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ON THE EXISTENCE OF THE HARMONIC VARIATIONAL FLOW SUBJECT TO THE TWO–SIDED CONDITIONS

0. Introduction

The aim of this paper consists in investigating the existence of the variational flow related to the quadratic functional with the initial and boundary data from some Sobolev space, obeying the two-sided conditions.

In this context, M. Struwe [10] has established the global a-priori Hölder estimate of the weak solutions to the equations of the same type as described here, but not necessarily restricted to the variational structure, and, in the joint work [3] with M. Giaquinta, has weakened the condition on the parabolic constant to be supposed there. Further, also, improving the condition on the parabolic constant needed in [3], P. Tolksdorff [11] has achieved the a-priori Hölder estimates and, hence, the existence result. As long as the dimension of the definition domain is limited to two, with the less restricted condition, what is called the one-sided condition, W. Wieser [12] has demonstrated the existence of the strong solutions to the equations imposing the variational structure similar to ours but that he has moreover assumed to place the Lipshitz continuity on the coefficients in the main part of equations.

All the keys leading to these results are the a-priori Hölder estimates up to the boundary. The authors of the works cited above have essentially used the Hölder continuity of the initial and boundary data. On the other hand, our situation doesn't allow such estimation at the vicinity of the boundary in view of the lack of the hypothesis of the Hölder continuity of them. Therefore, we have to look for a way to overcome this difficulty.

For this purpose, we shall present a scheme which adds Rothe's approximation method to the direct method in the calculus of variations. To put it more concretely, we shall introduce a family of functionals whose first variations take form of the difference partial differential equations representing Rothe's approximation to the differential equations prescribing the variational flow under consideration.

Apart from the *routine* treatment, our approach is characteristic in some respects that we shall resort to the minimality of the functionals and that we shall perform the local estimates for the solutions in question. See [5], [6], and [7] in this regard.

These two devices are main tools for our reasoning, and do work effectively. In fact, the first one permits, despite of merely imposing the measurability on the coefficients standing in the principal part of the equations, the energy inequality for those solutions as usual known for the heat flow between two manifolds [1]. The legitimacy of the energy inequality is usually ensured on the basis of the smoothness of the coefficients. For instance, compare Lemma 2 with [1] and [12]. The second one enables us to achieve the higher integrability of the gradients of approximate solutions and to construct weak solutions under the more relaxed initial condition without the Hölder continuity than is involved in [11] and [12], while the requirements on the parabolic constant are both weaker than ours. This leaves us a suggestion of the possibility of some extension of this paper. For more details, see Remark 2.1 and the subsequent paragraph.

To make our explanations clear, we shall start by preparing for it. Let Ω be a bounded domain in $\mathbf{R}^m, m \geq 2$, with Lipschitz boundary $\partial\Omega$. $W^{1,p}(\Omega, \mathbf{R}^M)$,$W_0^{1,p}(\Omega, \mathbf{R}^M)$, $M \geq 1$, denote the usual Sobolev spaces. Particularly, we write $H^1(\Omega, \mathbf{R}^M) = W^{1,2}(\Omega, \mathbf{R}^M)$, $H_0^1(\Omega, \mathbf{R}^M) = W_0^{1,2}(\Omega, \mathbf{R}^M)$.

Let T and B be arbitrary assigned positive numbers. In what follows, we use the notations:

$$Q = (0, T) \times \Omega \quad \text{and} \quad S_B = \{u \in \mathbf{R}^M; |u| < B\}.$$

Throughout this paper, we shall fix u_0 as a mapping belonging to $H^1 \cap L^\infty(\Omega, \mathbf{R}^M)$ with $\|u_0\|_{\infty,\Omega} \leq B$.

For this u_0, we introduce the function spaces:

$$\begin{aligned} H^1_{u_0}(\Omega, \mathbf{R}^M) &= \{u \in H^1(\Omega, \mathbf{R}^M); u - u_0 \in H^1_0(\Omega, \mathbf{R}^M)\}, \\ H^1_{u_0,B}(\Omega, \mathbf{R}^M) &= \{u \in H^1_{u_0}(\Omega, \mathbf{R}^M); \|u\|_{\infty,\Omega} \leq B\}, \end{aligned}$$

which are convex and closed in the space $H^1(\Omega, \mathbf{R}^M)$.

