**S. I. Repin**

# A POSTERIORI ERROR ESTIMATION FOR NONLINEAR VARIATIONAL PROBLEMS BY DUALITY THEORY

ABSTRACT. In this paper we use duality theory of the calculus of variations for deriving a posteriori error estimates. We obtain a general form of this (duality) error estimate and show that known classes of a posteriori error estimates are its particular cases.

**Dedicated to memory of**
**A. P. Oskolkov**

## 1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. In the domain $\Omega$ we consider a variational problem:

**Problem $\mathcal{P}$.** *Find $u \in V_0 + u_0$ such that*

$$J(u) = \inf \mathcal{P} := \inf_{v \in V_0 + u_0} J(v), \; J(v) = \int_\Omega g(\nabla v)\, dx - \int_\Omega f v \, dx$$

Here $V_0$ is a closed subspace of a reflexive Banach space $V$ equipped with norm $\|.\|_V$, $u_0 \in V$ is a given function, $g : \mathbb{R}^n \to \mathbb{R}^+$ is a convex continuous function and $f \in U \subset V^*$, where $V^*$ is the space topologically dual to $V$ and $U$ is a Banach space such that $V$ is continuously embedded in $U$ and $V_0$ is dense in $U$. We suppose that $g$ has superlinear growth with respect to $| \nabla v |$ so that the functional $J$ is continuous and coercive on $V$.

Let $v \in V_0 + u_0$ be an approximate solution of Problem $\mathcal{P}$, then $e = v - u$ is the approximation error. In this paper we analyse methods of getting upper bound of $\|e\|_V$. There are two classes of such estimates: a priori and a posteriori. A priori estimates give an information on asymptotic behavior of errors as dimension of the corresponding discrete problems tends to infinity. These estimates involve unknown constants which depend on the exact solution $u$. Therefore, these estimates cannot be calculated explicitly, or they are much too pessimistic to be of practical relevance. A posteriori error estimate gives upper bound of the error using an approximate solution $v$ only. In this case we have to find a function $\mathbb{M}$ depending on $v$ and external data such that:

$$\|v - u\| \leq \mathbb{M}(v) \tag{1.1}$$

$$\mathbb{M}(v_k) \to 0 \text{ for any sequence } \{v_k\} \text{ such that } v_k \to u \text{ in } V \tag{1.2}$$

Methods of a posteriori error estimation for partial differential equations started receiving attention more than two decades ago (see e.g. [2, 3, 15]) and have been developed mostly for linear problems. The literature on this subject is vast. However, almost all methods may be collected in the following three main groups:

(A) the residual method,

(B) the gradient averaging method,

(C) the equilibrated data method.

In the residual method (see e.g. [1, 3, 10, 20]) a weak norm of the residual function is used in the capacity of $\mathbb{M}$. Methods of the group (B) (see e.g. [4, 21]) are based on smoothening of approximate solutions obtained by the finite element method and superconvergence phenomenon. The use of complementary energy principles for getting error estimates was suggested in [8, 12, 13]. These papers formed the basis of methods (C) which apply special post-processing procedures designed for constructing the so-called "equilibrated data" (i.e., functions exactly satisfying equilibrium equations) (see e.g. [1, 5, 13]). These methods are known as independent ways of a posteriori error estimation that have been proposed on different grounds and have different levels of mathematical justification.

In this paper we focus attention on a way for estimating errors of approximate solutions that follows from duality theory of the calculus of variations. In earlier papers we used duality theory to obtain a priori error estimates for Hencky plasticity problems [17, 18] and a posteriori estimates for plasticity problems with linear hardening [19]. Now we represent a general method for deriving a posteriori error estimates based on duality theory. We obtain a *duality error estimate* (2.4) and show that it encompasses (A) – (C) as *particular cases.*

The outline of the paper is as follows. In Section 2 we derive a general form of the duality error estimate represented by the function $\mathbb{M}_1$. This function is a sum of two parts measuring errors in the *duality relations* and *differential equation for the dual variable*, respectively (Theorem 2.2). Another form of the error majorant ($\mathbb{M}_2$) is given by (2.9) (Theorem 2.3). We show that introduced error majorants are consistent (i.e., they tends to zero for any sequences $\{v_k\}$ and $\{\tau_k^*\}$ which tend to the exact solutions of the primal and dual problems, respectively) provided that the operators formed by the duality relations possesses necessary continuity properties. (Theorem 2.4).

In Section 3 we analyse particular cases of the estimate (2.9) when the dual variable is defined by the duality relations as $\tau^* = \Lambda\nabla v$. By projecting $\tau^*$ on embedded sets $Q_f^* \subset Q^* \subset \Sigma^*$ we obtain the error estimates (3.5) – (3.7). Taking linear elliptic equations as an example we relate (3.5) – (3.7) with the recognized types of a posteriori error estimates what, in fact, justifies them via the duality theory.

