ell)$, $v_\ell = T_{g_\ell}^{-1}(u)$. These definitions imply the identities

$$Lv_\ell + P\Phi(v_\ell + g_\ell(v_\ell)) = u, \qquad \ell = 1, 2, \tag{2}$$

$$Lg_\ell(v_\ell) + Q\Phi(v_\ell + g_\ell(v_\ell)) = f_\ell(u), \quad \ell = 1, 2, \tag{3}$$

From (2) we have

$$\begin{aligned}(\lambda + \delta)\|v_1 - v_2\| &\le p\sigma\|v_1 - v_2\| + p\sigma\|g_1(v_1) - g_2(v_2)\| \\ &\le 2p\sigma\|v_1 - v_2\| + p\sigma\|g_1(v_1) - g_2(v_1)\|,\end{aligned}$$

and therefore

$$\|v_1 - v_2\| \le \frac{p\sigma}{\lambda + \delta - 2p\sigma}\|g_1(v_1) - g_2(v_1)\|. \tag{4}$$

Making use of the identities (3) one obtains

$$\begin{aligned}\|f_1(u) - f_2(u)\| &\le (\lambda - \delta + p\sigma)\|g_1(v_1) - g_2(v_2)\| + p\sigma\|v_1 - v_2\| \\ &\le (\lambda - \delta + p\sigma)\|g_1(v_1) - g_2(v_1)\| + (\lambda - \delta + 2p\sigma)\|v_1 - v_2\|,\end{aligned}$$

which along with (4) yields

$$\|f_1(u) - f_2(u)\| \le (\lambda - \delta + 2p\sigma)\|g_1(v_1) - g_2(v_1)\|.$$

Finally, taking into account that, by Proposition 1, $\|u\| \ge (\lambda + \delta - 2p\sigma)\|v_1\|$, we have

$$\|f_1 - f_2\|_{\mathcal{G}} = \sup_u \frac{\|f_1(u) - f_2(u)\|}{\|u\|}$$

$$\le \frac{\lambda - \delta + 2p\sigma}{\lambda + \delta - 2p\sigma} \sup_{v_1} \frac{\|g_1(v_1) - g_2(v_1)\|}{\|v_1\|} = \frac{\lambda - \delta + 2p\sigma}{\lambda + \delta - 2p\sigma}\|g_1 - g_2\|_{\mathcal{G}},$$

and therefore G is a contraction on $\mathcal{G}_0$. $\square$

From now on we denote by g the solution of the equation $G(g) = g$ just obtained. Using Proposition 2 one shows that $4\gamma_g/(1+\gamma_g)^2 \le 2p\sigma/\delta$. By Proposition 3, $S(M_g) = M_g$, and $S : M_g \to M_g$ is a one-to-one map. In Propositions 6 and 8 we shall show that $M_g = \mathcal{M}_\lambda(S)$.

Proposition 5. *If $u \in M_g$, then $\|PS(u)\| \geq (\lambda + \delta - 2p\sigma)\|Pu\|$.*

Proof. Taking into account that $u = Pu + g(Pu)$, we have $PS(u) = T_g(Pu)$. The desired inequality then follows from Proposition 1. $\square$

Proposition 6. $M_g \subset \mathcal{M}_\lambda(S)$.

Proof. By the invariance of M_g, for each $u \in M_g$ there exists a sequence $u_k \in M_g$, $k = 0, 1, \ldots$, such that $u_0 = u$ and $u_k = S(u_{k+1})$, $\forall k$. By Proposition 5, $\|Pu_k\| \leq (\lambda + \delta - 2p\sigma)^{-k}\|Pu_0\|$ and hence $\lambda^k\|Pu_k\| \to 0$ as $k \to \infty$. The observation that $\|u_k\| = \|Pu_k + g(Pu_k)\| \leq 2\|Pu_k\|$ completes the proof. $\square$

Proposition 7. *For every $u \in X$ the inequality $\|QS(u) - g(PS(u))\| \leq (\lambda - \delta + 2p\sigma)\|Qu - g(Pu)\|$ holds.*

Proof. Fix some $u \in X$ and denote $v = Pu + g(Pu) \in M_g$. The invariance of M_g implies $QS(v) = g(PS(v))$, and therefore

$$\begin{aligned}
\|QS(u) - g(PS(u))\| &\leq \|QS(u) - QS(v)\| + \|g(PS(u)) - g(PS(v))\| \\
&\leq \|QS(u) - QS(v)\| + \|PS(u) - PS(v)\| \\
&\leq \|L(u-v)\| + \|Q\Phi(u) - Q\Phi(v)\| + \|P\Phi(u) - P\Phi(v)\| \\
&\leq (\lambda - \delta + 2p\sigma)\|u - v\| = (\lambda - \delta + 2p\sigma)\|Qu - g(Pu)\|
\end{aligned}$$

(here we also used the fact that $PL(u - v) = 0$). $\square$

Proposition 8. $\mathcal{M}_\lambda(S) \subset M_g$.

Proof. By the definition of $\mathcal{M}_\lambda$, for each $u \in \mathcal{M}_\lambda(S)$ there exists a sequence $u_k \in X$, $k = 0, 1, \ldots$, such that $u_0 = u$, $u_k = S(u_{k+1})$, $\forall k$, and $\lambda^k\|u_k\| \to 0$ as $k \to \infty$. On the other hand, by Proposition 7

$$\begin{aligned}
\|Qu - g(Pu)\| &\leq (\lambda - \delta + 2p\sigma)^k\|Qu_k - g(Pu_k)\| \\
&\leq 2p(\lambda - \delta + 2p\sigma)^k\|u_k\| < 2p\lambda^k\|u_k\|
\end{aligned}$$

for all k, and therefore $Qu - g(Pu) = 0$, i.e., $u \in M_g$. $\square$

This completes the proof of Theorem A.