We shall concern ourselves with a study of the existence of the weak solutions to parabolic systems of the form

$$\frac{\partial u^i}{\partial t} = D_\alpha(A^{\alpha\beta}(x,u)D_\beta u^i) - \frac{1}{2}A^{\alpha\beta}_{u^i}(x,u)D_\alpha u^j D_\beta u^j \quad \text{in } Q \qquad (i = 1, 2, \cdots, M) \tag{0.1}$$

with the initial and boundary conditions

$$u = u_0 \quad \text{on } \{0\} \times \Omega \cup (0,T) \times \partial\Omega,$$

where $D_\alpha = \partial/\partial x_\alpha$ $(\alpha = 1, 2, \cdots, m)$, while $A^{\alpha\beta}_{u^i} = \partial A^{\alpha\beta}/\partial u^i$.

Here and elsewhere, the summation convention is used.

The coefficients $\{A^{\alpha\beta}\}$ are assumed to satisfy the conditions:
(1) $A^{\alpha\beta} = A^{\alpha\beta}(x,u) : \Omega \times S_B \to \mathbf{R}$ is measurable in $x \in \Omega$ and continuously differentiable in $u \in S_B$ with $A^{\alpha\beta} = A^{\beta\alpha}$.
(2) There exist positive numbers L, λ and a such that

$$\begin{aligned} &|A^{\alpha\beta}(x,u)| \le L, \\ &A^{\alpha\beta}(x,u)\,\xi^i_\alpha \xi^i_\beta \ge \lambda|\xi|^2, \\ &\frac{1}{2}\left|A^{\alpha\beta}_{u^i}(x,u)\,\xi^j_\alpha \xi^j_\beta\right| \le a|\xi|^2 \quad (i = 1, 2, \cdots M) \end{aligned} \tag{0.2}$$

for almost every $x \in \Omega$ and every $u \in S_B$, and for all $\xi = (\xi^i_\alpha) \in \mathbf{R}^{mM}$.

We take a positive integer N and put

$$h = T/N \quad \text{and} \quad t_n = nh \quad (n = 0, 1, \cdots, N).$$

For our investigation, we shall introduce a family of functionals:

$$F_n(u) = \frac{1}{h}\int_\Omega |u - u_{n-1}|^2\,dx + F(u) \quad \text{in } H^1_{u_0,B}(\Omega, \mathbf{R}^M) \quad (n = 1, 2, \cdots N), \tag{0.3}$$

where

$$F(u) = \int_\Omega A^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^i\,dx.$$

The Euler-Lagrange operator of $F(u)$ is the right-hand side of (0.1).

Beginning with the initial data u_0, we shall inductively determine u_n as a minimizer of the functional F_n in $H^1_{u_0,B}(\Omega, \mathbf{R}^M)$, the existence of which is assured by the sequentially lower semi-continuity of F_n with respect to the weak convergence and the coercivity of F_n in $H^1_{u_0}(\Omega, \mathbf{R}^M)$ and, in addition, $H^1_{u_0,B}(\Omega, \mathbf{R}^M)$ being convex and closed in $H^1(\Omega, \mathbf{R}^M)$.

Henceforce, we shall use the compact expression for A, there being not any confusion likely to arise.

The minimizer u_n of F_n in $H^1_{u_0,B}(\Omega, \mathbf{R}^M)$ will satisfy the Euler-Lagrange equations of functional F_n:

$$\begin{aligned}&\int_\Omega \Big(\frac{1}{h}(u_n - u_{n-1})\varphi + A(x, u_n)\,(Du_n, D\varphi)\Big)\,dx \\ &\qquad = -\frac{1}{2}\int_\Omega \varphi A_u(x, u_n)\,(Du_n, Du_n)\,dx \quad (n = 1, 2, \cdots, N)\end{aligned} \tag{0.4}$$

for any $\varphi \in L^\infty \cap H^1_0(\Omega, \mathbf{R}^M)$ such that $u_n \pm \varepsilon\varphi \in S_B$ for sufficiently small $\varepsilon > 0$.