## 2. General form of the duality error estimate

We begin by introducing some notations. We denote by $\Sigma$ the reflexive Banach space that contain gradients of functions $v \in V$ and by $\Sigma^*$ its topologically dual space. The corresponding norms are designated $||.||$ and $||.||_*$, respectively. Throughout the reminder of this paper the dot denotes scalar product in $\mathbb{R}^n$ and the symbol := means "equal by definition".

In duality theory Problem $\mathcal{P}$ is investigated together with its dual:

**Problem $\mathcal{P}^*$..** *Find $\sigma^* \in \Sigma^*$ such that*

$$G^*(\sigma^*) = \sup_{\tau^* \in \Sigma^*} G^*(\tau^*)$$

*where*

$$G^*(\tau^*) = \int_\Omega (\tau^* \cdot \nabla u_0 - g^*(\tau^*) + f u_0)\, dx,$$

$$Q_f^* := \left\{ \tau^* \in \Sigma^* \;\; || \int_\Omega \tau^* \cdot \nabla w \, dx = \int_\Omega f w \, dx \; \forall w \in V_0 \right\}$$

*and $g^*$ is a function conjugate to $g$.*

The existence theorem for Problems $\mathcal{P}$ and $\mathcal{P}^*$ readily follows from known results of the calculus of variations (see e.g. [6]).

**Theorem 2.1.** *Under the above assumptions Problems $\mathcal{P}$ and $\mathcal{P}^*$ are uniquely solvable and their solutions $u$ and $\sigma^*$ satisfy the following relations:*

$$
\begin{aligned}
&(a)\ J(v) \geq J(u) = G^*(\sigma^*)\ \geq G^*(\eta^*)\\
&(b)\ \mathbb{F}(u, \sigma^*) = 0
\end{aligned}
\tag{2.1}
$$

*where* $\mathbb{F}(v, \tau^*) := \int_\Omega (g(\nabla v) + g^*(\tau^*) - \tau^* \cdot \nabla v)\, dx$.

We note that (2.1)(b) is equivalent to the relations

$$
\nabla u(x) = \Lambda^* \sigma^*(x), \quad \sigma^*(x) = \Lambda \nabla u(x) \ \text{ for a.e. } x \in \Omega \tag{2.2}
$$

where

$$
\Lambda^* z := g^{*\prime}(z) \quad \Lambda z := g'(z) \ \ \forall\, z \in \mathbb{R}^n
$$

In the sequel we assume that $g(z)$ and $g^*(z)$ are Gateaux differentiable for any $z \in \mathbb{R}^n$, $g'(0) = g^{*\prime}(0) = 0$ and the operators $\Lambda^*$ and $\Lambda$ are continuous in the following sense: a sequence of functions $\Lambda^* \tau_k^*$ tends to $\Lambda^* \tau^*$ in $\Sigma$ if $\tau_k^* \to \tau^*$ in $\Sigma^*$ and a sequence $\Lambda \tau_k$ tends to $\Lambda \tau$ in $\Sigma^*$ if $\tau_k \to \tau$ in $\Sigma$.

The variational method of getting a posteriori error estimates is based on that in many cases one can find a continuous nondecreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Phi(0) = 0$ and

$$
\Phi(\|e\|_V) \leq J(v) - inf\,\mathcal{P}. \tag{2.3}
$$

In particular (2.3) is valid in some neighborhood of $u$ when $J$ is a uniformly convex functional [9, 16]). Since the value of $inf\,\mathcal{P}$ can be estimated from below by the duality relations (2.1) we see that the difference $J(v) - G^*(\tau^*)$, where $\tau^*$ is an arbitrary element of the dual functional space $\Sigma^*$, gives an estimate of the error. It is worthwhile to remark that the above difference is finite only for $\tau^* \in Q_f^*$, so that a straightforward attempt to use it for measuring the error necessitates solving Problem $\mathcal{P}^*$ with all the ensuing difficulties. In this paper we use another way and find a majorant of this difference which is the sum of two physically meaningful parts that can be calculated without directly solving the dual problem. By virtue of this function we obtain a *duality error estimate* (2.4).