In the rest of the note we show that the λ-relatively unstable invariant set $\mathcal{M}_\lambda(S)$ continuously, in a proper sense, depends on the operator S.

Theorem B. *Let $S_1, S_2 : X \to X$ be Lipscitz mappings and let $S_1(0) = S_2(0) = 0$. Assume there exists a bounded linear operator $L : X \to X$ satisfying* (L1) *and* (L2) *and a positive number σ satisfying $2p\sigma < \delta$, such* *that* $\mathrm{Lip}\,(\Phi_\ell) \le \sigma$, $\ell = 1,2$, *where* $\Phi_\ell(\cdot) = S_\ell(\cdot) - L$. *Finally assume that* $\|\Phi_1(v) - \Phi_2(v)\| \le \sigma\varepsilon\|v\|$ *for some* $\varepsilon > 0$ *and all* $v \in \mathcal{M}_\lambda(S_1)$. *Then*

$$\|g_1 - g_2\|_{\mathcal{G}} \le \frac{2p\sigma\varepsilon}{\delta - 2p\sigma},$$

where the functions $g_1,\, g_2 : PX \to QX$ *are those provided by Theorem A, i.e., such that* $\{y + g_\ell(y),\ y \in PX\} = \mathcal{M}_\lambda(S_\ell)$, $\ell = 1, 2$.

Proof. The proof is similar to the one of Proposition 4. Fix some $u \in PX$ and for each of $\ell = 1,2$ denote by v_ℓ the solution of the equation $PS_\ell(v_\ell + g_\ell(v_\ell)) = u$. By Proposition 1 such v_ℓ exists, is unique, and

$$\|u\| \ge (\lambda + \delta - 2p\sigma)\|v_\ell\|, \quad \ell = 1, 2. \tag{5}$$

From the invariance of the sets $\mathcal{M}_\lambda(S_\ell) = \{y + g_\ell(y),\ y \in PX\}$ it follows that $QS_\ell(v_\ell + g_\ell(v_\ell)) = g_\ell(u)$. Thus we have

$$Lv_\ell + P\Phi_\ell(v_\ell + g_\ell(v_\ell)) = u, \qquad \ell = 1, 2, \tag{6}$$
$$Lg_\ell(v_\ell) + Q\Phi_\ell(v_\ell + g_\ell(v_\ell)) = g_\ell(u), \quad \ell = 1, 2, \tag{7}$$

For further reference we display here the inequality

$$\begin{aligned} &\|\Phi_1(v_1 + g_1(v_1)) - \Phi_2(v_2 + g_2(v_2))\| \\ \leqslant\ &\|\Phi_1(v_1 + g_1(v_1)) - \Phi_2(v_1 + g_1(v_1))\| + \sigma\|v_1 - v_2\| + \sigma\|g_1(v_1) - g_2(v_2)\| \\ \le\ &2\sigma\varepsilon\|v_1\| + 2\sigma\|v_1 - v_2\| + \sigma\|g_1(v_1) - g_2(v_1)\|, \end{aligned} \tag{8}$$

which we shall need twice. The identities (6) along with (8) yield

$$\begin{aligned} (\lambda + \delta)\|v_1 - v_2\| &\le p\|\Phi_1(v_1 + g_1(v_1)) - \Phi_2(v_2 + g_2(v_2))\| \\ &\le 2p\sigma\varepsilon\|v_1\| + 2p\sigma\|v_1 - v_2\| + p\sigma\|g_1(v_1) - g_2(v_1)\|, \end{aligned}$$

and therefore

$$\|v_1 - v_2\| \le \frac{p\sigma}{\lambda + \delta - 2p\sigma}\big(2\varepsilon\|v_1\| + \|g_1(v_1) - g_2(v_1)\|. \tag{9}$$

From (7), again using (8), we have

$$\begin{aligned} &\|g_1(u) - g_2(u)\| \\ &\le (\lambda - \delta)\|g_1(v_1) - g_2(v_2)\| + p\|\Phi_1(v_1 + g_1(v_1)) - \Phi_2(v_2 + g_2(v_2))\| \\ &\le (\lambda - \delta + p\sigma)\|g_1(v_1) - g_2(v_1)\| + 2p\sigma\varepsilon\|v_1\| + (\lambda - \delta + 2p\sigma)\|v_1 - v_2\|, \end{aligned}$$

or, together with (9),

$$\|g_1(u) - g_2(u)\| \leq (\lambda - \delta + 2p\sigma)\|g_1(v_1) - g_2(v_1)\| + 4p\sigma\varepsilon\|v_1\|.$$

Divide the obtained estimates by $\|v_1\|$ and take (5) into account. Finally taking supremum over $v_1 \in PX$ one obtains

$$(\lambda + \delta - 2p\sigma)\|g_1 - g_2\|_{\mathcal{G}} \leq (\lambda - \delta + 2p\sigma)\|g_1 - g_2\|_{\mathcal{G}} + 4p\sigma\varepsilon,$$

which is equivalent to the desired estimate. $\square$

References

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