By an *approximate solution to system* (0.1) *with the initial and boundary data* u_0, we will mean an *h-step* mapping $u_{(h)}(t) \in H^1_{u_0}(\Omega, \mathbf{R}^M)$, $0 \le t \le T$, connected by the relations

$$u_{(h)}(0) = u_0, \quad u_{(h)}(t) = u_n \quad \text{for} \quad t_{n-1} < t \le t_n \quad (n = 1, 2, \cdots, N), \tag{0.5}$$

where u_n is a minimizer of functional F_n in $H^1_{u_0,B}(\Omega, \mathbf{R}^M)$.

In this response, we shall define a Cauchy-Euler polygon

$$u^*_{(h)}(t) = \frac{t_n - t}{h}\, u_{n-1} + \frac{t - t_{n-1}}{h}\, u_n \quad \text{for} \quad t_{n-1} \le t \le t_n \ (n = 1, 2, \cdots, N)$$

and a mapping shifted backward in t

$$\widetilde{u}_{(h)}(t) = u_{(h)}(t - h).$$

This way, identity (0.4), which is of discrete form, is representable in the continuous expression:

$$\begin{aligned}&\iint_Q \Big(\frac{\partial u^*_{(h)}}{\partial t}\varphi_{(h)} + A(x, u_{(h)})\,(Du_{(h)}, D\varphi_{(h)})\Big)\,dt dx \\ &\qquad = -\frac{1}{2}\iint_Q \varphi_{(h)} A_u(x, u_{(h)})\,(Du_{(h)}, Du_{(h)})\,dt dx\end{aligned} \tag{0.6}$$

for any *h-step* mapping $\varphi_{(h)}$ generated by $\varphi_n \in L^\infty \cap H^1_0(\Omega, \mathbf{R}^M)$ in the same manner as in (0.5) such that $u_{(h)} \pm \varepsilon\varphi_{(h)} \in S_B$ for small enough $\varepsilon > 0$.

To proceed our scheme, it is required that two-sided conditions

$$\lambda > 2aB \tag{0.7}$$

be placed on the elliptic constant λ from (0.2), where a is a positive number from (0.2), while B is such that $\|u_0\|_{\infty,\Omega} < B$.

Throughout this paper, the same letter C will be used to denote different constants depending on the same set of arguments, particularly not on h.

1. Main result

We shall give the definition of a weak solution to system (0.2) with the initial and boundary data u_0.

By a *weak solution u to system* (0.1) *with the initial and boundary data* u_0, we will mean a mapping

$$u \in L^\infty(0,T,H^1_{u_0}(\Omega,\mathbf{R}^M)) \cap H^1(0,T,L^2(\Omega,\mathbf{R}^M))$$

which satisfies

$$\iint_Q \Big(\frac{\partial u}{\partial t}\varphi + A(x,u)\,(Du,D\varphi)\Big)\,dtdx = -\frac{1}{2}\iint_Q \varphi A_u(x,u)\,(Du,Du)\,dtdx \tag{1.1}$$

for every $\varphi \in L^\infty(Q,\mathbf{R}^M) \cap L^2(0,T,H^1_0(\Omega,\mathbf{R}^M))$,
and

$$\lim_{\substack{t\to 0\\ t>0}} u(t) = u_0 \quad \text{in } L^2(\Omega,\mathbf{R}^M).$$

We are now in a position to state the main result.

Theorem. *Suppose $\lambda > 2aB$ as in* (0.7) *and let u_0 be a mapping belonging to $L^\infty \cap H^1(\Omega,\mathbf{R}^M)$ with the trace $u_0|_{\partial\Omega} = \hat{u}_0|_{\partial\Omega}$ for some mapping $\hat{u}_0 \in W^{1,p_0}(\Omega,\mathbf{R}^M)$ with some $p_0 > 2$. Then, there exists at least one weak solution to system* (0.1) *with the initial and boundary data u_0. Furthermore, this solution becomes of class $C^\alpha(Q;\delta)$ for some $0 < \alpha < 1$, where δ is the parabolic metric*:

$$\delta(z_1,z_2) = \max\{|t_1-t_2|^{1/2}, |x_1-x_2|\}, \quad z_i = (t_i,x_i) \ \ (i=1,2).$$

2. Preliminaries

In this section we shall derive some auxiliary lemmata.