**Theorem 2.2.** *Let $\mathbb{H} : \Sigma \to \mathbb{R}^+$ and $\mathbb{H}^* : \Sigma^* \to \mathbb{R}^+$ be a pair of convex functionals which are conjugate to each other and such that $\mathbb{H}(0) = 0$, $\mathbb{H}^*(0) = 0$. Then for any $\tau^* \in \Sigma^*$ the following estimate holds*

$$\Phi(\|e\|_V) \le \mathbb{M}_1(v,\tau^*) := \mathbb{E}_\Lambda(v,\tau^*) + \mathbb{E}_Q(\tau^*), \tag{2.4}$$

*where*

$$\mathbb{E}_\Lambda(v,\tau^*) = \mathbb{F}(v,\tau^*) + \mathbb{H}(\nabla v - \Lambda^*\tau^*)$$

$$\mathbb{E}_Q(\tau^*) = \inf_{\eta^*\in Q_f^*}\left(\mathbb{H}^*(\eta^*-\tau^*) + \int_\Omega (\Lambda^*\eta^* - \Lambda^*\tau^*)\cdot(\eta^*-\tau^*)\,dx\right)$$

**Proof.** Making use of (2.1)(a) we see

$$\begin{aligned} J(v) - J(u) \le\ & J(v) - G^*(\tau^*) + G^*(\tau^*) - G^*(\eta^*) = \\ & = \int_\Omega (g(\nabla v) + g^*(\tau^*) - f(v-u_0) - \tau^*\cdot\nabla u_0)\ dx = \\ & + \int_\Omega (g^*(\eta^*) - g^*(\tau^*) + \nabla u_0\cdot(\tau^*-\eta^*))\ dx = \\ & = \mathbb{F}(v,\tau^*) + \int_\Omega (\tau^*\cdot\nabla(v-u_0) - f(v-u_0))\ dx + \\ & \int_\Omega (g^*(\eta^*) - g^*(\tau^*) + \nabla u_0\cdot(\tau^*-\eta^*))\ dx. \end{aligned} \tag{2.5}$$

Since

$$\int_\Omega \eta^*\cdot\nabla(v-u_0)\,dx = \int_\Omega f(v-u_0)\,dx\ \forall\eta^*\in Q_f^*$$

the inequality (2.5) can be represented as

$$J(v) - J(u) \le \mathbb{F}(v,\tau^*) + \int_\Omega (g^*(\eta^*) - g^*(\tau^*) + (\tau^*-\eta^*)\cdot\nabla v)\,dx.$$

Recalling that $g^*$ is a convex functional we get

$$g^*(\eta^*) - g^*(\tau^*) \le \Lambda^*\eta^*\cdot(\eta^*-\tau^*)$$

whence

$$\begin{aligned} J(v) - J(u) \le \mathbb{F}(v,\tau^*) + \int_\Omega (\Lambda^*\tau^* - \nabla v)\cdot(\eta^*-\tau^*)\,dx + \\ + \int_\Omega (\Lambda^*\eta^* - \Lambda^*\tau^*)\cdot(\eta^*-\tau^*)\,dx \end{aligned} \tag{2.6}$$

By applying Young-Fenchel inequality [7] we infer

$$\int_\Omega (\Lambda^*\tau^* - \nabla v)\cdot(\eta^* - \tau^*)\,dx \le \mathbb{H}(\Lambda^*\tau^* - \nabla v) + \mathbb{H}^*(\eta^* - \tau^*). \quad (2.7)$$

which together with (2.6) results in the desired (2.4).

**Remark 1.** We note that the function $\mathbb{E}_\Lambda(v,\tau^*)$ is nonnegative and equal to zero if and only if $\nabla v$ and $\tau^*$ satisfy the duality relations (2.1)(b). The function $\mathbb{E}_Q(\tau^*)$ is nonnegative and equal to zero if and only if $\tau^* \in Q^*_f$. Therefore, $\mathbb{E}_\Lambda(v,\tau^*)$ is a measure of the error in the duality relations for the pair $(v,\tau^*)$ and $\mathbb{E}_Q(\tau^*)$ is a measure of the distance from $\tau^*$ to the set $Q^*_f$ which contains the exact solution of the dual problem.

Let $\varphi^* : \mathbb{R} \to \mathbb{R}^+$ be an even convex continuous function such that $\varphi^*(0) = 0$, $c^*_- t^{\alpha^*} \le \varphi^*(t) \le c^*_+ t^{\alpha^*}$ with some constants $\alpha^* > 1$, $c^*_+ \ge c^*_- > 0$ and

$$\int_\Omega (\Lambda^*\eta^* - \Lambda^*\tau^*)\cdot(\eta^* - \tau^*))\,dx \le \varphi^*(\|\eta^* - \tau^*\|_*)\ \forall \eta^*, \tau^* \in \Sigma^* \quad (2.8)$$

In the sequel we consider the case when

$$\mathbb{H}^*(\tau^*) = \varphi^*(\|\tau^*\|_*) \quad \mathbb{H}(\tau) = \varphi(\|\tau\|)$$

where $\varphi : \mathbb{R} \to \mathbb{R}^+$ is a function conjugate to $\varphi^*$. It is not difficult to verify that $\varphi(0) = 0$ and $c_- t^\alpha \le \varphi(t) \le c_+ t^\alpha$ with some $c_+ \ge c_- > 0$ and $\alpha = \frac{\alpha^*}{\alpha^* - 1} > 1$.