Based on the maximum principle for difference equations which is a modification of Lemma 4 in [8], one can provide the following assertion:

Lemma 1 (Euler-Lagrange equations). *Suppose the so-called one-sided condition*:

$$-\frac{1}{2}uA_u(x,u)(\xi,\xi) \leq \lambda^*|\xi|^2 \quad \textit{with} \quad \lambda > \lambda^*, \tag{2.1}$$

for almost every $x \in \Omega$ and every $u \in S_B$, and for all $\xi \in \mathbf{R}^{mM}$. And let $u_{(h)}$ be an approximate solution to (0.1) *with the initial and boundary data u_0, which is generated by minimizers $\{u_n\}$ of $\{F_n\}$ in $H^1_{u_0}(\Omega,\mathbf{R}^M)$. Then there is valid the relation*

$$\iint_Q \Big(\frac{\partial u^*_{(h)}}{\partial t}\varphi_{(h)} + A(x,u_{(h)})\,(Du_{(h)}, D\varphi_{(h)})\Big)\,dtdx$$
$$= -\frac{1}{2}\iint_Q \varphi_{(h)}A_u(x,u)\,(Du_{(h)}, Du_{(h)})\,dtdx$$

for any h-step mapping $\varphi_{(h)}$ generated by $\varphi_n \in L^\infty \cap H^1_0(\Omega,\mathbf{R}^M))$ in the same fashion as in (0.5).

Proof. Since

$$u_n - \varepsilon\eta u_n = (1-\varepsilon\eta)u_n \in S_B$$

for every non-negative $\eta \in C_0^\infty(\Omega,\mathbf{R}^1)$, and for all sufficiently small $\varepsilon > 0$, it follows that

$$\begin{aligned}
0 &\leq \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{F_n(u_n-\varepsilon\eta u_n) - F_n(u_n)}{\varepsilon} \\
&= -\int\limits_Q \Big(\eta\,\frac{1}{h}\,(u_n-u_{n-1})\,u_n + A(x,u_n)\,(Du_n, D\eta u_n) \\
&\qquad + \eta A(x,u_n)\,(Du_n,Du_n) + \frac{1}{2}\,\eta u_n A_u(x,u_n)\,(Du_n,Du_n)\Big)\,dx \\
&\leq -\int\limits_\Omega \Big(\eta\,\frac{1}{2h}\,(|u_n|^2 - |u_{n-1}|^2) + \frac{1}{2}A(x,u_n)\,(D|u_n|^2, D\eta) + \\
&\quad + (\lambda-\lambda^*)\eta|Du_n|^2\Big)\,dx,
\end{aligned}$$

the first and the last inequality arising from the minimality of F_n and assumption (2.1) respectively. Whereupon,

$$\int_Q \left(\eta \frac{1}{h} (v_n - v_{n-1}) + A(x, u_n) (Dv_n, D\eta) \right) dx \le 0 \quad \text{with} \quad v_n = |u_n|^2, \tag{2.2}$$

for any non-negative $\eta \in C_0^\infty(\Omega, \mathbf{R}^1)$. Inserting into (2.2) as a test function η

$$v_n^{(k)} = \max(v_n - k, 0), \quad |u_0|_\infty^2 \le k < B,$$

and taking a summation with respect to n from 1 to l, we find

$$\int_\Omega |v_l^{(k)}|^2 \, dx \le \int_\Omega |v_0^{(k)}|^2 \, dx = 0 \quad (l = 1, 2, \cdots, N).$$

Immediately,

$$|u_{(h)}|_\infty \le |u_0|_\infty < B. \tag{2.3}$$

Hence, we get

$$u_n \pm \varepsilon\varphi \in S_B$$

for every $\varphi \in L^\infty \cap H_0^1(\Omega, \mathbf{R}^1)$, and for all small enough ε. Eventually, we arrive at the relation

$$\frac{d}{d\varepsilon} F_n(u_n + \varepsilon\varphi) \Big|_{\varepsilon = 0} = 0,$$

as required.