Now we are in a position to represented another form of the error majorant given by the following theorem.

**Theorem 2.3.** *Let the assumptions of Theorem 2.2 and the condition (2.8) are satisfied. Then for any $\tau^* \in \Sigma^*$ the following estimates holds*

$$\begin{aligned} &(a)\ \ \Phi(\|e\|_V) \le \mathbb{M}_2^{(1)}(v,\tau^*) := \mathbb{E}_\Lambda(v,\tau^*) + \mathbb{E}_Q^{(1)}(\tau^*) \\ &(b)\ \ \Phi(\|e\|_V) \le \mathbb{M}_2^{(2)}(v,\tau^*) := \mathbb{E}_\Lambda(v,\tau^*) + \mathbb{E}_Q^{(2)}(\tau^*) \end{aligned} \quad (2.9)$$

*where*

$$\mathbb{E}_\Lambda(v,\tau^*) = \mathbb{F}(v,\tau^*) + \varphi(\|\nabla v - \Lambda^*\tau^*\|)$$
$$\mathbb{E}_Q^{(1)}(\tau^*) = 2 \inf_{\eta^*\in Q_f^*} \varphi^*(\|\eta^* - \tau^*\|_*)$$
$$\mathbb{E}_Q^{(2)}(\tau^*) = -2 \inf_{w\in V_0} \int\limits_\Omega (\tau^*\cdot\nabla w - fw)\,dx + \varphi(\|\nabla w\|).$$

**Proof.** The estimate (2.9)(a) is a consequence of (2.4). For proving (2.9)(b) we note that

$$\begin{aligned}\mathbb{E}_Q^{(1)}(\tau^*) &= 2 \inf_{\eta^*\in Q_f^*} \varphi^*(\|\eta^* - \tau^*\|_*) = \\ &= -2 \sup_{\eta^*\in\Sigma^*} \inf_{w\in V_0} \int\limits_\Omega (\eta^*\cdot\nabla w - fw) - \varphi^*(\|\eta^* - \tau^*\|_*) = \\ &= -2 \sup_{\eta^*\in\Sigma^*} \inf_{w\in V_0} \mathcal{L}(w,\tau^*)\end{aligned} \tag{2.10}$$

where the Lagrangian

$$\mathcal{L}(w,\tau^*) = \int\limits_\Omega (\tau^*\cdot\nabla w - fw + \eta^*\cdot\nabla w)\ dx - \varphi^*(\|\eta^*\|_*)$$

possesses the following properties:

$\forall\eta^*\in\Sigma^*$ the function $w\to\mathcal{L}_0(w,\eta^*)$ is convex and continuous in $V$,

$\forall w\in V_0$ the function $\eta^*\to\mathcal{L}_0(w,\eta^*)$ is concave and continuous in $\Sigma^*$,

$\mathcal{L}_0(w,\eta^*)\to-\infty$ if $\|\eta^*\|_*\to+\infty$ $\forall w\in V_0$.

Then known results of the convex analysis (see e.g. [6]) yield the equality

$$\begin{aligned}\sup_{\eta^*\in\Sigma^*} \inf_{w\in V_0} \mathcal{L}(w,\tau^*) &= \inf_{w\in V_0} \sup_{\eta^*\in\Sigma^*} \mathcal{L}(w,\tau^*) = \\ &= \varphi(\|\nabla w\|) + \int\limits_\Omega (\tau^*\cdot\nabla w - fw)\ dx\end{aligned} \tag{2.11}$$

Hence, we see that $\mathbb{E}_Q^{(1)}(\tau^*) = \mathbb{E}_Q^{(2)}(\tau^*)$ and deduce (2.9)(b) from (2.9)(a).

**Remark 2.** The two forms of the function $\mathbb{M}_2$ are equivalent, i.e., $\mathbb{M}_2^{(1)}(v,\tau^*) = \mathbb{M}_2^{(2)}(v,\tau^*)$. For this reason we use hereafter the simplified notation $\mathbb{M}_2(v,\tau^*)$.

The consistency of the estimates (2.9) is justified by the following assertion.

**Theorem 2.4.** *If $v_k \to u$ in $V$ and $\tau_k^* \to \sigma^*$ in $\Sigma^*$ as $k \to +\infty$ then*

$$\mathbb{M}_2(v_k, \tau_k^*) \to 0 \tag{2.12}$$

**Proof.** It is not difficult to see that

$$\begin{aligned}\mathbb{F}(v_k, \tau_k^*) &= \int\limits_\Omega (g(\nabla v_k) + g^*(\tau_k^*) - f(v_k - u_0) - \nabla u_0 \cdot \tau_k^*)\, dx + \\ &+ \int\limits_\Omega (\sigma^* \cdot \nabla(v_k - u_0) + \nabla u_0 \cdot \tau_k^* - \nabla v_k \cdot \tau_k^*)\, dx = \\ &= J(v_k) - G^*(\tau_k^*) + \int\limits_\Omega (\sigma^* - \tau_k^*) \cdot \nabla(v_k - u_0) dx\end{aligned}$$

Therefore, $\mathbb{F}(v_k, \tau_k^*) \to J(u) - G^*(\sigma^*) = 0$. It can be obtained readily that

$$\mathbb{E}_Q^{(2)}(\tau_k^*) = \mathbb{E}_Q^{(1)}(\tau_k^*) \leq 2\varphi^*(\|\tau_k^* - \sigma^*\|_*) \to 0$$

Finally, we have

$$\varphi(\|\nabla v_k - \Lambda^* \tau_k^*\|) \leqslant \varphi(\|\nabla(v_k - u)\| + \|\Lambda^*(\tau_k^* - \sigma^*)\|) \to 0$$

and arrive at (2.12).

## 3. Particular cases of the duality error estimates

In this section we investigate particular forms of the introduced error majorants when the dual variable $\tau^*$ is chosen in some special way. To this end we define

$$\tau^*(v)(x) = \Lambda \nabla v(x) \tag{3.1}$$

and the set

$$Q^* := \{\xi^* \in \Sigma^* \;\|\; \nabla \cdot \xi^* \in U\}\,.$$

Let $\Pi_1 : \Sigma^* \to Q^*$ and $\Pi_2 : \Sigma^* \to Q_f^*$ be continuous projection operators. We analyse three ways of determining the dual variable:

$$\tau_1^* = \tau^*(v) \in \Sigma^* \tag{3.2}$$

$$\tau_2^* = \Pi_1(\tau^*(v)) \in Q^* \tag{3.3}$$

$$\tau_3^* = \Pi_2(\tau^*(v)) \in Q_f^* \tag{3.4}$$

It is worth remarking that $Q^*_f \subset Q^* \subset \Sigma^*$ so that $\tau^*_i$, $i = 1, 2, 3$ are projections on the consequently embedded sets. Substitution $\tau^*_i$ into (2.9) yields the following error majorants:

$$\mathbb{M}_2(v, \tau^*_1) = -2 \inf_{w \in V_0} \int_\Omega (\tau^*_1 \cdot \nabla w - fw + \varphi(\|\nabla w\|))\, dx \tag{3.5}$$

$$\mathbb{M}_2(v, \tau^*_2) = \mathbb{F}(v, \tau^*_2) - 2 \inf_{w \in V_0} \int_\Omega \left(\varphi(\|\nabla w\|) - \widetilde{f} w\right) dx \tag{3.6}$$

where $\widetilde{f} = \nabla \cdot \tau^*_2 + f \in U$.

By taking in (2.4) $\tau^* = \tau^*_3 \in Q^*_f$, $\eta^* = \tau^*_3$ and $\mathbb{H}(\tau) = 0$ we get

$$\mathbb{M}_1(v, \tau^*_3) = \mathbb{F}(v, \tau^*_3) \tag{3.7}$$

In the rest of this article we show that for variational problems with quadratic functionals (e.g. for linear elliptic problems) the estimates (3.5), (3.6) and (3.7) lead to a posteriori error estimates (A), (B) and (C), respectively.

Let

$$g(\nabla v) = \frac{1}{2} a(\nabla v, \nabla v) \;\; \forall v \in V := \mathrm{W}^1_2(\Omega),$$

where $a(z_1, z_2) := \mathrm{A} z_1 \cdot z_2 \; \forall z_1, z_2 \in \mathbb{R}^n$. We assume that the matrix $\mathrm{A} = a_{ij}(x)$ $x \in \Omega$ is symmetric and $a_{ij}(x)$ are bounded measurable functions such that the following double inequality holds

$$\nu_1 \mid z \mid^2 \leq \mathrm{A} z \cdot z \leq \nu_2 \mid z \mid^2 \quad \forall\, z \in \mathbb{R}^n \tag{3.8}$$