The following is a consequence of directly working with the functional F_n from (0.3), which is strictly related to the minimality of the functional.

Lemma 2 (Energy inequality). *Let $u_{(h)}$ be an approximate solution to (0.1) with the initial and boundary data u_0, which is generated by minimizers $\{u_n\}$ of $\{F_n\}$ in $H^1_{u_0,B}(\Omega, \mathbf{R}^M)$. Then there exists a positive absolute constant C such that*

$$\sup_{t \in (0,T)} \int_\Omega (|u^*_{(h)}(t)| + |Du^*_{(h)}(t)|)^2 \, dx \le C \int_\Omega (|u_0| + |Du_0|)^2 \, dx,$$

and

$$\iint_Q \left| \frac{\partial u^*_{(h)}}{\partial t} \right|^2 dt dx \le C \int_\Omega |Du_0|^2 \, dx.$$

That is, the family of $\{u^*_{(h)}\}_{h>0}$ *is bounded in* $L^\infty(0,T,H^1_{u_0}(\Omega,\mathbf{R}^M))\cap H^1(0,T,L^2(\Omega,\mathbf{R}^M))$.

Proof. Upon comparing u_{n-1} with a minimizer u_n of F_n in $H^1_{u_0,B}(\Omega,\mathbf{R}^M)$, we infer

$$\int_\Omega \Big(\frac{1}{h}\,|u_n-u_{n-1}|^2+A(x,u_n)\,(Du_n,Du_n)\Big)\,dx\le$$
$$\le\int_\Omega A(x,u_{n-1})\,(Du_{n-1},Du_{n-1})\,dx,$$

from which

$$\begin{aligned}\int_\Omega A(x,u_n)\,(Du_n,Du_n)\,dx &\le \int_\Omega A(x,u_{n-1})\,(Du_{n-1},Du_{n-1})\,dx\\ &\le \int_\Omega A(x,u_0)\,(Du_0,Du_0)\,dx\end{aligned}$$

and

$$\begin{aligned}&\sum_{n=1}^N h\int_\Omega \Big|\frac{u_n-u_{n-1}}{h}\Big|^2\,dx\\ &\le \sum_{n=1}^N\int_\Omega \big(A(x,u_{n-1})\,(Du_{n-1},Du_{n-1})-A(x,u_n)\,(Du_n,Du_n)\big)\,dx\\ &\le \int_\Omega A(x,u_0)\,(Du_0,Du_0)\,dx.\end{aligned}$$

Namely,

$$\sup_{t\in(0,T)}\int_\Omega |Du^*_{(h)}(t)|^2\,dx\le C\int_\Omega |Du_0|^2\,dx,$$

and

$$\int_\Omega \Big|\frac{\partial u^*_{(h)}}{\partial t}\Big|^2\,dtdx\le C\int_\Omega |Du_0|^2\,dx.$$

Poincare's inequality yields

$$\sup_{t\in(0,T)}\int_\Omega |u^*_{(h)}(t)|^2\,dx \le 2\sup_{t\in(0,T)}\int_\Omega (|u_0|^2+|u^*_{(h)}(t)-u_0|^2)\,dx$$
$$\le C\sup_{t\in(0,T)}\int_\Omega (|u_0|^2+|Du_{(h)}(t)|^2+|D\widetilde{u}_{(h)}(t)|^2)\,dx$$
$$\le C\int_\Omega (|u_0|+|Du_0|)^2\,dx.$$

Summing up these results, we can draw a desirable conclusion.

Remark 2.1. The coefficients $A^{\alpha\beta}(x)=A^{\alpha\beta}(x,u(x))$ are measurable in $x\in\Omega$; nevertheless we have achieved Lemma 2. This assertion can be assured thanks to the minimality of the functionals (0.3). On the other hand, as to the heat flows between two manifolds M and N, as well known, analogous estimates (for example, see [1]) can be derived owing to the smoothness of the coefficients, say, the metric $\gamma^{\alpha\beta}(x)$ of M in this situation, whose legitimation follows with the aid of the linear theory [Theorem 9.1, 8] for parabolic differential equations. Furthermore, for our situation, one usually needs the a-priori Hölder estimations for the solutions in question if one wants the a-priori energy estimates, because [Theorem 9.1, 8] is the estimation affecting on the modulus of the continuity of $A^{\alpha\beta}(x)=A^{\alpha\beta}(x,u(x))$. See [12]. But, such a-priori Hölder estimations will break down in a neighbourhood of the boundary, unless the initial and boundary data possess the Hölder continuity, corresponding to which so will do the a-priori energy estimates. However, our procedure will be applicable to it without the Hölder continuity of it and even also without the smoothness of $A^{\alpha\beta}(x,u)$ with respect to $x\in\Omega$ and $u\in\mathbf{R}^M$. In general, one might not perform local estimates of solutions in dealing with difference partial differential equations. But, we have encountered the need to do so in [5]. And this enables us to construct weak solutions to system (0.1), assuming the weaker requirement on the initial condition than is the Hölder regularity (refer to [11] and [12]). To attain our object, local estimations done in [5] are indispensable for our reasoning.

To be explicit, we now recall the result in [5].

Maintain the same conditions as in Theorem. Let $\hat{\hat{u}}_0$ be a weak solution to $\Delta\hat{\hat{u}}_0=0$ in Ω, and $\hat{\hat{u}}_0=\hat{u}_0$ on $\partial\Omega$. Then, for any $\varepsilon>0$, there exist positive constants C and ε_0 such that, for every $h > 0$, the estimate

$$
\begin{aligned}
&\Big(\iint_{Q_R(z_0)} |Du_h|^p \, dtdx\Big)^{2/p} \le \\
&\le C_\varepsilon \iint_{Q_{2R}(z_0)} |Du_h|^2 \, dtdx + C\varepsilon \iint_{Q_{2R}(z_0)} |D\widetilde{u}_h|^2 \, dtdx \\
&\quad + C_\varepsilon \Big(\int_{\Omega_{2R}(x_0)} |D\hat{\hat{u}}_0|^p \, dx\Big)^{2/p}
\end{aligned}
$$

is fulfilled for any $z_0 = (t_{n_0}, x_0) \in \overline{Q}$ $(n_0 = 1, 2, \cdots, N)$, $0 < R <$ diam Q and $2 \le p < \min\{2 + \varepsilon_0, p_0\}$, where C_ε is a positive number approaching infinity as ε goes to zero.

In it, we have used the notations

$$
\Omega_R(x_0) = \Omega \cap \{x \in R^m; |x - x_0| < R\},
$$
$$
Q_R(z_0) = \{t \in (0, T); t_{n_0} - R^2 < t < t_{n_0}\} \times \Omega_R(x_0).
$$

The symbol $\int_A$ stands for an averaging of the integration over the domain A.

A direct consequence of this is

Lemma 3. *Let $u_{(h)}$ be an approximate solution to* (0.1) *with the initial and boundary data u_0. Suppose $\lambda > 2aB$ and $u_0 \in L^\infty \cap H^1(\Omega, \mathbf{R}^M)$ with the trace $u_0|_{\partial\Omega} = \hat{u}_0|_{\partial\Omega}$ for some mapping $\hat{u}_0 \in W^{1,p_0}(\Omega, \mathbf{R}^M)$ with some $p_0 > 2$. Then there exist positive numbers C, ε_0 depending on $a, m, T, B, \lambda, \Omega$ and p_0 such that*

$$
\iint_Q |Du_{(h)}|^p \, dtdx \le C \int_\Omega |D\hat{u}_0|^p \, dx
$$

holds for every $h > 0$ and for all $p \in [\,2, \min\{2 + \varepsilon, p_0\}\,)$.

Remark 2.2. Except for Lemma 3, our argument remains available for the case of the one-sided condition.