with some positive constants $\nu_1$ i $\nu_2$. The function $g^*(\tau^*) = \frac{1}{2} a^*(\tau^*, \tau^*) = \frac{1}{2} \mathrm{A}^* \tau^* \cdot \tau^*$ is conjugate to $g$ if $\mathrm{A}^*$ is inverse of the matrix A. We identify $\Sigma$ and $\Sigma^*$ as spaces of square summable vector valued functions with the norms $\|\tau\|^2 := \int_\Omega \mathrm{A}\tau \cdot \tau dx$ and $\|\tau^*\|^2_* := \int_\Omega \mathrm{A}^* \tau^* \cdot \tau^* dx$, respectively. Lastly, $f \in U$ where $U$ is the space $L_2(\Omega)$ with the norm $\|.\|_{2,\Omega}$.

In this case a minimizer $u$ to Problem $\mathcal{P}$ is a generalized solution of some linear elliptic equation. Under the above assumptions there exists a unique function $u$ which satisfies the integral identity (see [14]):

$$\int_\Omega a(\nabla u, \nabla v) dx = \int f v\, dx \quad \forall v \in V_0 := \{w \in V \;\; \| \; w = 0 \text{ on } \partial\Omega\} . \tag{3.9}$$

By (3.9) we obtain the inequality

$$\frac{1}{2}\|e\|^2 = \frac{1}{2}\int\limits_\Omega a(\nabla(v-u),\nabla(v-u)) + [a(\nabla u,\nabla(v-u)) + f(v-u)]\,dx = \\ = J(v) - J(u) = J(v) - \inf \mathcal{P}. \tag{3.10}$$

which shows that in this case the left–hand side of (2.9) is $\frac{1}{2}\|e\|^2$. The operators $\Lambda^*$ and $\Lambda$ coincide with $\mathrm{A}^*$ and $\mathrm{A}$, respectively and $\varphi^*(t) = t^2$, $\varphi(t) = \frac{1}{4}t^2$. Hence, we obtain

$$\mathbb{E}_\Lambda(v,\tau^*) \le c_1 \|\mathrm{A}\nabla v - \tau^*\|^2_{2,\Omega} \tag{3.11}$$

$$\mathbb{E}_Q(\tau^*) = -\inf_{w\in V_0}\int\limits_\Omega \left(\frac{1}{2}\mathrm{A}\nabla w\cdot\nabla w + 2(\tau^*\cdot\nabla w - fw)\right)dx \tag{3.12}$$

where $c_1 = {\nu_1}^{-1}$.

**3.1. A residual error estimate.** When the coefficients $a_{ij}$ are smooth and $v$ is regular enough we define the residual of the differential equation as

$$R(v) := \nabla\cdot(A\nabla v) + f. \tag{3.13}$$

If $\tau_1^*(v) = A\nabla v \in \Sigma^*$ the residual $R(v) = \nabla\cdot\tau_1^* + f$ should be understood in the sense of distributions. To this end we define the quantity

$$\|R(v)\|_- = \sup_{\substack{w\in V_0\\ w\neq 0}} \frac{\int_\Omega (\tau_1^*(v)\cdot\nabla w - fw)\,dx}{\|\nabla w\|_{2,\Omega}}$$

which is finite for any $\tau_1^*(v)$. Since $\mathbb{E}(v,\tau_1^*) = 0$ we have the inequality

$$\frac{1}{2}\|e\|^2 \le -\inf_{w\in V_0}\int\limits_\Omega \left(\frac{1}{2}\mathrm{A}\nabla w\cdot\nabla w + 2(\tau_1^*\cdot\nabla w - fw)\right)dx \le \\ \le \inf_{w\in V_0}\left(\frac{\nu_1}{2}\|\nabla w\|^2_{2,\Omega} - 2\|R(v)\|_-\|\nabla w\|_{2,\Omega}\right) \le \frac{2}{\nu_1}\|R(v)\|^2_-, \tag{3.14}$$

which gives the estimate

$$\|e\|_a \le c_2\|R(v)\|_- . \tag{3.15}$$

For a function $v$ obtained by the finite element method calculation of $\|R(v)\|_-$ leads to the known residual error estimates (see e.g. [20]).