3. Proof of Theorem

Proof. By virtue of Lemma 2, we can select a subsequence $\{u_{(h_j)}\}$ with a suitable zero sequence $\{h_j\}$, and find a limit mapping $u \in$

$L^\infty(0,T,H^1_{u_0}(\Omega,\mathbf{R}^M))\cap H^1(0,T,L^2(\Omega,\ \mathbf{R}^M))$ such that

$$\begin{aligned}\lim_{j\to\infty} u^*_{(h_j)} &= u \quad \text{weakly}^* \ \text{ in } \ L^\infty(0,T,H^1_{u_0}(\Omega,\mathbf{R}^M)),\\ \lim_{j\to\infty} u^*_{(h_j)} &= u \quad \text{weakly in } \ H^1(0,T,L^2(\Omega,\mathbf{R}^M))\end{aligned} \tag{3.1}$$

and hence, due to Rellich's theorem,

$$\lim_{j\to\infty} u^*_{(h_j)} = u \quad \text{strongly in } L^2(Q,\mathbf{R}^M). \tag{3.2}$$

By means of Lemma 2, we readily see

$$\begin{aligned}&\iint_Q |u_{(h)}-u^*_{(h)}|^2\,dtdx \le \iint_Q |u_{(h)}-\widetilde{u}_{(h)}|^2\,dtdx\\ &\le\ h^2\iint_Q \Big|\frac{\partial u^*_{(h)}}{\partial t}\Big|^2\,dtdx \le Ch^2\int_\Omega |Du_0|^2\,dx,\end{aligned} \tag{3.3}$$

which, together with the property (3.2), implies

$$\lim_{j\to\infty} u_{(h_j)} = u \quad \text{strongly in } L^2(Q,\mathbf{R}^M). \tag{3.4}$$

We carry out an estimation of the following identity by dividing it into three parts:

$$\begin{aligned}&\iint_Q A(x,u_{(h_j)})\,(D(u_{(h_j)}-u),D(u_{(h_j)}-u))\,dtdx \qquad (3.5)\\ =&\iint_Q A(x,u_{(h_j)})\,(Du_{(h_j)},D(u_{(h_j)}-u))\,dtdx\\ &-\iint_Q A(x,u)\,(Du,D(u_{(h_j)}-u))\,dtdx\\ &+\iint_Q (A(x,u)-A(x,u_{(h_j)}))\,(Du,D(u_{(h_j)}-u))\,dtdx\\ =:&\ I_j+\mathit{II}_j+\mathit{III}_j\,.\end{aligned}$$

In light of Lemma 1, we can transform the expression I_j into a more convenient form

$$I_j = -\iint\Big\{\frac{\partial u^*_{(h_j)}}{\partial t}(u_{(h_j)}-u)+\frac{1}{2}(u_{(h_j)}-u)A_u(x,u_{(h_j)})(Du_{(h_j)},Du_{(h_j)})\Big\}dtdx.$$

We take advantage of Lemma 2 and 3 to deduce

$$\begin{aligned}|I_j| &\le \Big(\iint_Q \Big|\frac{\partial u^*_{(h_j)}}{\partial t}\Big|^2\,dtdx\Big)^{1/2}\Big(\iint_Q |u_{(h_j)}-u|^2\,dtdx\Big)^{1/2}\\ &\quad + C\Big(\iint_Q |Du_{(h_j)}|^p\,dtdx\Big)^{2/p}\Big(\iint_Q |u_{(h_j)}-u|^{p/(p-2)}\,dtdx\Big)^{1-2/p}\\ &\le C\Big(\int_\Omega |Du_0|^2\,dx\Big)^{1/2}\Big(\iint_Q |u_{(h_j)}-u|^2\,dtdx\Big)^{1/2}\\ &\quad + C\Big(\int_\Omega |D\hat{u}_0|^p\,dx\Big)^{2/p}\Big(\iint_Q |u_{(h_j)}-u|^2\,dtdx\Big)^{1-2/p},\end{aligned}$$

where p is an index from Lemma 3, while we have utilized the a-priori boundedness of $u_{(h_j)}$ from (2.3) in Lemma 1. Thereby, it turns out that

$$\lim_{j\to\infty} I_j = 0,$$

because of (3.4). The property (3.1) with (3.3) directly admits

$$\lim_{j\to\infty} II_j = 0,$$

according to the very definition regarding the weak convergence. Via Young's inequality, we are led to the inequality

$$|III_j| \le \frac{\lambda}{2}\iint_Q |D(u_{(h_j)}-u)|^2\,dtdx + IV_j, \tag{3.6}$$

where

$$IV_j = C\iint_Q |A(x,u)-A(x,u_{(h_j)})|^2|Du|^2\,dtdx.$$

The first term of the right hand side of (3.6) can be absorbed into the left-hand side of (3.5) on account of the parabolicity (0.2), while it follows that

$$\lim_{j\to\infty} IV_j = 0,$$

in view of (3.4) and Lebesgue's dominated convergence theorem. Substituting (3.6) into (3.5), we accomplish the inequality

$$\iint_Q |D(u_{(h_j)}-u)|^2\,dtdx \le C(|I_j|+|II_j|+IV_j).$$

After thus, summarize these considerations, and this brings us the relation

$$\lim_{j\to\infty} u_{(h_j)} = u \quad \text{in } L^2(0,T,H^1_{u_0}(\Omega,\mathbf{R}^M)).$$

Consequently, letting j tend to infinity in the relation (0.6) in place of h with h_j and in place of $\varphi_{(h)}$ with

$$\varphi_{(h)}(t) = \frac{1}{h}\int_{t_{n-1}}^{t_n} \varphi(t)\,dt \quad \text{for} \quad t_{n-1} < t \le t_n \ \ (n = 1,\cdots,N)$$

for $\varphi \in L^\infty(Q,\mathbf{R}^M)\cap L^2(0,T,H^1_0(\Omega,\mathbf{R}^M))$, we reach the conclusion that u satisfies the relation (1.1).

It remains for us to verify that there shall be well-defined the initial and boundary conditions. An inspection of the fact $u_{(h_j)} \in H^1_{u_0}(\Omega,\mathbf{R}^M)$ apparently shows $u \in H^1_{u_0}(\Omega,\mathbf{R}^M)$ through (3.1) and (3.2). On the other hand, we have, with aid of Lemma 2,

$$\|u^*_{(h)}(s)-u^*_{(h)}(t)\|_{L^2(\Omega)} \le \sqrt{|s-t|}\,\left\|\frac{\partial u^*_{(h)}}{\partial t}\right\|_{L^2(Q)} \le C\sqrt{|s-t|}\,\|Du_0\|_{L^2(\Omega)} \tag{3.7}$$

for any s and $t\in[0,T]$. Letting j tend to infinity in (3.7) with h_j in place of h, this inequality will take the form

$$\|u(s)-u(t)\|_{L^2(\Omega)} \le C\sqrt{|s-t|}\,\|Du_0\|_{L^2(\Omega)}$$

for almost all $s,t\in(0,T)$, owing to (3.1). As a result, the limit function u is equivalent in Q to a function that is continuous in all $t\in[0,T]$ in the norm of $L^2(\Omega)$, which we call u again. In particular, one gains

$$\lim_{\substack{t\to 0\\ t>0}} u(t) = u_0 \quad \text{in } L^2(\Omega),$$

as desired.

We shall close the paper by remarking that the property $u\in C^\alpha(Q,\delta)$ is derived from the observation

$$\liminf_{\substack{\rho\to 0\\ \rho>0}} \rho^{-m} \iint_{Q^\sigma_\rho(t,x)} |Du|^2\,dtdx = 0 \tag{3.8}$$

for any $0<\sigma<1$, and any (t,x) not lying on the parabolic boundary of Q, where

$$Q^\sigma_\rho(t,x) = \{s\in(0,T); t-\rho^2 < s < t-\sigma^2\rho^2\}\times B_\rho(x).$$

There shall be retained the validity of relation (3.8) from the one-sided condition, especially from the two-sided condition. In this regard, refer to the contributions [3, 4, 10] and [12].

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Haga J., Kikuchi N. On the existence of the harmonic variational flow subject to the two-sided conditions.

We shall devote ourselves to a study of constructing the weak solutions to parabolic systems of the variational flow type associated with the quadratic functional with the initial and boundary data from a suitable Sobolev space, subject to the two-sided conditions. In order to do this, we shall present an approach made by extending Rothe's method which is useful for solving the Cauchy problem and the initial- boundary-value problems for many sorts of equations.

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