**3.2. Error estimates based on smoothening of the dual variable.** The purpose of this subsection is to show that the error majorant $\mathbb{M}_2(v, \tau_2^*(v))$ yields the estimate (3.16). The latter is a general form of a posteriori estimate of the group (B) valid for various types of gradient averaging operators $\Pi_1$.

$$\tfrac{1}{2}\|e\|_a^2 \leq \mathbb{M}_2(v, \tau_2^*) \leq c_1 \|\mathrm{A}\nabla v - \Pi_1(\mathrm{A}\nabla v)\|_{2,\Omega}^2 + \frac{2}{\lambda^2 \nu_1}\left\|\widetilde{f}\right\|_{2,\Omega}^2 \tag{3.16}$$

where

$$\lambda^2 = \inf_{w \in V_0} \frac{\int\limits_\Omega |\nabla w|^2 \, dx}{\int\limits_\Omega w^2 \, dx}$$

Really, we see that

$$\mathbb{E}_Q(\tau_2^*) = -\inf_{w \in V_0} \int\limits_\Omega \left(\frac{1}{2}\mathrm{A}\nabla w \cdot \nabla w - 2\widetilde{f}w\right) dx$$

Since

$$\inf_{w \in V_0} \int\limits_\Omega \left(\frac{1}{2}\mathrm{A}\nabla w \cdot \nabla w - 2\widetilde{f}w\right) dx \geq$$
$$\geq \inf_{w \in V_0}\left(\frac{\nu_1}{2}\|\nabla w\|_{2,\Omega}^2 - \frac{2}{\lambda}\left\|\widetilde{f}\right\|_{2,\Omega}\|\nabla w\|_{2,\Omega}\right) \geq -\frac{2}{\lambda^2\nu_1}\left\|\widetilde{f}\right\|_{2,\Omega}^2$$

we find that

$$\mathbb{E}_Q(\tau_2^*) \leq \frac{2}{\lambda^2 \nu_1}\left\|\widetilde{f}\right\|_{2,\Omega}^2 \tag{3.17}$$

For $\mathbb{E}_\Lambda$ we have the estimate

$$\begin{aligned}\mathbb{E}_\Lambda(v, \tau_2^*) &= \int\limits_\Omega \left(\frac{1}{2}\mathrm{A}\nabla v \cdot \nabla v + \frac{1}{2}\mathrm{A}^*\tau_2^* \cdot \tau_2^* - \nabla v \cdot \tau_2^*\right) dx + \\ &+ \int\limits_\Omega \frac{1}{4}\mathrm{A}(\nabla v - \mathrm{A}^*\tau_2^*) \cdot (\nabla v - \mathrm{A}^*\tau_2^*) dx \leq \\ &\leq \|\mathrm{A}\nabla v - \tau_2^*\|_*^2 \leq \frac{1}{\nu_1}\int\limits_\Omega |\mathrm{A}\nabla v - \Pi_1(\mathrm{A}\nabla v)|^2 \, dx\end{aligned} \tag{3.18}$$

Now (3.16) follows from (2.9)b, (3.17) and (3.18).

**Remark 3.** The first term in (3.16) coincides with the well known error estimates of the group (B) (see e.g. [21]). When $\tau^*$ is close to the exact solution we can assume that the second term is very small and, thus, can be neglected. However, in general, this assumption may be not correct and rigorous estimate should consider both terms. It is worth remarking that unlike the first explicitly computable term the second one includes unknown constant $\lambda$, which exact value is not easy to find. Nevertheless, we can circumvent this difficulty if replace $\lambda$ by $\lambda_0$ where

$$\lambda_0^2 = \inf_{w \in V_0} \frac{\int_\omega \mid \nabla w \mid^2 \, dx}{\int_\omega w^2 \, dx}$$

and $\Omega \subset \omega$. In this case $\lambda_0 \leq \lambda$ and the estimate (3.16) holds with $\lambda_0$. It is left to note that for some simple domains $\omega$ (i.e., rectangular, circle) exact values of $\lambda_0$ can be easily found and used in the estimate (3.16).

**3.3. A posteriori estimates based on equilibration of the dual variable.** When a projection operator $\Pi_2$ is known we can find $\tau_3^* = \Pi_2\,(\mathrm{A}\nabla v)$ and apply the estimate (3.7) which yields the inequality

$$\tfrac{1}{2}\, \|e\|_a^2 \leq c_1\, \|A\nabla v - \tau_3^*\|_{2,\Omega}^2 \,. \tag{3.19}$$

We see that (3.19) gives the $\mathrm{L}_2$ norm of the error in duality relations for $(v, \tau_3^*)$. Hence, known estimates based on equilibrated data and a concept of "errors in constitutive relations" (see [13]) can be regarded as particular forms of the general duality error estimate (2.4).

**3.4. Final remarks**

It is worthwhile to add some remarks on the scope of the methods analyzed in this section. The general estimate (2.9) is valid for all pairs of functions $(v, \tau^*) \in (\, V_0 + u_0\,) \times \Sigma^*$. Restricting this set to $(v, \Lambda\nabla v)$, $v \in V_0 + u_0$ we get the form of (2.9) when $\mathbb{E}_\Lambda = 0$ (see (3.5)). The latter provides a basis for methods of the group (A). In a sense opposite case is represented by (3.7). Here $(v, \tau^*) \in (\, V_0 + u_0\,) \times Q_f^*$ so that $\mathbb{E}_Q = 0$ and total error is expressed by the only one term $\mathbb{E}_\Lambda$. This way leads to error estimates of the group (C). A "mixed" case, when both parts of the duality error estimate $\mathbb{E}_\Lambda$ and $\mathbb{E}_Q$ are not zero, is given by (3.6) which follows form (2.9) when $(v, \tau^*) \in V_0 + u_0 \times Q^*$. This estimate is, in fact, a *general form* of a posteriori error estimates based on averaging of the dual variable.

**Acknowledgement.** This work was supported by Russian Foundation of Fundamental Researches under grant number 96-01-01664 and by Ministry of Higher Education of Russia under grant number 95-0-2.2-14.

## References


1. M. Ainsworth and J. T. Oden, *A unified approach to a posteriori error estimation using element residual methods.* — Numer. Math. **65** (1993), 23–50.
2. J. P. Aubin, H. G. Burchard, *Some aspects of the method of hypercicle applied to elliptic variational problems.* in Numerical Solutions of Partial Differential Equations – II, SYNSPADE 1970, Ed. B. Hubbard, (Academic Press, New York, London, 1971)..
3. I. Babuška and W. C. Rheinboldt, *Error estimates for adaptive finite element computations.* — SIAM J. Numer. Anal. **18** (1978), 736–754.
4. I. Babuška and R. Rodriguez, *The problem of the selection of an a posteriori error indicator based on smoothening techniques.* — Internat. J. Numer. Meth. Engrg. **36** (1993), 539–567.
5. R. E. Bank and W. Weiser, *Some a posteriori error estimators for elliptic partial differential equations.* — Math. Comp. **44** (1985), 283-301.
6. I. Ekeland, R. Temam, *Convex analysis and variational problems.* Amsterdam, North-Holland, 1976.
7. W. Fenchel, *On the conjugate convex functions.* — Canad. J. Math. **1** (1949), 73–77.
8. B. Fraeijs de Veubeke, *Displacement and equilibrium models in the finite element method.* Stress analysis - Ed. 0.C. Zienkiewicz and G.S. Holister, London, New York, Wiley.
9. R. Glowinski, *Numerical methods for nonlinear variational problems.* New-York, Springer–Verlag, 1984.
10. C. Johnson, P. Hansbo, *Adaptive finite elements in computational mechanics.* — Comput. Methods Appl. Mech. Engrg. **101** (1992), 143–181.
11. C. Johnson, A. Szepessy, *Adaptive finite element methods for conservation laws based on a posteriori error estimates.* — Commun. Pure and Appl. Math. **XLVIII** (1995), 199–234.
12. D. W. Kelly, *The self-equilibration residuals and complementary a posteriori estimates in the finite element method.* — Internat. J. Numer. Methods Engng. **20** (1984), 1491–1506.
13. P. Ladeveze and D. Leguillon, *Error estimate procedure in the finite element method and applications.* — SIAM J. Numer. Anal. **20** (1983), 485–509.
14. O. A. Ladyzenskaya, (in Russian, English translation: Springer 1985), *The boundary value problems of mathematical physics.* Moscow, Nauka, 1971.
15. S.G.. G. Mikhlin, *Variational methods in mathematical physics.* Oxford, Pergamon, 1964.
16. P. P. Mosolov, V. P. Myasnikov, *Mechanics of rigid plastic media.* Moscow, Nauka, 1981.
17. S. I. Repin, *Errors of finite element method for perfectly elasto–plastic problems.* — Mathematical Models and Methods in Applied Sciences **6 5** (1996), 587–604.
18. S. I. Repin and G. A. Seregin, *Error estimates for stresses in the finite element analysis of the two–dimensional elasto–plastic problems.* — Int. J. Engrg. Sci. **33 2**, 255–268.
19. S. I. Repin and L. S. Xanthis, *A posteriori error estimation for elasto–plastic problems based on duality theory.* — Comput. Methods Appl. Mech. Engrg. (to

appear).
20. R. Verfürth, *A review of a posteriori error estimation and adaptive mesh–refinement techniques.* New-York, Wiley, Teubner, 1996.
21. O. C. Zienkiewicz and J. Z. Zhu, *A simple error estimator and adaptive procedure for practical engineering analysis.* — Internat. J. Numer. Meth. Engrg. **24** (1987), 337–357.

St. Petersburg State Technical University

Поступило 9 сентября 1996г.